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Czelakowski introduced the Suszko operator as a basis for the development of a hierarchy of non-protoalgebraic logics, paralleling the well-known abstract algebraic hierarchy of protoalgebraic logics based on the Leibniz operator of Blok and Pigozzi. The scope of the theory of the Leibniz operator was recently extended to cover the case of, the so-called, protoalgebraic π-institutions. In the present work, following the lead of Czelakowski, an attempt is made at lifting parts of the theory of the Suszko operator to the π-institution framework.

1 Introduction

In [3], Blok and Pigozzi, following work of Czelakowski [5] and their own previous work [2], introduced, for the first time, the notion of an algebraizable logic in an attempt to capture the class of those finitary logics whose properties are very intimately related to corresponding properties of appropriately chosen classes of algebras. In their efforts to give an intrinsic characterization of the class of algebraizable logics, they were led to the introduction of the Leibniz operator, an operator that assigns to every theory of the logic the largest congruence on the formula algebra that is compatible with the theory. In one of the main theorems of [3], Blok and Pigozzi showed that a finitary sentential logic is algebraizable if and only if the Leibniz operator is injective and continuous on the theories of the logic (see also [6, Theorem 4.6.2]).

Many algebraic logicians have subsequently followed Blok and Pigozzi's footsteps and, as a result, the Leibniz operator has been very heavily studied. These studies led to the establishment of an algebraic hierarchy of logics, based on properties of the Leibniz operator that may or may not be satisfied by particular logics. Justifiably so, this hierarchy is sometimes called the Leibniz hierarchy. It consists, mainly, of the classes of protoalgebraic logics [2], equivalential logics [18, 5], weakly algebraizable logics [8], and algebraizable logics [3, 14, 15, 16]. The following diagram depicts the inclusion relationships between these four major classes.

$$\begin{tikzcd}
 & \text{algebraizable} \arrow[dl] \arrow[dr] & \\
\text{equivalential} \arrow[dr] & & \text{weakly algebraizable} \arrow[dl] \\
 & \text{protoalgebraic} &
\end{tikzcd}$$

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At the very bottom of the hierarchy is the class of protoalgebraic logics that is characterized by the monotonicity of the Leibniz operator on the theories of the logics in the class. It has been widely believed to be the largest class consisting of sentential logics that are amenable to universal algebraic study techniques.

Since the success, in terms of both classifying power and wide applicability, of the theory of algebraizability of sentential logics based on the Leibniz operator has been broadly acknowledged, Czelakowski initiated in [7] an effort to use an alternative operator to the Leibniz operator in order to be able to classify, in a similar way, non-protoalgebraic logics. These logics are outside the scope of the traditional Leibniz hierarchy. Besides the desire to create a theory that would comprehend all sentential logics, the other important requirement that Czelakowski imposed on the new operator is that the resulting theory must coincide with the old one when relativized to the class of protoalgebraic logics. It turns out that the operator that satisfies to a large extent these requirements is the Suszko operator. The Suszko operator Σ^C of a sentential logic $\mathcal{S} = \langle \mathcal{L}, C \rangle$ associates with a given theory T of the logic the largest congruence $\Sigma^C(T)$ on the formula algebra of the logic that is compatible with every theory T' of $\mathcal{S}$ that includes T. Equivalently, $\Sigma^C(T) = \bigcap\{\Omega(T') \,:\, T' \in \mathrm{Th}(\mathcal{S}) \text{ such that } T \subseteq T'\}$, where $\Omega(T')$ denotes, as usual, the Leibniz congruence associated with the theory T' of $\mathcal{S}$. In [7, Theorem 1.8], Czelakowski characterizes the Suszko operator as being the largest operator from theories to congruences that is monotone and compatible with the theories of the logic. And in [7, Theorem 1.10] he shows that, indeed, when the logic $\mathcal{S}$ is protoalgebraic, the Suszko operator coincides with the Leibniz operator of Blok and Pigozzi and, as a consequence, the two theories provide identical results when classifying protoalgebraic logics. Czelakowski studies a wide variety of properties concerning the Suszko operator and promises a second installment to appear exploring even more aspects of the theory.

The theory of the Leibniz operator has recently been abstracted by the author [23] to cover logics that are formalized as π-institutions. A detailed account of the work that led to these developments, together with some subsequent related investigations, is given in the series of papers [19] – [25], some of which contain abstractions of the classes of the Leibniz hierarchy, discussed above, to corresponding classes of π-institutions. In [23], an analog of the Suszko operator of Czelakowski was introduced at the π-institution level and it was shown that a π-institution is protoalgebraic if and only if this categorical version of the Suszko operator coincides with a similarly abstracted version of the Leibniz operator. This fact, understood in the context of Czelakowski's work, triggers the idea that the categorical operator may be appropriate, in this context as well, for extending a hierarchy of protoalgebraic π-institutions to the non-protoalgebraic ones.

Motivated by the ideas and by the work of Czelakowski in tackling the algebraizability status of the non-protoalgebraic logics using the Suszko operator, and in anticipation of the second part of his studies on the Suszko operator, an exposition is provided in this paper of some of the most elementary properties of the categorical analog of the Suszko operator. We follow [7] and provide analogs at the level of π-institutions of many of the properties that were shown to hold in the sentential logic case.

In the remainder of this section, a brief overview of the contents of the present paper will be provided.

Recall that, given a category $\mathbf{Sign}$ and a functor SEN : $\mathbf{Sign} \longrightarrow \mathbf{Set}$, the clone of all natural transformations on SEN is defined to be the locally small category with collection of objects $\{\mathrm{SEN}^\alpha \,:\, \alpha \text{ an ordinal}\}$ and collection of morphisms $\tau : \mathrm{SEN}^\alpha \longrightarrow \mathrm{SEN}^\beta$ β-sequences of natural transformations $\tau_i : \mathrm{SEN}^\alpha \longrightarrow \mathrm{SEN}$ [23]. Composition

$$\mathrm{SEN}^\alpha \xrightarrow{\langle \tau_i \,:\, i < \beta\rangle} \mathrm{SEN}^\beta \xrightarrow{\langle \sigma_j \,:\, j < \gamma\rangle} \mathrm{SEN}^\gamma$$

is defined by

$$\langle \sigma_j \,:\, j < \gamma\rangle \circ \langle \tau_i \,:\, i < \beta\rangle = \langle \sigma_j(\langle \tau_i \,:\, i < \beta\rangle) \,:\, j < \gamma\rangle.$$

A subcategory N of this category whose objects are all objects of the form SEN^k for $k < \omega$ that contains all projection morphisms $p^{k,i} : \mathrm{SEN}^k \longrightarrow \mathrm{SEN}$, $i < k$, $k < \omega$, with $p^{k,i}_\Sigma : \mathrm{SEN}(\Sigma)^k \longrightarrow \mathrm{SEN}(\Sigma)$ given by

$$p^{k,i}_\Sigma(\vec{\varphi}) = \varphi_i, \quad \text{for all } \vec{\varphi} \in \mathrm{SEN}(\Sigma)^k,$$

and is such that, for every family $\{\tau_i : \mathrm{SEN}^k \longrightarrow \mathrm{SEN} \,:\, i < l\}$ of natural transformations in N, the sequence

$$\langle \tau_i \,:\, i < l\rangle : \mathrm{SEN}^k \longrightarrow \mathrm{SEN}^l$$

is also in N, is referred to as a *category of natural transformations on* SEN.

Suppose $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution and N a category of natural transformations on SEN. Given a theory family $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ of $\mathcal{I}$, the Suszko N-congruence system $\Theta^N(T)$ associated with T is defined, for all $\Sigma \in |\mathbf{Sign}|$, by setting, for all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle\varphi,\psi\rangle \in \Theta^N_\Sigma(T)$ if and only if

$$C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi})\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})\}), \tag{1}$$

for all $\sigma : \mathrm{SEN}^k \longrightarrow \mathrm{SEN}$ in N, all $\Sigma' \in |\mathbf{Sign}|$, all $f \in \mathbf{Sign}(\Sigma,\Sigma')$, and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$, where the understanding is that $\mathrm{SEN}(f)(\varphi)$ and $\mathrm{SEN}(f)(\psi)$ may appear in any position of σ, but that they must appear in the same position in both sides of the equation (1). This convention has been followed repeatedly in similar contexts in categorical abstract algebraic logic because it helps simplify the notation considerably.

$\Theta^N(T)$ is an N-congruence system of $\mathcal{I}$ and is compatible with the theory family T, whence we immediately have that $\Theta^N(T) \leq \Omega^N(T)$. In fact, it is shown in Proposition 2.1 that $\Theta^N(T) = \bigcap_{T\leq T'} \Omega^N(T')$, where the intersection is taken over all theory families of $\mathcal{I}$ $\leq$-including T. Moreover, it may be shown that $\Theta^N(T) = \Omega^N(T)$ if $\mathcal{I}$ is an N-protoalgebraic π-institution, i. e., the Suszko and the Leibniz operators coincide in the case of N-protoalgebraic π-institutions, as happens in the framework of sentential logics.

In Section 3, the notion of a deductive semi-interpretation is introduced between π-institutions, following the corresponding notion between logical matrices of [7]. The same notion is also defined in a slightly different way for what are called matrix system models of a given π-institution. It is shown in Proposition 3.4 that kernels of deductive logical morphisms roughly correspond to logical N-congruence systems, as defined in [19]. The bilogical morphisms of [19] are also special cases of deductive logical morphisms, whence the canonical projection morphism from a π-institution $\mathcal{I}^T$ to the Suszko-reduction $\mathcal{I}^T/\Theta^N(T)$ is a deductive logical morphism (actually a bilogical morphism). Similar results are also formulated in the context of matrix system models of π-institutions. In Proposition 3.8 an analog of the Correspondence Property for Deductive Homomorphisms of sentential logics is shown to hold for π-institutions. More precisely, given π-institutions

$$\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle, \quad \mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle,$$

with N and N' categories of natural transformations on SEN and SEN$'$, respectively, and $\langle F,\alpha\rangle : \mathcal{I} \vdash^{\mathrm{se}} \mathcal{I}'$ a deductive (N,N')-logical morphism, $\alpha_\Sigma^{-1}(\alpha_\Sigma(T)) = T$, for every Σ-theory T of $\mathcal{I}$, $\Sigma \in |\mathbf{Sign}|$. Based on this result, it is proven in Corollary 3.12 that if $\langle F,\alpha\rangle : \mathcal{I} \vdash^{\mathrm{se}} \mathcal{I}'$ is an (N,N')-bilogical morphism, then, for all theory families T' of $\mathcal{I}'$, $\Theta^N(\alpha^{-1}(T')) = \alpha^{-1}(\Theta^{N'}(T'))$. This result is used to prove a transfer property, stating, roughly speaking, that the continuity of the Suszko operator on a given π-institution is preserved by every bilogical morphism. This is the content of Proposition 3.14.

In Section 4, the notions of a Suszko reduced π-institution with respect to a given theory family and that of a Suszko reduced matrix system model of a given π-institution are introduced. It is shown that a π-institution $\mathcal{I}$ is Suszko reduced with respect to given theory system T if and only if the π-institution $\mathcal{I}^T$ is isomorphic to a quotient π-institution of the form $\mathcal{I}'^{T'}/\Theta^{N'}(T')$, for some π-institution $\mathcal{I}'$ and some theory system T' of $\mathcal{I}'$.

In Section 5, the investigations on properties of the Suszko operator are continued with the introduction of some categorical analogs of the determinator systems of Czelakowski in the π-institution framework.

Finally, in Section 6, an analog of [7, Proposition 5.1] asserting that every countable Suszko-reduced model of a given sentential logic is a strict homomorphic image of the Suszko reduction of a Lindenbaum matrix is provided in the categorical framework.

For all unexplained categorical terminology and notation the reader is referred to any of [1, 4, 17]. For the definitions pertaining to institutions see [12, 13], whereas π-institutions were introduced in [9]. For background on the theory of abstract algebraic logic and discussion of the classes of the abstract algebraic hierarchy, some of which were mentioned in this introduction, the reader is referred to the review article [11], the monograph [10], and the comprehensive treatise [6].

2 The categorical Suszko operator

Before starting the exposition, a notational clarification is in order: Czelakowski [6, 7] uses the symbol Σ (as was also done, following his example, in the introduction) to denote the Suszko operator. Since Σ is heavily used in the present context (and, more generally, in institution theory) to denote an arbitrary signature in the category $\mathbf{Sign}$

of signatures of a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, the letter Θ, rather than the letter Σ, will be used to denote the institutional operator corresponding to the Suszko operator of Czelakowski in the present work.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, let N be a category of natural transformations on SEN, and let $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ be a theory family of $\mathcal{I}$. Define the family of binary relations

$$\Theta^N(T) = \{\Theta^N_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|}$$

by letting, for all $\Sigma \in |\mathbf{Sign}|$, $\varphi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle\varphi, \psi\rangle \in \Theta^N_\Sigma(T)$ if and only if

$$\begin{aligned}(2)\qquad & C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\chi_0, \ldots, \chi_{i-1}, \mathrm{SEN}(f)(\varphi), \chi_{i+1}, \ldots, \chi_{k-1})\}) \\ &\quad = C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\chi_0, \ldots, \chi_{i-1}, \mathrm{SEN}(f)(\psi), \chi_{i+1}, \ldots, \chi_{k-1})\}),\end{aligned}$$

for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\sigma : \mathrm{SEN}^k \longrightarrow \mathrm{SEN}$ in N, $\vec{\chi} \in \mathrm{SEN}(\Sigma')^k$, and $i < k$.

Following a convention, introduced previously in a similar context in [20], equation (2) will sometimes be abbreviated in the form

$$C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi}\,)\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}\,)\}),$$

with the understanding that $\mathrm{SEN}(f)(\varphi)$ and $\mathrm{SEN}(f)(\psi)$ may be appearing in any of the k argument positions of $\sigma_{\Sigma'}$ and not just the first, but that they must appear in the same position on both sides of the equation.

It was proven in [23, Proposition 6.29] that $\Theta^N(T)$, as defined above, is an N-congruence system on SEN that is compatible with the theory family T. Since $\Omega^N(T)$ is the largest N-congruence system that is compatible with the theory family T, this yields immediately that $\Theta^N(T) \leq \Omega^N(T)$, for every theory family $T = \{T_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ of $\mathcal{I}$.

Moreover, it may be shown that, for every theory family T of $\mathcal{I}$,

$$\Theta^N(T) = \textstyle\bigcap_{T\leq T'} \Omega^N(T'),$$

i. e., that, for all $\Sigma \in |\mathbf{Sign}|$, $\Theta^N_\Sigma(T) = \bigcap_{T\leq T'} \Omega^N_\Sigma(T')$.

Proposition 2.1 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Then, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Theta^N(T) = \bigcap_{T\leq T'} \Omega^N(T')$, where the signature-wise intersection is taken over all theory families of $\mathcal{I}$ $\leq$-including T.*

P r o o f. Suppose, first, that $\Sigma \in |\mathbf{Sign}|$, $\varphi, \psi \in \mathrm{SEN}(\Sigma)$ such that $\langle\varphi, \psi\rangle \in \bigcap_{T\leq T'} \Omega^N_\Sigma(T')$. Then we have that, by [23, Proposition 2.4], for all $T' \geq T$, all $\sigma : \mathrm{SEN}^n \longrightarrow \mathrm{SEN}$ in N, all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{n-1}$,

$$\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi}\,) \in T'_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}\,) \in T'_{\Sigma'}.$$

Consider the theory family $T' = T^{[\langle\Sigma', \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}\,)\rangle]}$ as defined in the proof of [23, Lemma 3.8]. Then the equivalence above yields that $\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi}\,) \in T^{[\langle\Sigma', \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}\,)\rangle]}_{\Sigma'}$, whence we get that

$$C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi}\,)\}) \subseteq C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}\,)\}).$$

Hence, by symmetry, we get that

$$C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi}\,)\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}\,)\})$$

and, therefore, $\langle\varphi, \psi\rangle \in \Theta^N_\Sigma(T)$. Thus, we have that $\bigcap_{T\leq T'} \Omega^N(T') \leq \Theta^N(T)$.

Suppose, conversely, that $\Sigma \in |\mathbf{Sign}|$, $\varphi, \psi \in \mathrm{SEN}(\Sigma)$ such that $\langle\varphi, \psi\rangle \in \Theta^N_\Sigma(T)$. Thus, we obtain that for all $\sigma : \mathrm{SEN}^n \longrightarrow \mathrm{SEN}$ in N, all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, and all $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{n-1}$,

$$C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi}\,)\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}\,)\}).$$

Now let $T' \in \mathrm{ThFam}(\mathcal{I})$ such that $T \leq T'$, and assume that $\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi}\,) \in T'_{\Sigma'}$. Then the equality above yields that

$$\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}\,) \in C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi}\,)\}) \subseteq C_{\Sigma'}(T'_{\Sigma'}) = T'_{\Sigma'}.$$

By symmetry, we obtain that

$$\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi}\,) \in T'_{\Sigma'} \quad \text{iff} \quad \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi}\,) \in T'_{\Sigma'},$$

whence, again by [23, Proposition 2.4], $\langle \varphi, \psi \rangle \in \Omega^N_\Sigma(T')$. Therefore, we get that $\langle \varphi, \psi \rangle \in \bigcap_{T \leq T'} \Omega^N_\Sigma(T')$, and thus $\Theta^N(T) \leq \bigcap_{T \leq T'} \Omega^N(T')$. □

The Suszko operator, unlike the Leibniz operator, and similarly with the deductive system framework, is always monotone on theory families of a π-institution, i. e., $\Theta^N(T^1) \leq \Theta^N(T^2)$, for all theory families T^1, T^2 of $\mathcal{I}$ such that $T^1 \leq T^2$. This was the content of [23, Proposition 6.31].

Similarly with [6, Theorem 1.5.3], the N-Suszko operator Θ^N of a π-institution $\mathcal{I}$ may be characterized as the operator yielding, for every theory family T, the largest N-congruence system such that, when it identifies two Σ-sentences φ and ψ, then φ and ψ satisfy $C_\Sigma(T_\Sigma \cup \{\varphi\}) = C_\Sigma(T_\Sigma \cup \{\psi\})$. More precisely, it was shown in [23, Theorem 6.32] that, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, with N a category of natural transformations on SEN, and such that, for every theory family T of $\mathcal{I}$, $\mathcal{O}^N(T)$ is an N-congruence system on SEN such that, for all $\Sigma \in |\mathbf{Sign}|$, $\varphi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \varphi, \psi \rangle \in \mathcal{O}^N_\Sigma(T) \quad \text{implies} \quad C_\Sigma(T_\Sigma \cup \{\varphi\}) = C_\Sigma(T_\Sigma \cup \{\psi\}),$$

then $\mathcal{O}^N(T) \leq \Theta^N(T)$, for all theory families T of $\mathcal{I}$.

Finally, in [23, Proposition 6.33], it was shown that a π-institution $\mathcal{I}$ is N-protoalgebraic if and only if

$$\Omega^N(T) = \Theta^N(T), \quad \text{for all } T \in \mathrm{ThFam}(\mathcal{I}).$$

This result is very significant in that it shows that, in agreement with the case of sentential logics, the categorical N-Suszko operator will have a role to play in the case of non N-protoalgebraic π-institutions, while its theory reduces to the theory of the N-Leibniz operator in the context of N-protoalgebraic π-institutions.

3 Deductive translations and bilogical morphisms

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C' \rangle$ be π-institutions. Then we say that a singleton semi-interpretation $\langle F, \alpha \rangle : \mathcal{I} \succ^{\mathrm{s}} \mathcal{I}'$ is *deductive* if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\alpha_\Sigma(\varphi) = \alpha_\Sigma(\psi) \quad \text{implies} \quad C_\Sigma(\varphi) = C_\Sigma(\psi).$$

If, in addition, N, N' are categories of natural transformations on $\mathrm{SEN}, \mathrm{SEN}'$, respectively, then an (N, N')-logical morphism $\langle F, \alpha \rangle : \mathcal{I} \succ^{\mathrm{se}} \mathcal{I}'$ is said to be *deductive* if it is deductive as a semi-interpretation.

Note that every singleton interpretation $\langle F, \alpha \rangle : \mathcal{I} \vdash^{\mathrm{s}} \mathcal{I}'$ is deductive. Indeed, if $\Sigma \in |\mathbf{Sign}|$, $\varphi, \psi \in \mathrm{SEN}(\Sigma)$ such that $\alpha_\Sigma(\varphi) = \alpha_\Sigma(\psi)$, then $\alpha_\Sigma(\psi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\varphi))$ and $\alpha_\Sigma(\varphi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\psi))$, whence, by the hypothesis, $\psi \in C_\Sigma(\varphi)$ and $\varphi \in C_\Sigma(\psi)$, yielding $C_\Sigma(\varphi) = C_\Sigma(\psi)$, i. e., $\langle F, \alpha \rangle$ is deductive. As a consequence, every strong (N, N')-logical morphism $\langle F, \alpha \rangle : \mathcal{I} \succ^{\mathrm{se}} \mathcal{I}'$ is deductive. Thus the following has been shown:

Lemma 3.1 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$, $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C' \rangle$ be π-institutions. Every singleton interpretation $\langle F, \alpha \rangle : \mathcal{I} \vdash^{\mathrm{s}} \mathcal{I}'$ is deductive.*

The following proposition, on the other hand, characterizes those surjective and deductive singleton semi-interpretations that are interpretations.

Recall the convention followed in [10] whereby, given a closure operator C on some set X, the corresponding closed set system is denoted by $\mathcal{C}$.

Proposition 3.2 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ and $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C' \rangle$ be π-institutions. Then a surjective deductive singleton semi-interpretation $\langle F, \alpha \rangle : \mathcal{I} \succ^{\mathrm{s}} \mathcal{I}'$ is an interpretation if and only if, for all $\Sigma \in |\mathbf{Sign}|$, $\alpha_\Sigma(\mathcal{C}_\Sigma) = \mathcal{C}'_{F(\Sigma)}$.*

Proof. Suppose, first, that $\langle F,\alpha\rangle : \mathcal{I} \succ^{\mathrm{s}} \mathcal{I}'$ is a surjective deductive singleton semi-interpretation, that is an interpretation. It will be shown that $\alpha_\Sigma(\mathcal{C}_\Sigma) = \mathcal{C}'_{F(\Sigma)}$. For the left-to-right inclusion, suppose that $T \in \mathcal{C}_\Sigma$ and let $\varphi' \in \mathrm{SEN}'(F(\Sigma))$ such that $\varphi' \in C'_{F(\Sigma)}(\alpha_\Sigma(T))$. By the surjectivity of $\langle F,\alpha\rangle$, there exists $\varphi \in \mathrm{SEN}(\Sigma)$ such that $\alpha_\Sigma(\varphi) = \varphi'$. Thus $\alpha_\Sigma(\varphi) \in C'_{F(\Sigma)}(\alpha_\Sigma(T))$, whence, because $\langle F,\alpha\rangle$ is an interpretation, $\varphi \in C_\Sigma(T) = T$. Thus, $\varphi' = \alpha_\Sigma(\varphi) \in \alpha_\Sigma(T)$. This shows that $\alpha_\Sigma(T) \in \mathcal{C}'_{F(\Sigma)}$. Conversely, if $T' \in \mathcal{C}'_{F(\Sigma)}$, then $\alpha_\Sigma^{-1}(T') \in \mathcal{C}_\Sigma$ and $T' = \alpha_\Sigma(\alpha_\Sigma^{-1}(T')) \in \alpha_\Sigma(\mathcal{C}_\Sigma)$, where the equality follows by the surjectivity of $\langle F,\alpha\rangle$.

Now it is sufficient to show the following: if $\alpha_\Sigma(\mathcal{C}_\Sigma) = \mathcal{C}'_{F(\Sigma)}$, for all $\Sigma \in |\mathbf{Sign}|$, then, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$,

$$\alpha_\Sigma(\varphi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \quad \text{implies} \quad \varphi \in C_\Sigma(\Phi).$$

In fact, if T is a Σ-theory of $\mathcal{I}$ such that $\Phi \subseteq T$, then $\alpha_\Sigma(T)$ is, by our hypothesis, an $F(\Sigma)$-theory of $\mathcal{I}'$ such that $\alpha_\Sigma(\Phi) \subseteq \alpha_\Sigma(T)$, whence, since $\alpha_\Sigma(\varphi) \in C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$, $\alpha_\Sigma(\varphi) \in \alpha_\Sigma(T)$. But, then, since $\langle F,\alpha\rangle$ is deductive, we get that there exists $\psi \in T$ such that $C_\Sigma(\varphi) = C_\Sigma(\psi)$. Therefore

$$\varphi \in C_\Sigma(\varphi) = C_\Sigma(\psi) \subseteq C_\Sigma(T) = T.$$

Since T was arbitrary, it follows that $\varphi \in C_\Sigma(\Phi)$, as was to be shown. □

For the previous notions there are corresponding counterparts in the case of logical matrix systems. Let a functor $\mathrm{SEN} : \mathbf{Sign} \longrightarrow \mathbf{Set}$ be given, where N is a category of natural transformations on SEN. Then an *N-matrix system $\langle\langle \mathrm{SEN}', \langle F,\alpha\rangle\rangle, T'\rangle$ for* SEN consists of a functor $\mathrm{SEN}' : \mathbf{Sign}' \longrightarrow \mathbf{Set}$, with N' a category of natural transformations on SEN', and a surjective (N,N')-epimorphic translation $\langle F,\alpha\rangle : \mathrm{SEN} \longrightarrow^{\mathrm{se}} \mathrm{SEN}'$, together with an axiom family T' of SEN'. On the other hand, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, a triple $\langle\langle \mathrm{SEN}', \langle F,\alpha\rangle\rangle, T'\rangle$ is said to be an *N-matrix system model of $\mathcal{I}$* if it is an N-matrix system for SEN and $T' \in \mathrm{ThFam}_{\mathcal{I}}^{\langle F,\alpha\rangle}(\mathrm{SEN}')$, i. e., T' is a theory family of the $\langle F,\alpha\rangle$-min (N,N')-model of $\mathcal{I}$ on SEN' (see [20, Section 5]). An (N',N'')-epimorphic translation

$$\langle G,\beta\rangle : \mathrm{SEN}' \longrightarrow^{\mathrm{se}} \mathrm{SEN}''$$

is said to be a *matrix system morphism* from the N-matrix system model $\langle\langle \mathrm{SEN}', \langle F,\alpha\rangle\rangle, T'\rangle$ to the N-matrix system model $\langle\langle \mathrm{SEN}'', \langle GF,\beta_F\alpha\rangle\rangle, T''\rangle$ of $\mathcal{I}$ if

$$\begin{tikzcd} & \mathcal{I} \arrow[dl, "\langle F,\alpha\rangle"'] \arrow[dr, "\langle GF,\beta_F\alpha\rangle"] & \\ \mathrm{SEN}' \arrow[rr, "\langle G,\beta\rangle"'] & & \mathrm{SEN}'' \end{tikzcd}$$

for all $\Sigma \in |\mathbf{Sign}'|$ and all $\varphi \in \mathrm{SEN}'(\Sigma)$,

$$\varphi \in T'_\Sigma \quad \text{implies} \quad \beta_\Sigma(\varphi) \in T''_{G(\Sigma)}.$$

$\langle G,\beta\rangle : \mathrm{SEN}' \longrightarrow^{\mathrm{se}} \mathrm{SEN}''$ is said to be a *strict matrix system morphism* if, for all $\Sigma \in |\mathbf{Sign}'|$, $\varphi \in \mathrm{SEN}'(\Sigma)$,

$$\varphi \in T'_\Sigma \quad \text{iff} \quad \beta_\Sigma(\varphi) \in T''_{G(\Sigma)}.$$

Finally, $\langle G,\beta\rangle$ is said to be a *deductive matrix system morphism* if, for all $\Sigma \in |\mathbf{Sign}|$, $\varphi,\psi \in \mathrm{SEN}'(\Sigma)$,

$$\beta_\Sigma(\varphi) = \beta_\Sigma(\psi) \quad \text{implies} \quad C'^{\min}_\Sigma(T'_\Sigma \cup \{\varphi\}) = C'^{\min}_\Sigma(T'_\Sigma \cup \{\psi\}),$$

where by $C'^{\min}$ is denoted the closure system of the $\langle F,\alpha\rangle$-min (N,N')-model of $\mathcal{I}$ on SEN'.

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, where N is a category of natural transformations on SEN, recall from [19] that an N-congruence system $\theta = \{\theta_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is said to be a *logical N-congruence system of $\mathcal{I}$* if and only if, for all $\Sigma \in |\mathbf{Sign}|$, $\varphi,\psi \in \mathrm{SEN}(\Sigma)$,

$$\langle\varphi,\psi\rangle \in \theta_\Sigma \quad \text{implies} \quad C_\Sigma(\varphi) = C_\Sigma(\psi).$$

Recall, also, from [22, Proposition 2] the construction of the π-institution $\mathcal{I}^T = \langle \mathbf{Sign}, \mathrm{SEN}, C^T\rangle$ out of a given π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ and of a given theory system $T \in \mathrm{ThSys}(\mathcal{I})$. The closure system C^T of $\mathcal{I}^T$ is defined, for all $\Sigma \in |\mathbf{Sign}|$ and all $\Phi \cup \{\varphi\} \subseteq \mathrm{SEN}(\Sigma)$, by

$$\varphi \in C^T_\Sigma(\Phi) \quad \text{iff} \quad \varphi \in C_\Sigma(T_\Sigma \cup \Phi).$$

This construction may be carried out *only* in the case when T is a theory system and *not* in the case of an arbitrary theory family. If T is an arbitrary theory family, then the family of closure operators $\{C_\Sigma\}_{\Sigma\in|\mathbf{Sign}|}$ may not form a closure system on SEN, since it may not be structural.

Proposition 3.3 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN*, and $T \in \mathrm{ThSys}(\mathcal{I})$. $\Theta^N(T)$ is the largest logical N-congruence system of $\mathcal{I}^T = \langle \mathbf{Sign}, \mathrm{SEN}, C^T\rangle$, i. e., $\Theta^N(T) = \widetilde{\Omega}^N(\mathcal{I}^T)$.*

Proof. This follows immediately by comparing the definition of $\Theta^N(T)$ and the characterization of $\widetilde{\Omega}^N(\mathcal{I}^T)$, given in [19, Theorem 4]. □

An analog of [7, Proposition 2.1] that relates deductive homomorphisms and deductive congruences is given next. It relates deductive logical morphisms with logical congruence systems.

Proposition 3.4 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ be π-institutions and N, N' categories of natural transformations on* SEN, SEN′*, respectively.*

1. *If θ is an N-congruence system on* SEN*, then the canonical (N, N^θ)-epimorphic translation*

$$\langle \mathrm{I}_{\mathbf{Sign}}, \pi^\theta\rangle : \mathrm{SEN} \longrightarrow^{\mathrm{se}} \mathrm{SEN}^\theta$$

is a deductive (N, N^θ)-logical morphism $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^\theta\rangle : \mathcal{I} \succ^{\mathrm{se}} \mathcal{I}^\theta$ if and only if θ is a logical N-congruence system of $\mathcal{I}$.

2. *Given an (N, N')-logical morphism $\langle F, \alpha\rangle : \mathcal{I} \succ^{\mathrm{se}} \mathcal{I}'$, $\langle F, \alpha\rangle$ is deductive if and only if the kernel*

$$\theta^{\langle F,\alpha\rangle} := \mathrm{Ker}(\langle F, \alpha\rangle)$$

is a logical N-congruence system of $\mathcal{I}$.

Proof.

1. If θ is a logical N-congruence system of $\mathcal{I}$, then by [19, Proposition 23], $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^\theta\rangle : \mathrm{SEN} \longrightarrow^{\mathrm{se}} \mathrm{SEN}^\theta$ is an (N, N^θ)-bilogical morphism $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^\theta\rangle : \mathcal{I} \vdash^{\mathrm{se}} \mathcal{I}^\theta$. Thus, it is deductive, by Lemma 3.1.

If, conversely, $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^\theta\rangle : \mathcal{I} \succ^{\mathrm{se}} \mathcal{I}^\theta$ is a deductive (N, N^θ)-logical morphism, then, for all $\Sigma \in |\mathbf{Sign}|$, and all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \varphi, \psi\rangle \in \theta_\Sigma \quad \text{iff} \quad \varphi/\theta_\Sigma = \psi/\theta_\Sigma \quad \text{iff} \quad \pi^\theta_\Sigma(\varphi) = \pi^\theta_\Sigma(\psi)$$

and

$$\pi^\theta_\Sigma(\varphi) = \pi^\theta_\Sigma(\psi) \quad \text{implies} \quad C_\Sigma(\varphi) = C_\Sigma(\psi).$$

Therefore, θ is a logical N-congruence system of $\mathcal{I}$.

2. Suppose next that $\langle F, \alpha\rangle : \mathcal{I} \succ^{\mathrm{se}} \mathcal{I}'$ is an (N, N')-logical morphism.

Let $\langle F, \alpha\rangle$ be deductive and let $\Sigma \in |\mathbf{Sign}|$, $\varphi, \psi \in \mathrm{SEN}(\Sigma)$ such that $\langle \varphi, \psi\rangle \in \theta^{\langle F,\alpha\rangle}_\Sigma$. Then $\alpha_\Sigma(\varphi) = \alpha_\Sigma(\psi)$, whence $C_\Sigma(\varphi) = C_\Sigma(\psi)$ and, therefore, $\theta^{\langle F,\alpha\rangle}$ is a logical N-congruence system.

Conversely, let $\theta^{\langle F,\alpha\rangle}$ be a logical N-congruence system. Then if $\Sigma \in |\mathbf{Sign}|$ and $\varphi, \psi \in \mathrm{SEN}(\Sigma)$ are such that $\alpha_\Sigma(\varphi) = \alpha_\Sigma(\psi)$, then $\langle \varphi, \psi\rangle \in \theta^{\langle F,\alpha\rangle}_\Sigma$, whence $C_\Sigma(\varphi) = C_\Sigma(\psi)$ and $\langle F, \alpha\rangle$ is a deductive (N, N')-logical morphism. □

The following proposition exhibits an important example of a deductive logical morphism. It is the canonical $(N, N^{\Theta^N(T)})$-epimorphic projection $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\Theta^N(T)}\rangle : \mathrm{SEN} \longrightarrow^{\mathrm{se}} \mathrm{SEN}^{\Theta^N(T)}$ from the sentence functor SEN of a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ to its quotient $\mathrm{SEN}^{\Theta^N(T)}$ by the Suszko N-congruence of a given theory system T of $\mathcal{I}$. Of course, $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\Theta^N(T)}\rangle$ is not only a deductive logical morphism from $\mathcal{I}^T$ to $\mathcal{I}^T/\Theta^N(T)$, but in fact a bilogical morphism.

Proposition 3.5 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN*, and $T \in \mathrm{ThSys}(\mathcal{I})$. Then the canonical $(N, N^{\Theta^N(T)})$-epimorphic projection*

$$\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\Theta^N(T)}\rangle : \mathrm{SEN} \longrightarrow^{\mathrm{se}} \mathrm{SEN}^{\Theta^N(T)}$$

is an $(N, N^{\Theta^N(T)})$-bilogical morphism $\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\Theta^N(T)}\rangle : \mathcal{I}^T \vdash^{\mathrm{se}} \mathcal{I}^T/\Theta^N(T)$.

P r o o f. Follows by [19, Proposition 23] and Proposition 3.3. □

Going back to the matrix system framework, consider a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, and $\mathfrak{M} = \langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle$ an N-matrix system model of $\mathcal{I}$. An N'-congruence system θ on SEN' is a *deductive N'-congruence system on $\mathfrak{M}$* if, for all $\Sigma \in |\mathbf{Sign}'|$, $\varphi, \psi \in \mathrm{SEN}'(\Sigma)$,

$$\langle \varphi, \psi\rangle \in \theta_\Sigma \quad \text{implies} \quad C'^{\min}_\Sigma(T'_\Sigma \cup \{\varphi\}) = C'^{\min}_\Sigma(T'_\Sigma \cup \{\psi\}),$$

where, again, by $C'^{\min}$ is denoted the closure system of the $\langle F, \alpha\rangle$-min (N, N')-model of $\mathcal{I}$ on SEN'.

For matrix systems Proposition 3.4 takes the following form.

Proposition 3.6 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN*, and $\langle F, \alpha\rangle : \mathrm{SEN} \longrightarrow^{\mathrm{se}} \mathrm{SEN}'$ an (N, N')-epimorphic translation.*

1. *Let θ be an N'-congruence system on SEN' and*

$$\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^\theta\rangle : \langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle \longrightarrow \langle\langle \mathrm{SEN}'^\theta, \langle F, \pi^\theta_F \alpha\rangle\rangle, T'/\theta\rangle$$

a matrix system morphism. Then $\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^\theta\rangle$ is a deductive matrix system morphism

$$\mathcal{I} \xrightarrow{\langle F, \alpha\rangle} \mathrm{SEN}' \xrightarrow{\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^\theta\rangle} \mathrm{SEN}'^\theta$$

if and only if θ is a deductive N'-congruence system on $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle$.

2. *Suppose that $\langle G, \beta\rangle : \mathrm{SEN}' \longrightarrow^{\mathrm{se}} \mathrm{SEN}''$ is an (N', N'')-epimorphic translation such that $\langle G, \beta\rangle$ is a matrix system morphism from $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle$ to $\langle\langle \mathrm{SEN}'', \langle GF, \beta_F \alpha\rangle\rangle, T''\rangle$. Then $\langle G, \beta\rangle$ is a deductive matrix system morphism*

$$\mathcal{I} \xrightarrow{\langle F, \alpha\rangle} \mathrm{SEN}' \xrightarrow{\langle G, \beta\rangle} \mathrm{SEN}''$$

if and only if $\theta^{\langle G, \beta\rangle}$ is a deductive N'-congruence system on $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle$.

P r o o f.

1. Suppose, first, that $\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^\theta\rangle$ is a deductive matrix system morphism. Then we have, for all $\Sigma \in |\mathbf{Sign}'|$, $\varphi, \psi \in \mathrm{SEN}'(\Sigma)$ such that $\langle \varphi, \psi\rangle \in \theta_\Sigma$, that $\pi^\theta_\Sigma(\varphi) = \pi^\theta_\Sigma(\psi)$, which yields that

$$C'^{\min}_\Sigma(T'_\Sigma \cup \{\varphi\}) = C'^{\min}_\Sigma(T'_\Sigma \cup \{\psi\}),$$

i. e., θ is indeed deductive. If, conversely, θ is deductive on $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle$, $\Sigma \in |\mathbf{Sign}'|$, $\varphi, \psi \in \mathrm{SEN}'(\Sigma)$ such that $\pi^\theta_\Sigma(\varphi) = \pi^\theta_\Sigma(\psi)$, then we have $\varphi/\theta_\Sigma = \psi/\theta_\Sigma$, i. e., that $\langle \varphi, \psi\rangle \in \theta_\Sigma$, which yields that

$$C'^{\min}_\Sigma(T'_\Sigma \cup \{\varphi\}) = C'^{\min}_\Sigma(T'_\Sigma \cup \{\psi\}).$$

Thus, $\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^\theta\rangle$ is deductive.

2. Suppose, now, that $\theta^{\langle G, \beta\rangle}$ is deductive and let $\Sigma \in |\mathbf{Sign}'|$ and $\varphi, \psi \in \mathrm{SEN}'(\Sigma)$ such that $\beta_\Sigma(\varphi) = \beta_\Sigma(\psi)$. Then $\langle \varphi, \psi\rangle \in \theta^{\langle G, \beta\rangle}_\Sigma$, whence we get that $C'^{\min}_\Sigma(T'_\Sigma \cup \{\varphi\}) = C'^{\min}_\Sigma(T'_\Sigma \cup \{\psi\})$ and $\langle G, \beta\rangle$ is in fact a deductive matrix system morphism. Suppose, conversely, that $\langle G, \beta\rangle$ is deductive. If $\Sigma \in |\mathbf{Sign}'|$, $\varphi, \psi \in \mathrm{SEN}'(\Sigma)$ are such that $\langle \varphi, \psi\rangle \in \theta^{\langle G, \beta\rangle}_\Sigma$, then $\beta_\Sigma(\varphi) = \beta_\Sigma(\psi)$, whence $C'^{\min}_\Sigma(T'_\Sigma \cup \{\varphi\}) = C'^{\min}_\Sigma(T'_\Sigma \cup \{\psi\})$ and $\theta^{\langle G, \beta\rangle}$ is deductive. □

The paradigm for a surjective deductive morphism is the canonical mapping from a matrix system model to its quotient by the Suszko congruence system of its designated theory family.

Proposition 3.7 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Let also $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle$ be an N-matrix system model of $\mathcal{I}$ and $\Theta^{N'}(T')$ the N'-Suszko congruence of T' in the $\langle F, \alpha\rangle$-min (N, N')-model*

$$\mathcal{I}'^{\min} = \langle \mathbf{Sign}', \mathrm{SEN}', C'^{\min}\rangle$$

of $\mathcal{I}$ on SEN′. *If $T'/\Theta^{N'}(T') \in \mathrm{ThFam}_{\mathcal{I}}^{\langle F, \pi_F^{\Theta^{N'}(T')}\alpha\rangle}(\mathrm{SEN}'^{\Theta^{N'}(T')})$, then the canonical $(N', N'^{\Theta^{N'}(T')})$-epimorphic projection $\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^{\Theta^{N'}(T')}\rangle : \mathrm{SEN}' \longrightarrow^{\mathrm{se}} \mathrm{SEN}'^{\Theta^{N'}(T')}$ is a surjective, strict, and deductive matrix system morphism from $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle$ to $\langle\langle \mathrm{SEN}'^{\Theta^{N'}(T')}, \langle F, \pi_F^{\Theta^{N'}(T')}\alpha\rangle\rangle, T'/\Theta^{N'}(T')\rangle$.*

Proof. It is clear that $\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^{\Theta^{N'}(T')}\rangle$ is a surjective translation. To show that it is also deductive, suppose that $\Sigma \in |\mathbf{Sign}'|$, $\varphi, \psi \in \mathrm{SEN}'(\Sigma)$ such that $\pi_\Sigma^{\Theta^{N'}(T')}(\varphi) = \pi_\Sigma^{\Theta^{N'}(T')}(\psi)$. Then $\langle \varphi, \psi\rangle \in \Theta_\Sigma^{N'}(T')$. Thus, by the definition of $\Theta^{N'}(T')$, we have that $C_\Sigma'^{\min}(T'_\Sigma \cup \{\varphi\}) = C_\Sigma'^{\min}(T'_\Sigma \cup \{\psi\})$ and, therefore, $\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^{\Theta^{N'}(T')}\rangle$ is a deductive matrix system morphism. To show that it is also strict, suppose that $\Sigma \in |\mathbf{Sign}'|$, $\varphi \in \mathrm{SEN}'(\Sigma)$ such that $\pi_\Sigma^{\Theta^{N'}(T')}(\varphi) \in T'_\Sigma/\Theta_\Sigma^{N'}(T')$. Then there exists $\psi \in \mathrm{SEN}'(\Sigma)$ such that $\psi \in T'_\Sigma$ and $\langle \varphi, \psi\rangle \in \Theta_\Sigma^{N'}(T')$. This shows that $\varphi \in C_\Sigma'^{\min}(T'_\Sigma \cup \{\varphi\}) = C_\Sigma'^{\min}(T'_\Sigma \cup \{\psi\}) = T'_\Sigma$. Therefore $\langle \mathrm{I}_{\mathbf{Sign}'}, \pi^{\Theta^{N'}(T')}\rangle$ is also strict. □

Next, an analog of the Correspondence Property for Deductive Homomorphisms [7, Proposition 2.3] will be shown to hold in the framework of π-institutions. It says, roughly speaking, that pushing forth and then pulling back a closed set along a deductive logical morphism leaves the closed set unchanged.

Proposition 3.8 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ are π-institutions and N, N' categories of natural transformations on* SEN, SEN′, *respectively. If $\langle F, \alpha\rangle : \mathcal{I} \succ^{\mathrm{se}} \mathcal{I}'$ is a deductive (N, N')-logical morphism, $\Sigma \in |\mathbf{Sign}|$, and T is a Σ-theory, then $T = \alpha_\Sigma^{-1}(\alpha_\Sigma(T))$.*

Proof. We always have, set-theoretically, that

$$T \subseteq \alpha_\Sigma^{-1}(\alpha_\Sigma(T)).$$

To prove the reverse inclusion, suppose that $\varphi \in \mathrm{SEN}(\Sigma)$ such that $\varphi \in \alpha_\Sigma^{-1}(\alpha_\Sigma(T))$. Then $\alpha_\Sigma(\varphi) \in \alpha_\Sigma(T)$. Therefore, there is $\psi \in T$ such that $\alpha_\Sigma(\varphi) = \alpha_\Sigma(\psi)$. But $\langle F, \alpha\rangle$ is deductive, whence $C_\Sigma(\varphi) = C_\Sigma(\psi)$. Now we have $\varphi \in C_\Sigma(\varphi) = C_\Sigma(\psi) \subseteq C_\Sigma(T) = T$. □

It was shown that a deductive logical morphism preserves theories that are pushed forth and then pulled back. If the morphism happens, in addition, to be surjective and strong, i. e., a (deductive) bilogical morphism, then the stronger conclusion that theories are in fact mapped to theories may be reached.

Proposition 3.9 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ are π-institutions and N, N' categories of natural transformations on* SEN, SEN′, *respectively. If $\langle F, \alpha\rangle : \mathcal{I} \vdash^{\mathrm{se}} \mathcal{I}'$ is an (N, N')-bilogical morphism, $\Sigma \in |\mathbf{Sign}|$, and T is a Σ-theory of $\mathcal{I}$, then $\alpha_\Sigma(T)$ is an $F(\Sigma)$-theory of $\mathcal{I}'$.*

Proof. Immediate by Lemma 3.1 and Proposition 3.2. □

Proposition 3.9 has several corollaries.

Corollary 3.10 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, N a category of natural transformations on* SEN, *and $T \in \mathrm{ThSys}(\mathcal{I})$. Then for each theory family $T' \in \mathrm{ThFam}(\mathcal{I}^T)$, $\pi^{\Theta^N(T)}(T') \in \mathrm{ThFam}(\mathcal{I}^T/\Theta^N(T))$, where by $\pi^{\Theta^N(T)}(T')$ is denoted the theory family $\{\pi_\Sigma^{\Theta^N(T)}(T'_\Sigma)\}_{\Sigma \in |\mathbf{Sign}|}$.*

Proof. This follows from Proposition 3.5 in conjunction with Proposition 3.9. □

Corollary 3.11 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ are π-institutions and N, N' categories of natural transformations on* SEN, SEN′, *respectively. If $\langle F, \alpha\rangle : \mathcal{I} \vdash^{\mathrm{se}} \mathcal{I}'$ is an (N, N')-bilogical morphism, then, for all $\Sigma \in |\mathbf{Sign}|$, $\Phi \subseteq \mathrm{SEN}(\Sigma)$,*

$$\alpha_\Sigma(C_\Sigma(\Phi)) = C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)).$$

Proof. The fact that $\alpha_\Sigma(C_\Sigma(\Phi)) \subseteq C'_{F(\Sigma)}(\alpha_\Sigma(\Phi))$ follows from the definition of an (N, N')-logical morphism. For the reverse inclusion, it is clear that $\alpha_\Sigma(\Phi) \subseteq \alpha_\Sigma(C_\Sigma(\Phi))$. But, by Proposition 3.9, $\alpha_\Sigma(C_\Sigma(\Phi))$ is an $F(\Sigma)$-theory, whence $C'_{F(\Sigma)}(\alpha_\Sigma(\Phi)) \subseteq \alpha_\Sigma(C_\Sigma(\Phi))$. □

Corollary 3.12 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ are π-institutions and N, N' categories of natural transformations on* SEN, SEN′*, respectively.*

1. *If $\langle F, \alpha\rangle : \mathcal{I} \vdash^{\mathrm{se}} \mathcal{I}'$ is an (N, N')-bilogical morphism, then, for all $T \in \mathrm{ThFam}(\mathcal{I}')$,*

$$\Theta^N(\alpha^{-1}(T)) = \alpha^{-1}(\Theta^{N'}(T)).$$

2. *If, moreover, $F : \mathbf{Sign} \longrightarrow \mathbf{Sign}'$ is an isomorphism, the mapping defined by $T' \longmapsto \alpha(T')$ is an isomorphism between the complete lattice $\mathbf{ThFam}(\mathcal{I}^T)$ and the complete lattice $\mathbf{ThFam}(\mathcal{I}'^{\alpha(T)})$, for every theory system $T \in \mathrm{ThSys}(\mathcal{I})$.*

Proof.

1. We have, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{aligned}
\alpha_\Sigma^{-1}(\Theta^{N'}_{F(\Sigma)}(T)) &= \alpha_\Sigma^{-1}(\textstyle\bigcap\{\Omega^{N'}_{F(\Sigma)}(T') : T' \in \mathrm{ThFam}(\mathcal{I}') \text{ and } T \leq T'\}) && \text{(by Proposition 2.1)}\\
&= \textstyle\bigcap\{\alpha_\Sigma^{-1}(\Omega^{N'}_{F(\Sigma)}(T')) : T' \in \mathrm{ThFam}(\mathcal{I}') \text{ and } T \leq T'\}\\
&= \textstyle\bigcap\{\Omega^N_\Sigma(\alpha^{-1}(T')) : T' \in \mathrm{ThFam}(\mathcal{I}') \text{ and } T \leq T'\} && \text{(by [23, Lemma 5.26])}\\
&= \textstyle\bigcap\{\Omega^N_\Sigma(T'') : T'' \in \mathrm{ThFam}(\mathcal{I}) \text{ and } \alpha^{-1}(T) \leq T''\}\\
&= \Theta^N_\Sigma(\alpha^{-1}(T)) && \text{(by Proposition 2.1),}
\end{aligned}$$

where the second equation before the end is justified by the fact that

$$\{\alpha^{-1}(T') : T' \in \mathrm{ThFam}(\mathcal{I}') \text{ and } T \leq T'\} = \{T'' : T'' \in \mathrm{ThFam}(\mathcal{I}) \text{ and } \alpha^{-1}(T) \leq T''\},$$

which follows by Corollary 3.11.

2. Let $T \in \mathrm{ThSys}(\mathcal{I})$ and consider the mapping $T' \longmapsto \alpha(T')$, for $T' \in \mathrm{ThFam}(\mathcal{I}^T)$, from $\mathrm{ThFam}(\mathcal{I}^T)$ to $\mathrm{ThFam}(\mathcal{I}'^{\alpha(T)})$. By Proposition 3.9, this mapping is well-defined, since $\alpha_\Sigma(T'_\Sigma)$ is an $F(\Sigma)$-theory of $\mathcal{I}'$ that includes $\alpha_\Sigma(T_\Sigma)$, for every $\Sigma \in |\mathbf{Sign}|$. To show that it is one-to-one, suppose that $T', T'' \in \mathrm{ThFam}(\mathcal{I}^T)$ such that $\alpha(T') = \alpha(T'')$. Then

$$\alpha^{-1}(\alpha(T')) = \alpha^{-1}(\alpha(T'')).$$

So it suffices to show that, for all $T' \in \mathrm{ThFam}(\mathcal{I}^T)$, $\alpha^{-1}(\alpha(T')) = T'$. The right-to-left inclusion is obvious. For the left-to-right inclusion, suppose that $\varphi \in \alpha_\Sigma^{-1}(\alpha_\Sigma(T'_\Sigma))$. Then $\alpha_\Sigma(\varphi) \in \alpha_\Sigma(T'_\Sigma)$. Thus, there is $\psi \in T'$ with $\alpha_\Sigma(\varphi) = \alpha_\Sigma(\psi)$. Thus, since, by Lemma 3.1, $\langle F, \alpha\rangle$ is deductive, $C_\Sigma(\varphi) = C_\Sigma(\psi) \in T'_\Sigma$. Hence $\varphi \in T'_\Sigma$. This concludes the proof that $\alpha^{-1}(\alpha(T')) = T'$ and shows that $T' \longmapsto \alpha(T')$ is one-to-one. To show that it is onto, suppose now that $T'' \in \mathrm{ThFam}(\mathcal{I}'^{\alpha(T)})$. Then $\alpha^{-1}(T'') \in \mathrm{ThFam}(\mathcal{I}^T)$ and, by the surjectivity of $\langle F, \alpha\rangle$, $\alpha(\alpha^{-1}(T'')) = T''$. Thus $T' \longmapsto \alpha(T')$ is also onto. Since it is obviously order-preserving, it is an isomorphism from $\mathbf{ThFam}(\mathcal{I}^T)$ to $\mathbf{ThFam}(\mathcal{I}'^{\alpha(T)})$. □

Next, an analog of Proposition 3.8 will be shown to hold in the framework of matrix system models. It parallels more closely the Correspondence Property for Deductive Homomorphisms [7, Proposition 2.3].

Proposition 3.13 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN, *$\langle F, \alpha\rangle : \mathrm{SEN} \longrightarrow^{\mathrm{se}} \mathrm{SEN}'$ an (N, N')-epimorphic translation, and let $\langle G, \beta\rangle : \mathrm{SEN}' \longrightarrow^{\mathrm{se}} \mathrm{SEN}''$ be an (N', N'')-epimorphic translation, $\langle G, \beta\rangle$ also a deductive matrix system morphism from $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle$ to $\langle\langle \mathrm{SEN}'', \langle GF, \beta_F\alpha\rangle\rangle, T''\rangle$. If $T''' \in \mathrm{ThFam}^{\langle F,\alpha\rangle}_{\mathcal{I}}(\mathrm{SEN}')$ such that $T' \leq T'''$, then $\beta_\Sigma^{-1}(\beta_\Sigma(T'''_\Sigma)) = T'''_\Sigma$ for all $\Sigma \in |\mathbf{Sign}'|$.*

P r o o f. Obviously, we have $T'''_\Sigma \subseteq \beta_\Sigma^{-1}(\beta_\Sigma(T'''_\Sigma))$, for all $\Sigma \in |\mathbf{Sign}'|$. To show the reverse inclusion, suppose that $\Sigma \in |\mathbf{Sign}'|$ and $\varphi \in \mathrm{SEN}'(\Sigma)$ are such that $\varphi \in \beta_\Sigma^{-1}(\beta_\Sigma(T'''_\Sigma))$. Then $\beta_\Sigma(\varphi) \in \beta_\Sigma(T'''_\Sigma)$. Thus, there exists $\psi \in T'''_\Sigma$ such that $\beta_\Sigma(\varphi) = \beta_\Sigma(\psi)$. Since $\langle G, \beta\rangle$ is deductive and $T' \leq T'''$, we now obtain

$$\varphi \in C'^{\min}_\Sigma(T'_\Sigma \cup \{\varphi\}) = C'^{\min}_\Sigma(T'_\Sigma \cup \{\psi\}) \subseteq T'''_\Sigma$$

and therefore $\beta_\Sigma^{-1}(\beta_\Sigma(T'''_\Sigma)) \subseteq T'''_\Sigma$, for all $\Sigma \in |\mathbf{Sign}'|$. □

An interesting application of Corollary 3.12 concerns the property of continuity of the Suszko operator.

Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on SEN. The N-Suszko operator Θ^N is said to be *continuous* if, for every directed system $\{T^i : i \in I\}$ of theory families of $\mathcal{I}$ such that $\bigcup_{i\in I} T^i$ is also a theory family,

$$\Theta^N(\textstyle\bigcup_{i\in I} T^i) = \bigcup_{i\in I} \Theta^N(T^i).$$

The following proposition expresses a “transfer” property concerning the continuity of the Suszko operator. It says that if the N-Suszko operator of a π-institution $\mathcal{I}$ is continuous, then the N'-Suszko operator of every (N, N')-model $\mathcal{I}'$ of $\mathcal{I}$ via an (N, N')-bilogical morphism is also continuous.

Proposition 3.14 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Let, also, $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$, with N' a category of natural transformations on* SEN′*, be a model of $\mathcal{I}$ via an (N, N')-bilogical morphism $\langle F, \alpha\rangle : \mathcal{I} \vdash^{\mathrm{se}} \mathcal{I}'$. If the N-Suszko operator is continuous on* $\mathrm{ThFam}(\mathcal{I})$*, then the N'-Suszko operator is continuous on* $\mathrm{ThFam}(\mathcal{I}')$.

P r o o f. Let $\{T^i : i \in I\}$ be a directed system of theory families of $\mathcal{I}'$ such that $\bigcup_{i\in I} T^i$ is also a theory family of $\mathcal{I}'$. We have, using Corollary 3.12, for all $\Sigma \in |\mathbf{Sign}|$,

$$\begin{aligned}
\alpha_\Sigma^{-1}(\Theta^{N'}_{F(\Sigma)}(\textstyle\bigcup_{i\in I} T^i)) &= \Theta^N_\Sigma(\alpha^{-1}(\textstyle\bigcup_{i\in I} T^i)) && \text{(by Corollary 3.12)}\\
&= \Theta^N_\Sigma(\textstyle\bigcup_{i\in I} \alpha^{-1}(T^i)) && \text{(set-theoreticallly)}\\
&= \textstyle\bigcup_{i\in I} \Theta^N_\Sigma(\alpha^{-1}(T^i)) && \text{(by the hypothesis)}\\
&= \textstyle\bigcup_{i\in I} \alpha_\Sigma^{-1}(\Theta^{N'}_{F(\Sigma)}(T^i)) && \text{(by Corollary 3.12)}\\
&= \alpha_\Sigma^{-1}(\textstyle\bigcup_{i\in I} \Theta^{N'}_{F(\Sigma)}(T^i)) && \text{(set-theoretically),}
\end{aligned}$$

which yields, by the surjectivity of $\langle F, \alpha\rangle$, that $\Theta^{N'}_{F(\Sigma)}(\bigcup_{i\in I} T^i) = \bigcup_{i\in I} \Theta^{N'}_{F(\Sigma)}(T^i)$. □

4 Suszko-reduced π-institutions

Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution and N a category of natural transformations on SEN. Given a theory family $T \in \mathrm{ThFam}(\mathcal{I})$, $\mathcal{I}$ is said to be *N-Suszko reduced with respect to* T if $\Theta^N(T) = \Delta^{\mathrm{SEN}}$, the identity N-congruence system on SEN, where Θ^N is the N-Suszko operator of $\mathcal{I}$.

Note that if $T \in \mathrm{ThSys}(\mathcal{I})$, then $\mathcal{I}$ is N-Suszko reduced with respect to the theory system T if and only if $\mathcal{I}^T$ is N-Tarski reduced, since, by Proposition 3.3, $\widetilde{\Omega}^N(\mathcal{I}^T) = \Theta^N(T)$.

Similarly, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, and an N-matrix system model $M = \langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle$ of $\mathcal{I}$, M will be said to be *N'-Suszko reduced* if, in the $\langle F, \alpha\rangle$-min (N, N')-model $\mathcal{I}'^{\min}$ of $\mathcal{I}$ on SEN′, $\Theta^{N'}(T') = \Delta^{\mathrm{SEN}'}$.

Compare this notion to the notion of $\langle\langle \mathrm{SEN}', \langle F, \alpha\rangle\rangle, T'\rangle$ being *N'-Leibniz reduced*, which means that

$$\Omega^{N'}(T') = \Delta^{\mathrm{SEN}'}$$

and which is *absolutely* defined, rather than *relatively* defined with respect to a specific closure system on SEN′.

Theorem 4.1 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution and N a category of natural transformations on* SEN. *Given a theory system $T \in \mathrm{ThSys}(\mathcal{I})$, the following statements are equivalent:*

1. *$\mathcal{I}$ is N-Suszko reduced with respect to T.*

2. *$\mathcal{I}^T$ is isomorphic to a π-institution $\mathcal{I}'^{T'}/\Theta^{N'}(T')$, for some π-institution $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$, with N' a category of natural transformations on* SEN′*, and some theory system $T' \in \mathrm{ThSys}(\mathcal{I}')$.*

P r o o f. If $\mathcal{I}$ is N-Suszko-reduced with respect to T, then $\Theta^N(T) = \Delta^{\mathrm{SEN}}$, which implies that $\mathcal{I}^T$ is isomorphic to $\mathcal{I}^T/\Theta^N(T)$ and hence, 1. implies 2.

Suppose, conversely, that $\mathcal{I}^T$ is isomorphic to the π-institution $\mathcal{I}'^{T'}/\Theta^{N'}(T')$, where $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$ is a π-institution with N' a category of natural transformations on SEN$'$ and some theory system $T' \in \mathrm{ThSys}(\mathcal{I}')$. It is shown first that $\mathcal{I}'^{T'}/\Theta^{N'}(T')$ is $N'^{\Theta^{N'}(T')}$-Tarski reduced, i. e., $N'^{\Theta^{N'}(T')}$-Suszko reduced with respect to the theory system $T'/\Theta^{N'}(T')$. To this end, let $\Sigma \in |\mathbf{Sign}'|$, $\varphi, \psi \in \mathrm{SEN}'(\Sigma)$ be such that, for all $\Sigma' \in |\mathbf{Sign}'|$, $f \in \mathbf{Sign}'(\Sigma, \Sigma')$, $\sigma' : \mathrm{SEN}'^n \longrightarrow \mathrm{SEN}'$ in N', $\vec{\chi} \in \mathrm{SEN}'(\Sigma')^{n-1}$,

$$\begin{aligned} &C'^{T'^{\Theta^{N'}(T')}}_{\Sigma'}(\sigma'^{\Theta^{N'}(T')}_{\Sigma'}(\mathrm{SEN}'^{\Theta^{N'}(T')}(f)(\varphi/\Theta^{N'}_{\Sigma}(T')), \vec{\chi}/\Theta^{N'}_{\Sigma'}(T')))\\ &\quad = C'^{T'^{\Theta^{N'}(T')}}_{\Sigma'}(\sigma'^{\Theta^{N'}(T')}_{\Sigma'}(\mathrm{SEN}'^{\Theta^{N'}(T')}(f)(\psi/\Theta^{N'}_{\Sigma}(T')), \vec{\chi}/\Theta^{N'}_{\Sigma'}(T'))). \end{aligned}$$

Then, unraveling the definitions of $C'^{T'^{\Theta^{N'}(T')}}$, $\sigma'^{\Theta^{N'}(T')}$, and $\mathrm{SEN}'^{\Theta^{N'}(T')}(f)$, we obtain that

$$C'^{T'}_{\Sigma'}(\sigma'_{\Sigma'}(\mathrm{SEN}'(f)(\varphi), \vec{\chi})) = C'^{T'}_{\Sigma'}(\sigma'_{\Sigma'}(\mathrm{SEN}'(f)(\psi), \vec{\chi})).$$

But this gives that $\langle \varphi, \psi\rangle \in \Theta^{N'}_{\Sigma}(T')$, whence $\varphi/\Theta^{N'}_{\Sigma}(T') = \psi/\Theta^{N'}_{\Sigma}(T')$ and $\mathcal{I}'^{T'}/\Theta^{N'}(T')$ is $N'^{\Theta^{N'}(T')}$-Tarski reduced. This, combined with the hypothesis, shows that $\widetilde{\Omega}^N(\mathcal{I}^T) = \Delta^{\mathrm{SEN}}$, whence, by Proposition 3.3, it follows that $\Theta^N(T) = \Delta^{\mathrm{SEN}}$ and $\mathcal{I}$ is N-Suszko reduced with respect to T. □

Given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, and a theory system $T \in \mathrm{ThSys}(\mathcal{I})$, the quotient π-institution $\mathcal{I}^T/\Theta^N(T)$ is termed *the N-Suszko reduction of the π-institution $\mathcal{I}$ with respect to the theory system T*. Note that with this terminology at hand, Theorem 4.1 says that a π-institution $\mathcal{I}$ is N-Suszko reduced with respect to a theory system T if and only if $\mathcal{I}^T$ is isomorphic to the N'-Suszko reduction $\mathcal{I}'^{T'}/\Theta^{N'}(T')$ of some π-institution $\mathcal{I}'$ with respect to its theory system T'.

Proposition 4.2 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN*. If $T \in \mathrm{ThSys}(\mathcal{I})$ and θ is an N-congruence system on* SEN *such that $\mathcal{I}^T/\theta$ is N^θ-Suszko reduced with respect to T/θ, then $\Theta^N(T) \le \theta$.*

P r o o f. Suppose that $\Sigma \in |\mathbf{Sign}|$, $\varphi, \psi \in \mathrm{SEN}(\Sigma)$ are such that $\langle \varphi, \psi\rangle \in \Theta^N_\Sigma(T)$. Thus, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\sigma : \mathrm{SEN}^n \longrightarrow \mathrm{SEN}$ in N, and $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{n-1}$,

$$C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi})\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})\}),$$

i. e.,

$$C^T_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi})) = C^T_{\Sigma'}(\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})).$$

This, now, implies, by passing to the quotient $\mathcal{I}^T/\theta$, that

$$C^{T^\theta}_{\Sigma'}(\sigma^\theta_{\Sigma'}(\mathrm{SEN}^\theta(f)(\varphi/\theta_\Sigma), \vec{\chi}/\theta_{\Sigma'})) = C^{T^\theta}_{\Sigma'}(\sigma^\theta_{\Sigma'}(\mathrm{SEN}^\theta(f)(\psi/\theta_\Sigma), \vec{\chi}/\theta_{\Sigma'})).$$

Thus, we obtain that $\langle \varphi/\theta_\Sigma, \psi/\theta_\Sigma\rangle \in \Theta^{N^\theta}_\Sigma(T/\theta)$. Since, however, $\mathcal{I}^T/\theta$ is, by hypothesis, N^θ-Suszko reduced with respect to T/θ, we get that $\varphi/\theta_\Sigma = \psi/\theta_\Sigma$, i. e., that $\langle \varphi, \psi\rangle \in \theta_\Sigma$. This proves that $\Theta^N(T) \le \theta$. □

Corollary 4.3 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, with N a category of natural transformations on* SEN*, and $T \in \mathrm{ThSys}(\mathcal{I})$. $\Theta^N(T)$ is the only logical N-congruence system θ of $\mathcal{I}^T$ such that $\mathcal{I}^T/\theta$ is N^θ-Suszko reduced with respect to T/θ.*

P r o o f. If θ is an N-congruence system on SEN such that $\mathcal{I}^T/\theta$ is N^θ-Suszko reduced with respect to T/θ, then, by Proposition 4.2, we obtain $\Theta^N(T) \le \theta$. On the other hand, if θ is a logical N-congruence system of $\mathcal{I}^T$, then $\theta \le \Theta^N(T)$, since $\Theta^N(T)$ is, by Proposition 3.3, the largest logical N-congruence system of $\mathcal{I}^T$. □

5 Determinator systems for the Suszko operator

In [7, Section 4], Czelakowski introduced determinator systems for matrix models of a given sentential logic $\mathcal{S}$. These are sets of pairs of sentential formulas that define, in a specific technical sense, the Suszko relation of the matrix model with reference to the closure system on the underlying algebra of the model consisting of all $\mathcal{S}$-filters. In this section the determinator systems for the Suszko operator are abstracted from the framework of sentential logics to the categorical level. Many of the results shown to hold for sentential logics in [7] will also be carried over without significant changes to the context of logics formalized as π-institutions.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on SEN. Suppose that $E = \{\langle \delta^i, \varepsilon^i\rangle : i \in I\}$ is a collection of pairs of natural transformations $\delta^i, \varepsilon^i : \mathrm{SEN}^{2+k_i} \longrightarrow \mathrm{SEN}$ in N. Given a theory family $T \in \mathrm{ThFam}(\mathcal{I})$, define on SEN the (binary) relation family $E(T) = \{E_\Sigma(T)\}_{\Sigma\in|\mathbf{Sign}|}$ by setting, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{aligned} \langle\varphi,\psi\rangle \in E_\Sigma(T) \quad \text{iff} \quad & C_{\Sigma'}(T_{\Sigma'} \cup \{\delta^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi), \vec{\chi})\}) \\ & \quad = C_{\Sigma'}(T_{\Sigma'} \cup \{\varepsilon^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi), \vec{\chi})\}) \\ & \text{for all } \Sigma' \in |\mathbf{Sign}|,\ f \in \mathbf{Sign}(\Sigma,\Sigma'),\ i \in I,\ \vec{\chi} \in \mathrm{SEN}(\Sigma')^{k_i}. \end{aligned}$$

We follow [7] in calling the relation family $E(T)$ *the analytical relation system on* SEN *determined by* T *and* E. This terminology is justified by the following lemma:

Lemma 5.1 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Let $E = \{\langle \delta^i, \varepsilon^i\rangle : i \in I\}$ be a collection of pairs of natural transformations $\delta^i, \varepsilon^i : \mathrm{SEN}^{2+k_i} \longrightarrow \mathrm{SEN}$ in N. For every theory family $T \in \mathrm{ThFam}(\mathcal{I})$, $E(T)$ is a relation system.*

Proof. Suppose that $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, $\varphi, \psi \in \mathrm{SEN}(\Sigma_1)$ are such that $\langle\varphi,\psi\rangle \in E_{\Sigma_1}(T)$. Then

$$C_{\Sigma'}(T_{\Sigma'} \cup \{\delta^i_{\Sigma'}(\mathrm{SEN}(g)(\varphi), \mathrm{SEN}(g)(\psi), \vec{\chi})\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\varepsilon^i_{\Sigma'}(\mathrm{SEN}(g)(\varphi), \mathrm{SEN}(g)(\psi), \vec{\chi})\}),$$

for all $\Sigma' \in |\mathbf{Sign}|$, $g \in \mathbf{Sign}(\Sigma_1, \Sigma')$, $i \in I$, $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k_i}$,

$$\begin{tikzcd} \Sigma_1 \arrow[rr, "f"] \arrow[dr, "g"'] & & \Sigma_2 \arrow[dl, "h"] \\ & \Sigma'. & \end{tikzcd}$$

This implies that, for all $\Sigma' \in |\mathbf{Sign}|$, $h \in \mathbf{Sign}(\Sigma_2, \Sigma')$, $i \in I$, $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k_i}$,

$$\begin{aligned} & C_{\Sigma'}(T_{\Sigma'} \cup \{\delta^i_{\Sigma'}(\mathrm{SEN}(h)(\mathrm{SEN}(f)(\varphi)), \mathrm{SEN}(h)(\mathrm{SEN}(f)(\psi)), \vec{\chi})\}) \\ & \quad = C_{\Sigma'}(T_{\Sigma'} \cup \{\varepsilon^i_{\Sigma'}(\mathrm{SEN}(h)(\mathrm{SEN}(f)(\varphi)), \mathrm{SEN}(h)(\mathrm{SEN}(f)(\psi)), \vec{\chi})\}). \end{aligned}$$

Therefore $\langle \mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi)\rangle \in E_{\Sigma_2}(T)$ and $E(T)$ is a relation system, as was to be shown. □

In a way very similar to the way an N-parameterized equivalence system behaves with respect to the N-Leibniz operator on π-institutions (see, e. g., [24, Proposition 4.1]), the analytical relation system $E(T)$ on SEN determined by E and T has the property that, if it is reflexive, then it contains the N-Suszko congruence system of T.

Lemma 5.2 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Let $E = \{\langle \delta^i, \varepsilon^i\rangle : i \in I\}$ be a collection of pairs of natural transformations $\delta^i, \varepsilon^i : \mathrm{SEN}^{2+k_i} \longrightarrow \mathrm{SEN}$ in N. For every theory family $T \in \mathrm{ThFam}(\mathcal{I})$, if the relation system $E(T)$ is reflexive on* SEN, *then*

$$\Theta^N(T) \leq E(T).$$

P r o o f. In fact, suppose that $\Sigma \in |\mathbf{Sign}|$, $\varphi, \psi \in \mathrm{SEN}(\Sigma)$ such that $\langle \varphi, \psi \rangle \in \Theta^N_\Sigma(T)$. This means that for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\sigma : \mathrm{SEN}^n \longrightarrow \mathrm{SEN}$ in N, $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{n-1}$,

$$C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi})\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})\}).$$

This implies that, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $i \in I$, $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k_i}$,

$$C_{\Sigma'}(T_{\Sigma'} \cup \{\delta^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi), \vec{\chi})\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\delta^i_{\Sigma'}(\mathrm{SEN}(f)(\psi), \mathrm{SEN}(f)(\psi), \vec{\chi})\}),$$

and

$$C_{\Sigma'}(T_{\Sigma'} \cup \{\varepsilon^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi), \vec{\chi})\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\varepsilon^i_{\Sigma'}(\mathrm{SEN}(f)(\psi), \mathrm{SEN}(f)(\psi), \vec{\chi})\}).$$

But $E(T)$ being reflexive also yields that, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $i \in I$, $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k_i}$,

$$C_{\Sigma'}(T_{\Sigma'} \cup \{\delta^i_{\Sigma'}(\mathrm{SEN}(f)(\psi), \mathrm{SEN}(f)(\psi), \vec{\chi})\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\varepsilon^i_{\Sigma'}(\mathrm{SEN}(f)(\psi), \mathrm{SEN}(f)(\psi), \vec{\chi})\}).$$

Thus, combining all three relations, we obtain that, for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $i \in I$, $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k_i}$,

$$C_{\Sigma'}(T_{\Sigma'} \cup \{\delta^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi), \vec{\chi})\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\varepsilon^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi), \vec{\chi})\}),$$

which shows that $\langle \varphi, \psi \rangle \in E_\Sigma(T)$. Therefore, $\Theta^N_\Sigma(T) \subseteq E_\Sigma(T)$. Since $\Sigma \in |\mathbf{Sign}|$ was arbitrary, we finally obtain $\Theta^N(T) \leq E(T)$. □

If T is a theory family of $\mathcal{I}$ and E is a collection of pairs of natural transformations in N as in Lemma 5.2 such that $\Theta^N(T) = E(T)$, then E is said to be an *N-(parameterized) determinator system for T*. Moreover, E will be called an *N-(parameterized) determinator system of the N-Suszko operator for $\mathcal{I}$* if and only if E is an N-parameterized determinator system for every theory family $T \in \mathrm{ThFam}(\mathcal{I})$.

It is shown, next, that the N-Suszko operator for every π-institution $\mathcal{I}$ possesses such an N-parameterized determinator. In fact, analogously with the case of sentential logics (see [7, Theorem 4.2]), the collection of all pairs of natural transformations in at least one variable yields an N-parameterized determinator for Θ^N.

Theorem 5.3 *For every π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on* SEN*, the N-Suszko operator for $\mathcal{I}$ possesses an N-parameterized determinator system.*

P r o o f. Let

$$Z = \{\zeta^i : \mathrm{SEN}^{1+k_i} \longrightarrow \mathrm{SEN} \,:\, i \in I\}$$

be the collection of all natural transformations $\zeta^i : \mathrm{SEN}^{1+k_i} \longrightarrow \mathrm{SEN}$ in N with at least one argument. Form the collection $E = \{\langle \delta^i, \varepsilon^i \rangle \,:\, i \in I\}$, where $\delta^i, \varepsilon^i : \mathrm{SEN}^{2+k_i} \longrightarrow \mathrm{SEN}$ are defined, for all $i \in I$, all $\Sigma \in |\mathbf{Sign}|$, and all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$, $\vec{\chi} \in \mathrm{SEN}(\Sigma)^{k_i}$, by

$$\delta^i_\Sigma(\varphi, \psi, \vec{\chi}) = \zeta^i_\Sigma(\varphi, \vec{\chi}) \quad \text{and} \quad \varepsilon^i_\Sigma(\varphi, \psi, \vec{\chi}) = \zeta^i_\Sigma(\psi, \vec{\chi}).$$

To show that E is an N-parameterized determinator of the N-Suszko operator for $\mathcal{I}$, follow the following string of equivalences, for all $T \in \mathrm{ThFam}(\mathcal{I})$, $\Sigma \in |\mathbf{Sign}|$, $\varphi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\begin{aligned}
&\langle \varphi, \psi \rangle \in \Theta^N_\Sigma(T)\\
&\quad \text{iff} \quad C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi})\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})\})\\
&\quad\phantom{\text{iff}}\quad \text{for all } \Sigma' \in |\mathbf{Sign}|,\ f \in \mathbf{Sign}(\Sigma, \Sigma'),\ \sigma : \mathrm{SEN}^n \longrightarrow \mathrm{SEN} \text{ in } N,\ \vec{\chi} \in \mathrm{SEN}(\Sigma')^{n-1}\\
&\quad \text{iff} \quad C_{\Sigma'}(T_{\Sigma'} \cup \{\zeta^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi})\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\zeta^i_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})\})\\
&\quad\phantom{\text{iff}}\quad \text{for all } \Sigma' \in |\mathbf{Sign}|,\ f \in \mathbf{Sign}(\Sigma, \Sigma'),\ i \in I,\ \vec{\chi} \in \mathrm{SEN}(\Sigma')^{k_i}\\
&\quad \text{iff} \quad C_{\Sigma'}(T_{\Sigma'} \cup \{\delta^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi), \vec{\chi})\})\\
&\quad\phantom{\text{iff}}\quad\qquad = C_{\Sigma'}(T_{\Sigma'} \cup \{\varepsilon^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi), \vec{\chi})\})\\
&\quad\phantom{\text{iff}}\quad \text{for all } \Sigma' \in |\mathbf{Sign}|,\ f \in \mathbf{Sign}(\Sigma, \Sigma'),\ i \in I,\ \vec{\chi} \in \mathrm{SEN}(\Sigma')^{k_i}\\
&\quad \text{iff} \quad \langle \varphi, \psi \rangle \in E_\Sigma(T).
\end{aligned}$$

□

Another way of constructing an N-parameterized determinator in the sentential logic framework was also provided by Czelakowski in [7, Proposition 4.3]. This new construction may be carried over as well to the π-institution framework. This is done in Proposition 5.4. Roughly speaking, instead of starting with the collection of all natural transformations in N in at least one argument, the collection of all natural transformations in N in at least two arguments is considered, and those pairs of natural transformations in two arguments that satisfy an extra property are collected together to form the new N-parameterized determinator.

Proposition 5.4 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Let E' be the collection of all pairs $\langle \delta^i, \varepsilon^i\rangle$ of natural transformations $\delta^i, \varepsilon^i : \mathrm{SEN}^{2+k_i} \longrightarrow \mathrm{SEN}$ in N such that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi \in \mathrm{SEN}(\Sigma)$,*

$$C_{\Sigma'}(\delta^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\varphi), \vec{\chi})) = C_{\Sigma'}(\varepsilon^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\varphi), \vec{\chi}))$$

for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k_i}$. Then E' is an N-parameterized determinator system of the N-Suszko operator for $\mathcal{I}$.

Proof. We follow Czelakowski's proof of [7, Proposition 4.3] in providing a shortcut to the direct verification that E' is an N-parameterized determinator system by using Lemma 5.2, an analog of [7, Lemma 4.1].

The determinator E constructed in Theorem 5.3 is included in the collection E'. It follows that $E'(T) \leq E(T)$, for every theory family T of $\mathcal{I}$. Moreover, by the definition of E', it follows that $E'(T)$ is a reflexive relation system on SEN. Therefore, by Lemma 5.2, we get that $\Theta^N(T) \leq E'(T)$, for every theory family T of $\mathcal{I}$. Putting these together, we, finally, obtain $\Theta^N(T) \leq E'(T) \leq E(T) = \Theta^N(T)$ and, therefore, $E'(T) = \Theta^N(T)$ and E' is indeed an N-parameterized determinator system for Θ^N. □

Using the definition of an N-parameterized determinator system for the N-Suszko operator of a π-institution and that of the N-Suszko reduction of a π-institution with respect to a theory family, we obtain

Corollary 5.5 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Let, also, E be an N-parameterized determinator system for the N-Suszko operator of $\mathcal{I}$. Then, for every theory system $T \in \mathrm{ThSys}(\mathcal{I})$, $\mathcal{I}$ is N-Suszko reduced with respect to T (i. e., $\mathcal{I}^T$ is N-Tarski reduced) if and only if $E(T) = \Delta^{\mathrm{SEN}}$, the identity relation system on* SEN.

Suppose now that $\mathrm{SEN} : \mathbf{Sign} \longrightarrow \mathbf{Set}$ is a functor and N a category of natural transformations on SEN. A relation system $R = \{R_\Sigma\}_{\Sigma \in |\mathbf{Sign}|}$ is said to be *compatible with* N if for all $\Sigma \in |\mathbf{Sign}|$, all $\sigma : \mathrm{SEN}^n \longrightarrow \mathrm{SEN}$ in N, and all $\vec{\varphi}, \vec{\psi} \in \mathrm{SEN}(\Sigma)^n$,

$$\langle \varphi_i, \psi_i\rangle \in R_\Sigma \text{ for } i < n \quad \text{imply} \quad \langle \sigma_\Sigma(\vec{\varphi}), \sigma_\Sigma(\vec{\psi})\rangle \in R_\Sigma.$$

If, on the other hand, $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, where N is a category of natural transformations on SEN, and T is a theory family of $\mathcal{I}$, then the relation system R is said to be *compatible with C-interderivability modulo* T if, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$,

$$\langle \varphi, \psi\rangle \in R_\Sigma \quad \text{implies} \quad C_\Sigma(T_\Sigma \cup \{\varphi\}) = C_\Sigma(T_\Sigma \cup \{\psi\}).$$

In the next lemma, it is shown that a reflexive relation system R on the sentence functor SEN of a given π-institution $\mathcal{I}$ that satisfies both compatibility with a category N of natural transformations on SEN and compatibility with C-interderivability modulo a theory family T of $\mathcal{I}$, must be included in the Suszko N-congruence system $\Theta^N(T)$ of T.

Lemma 5.6 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *If $T \in \mathrm{ThFam}(\mathcal{I})$ and R is a reflexive relation system on* SEN *satisfying the properties of compatibility with C-interderivability modulo T and of compatibility with N, then $R \leq \Theta^N(T)$.*

Proof. Let $\Sigma \in |\mathbf{Sign}|$, $\varphi, \psi \in \mathrm{SEN}(\Sigma)$ be such that $\langle \varphi, \psi\rangle \in R_\Sigma$. Now let $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $\sigma : \mathrm{SEN}^k \longrightarrow \mathrm{SEN}$ in N, and $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k-1}$. Then, since R is a relation system, we obtain that

$$\langle \mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi)\rangle \in R_{\Sigma'}$$

and since R is reflexive, we get that $\langle \chi_i, \chi_i \rangle \in R_{\Sigma'}$, for all $i < k$. Thus, since R satisfies compatibility with N, we obtain $\langle \sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi}), \sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})\rangle \in R_{\Sigma'}$. Therefore, by the compatibility with C-interderivability modulo T, we get that

$$C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \vec{\chi})\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\psi), \vec{\chi})\}).$$

This proves that $\langle \varphi, \psi \rangle \in \Theta^N_\Sigma(T)$. Therefore $R_\Sigma \subseteq \Theta^N_\Sigma(T)$ and, because Σ has been arbitrary, we, finally, obtain that $R \leq \Theta^N(T)$. □

We use Lemma 5.6 to prove Theorem 5.7, an analog of [7, Theorem 4.5], which provides a characterization of N-parameterized determinator systems for theory families T of a given π-institution $\mathcal{I}$.

Theorem 5.7 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Suppose, also, that $T \in \mathrm{ThFam}(\mathcal{I})$ and $E = \{\langle \delta^i, \varepsilon^i\rangle : i \in I\}$ is a collection of pairs of natural transformations $\delta^i, \varepsilon^i : \mathrm{SEN}^{2+k_i} \longrightarrow \mathrm{SEN}$, $i \in I$, in N. Then E is an N-parameterized determinator system for T if and only if the following three conditions are satisfied:*

(PD0) *The relation system $E(T)$ is reflexive.*

(PD1) *The relation system $E(T)$ is compatible with C-interderivability modulo T.*

(PD2) *The relation system $E(T)$ is compatible with N.*

P r o o f. Suppose, first, that E is an N-parameterized determinator system for T. Then $E(T) = \Theta^N(T)$. Now properties (PD0) – (PD2) obviously hold for $E(T)$ because they hold for $\Theta^N(T)$.

Suppose, conversely, that properties (PD0) – (PD2) hold for $E(T)$. Then property (PD0), combined with Lemma 5.2, yields that $\Theta^N(T) \leq E(T)$. But (PD0) – (PD2), combined with Lemma 5.6, yield that $E(T) \leq \Theta^N(T)$. Therefore, we obtain that $E(T) = \Theta^N(T)$ and, hence, E is an N-parameterized determinator system for T. □

We immediately obtain, by the definitions involved and Theorem 5.7:

Corollary 5.8 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on* SEN, *and $E = \{\langle \delta^i, \varepsilon^i\rangle : i \in I\}$ a collection of pairs of natural transformations $\delta^i, \varepsilon^i : \mathrm{SEN}^{2+k_i} \longrightarrow \mathrm{SEN}$, $i \in I$, in N. E is an N-parameterized determinator system of the N-Suszko operator of $\mathcal{I}$ if and only if the conditions* (PD0) – (PD2) *hold, for every theory family $T \in \mathrm{ThFam}(\mathcal{I})$.*

The next result shows, informally speaking, that the operator E on theory families commutes with inverse bilogical morphisms. This forms an analog at the level of π-institutions of [7, Lemma 4.9].

Lemma 5.9 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, N is a category of natural transformations on* SEN, *and $E = \{\langle \delta^i, \varepsilon^i\rangle : i \in I\}$ a collection of pairs of natural transformations $\delta^i, \varepsilon^i : \mathrm{SEN}^{2+k_i} \longrightarrow \mathrm{SEN}$, $i \in I$, in N. Suppose, also, that $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$, with N' a category of natural transformations on* SEN', *is an (N, N')-model of $\mathcal{I}$ via an (N, N')-bilogical morphism $\langle F, \alpha\rangle : \mathcal{I} \vdash^{\mathrm{se}} \mathcal{I}'$. Then, for all $T' \in \mathrm{ThFam}(\mathcal{I}')$, $E(\alpha^{-1}(T')) = \alpha^{-1}(E'(T'))$, where E' is the collection of pairs of natural transformations in N' corresponding to E via the (N, N')-epimorphic property.*

P r o o f. We have that, for all $\Sigma \in |\mathbf{Sign}|$ and all $\varphi, \psi \in \mathrm{SEN}(\Sigma)$, $\langle \varphi, \psi\rangle \in E_\Sigma(\alpha^{-1}(T'))$ if and only if for all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, $i \in I$, and $\vec{\chi} \in \mathrm{SEN}(\Sigma')^{k_i}$,

$$\begin{aligned}&C_{\Sigma'}(\alpha^{-1}_{\Sigma'}(T'_{F(\Sigma')}) \cup \{\delta^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi), \vec{\chi})\})\\ &\quad = C_{\Sigma'}(\alpha^{-1}_{\Sigma'}(T'_{F(\Sigma')}) \cup \{\varepsilon^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi), \vec{\chi})\})\end{aligned}$$

if and only if, since $\langle F, \alpha\rangle$ is an (N, N')-bilogical morphism,

$$\begin{aligned}&C'_{F(\Sigma')}(T'_{F(\Sigma')} \cup \{\alpha_{\Sigma'}(\delta^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi), \vec{\chi}))\})\\ &\quad = C'_{F(\Sigma')}(T'_{F(\Sigma')} \cup \{\alpha_{\Sigma'}(\varepsilon^i_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi), \vec{\chi}))\})\end{aligned}$$

if and only if, by the (N, N')-epimorphic property,

$$\begin{aligned}&C'_{F(\Sigma')}(T'_{F(\Sigma')} \cup \{\delta'^i_{F(\Sigma')}(\alpha^{2+k_i}_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi), \vec{\chi}))\})\\ &\quad = C'_{F(\Sigma')}(T'_{F(\Sigma')} \cup \{\varepsilon'^i_{F(\Sigma')}(\alpha^{2+k_i}_{\Sigma'}(\mathrm{SEN}(f)(\varphi), \mathrm{SEN}(f)(\psi), \vec{\chi}))\})\end{aligned}$$

if and only if, since α is a natural transformation,

$$C'_{F(\Sigma')}(T'_{F(\Sigma')} \cup \{\delta'^{i}_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\varphi)), \mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi)), \alpha^{k_i}_{\Sigma'}(\vec{\chi}))\}) \\ = C'_{F(\Sigma')}(T'_{F(\Sigma')} \cup \{\varepsilon'^{i}_{F(\Sigma')}(\mathrm{SEN}'(F(f))(\alpha_\Sigma(\varphi)), \mathrm{SEN}'(F(f))(\alpha_\Sigma(\psi)), \alpha^{k_i}_{\Sigma'}(\vec{\chi}))\})$$

if and only if, by the surjectivity of $\langle F, \alpha\rangle$, $\langle \alpha_\Sigma(\varphi), \alpha_\Sigma(\psi)\rangle \in E'_{F(\Sigma)}(T')$. □

Lemma 5.9 helps in establishing a preservation theorem for N-parameterized determinator systems of theory families along bilogical morphisms.

Theorem 5.10 *Suppose that $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ is a π-institution, N a category of natural transformations on* SEN*, and $E = \{\langle \delta^i, \varepsilon^i\rangle : i \in I\}$ a collection of pairs of natural transformations $\delta^i, \varepsilon^i : \mathrm{SEN}^{2+k_i} \longrightarrow \mathrm{SEN}$, $i \in I$, in N. Suppose, also, that $\mathcal{I}' = \langle \mathbf{Sign}', \mathrm{SEN}', C'\rangle$, with N' a category of natural transformations on* SEN′*, is an (N, N')-model of $\mathcal{I}$ via an (N, N')-bilogical morphism $\langle F, \alpha\rangle : \mathcal{I} \vdash^{\mathrm{se}} \mathcal{I}'$. If $T' \in \mathrm{ThFam}(\mathcal{I}')$, then E is an N-determinator system for the theory family $\alpha^{-1}(T') \in \mathrm{ThFam}(\mathcal{I})$ if and only if E' is an N'-determinator system for T', where E' is the collection of pairs of natural transformations in N' corresponding to E via the (N, N')-epimorphic property.*

P r o o f. We have that $E(\alpha^{-1}(T)) = \Theta^N(\alpha^{-1}(T))$ if and only if, by Corollary 3.12 and Lemma 5.9,

$$\alpha^{-1}(E'(T)) = \alpha^{-1}(\Theta^{N'}(T))$$

if and only if, by surjectivity of $\langle F, \alpha\rangle$, $E'(T) = \Theta^{N'}(T)$. Thus, E is an N-parameterized determinator system for $\alpha^{-1}(T)$ if and only if E' is an N'-parameterized determinator system for T. □

The following corollary easily follows from Theorem 5.10 and relates determinator systems of the Suszko operator of a given π-institution with determinator systems of the collection of all theory families of its Suszko reduct.

Corollary 5.11 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, N a category of natural transformations on* SEN*, and $E = \{\langle \delta^i, \varepsilon^i\rangle : i \in I\}$ a collection of pairs of natural transformations $\delta^i, \varepsilon^i : \mathrm{SEN}^{2+k_i} \longrightarrow \mathrm{SEN}$, $i \in I$, in N. Then E is an N-determinator system for the N-Suszko operator of $\mathcal{I}$ if and only if, for each $T \in \mathrm{ThSys}(\mathcal{I})$, $E^{\Theta^N(T)}$ is an $N^{\Theta^N(T)}$-determinator system for the $N^{\Theta^N(T)}$-Suszko operator of $\mathcal{I}^T/\Theta^N(T)$.*

6 A Lifting Theorem

In this last section of the paper, we present an analog of [7, Proposition 5.1]. [7, Proposition 5.1] asserts that every countable Suszko-reduced model of a given sentential logic is a strict homomorphic image of the Suszko reduction of a Lindenbaum matrix. Its analog in the π-institution framework, the Lifting Theorem 6.2, states, informally speaking, that, given a π-institution $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, with N a category of natural transformations on SEN, and T a theory system of $\mathcal{I}$ such that $\mathcal{I}$ is N-Suszko reduced with respect to T, there exist a theory system X of the π-institution $\mathcal{I}^t = \langle \mathbf{Sign}, \mathrm{Te}^N \circ \mathrm{SEN}, C^t\rangle$, whose sentences are N-terms in the sentences of $\mathcal{I}$, and an $(N^{t^{\Theta^{N^t}(X)}}, N)$-bilogical morphism $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha\rangle : \mathcal{I}^{t^X}/\Theta^{N^t}(X) \vdash^{\mathrm{se}} \mathcal{I}^T$, i. e., $\mathcal{I}^T$ is, in some sense, an epimorphic image of the Suszko reduction of $\mathcal{I}^{t^X}$ with respect to the theory system X.

Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$ be a π-institution, with N a category of natural transformations on SEN. Recall from [26] the construction of the functor $\mathrm{Te}^N \circ \mathrm{SEN} : \mathbf{Sign} \longrightarrow \mathbf{Set}$, of the category N' of natural transformations on $\mathrm{Te}^N \circ \mathrm{SEN}$, and of the singleton (N', N)-epimorphic translation $\langle \mathrm{I}_{\mathbf{Sign}}, \mu^N\rangle : \mathrm{Te}^N \circ \mathrm{SEN} \longrightarrow \mathrm{SEN}$. All these concepts are used now to provide an analog of [7, Proposition 5.1], which states that every countable Suszko-reduced model of a given sentential logic is a strict homomorphic image of the Suszko reduction of a Lindenbaum matrix. For the purpose of having a uniform notation below, the category of natural transformations on $\mathrm{Te}^N \circ \mathrm{SEN}$, that was denoted by N' in [26], will now be denoted by N^t.

First define, given $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C\rangle$, the triple $\mathcal{I}^t = \langle \mathbf{Sign}, \mathrm{Te}^N \circ \mathrm{SEN}, C^t\rangle$ by letting, for all $\Sigma \in |\mathbf{Sign}|$, $C^t_\Sigma : \mathcal{P}(\mathrm{Te}^N(\mathrm{SEN}(\Sigma))) \longrightarrow \mathcal{P}(\mathrm{Te}^N(\mathrm{SEN}(\Sigma)))$ be defined, for all $U \cup \{t\} \subseteq \mathrm{Te}^N(\mathrm{SEN}(\Sigma))$, by

$$t \in C^t_\Sigma(U) \quad \text{iff} \quad \mu^N_\Sigma(t) \in C_\Sigma(\mu^N_\Sigma(U)).$$

Then the following lemma is easy to verify:

Lemma 6.1 *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ be a π-institution, with N a category of natural transformations on* SEN. *Then $\mathcal{I}^t = \langle \mathbf{Sign}, \mathrm{Te}^N \circ \mathrm{SEN}, C^t \rangle$ is also a π-institution. Moreover, the singleton (N^t, N)-epimorphic translation $\langle \mathrm{I}_{\mathbf{Sign}}, \mu^N \rangle : \mathrm{Te}^N \circ \mathrm{SEN} \longrightarrow^{\mathrm{se}} \mathrm{SEN}$ is an (N^t, N)-bilogical morphism $\langle \mathrm{I}_{\mathbf{Sign}}, \mu^N \rangle : \mathcal{I}^t \vdash^{\mathrm{se}} \mathcal{I}$.*

Lemma 6.1 will be used now to prove Proposition 6.2, the promised analog of [7, Proposition 5.1].

Theorem 6.2 (Lifting Theorem) *Let $\mathcal{I} = \langle \mathbf{Sign}, \mathrm{SEN}, C \rangle$ be a π-institution, with N a category of natural transformations on* SEN, *and T a theory system of $\mathcal{I}$ such that $\mathcal{I}$ is N-Suszko reduced with respect to T, i. e., such that $\mathcal{I}^T$ is N-Tarski reduced. Then there exist a theory system X of $\mathcal{I}^t = \langle \mathbf{Sign}, \mathrm{Te}^N \circ \mathrm{SEN}, C^t \rangle$ and an $(N^{t^{\Theta^{N^t}(X)}}, N)$-bilogical morphism*

$$\langle \mathrm{I}_{\mathbf{Sign}}, \alpha \rangle : \mathcal{I}^{t^X} / \Theta^{N^t}(X) \vdash^{\mathrm{se}} \mathcal{I}^T$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{I}^{t^X} \arrow[rr, "\langle \mathrm{I}_{\mathbf{Sign}}, \mu^N \rangle"] \arrow[dr, "\langle \mathrm{I}_{\mathbf{Sign}}, \pi^{\Theta^{N^t}(X)} \rangle"'] & & \mathcal{I}^T \\
& \mathcal{I}^{t^X} / \Theta^{N^t}(X). \arrow[ur, "\langle \mathrm{I}_{\mathbf{Sign}}, \alpha \rangle"'] &
\end{tikzcd}$$

P r o o f. Define the theory system X by $X := (\mu^N)^{-1}(T)$. Furthermore, let

$$\alpha : \mathrm{Te}^N \circ \mathrm{SEN}/\Theta^{N^t}(X) \longrightarrow \mathrm{SEN}$$

be given, for all $\Sigma \in |\mathbf{Sign}|$ and all $t \in \mathrm{Te}^N(\mathrm{SEN}(\Sigma))$, by

$$\alpha_\Sigma(t/\Theta_\Sigma^{N^t}(X)) = \mu_\Sigma^N(t).$$

It will now be shown that α_Σ is well-defined. Suppose, to this end, that $\langle s, t \rangle \in \Theta_\Sigma^{N^t}(X)$. Then we have that for all $\sigma : \mathrm{SEN}^n \longrightarrow \mathrm{SEN}$ in N, all $\Sigma' \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma, \Sigma')$, and all $\vec{u} \in \mathrm{Te}^N(\mathrm{SEN}(\Sigma'))^{n-1}$,

$$C_{\Sigma'}^t(X_{\Sigma'} \cup \{\sigma_{\Sigma'}^t(\mathrm{Te}^N(\mathrm{SEN}(f))(s), \vec{u}\,)\}) = C_{\Sigma'}^t(X_{\Sigma'} \cup \{\sigma_{\Sigma'}^t(\mathrm{Te}^N(\mathrm{SEN}(f))(t), \vec{u}\,)\}).$$

This implies, by Lemma 6.1, that

$$\begin{aligned} &C_{\Sigma'}(\mu_{\Sigma'}^N(X_{\Sigma'}) \cup \{\mu_{\Sigma'}^N(\sigma_{\Sigma'}^t(\mathrm{Te}^N(\mathrm{SEN}(f))(s), \vec{u}\,))\}) \\ &\quad = C_{\Sigma'}(\mu_{\Sigma'}^N(X_{\Sigma'}) \cup \{\mu_{\Sigma'}^N(\sigma_{\Sigma'}^t(\mathrm{Te}^N(\mathrm{SEN}(f))(t), \vec{u}\,))\}). \end{aligned}$$

Hence, we obtain

$$\begin{aligned} &C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mu_{\Sigma'}^N(\mathrm{Te}^N(\mathrm{SEN}(f))(s)), \mu_{\Sigma'}^N(\vec{u}\,))\}) \\ &\quad = C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mu_{\Sigma'}^N(\mathrm{Te}^N(\mathrm{SEN}(f))(t)), \mu_{\Sigma'}^N(\vec{u}\,))\}), \end{aligned}$$

which gives

$$C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\mu_\Sigma^N(s)), \mu_{\Sigma'}^N(\vec{u}\,))\}) = C_{\Sigma'}(T_{\Sigma'} \cup \{\sigma_{\Sigma'}(\mathrm{SEN}(f)(\mu_\Sigma^N(t)), \mu_{\Sigma'}^N(\vec{u}\,))\}).$$

Thus, by the surjectivity of μ^N, we obtain that $\langle \mu_\Sigma^N(s), \mu_\Sigma^N(t) \rangle \in \Theta_\Sigma^N(T)$. But $\mathcal{I}$ was assumed to be N-Suszko reduced with respect to T, whence $\Theta^N(T) = \Delta^{\mathrm{SEN}}$, and, therefore, $\mu_\Sigma^N(s) = \mu_\Sigma^N(t)$, as was to be shown.

Next, it will be shown that $\alpha : \mathrm{Te}^N \circ \mathrm{SEN}/\Theta^{N^t}(X) \longrightarrow \mathrm{SEN}$ is a natural transformation. To this end, suppose that $\Sigma_1, \Sigma_2 \in |\mathbf{Sign}|$, $f \in \mathbf{Sign}(\Sigma_1, \Sigma_2)$, and $t \in \mathrm{Te}^N(\mathrm{SEN}(\Sigma_1))/\Theta^{N^t}_{\Sigma_1}(X)$. We have

$$\begin{tikzcd}
\mathrm{Te}^N(\mathrm{SEN}(\Sigma_1))/\Theta^{N^t}_{\Sigma_1}(X) \arrow[r, "\alpha_{\Sigma_1}"] \arrow[d, "\mathrm{Te}^N(\mathrm{SEN}(f))/\Theta^{N^t}(X)"'] & \mathrm{SEN}(\Sigma_1) \arrow[d, "\mathrm{SEN}(f)"] \\
\mathrm{Te}^N(\mathrm{SEN}(\Sigma_2))/\Theta^{N^t}_{\Sigma_2}(X) \arrow[r, "\alpha_{\Sigma_2}"'] & \mathrm{SEN}(\Sigma_2),
\end{tikzcd}$$

$$\begin{aligned}
\alpha_{\Sigma_2}(\mathrm{Te}^N(\mathrm{SEN}(f))/\Theta^{N^t}(X)(t/\Theta^{N^t}_{\Sigma_1}(X))) &= \alpha_{\Sigma_2}(\mathrm{Te}^N(\mathrm{SEN}(f))(t)/\Theta^{N^t}_{\Sigma_2}(X)) \\
&= \mu^N_{\Sigma_2}(\mathrm{Te}^N(\mathrm{SEN}(f))(t)) \\
&= \mathrm{SEN}(f)(\mu^N_{\Sigma_1}(t)) \\
&= \mathrm{SEN}(f)(\alpha_{\Sigma_1}(t/\Theta^{N^t}_{\Sigma_1}(X))).
\end{aligned}$$

Finally, it suffices to show that $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha\rangle$ is a bilogical morphism, since commutativity of the triangle is fairly obvious by the definition of $\langle \mathrm{I}_{\mathbf{Sign}}, \alpha\rangle$. In fact, for all $\Sigma \in |\mathbf{Sign}|$, $U \cup \{t\} \in \mathrm{Te}^N(\mathrm{SEN}(\Sigma))$, we have

$$\begin{aligned}
t/\Theta^{N^t}_{\Sigma}(X) \in C^{t^{X^{\Theta^{N^t}(X)}}}_{\Sigma}(U/\Theta^{N^t}_{\Sigma}(X)) \quad &\text{iff} \quad t \in C^{t^X}_{\Sigma}(\textstyle\bigcup U/\Theta^{N^t}_{\Sigma}(X)) \\
&\text{iff} \quad t \in C^{t}_{\Sigma}(X_\Sigma \cup \textstyle\bigcup U/\Theta^{N^t}_{\Sigma}(X)) \\
&\text{iff} \quad \mu^N_{\Sigma}(t) \in C_{\Sigma}(\mu^N_{\Sigma}(X_\Sigma) \cup \mu^N_{\Sigma}(\textstyle\bigcup U/\Theta^{N^t}_{\Sigma}(X))) \\
&\text{iff} \quad \alpha_{\Sigma}(t/\Theta^{N^t}_{\Sigma}(X)) \in C_{\Sigma}(T_\Sigma \cup \alpha_{\Sigma}(U/\Theta^{N^t}_{\Sigma}(X))) \\
&\text{iff} \quad \alpha_{\Sigma}(t/\Theta^{N^t}_{\Sigma}(X)) \in C^{T}_{\Sigma}(\alpha_{\Sigma}(U/\Theta^{N^t}_{\Sigma}(X))). \qquad \square
\end{aligned}$$

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