Applied Categorical Structures **12:** 203–223, 2004.

A Categorical Approach to Threshold Agent Networks ⋆

GEORGE VOUTSADAKIS
School of Mathematics and Computer Science, Lake Superior State University, 650 W. Easterday Avenue, Sault Sainte Marie, MI 49783, U.S.A. e-mail: gvoutsad@lssu.edu

(Received: 19 December 2002; accepted: 6 January 2004)

Abstract. Threshold agent networks (TANs) constitute a discretized modification of threshold (also known as neural) networks that are appropriate for modeling computer simulations. In this paper morphisms are introduced between TANs, thus forming a category **TAN**. Several of the properties of this category are explored. A generalization of TANs to allow more flexible interactions between the agents is then introduced. The new objects are also endowed with appropriate morphisms and the properties of the newly obtained category are compared with the properties of its older subcategory.

Mathematics Subject Classifications (2000): Primary: 93A05, 18B20; secondary: 92B20, 94Cxx, 18A35.

Key words: finite dynamical systems, automata networks, threshold networks, neural networks, threshold agent networks, categories of networks, products, equalizers, limits.

1. Introduction

Threshold or *neural networks* have been introduced in [10] to model some features of the nervous system and have since been developed to model several physical and biological phenomena. These models are now seen to constitute a special subclass of a wider class of systems called *automata networks* [5]. Besides simulating physical and biological phenomena, threshold networks are very successful in providing a framework for modeling and for analyzing properties of computer simulations. This point of view was taken in [13], where a discretized version of a threshold network, called *threshold agent network* (TAN) was introduced as an alternative approach to modeling computer simulations than the one provided using *sequential dynamical systems,* introduced in [2, 3]. The present paper considers TANs from a categorical viewpoint. Consideration of finite dynamical systems in this framework is motivated by the following two basic general categorical principles:

(1) A mathematical construct is best understood when viewed as an object related to other objects of the same or similar kind via morphisms in an appropri-

⋆ This research was supported by a grant from the U.S. Department of Defense.

ate category. This point of view was taken for sequential dynamical systems in [7, 8].

(2) Limits and colimits, and, in particular, products and coproducts, in this category (when they exist) are very useful theoretical tools in decomposing a complex system to simpler systems and in composing a complex system out of simpler ones. Decomposition in this context facilitates the analysis of properties of a system based on the analysis of its parts. Composition, on the other hand, enables building a complex system from smaller systems, whose design is often a more manageable task. This point of view is not new and it underlies, for instance, much of the work done in the area of coalgebraic specifications in theoretical computer science (see, e.g., [6] and [11, 12].) This second principle was also part of the considerations that led to the categorical explorations in [7, 8].

To relate TANs to each other, a new category is constructed by defining appropriate TAN morphisms. This forms the framework where the exploration for the existence of different kinds of limits and colimits takes place. The aforementioned basic principles provide the motivating force.

In the remainder of this section the notions of finite automata network, finite threshold network and threshold agent network will be briefly reviewed and some connections between them recalled. For all categorical terminology and notation in the following sections, the reader may consult any of [1, 4] or [9].

A *finite automata network* [5], defined on a finite set I of *agents*, is formally a triple $\mathcal{A} = \langle G, Q, (f_i)_{i\in I}\rangle$, where

- $G = \langle I, E\rangle$ is a digraph with set of vertices I,
- Q is a finite set of *states* and
- $f_i : Q^{|E_i|} \to Q$ is a *local update function*, where, for all $i \in I$, $E_i = \{j \in I : (j, i) \in E\}$.

The *global update function* $F : Q^I \to Q^I$ is now defined by combining the local update functions via some updating rule, which, for our purposes, will always be taken to be synchronous or parallel updating.

A *finite threshold* or *neural network* is a finite automata network whose digraph possesses two weighted structures. The first is a function $w : E \to \mathbb{R}$ that associates to every edge $(i, j) \in E$ its *weight* $w(i, j)$. If the weights on the pairs $(i, j) \notin E$ are defined to be zero, then w may be considered to be a function $w : I \times I \to \mathbb{R}$. The second is a function $t : I \to \mathbb{R}$. It associates to every agent $i \in I$ its *threshold* $t(i)$. The state set Q in the case of neural networks is assumed to be the two element set $Q = k = \{0, 1\}$. With the help of the two weighted structures w and t, the local update functions $f_i : k^{|E_i|} \to k$, $i \in I$, of the neural network may be defined as follows.

$$f_i(x_j : j \in E_i) = T\left(\sum_{j\in E_i} w(j, i)x_j - t(i)\right),$$

where

$$T(u) = \begin{cases} 1, & \text{if } u \geq 0, \\ 0, & \text{if } u < 0. \end{cases}$$

The finite neural network described here will be denoted by $\mathcal{N} = \langle I, E, w, t \rangle$.

A *threshold agent network* (TAN) is a finite threshold network, such that its threshold structure $t : I \to \mathbb{R}$ factors through the inclusion $i : \mathbf{Z} \hookrightarrow \mathbb{R}$ of the integers inside the reals, i.e., essentially, $t : I \to \mathbf{Z}$, and its weight structure $w : I \times I \to \mathbb{R}$ is completely determined by its threshold structure by

$$w(i, j) = \begin{cases} 1, & \text{if } (i, j) \in E \text{ and } t(j) \geq 0, \\ -1, & \text{if } (i, j) \in E \text{ and } t(j) < 0, \\ 0, & \text{otherwise,} \end{cases} \quad \text{for all } (i, j) \in I \times I.$$

Thus, the weight structure may be omitted altogether so that a TAN may be denoted by $A = \langle I, E, t \rangle$.

Writing the adjacency relation of the digraph G of the TAN A in a list form and emphasizing the individual agents as opposed to the whole structure of the TAN, from now on a TAN $A = \langle I, E, t \rangle$ will be given in the form $A = \{A_i\}_{i \in I}$, where each A_i, $i \in I$, is a pair $A_i = \langle k_i, P_i \rangle$, such that $k_i = t(i)$ is the *threshold* of the *agent* A_i and $P_i \subseteq \{1, \ldots, n\}$ is the *output set* of A_i, i.e., $P_i = \{j \in I : (i, j) \in E\}$.

With the notation $A = \{A_i\}_{i \in I}$ in place, the local update functions $f_i^A : k^I \to k$, $i \in I$, of the TAN can now be rewritten in the form

$$f_i^A(x) = \begin{cases} \chi_{|\{j : x_j = 1 \text{ and } i \in P_j\}| \geq k_i}(x), & \text{if } k_i \geq 0, \\ \chi_{|\{j : x_j = 1 \text{ and } i \in P_j\}| < -k_i}(x), & \text{if } k_i < 0, \end{cases} \quad \text{for all } x \in k^I,$$

where, given a property c of the elements in $k^{\mathbf{n}}$, by χ_c is denoted its characteristic function, i.e.,

$$\chi_c(x) = \begin{cases} 1, & \text{if } x \text{ satisfies } c, \\ 0, & \text{otherwise.} \end{cases}$$

Finally, the global dynamics is given by

$$f^A(x) = \langle f_i^A(x) : i \in I \rangle, \quad \text{for all } x \in k^I.$$

In Section 2, a natural notion of a state space is introduced and morphisms between state spaces are defined. Thus, a category **Stsp** with objects all state spaces is formed. Then TANs are defined formally and morphisms between TANs are introduced. With these morphisms between them the class of all TANs forms a category. A natural functor Sp is shown to exist from the category of TANs to the category of state spaces. The introduction of this functor enables one to formalize many notions of equivalence between systems that otherwise have to remain intuitive (see [13]). In Section 3 the category of TANs will be shown to have finite products but not equalizers or coproducts. Since having finite products and equalizers, i.e., together with having a terminal object, having finite limits, is a property that makes a category much nicer, it is natural to look for ways to find a category in which TANs may be embedded, i.e., a supercategory of the category of TANs, that will possess finite products and equalizers. To achieve this goal, in Section 4, a generalization of TANs to the, so-called, generalized TANs (GTANs) is introduced. This more general model has the feature that interactions between the agents in the model are more diverse than interactions in the original model. Morphisms of TANs are also generalized to morphisms between GTANs. Thus, a new category is formed with GTANs as its objects, that has the category of TANs as a subcategory. This new category is shown to have both finite products and equalizers but does not have (even finite) coproducts. Overall, all the constructions of finite products and equalizers, whenever they exist, are shown to be created by a natural functor from the category of TANs or GTANs to the category of sets.

2. The Categories of TANs and State Spaces

In this section the categories of TANs and of state spaces are defined formally. A functor from the first to the second that sends a TAN to its state space is also introduced. Finally, a functor from the category of TANs to the category **Set** of small sets is defined.

In the sequel, by k will be denoted the 2-element set $k = \{0, 1\}$, by $\mathbb{N}$ will be denoted the set of all natural numbers and by $\mathbf{n}$, $n \in \mathbb{N}$, the set $\mathbf{n} = \{1, 2, \ldots, n\}$.

2.1. THE CATEGORY OF STATE SPACES

A *state space* is a directed graph $G = \langle V, E\rangle$, where $V = k^{\mathbf{n}}$, for some $n \in \mathbb{N}$, such that, for all $x, y, z \in k^{\mathbf{n}}$,

$$\langle x, y\rangle \in E \quad \text{and} \quad \langle x, z\rangle \in E \quad \text{imply} \quad y = z.$$

Given two state spaces $G_1 = \langle k^{\mathbf{n}}, E_1\rangle$, $G_2 = \langle k^{\mathbf{m}}, E_2\rangle$, a *state space morphism* $f : G_1 \to G_2$ *from* G_1 *to* G_2 is a directed graph homomorphism from G_1 to G_2. State spaces with state space homomorphisms between them form a category, termed the *category of state spaces* and denoted by **Stsp**. It is clear that **Stsp** is a subcategory of the category **Dgr** of directed graphs and directed graph homomorphisms between them.

2.2. THE CATEGORY OF TANS

A *threshold agent network* (TAN) is a finite collection $A = \{A_i\}_{1\leq i\leq n}$ of *agents* A_i, $1 \leq i \leq n$, where each A_i is a pair $A_i = \langle k_i, P_i\rangle$ with k_i an integer, called the *threshold* of agent A_i and $P_i \subseteq \{1, 2, \ldots, n\}$ the *output set* of agent A_i. The dynamics of the TAN is generated by stipulating that agent i be active at time j

if at least k_i agents that have i in their output sets are active at time $j-1$. Note that if k_i is negative, then agent i will always be active at time j except if at least $-k_i$ agents that have i in their output sets are active at time $j-1$. P_i, $1 \leq i \leq n$, is the set of agents that will be affected by agent i at the end of each time step in case agent i's threshold has been attained at the end of the previous time step. This dynamical behavior is formally expressed by the function $f^A : k^{\mathbf{n}} \to k^{\mathbf{n}}$ defined as follows: First, given a condition c that an n-tuple $\langle x_1, \ldots, x_n \rangle \in k^{\mathbf{n}}$ may or may not satisfy, let $\chi_c : k^{\mathbf{n}} \to k$ be the characteristic function of c, i.e.,

$$\chi_c(\langle x_1, \ldots, x_n \rangle) = \begin{cases} 1, & \text{if } \langle x_1, \ldots, x_n \rangle \text{ satisfies } c, \\ 0, & \text{otherwise.} \end{cases}$$

Then define the functions $f_i^A : k^{\mathbf{n}} \to k$, $1 \leq i \leq n$, by

$$f_i^A(x) = \begin{cases} \chi_{|\{j : x_j = 1 \text{ and } i \in P_j\}| \geq k_i}(x), & \text{if } k_i \geq 0, \\ \chi_{|\{j : x_j = 1 \text{ and } i \in P_j\}| < -k_i}(x), & \text{if } k_i < 0, \end{cases} \qquad \text{for all } x \in k^{\mathbf{n}}.$$

Finally, set

$$f^A(x) = \langle f_1^A(x), \ldots, f_n^A(x) \rangle, \qquad \text{for all } x \in k^{\mathbf{n}}.$$

f^A is called the *dynamics* of the TAN A.

The notion of mapping between TANs is now introduced. A mapping preserves the TAN structure, i.e., the thresholds and the output sets of corresponding agents. In addition, a nice condition is imposed on the mappings so that they possess nice preservation properties with respect to the state spaces of the TANs they relate. More precisely, TAN morphisms, as defined here, induce morphisms between the corresponding state spaces of the TANs. In other words, existence of a morphism from one TAN to another guarantees a similarity between their dynamic behaviours.

Let A, B be two given TANs with sets of agents $A = \{A_i\}_{1 \leq i \leq n}$, $B = \{B_i\}_{1 \leq i \leq m}$, such that $A_i = \langle k_i, P_i \rangle$, $1 \leq i \leq n$, and $B_i = \langle l_i, Q_i \rangle$, $1 \leq i \leq m$, and dynamics $f^A : k^{\mathbf{n}} \to k^{\mathbf{n}}$ and $f^B : k^{\mathbf{m}} \to k^{\mathbf{m}}$, respectively. A *TAN morphism* $h : A \to B$ *from* the TAN A *to* the TAN B is a set mapping $h : \mathbf{m} \to \mathbf{n}$ satisfying, for all $1 \leq i \leq m$,

(1) $k_{h(i)} = l_i$,
(2) $h(Q_i) \subseteq P_{h(i)}$ and
(3) the following diagram commutes

$$\begin{array}{ccc} k^{\mathbf{n}} & \xrightarrow{f^A} & k^{\mathbf{n}} \\ {\scriptstyle h^*}\downarrow & & \downarrow{\scriptstyle h^*} \\ k^{\mathbf{m}} & \xrightarrow[f^B]{} & k^{\mathbf{m}} \end{array}$$

where $h^* : k^{\mathbf{n}} \to k^{\mathbf{m}}$ is defined by

$$h^*(x_1, \ldots, x_n) = \langle x_{h(1)}, \ldots, x_{h(m)} \rangle, \qquad \text{for all } \langle x_1, \ldots, x_n \rangle \in k^{\mathbf{n}}.$$

As an example, consider the TAN A given by the following table

Agents	Threshold	Output set
1	1	$\{2, 3\}$
2	-1	$\{1, 3\}$
3	1	$\{1, 2\}$

Consider also the TAN B given by the table

Agents	Threshold	Output set
1	-1	$\{2\}$
2	1	$\{1\}$

These two TANs are depicted in Figure 1.

The only candidate mappings $h : \{1, 2\} \to \{1, 2, 3\}$ for TAN morphisms are the mappings $1 \mapsto 2, 2 \mapsto 3$ and $1 \mapsto 2, 2 \mapsto 1$. However, it is not difficult to see that the function $h : \{1, 2\} \to \{1, 2, 3\}$, with $1 \mapsto 2, 2 \mapsto 3$ does not define a TAN morphism from A to B since it does not satisfy condition (3) of the definition. For instance $h^*(f^A(\langle 1, 0, 1\rangle)) = \langle 1, 0\rangle$, whereas $f^B(h^*(\langle 1, 0, 1\rangle)) = \langle 0, 0\rangle$. By symmetry, it can be concluded that the second function does not define a TAN morphism either.

As an example on the positive side, consider the TAN A given by the table

Agents	Thresholds	Output sets
1	-1	$\{2\}$
2	1	$\{1\}$

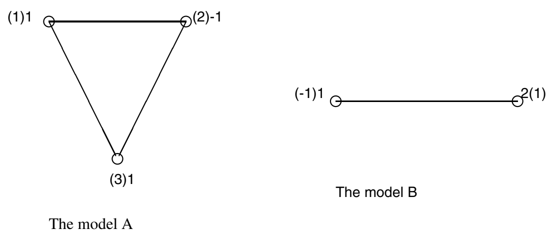

Figure 1. The two TANs A and B of the first example.

Consider also the TAN B given by

Agents	Thresholds	Output sets
1	-1	$\{2, 4\}$
2	1	$\{1, 3\}$
3	-1	$\{2, 4\}$
4	1	$\{1, 3\}$

These two TANs are depicted in Figure 2. It is not hard to check that the function $h : \{1, 2, 3, 4\} \to \{1, 2\}$, defined by $1, 3 \mapsto 1$ and $2, 4 \mapsto 2$ defines a TAN morphism $h : A \to B$.

Given a TAN A, with set of agents $A = \{A_i\}_{1 \leq i \leq n}$, the identity morphism $i_n : \mathbf{n} \to \mathbf{n}$ is a TAN morphism. Also, given two TAN morphisms $h_1 : A \to B$ and $h_2 : B \to C$, their composition as set functions is also a TAN morphism $h_1 \circ h_2 : A \to C$. Thus, the collection of all TANs with TAN morphisms between them forms a category. This category will be denoted by **TAN**.

This setting gives a well defined notion of isomorphism between TANs. Namely, two TANs A and B with sets of agents $A = \{A_i\}_{1 \leq i \leq n}$ and $B = \{B_i\}_{1 \leq i \leq m}$, such that $A_i = \langle k_i, P_i \rangle$, $1 \leq i \leq n$, and $B_i = \langle l_i, Q_i \rangle$, $1 \leq i \leq m$, and dynamics $f^A : k^{\mathbf{n}} \to k^{\mathbf{n}}$ and $f^B : k^{\mathbf{m}} \to k^{\mathbf{m}}$ are *isomorphic*, denoted $A \cong B$, if there exists a bijection $h : \mathbf{m} \to \mathbf{n}$ (in which case $n = m$) such that, for all $1 \leq i \leq n$,

- $k_{h(i)} = l_i$,
- $h(Q_i) = P_{h(i)}$ and
- the following diagram commutes

$$\begin{tikzcd} k^{\mathbf{n}} \arrow[r, "f^A"] \arrow[d, "h^*"'] & k^{\mathbf{n}} \arrow[d, "h^*"] \\ k^{\mathbf{n}} \arrow[r, "f^B"'] & k^{\mathbf{n}} \end{tikzcd}$$

It is not difficult to see that the following proposition holds

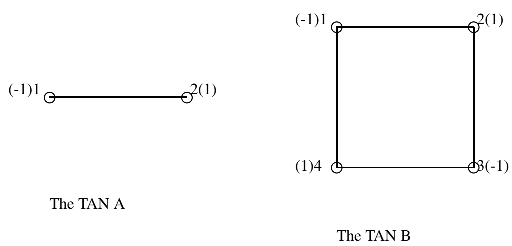

Figure 2. The two TANs A and B of the second example.

PROPOSITION 1. *Let A, B be TANs with sets of agents $A = \{A_i\}_{1 \leq i \leq n}$ and $B = \{B_i\}_{1 \leq i \leq m}$, such that $A_i = \langle k_i, P_i \rangle$, $1 \leq i \leq n$, and $B_i = \langle l_i, Q_i \rangle$, $1 \leq i \leq m$, respectively. $h : A \cong B$ if and only if h is a bijection and*

(1) $k_{h(i)} = l_i$,
(2) $h(Q_i) = P_{h(i)}$.

Proof. If $h : A \to B$ and $e : B \to A$ are inverse isomorphisms, then the first condition is obvious and the second follows from the relation $h(Q_i) \subseteq P_{h(i)}$ and the relation

$$P_{h(i)} = h(e(P_{h(i)})) \subseteq h(Q_{e(h(i))}) = h(Q_i).$$

Suppose, conversely, that $h : A \to B$ is a bijection that satisfies the two conditions of the proposition. It suffices to show commutativity of

$$\begin{tikzcd} k^{\mathbf{n}} \arrow[r, "f^A"] \arrow[d, "h^*"'] & k^{\mathbf{n}} \arrow[d, "h^*"] \\ k^{\mathbf{n}} \arrow[r, "f^B"'] & k^{\mathbf{n}} \end{tikzcd}$$

We have, for all $x \in k^{\mathbf{n}}$, $i \in \mathbf{n}$,

$$\begin{aligned} h^*(f^A(x))_i &= h^*(f_1^A(x), \ldots, f_n^A(x))_i \\ &= \begin{cases} \chi_{\{|\{j : x_j = 1 \wedge j \in P_{h(i)}\}| \geq k_{h(i)}\}}(x), & \text{if } k_{h(i)} \geq 0, \\ \chi_{\{|\{j : x_j = 1 \wedge j \in P_{h(i)}\}| < -k_{h(i)}\}}(x), & \text{if } k_{h(i)} < 0 \end{cases} \\ &= \begin{cases} \chi_{\{|\{j : x_j = 1 \wedge j \in h(Q_i)\}| \geq l_i\}}(x), & \text{if } l_i \geq 0, \\ \chi_{\{|\{j : x_j = 1 \wedge j \in h(Q_i)\}| < -l_i\}}(x), & \text{if } l_i < 0 \end{cases} \\ &= \begin{cases} \chi_{\{|\{h(j) : x_{h(j)} = 1 \wedge h(j) \in h(Q_i)\}| \geq l_i\}}(x_h), & \text{if } l_i \geq 0, \\ \chi_{\{|\{h(j) : x_{h(j)} = 1 \wedge h(j) \in h(Q_i)\}| < -l_i\}}(x_h), & \text{if } l_i < 0 \end{cases} \\ &= f^B(x_{h(1)}, \ldots, x_{h(n)})_i \\ &= f^B(h^*(x))_i. \end{aligned}$$

□

2.3. FUNCTORS FROM TANs TO STATE SPACES AND TO SETS

A functor Sp from the category of TANs to the category of state spaces is now defined. This functor provides a formal tool for connecting a TAN with its state space, i.e., with the dynamics graph that it generates.

Given a TAN A with set of agents $A = \{A_i\}_{1 \leq i \leq n}$, where $A_i = \langle k_i, P_i \rangle$, $1 \leq i \leq n$, and dynamics f^A, define the state space $\mathrm{Sp}(A) = \langle k^{\mathbf{n}}, E \rangle$, where, for all $x, y \in k^{\mathbf{n}}$,

$$\langle x, y \rangle \in E \quad \text{if and only if} \quad y = f^A(x).$$

Given a TAN morphism $h : A \to B$ from the TAN A to the TAN B, with $B = \{B_i\}_{1 \le i \le m}$, $B_i = \langle l_i, Q_i \rangle$ and dynamics f^B, define $\mathrm{Sp}(h) : \mathrm{Sp}(A) \to \mathrm{Sp}(B)$ to be the state space morphism given by

$$\mathrm{Sp}(h)(x) = h^*(x) = \langle x_{h(1)}, \ldots, x_{h(m)} \rangle, \quad \text{for all } x \in k^{\mathbf{n}}$$

and, for all $\langle x, y \rangle \in E$,

$$\mathrm{Sp}(\langle x, y \rangle) = \langle h^*(x), h^*(y) \rangle.$$

To show that this is well defined, it suffices to show that $f^B(h^*(x)) = h^*(y)$. We have

$$\begin{aligned} f^B(h^*(x)) &= h^*(f^A(x)) \quad \text{since } h : A \to B \\ &= h^*(y), \quad \text{since } \langle x, y \rangle \in E. \end{aligned}$$

It is not difficult to check that identity TAN morphisms are sent by Sp to identity state space morphisms and that Sp at the morphism level preserves composition of morphisms. Therefore, $\mathrm{Sp} : \mathbf{TAN} \to \mathbf{Stsp}$ is a functor from the category of TANs to the category of state spaces. It will be referred to as the (*state*) *space functor*.

Another functor that will be of interest in investigating the categorical properties of **TAN** is defined now.

Given a TAN A with set of agents $A = \{A_i\}_{1 \le i \le n}$, define the set

$$\mathrm{S}(A) = \mathbf{n} = \{1, 2, \ldots, n\},$$

i.e., $\mathrm{S}(A)$ is the underlying index set of the set of agents in A. Given another TAN B with set of agents $B = \{B_i\}_{1 \le i \le m}$ and a TAN morphism $h : A \to B$, define $\mathrm{S}(h) : \mathrm{S}(B) \to \mathrm{S}(A)$ by

$$\mathrm{S}(h)(i) = h(i), \quad \text{for all } i \in \mathbf{m}.$$

It is not hard to check that $\mathrm{S} : \mathbf{TAN} \to \mathbf{Set}^{\mathrm{op}}$ is a contravariant functor from the category of TANs to the category **Set** of all small sets. S will be referred to as the *underlying index set functor.*

3. Properties of TAN

3.1. TAN HAS FINITE PRODUCTS

In this section, it is shown that the category **TAN** has finite products. Since the TAN with an empty set of agents is clearly a terminal object in **TAN**, it suffices by [1], Exercise 5.3.20.1, to show that it has binary products.

Let A and B be two TANs with sets of agents $A = \{A_i\}_{1 \le i \le n}$ and $B = \{B_i\}_{1 \le i \le m}$, such that $A_i = \langle k_i, P_i \rangle$, $1 \le i \le n$, and $B_i = \langle l_i, Q_i \rangle$, $1 \le i \le m$, and dynamics $f^A : k^{\mathbf{n}} \to k^{\mathbf{n}}$ and $f^B : k^{\mathbf{m}} \to k^{\mathbf{m}}$, respectively. Construct the TAN $C = A \times B$ with set of agents $C = \{C_i\}_{1 \le i \le n+m}$, where

$$C_i = \begin{cases} \langle k_i, P_i \rangle, & \text{if } 1 \le i \le n, \\ \langle l_{i-n}, Q_{i-n} \rangle, & \text{if } n < i \le n+m \end{cases}$$

and dynamics $f^C : k^{\mathbf{n+m}} \to k^{\mathbf{n+m}}$, uniquely determined by

$$\begin{aligned} &\pi_i^{n+m}(f^C(x_1, \dots, x_{n+m})) \\ &\quad = \begin{cases} \pi_i^n(f^A(x_1, \dots, x_n)), & \text{if } 1 \le i \le n, \\ \pi_{n-i}^m(f^B(x_{n+1}, \dots, x_{n+m})), & \text{if } n < i \le n+m, \end{cases} \end{aligned}$$

where by $\pi_q^p : k^p \to k$, $1 \le q \le p$, is denoted the projection function of p variables, projecting onto the q-th variable. Note that the dynamics defined in this way is indeed the dynamics induced by the output sets and the thresholds of C.

Now consider the two set mapppings $\pi_A : \mathbf{n} \to \mathbf{n+m}$ and $\pi_B : \mathbf{m} \to \mathbf{n+m}$, such that $i \stackrel{\pi_A}{\mapsto} i$ and $i \stackrel{\pi_B}{\mapsto} n+i$. It is now shown that $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ are TAN morphisms. First for π_A, conditions (1) and (2) in the definition of a TAN morphism are clearly satisfied. For (3), the following diagram must be shown to commute

$$\begin{tikzcd} k^{\mathbf{n+m}} \arrow[r, "f^C"] \arrow[d, "\pi_A^*"'] & k^{\mathbf{n+m}} \arrow[d, "\pi_A^*"] \\ k^{\mathbf{n}} \arrow[r, "f^A"'] & k^{\mathbf{n}} \end{tikzcd}$$

$$\begin{aligned} &f^A(\pi_A^*(x_1, \dots, x_{n+m})) \\ &\quad = f^A(x_{\pi_A(1)}, \dots, x_{\pi_A(n)}) \\ &\quad = f^A(x_1, \dots, x_n) \\ &\quad = (\pi_1^n(f^A(x_1, \dots, x_n)), \dots, \pi_n^n(f^A(x_1, \dots, x_n))) \\ &\quad = (\pi_1^{n+m}(f^C(x_1, \dots, x_{n+m})), \dots, \pi_n^{n+m}(f^C(x_1, \dots, x_{n+m}))) \\ &\quad = (\pi_{\pi_A(1)}^{n+m}(f^C(x_1, \dots, x_{n+m})), \dots, \pi_{\pi_A(n)}^{n+m}(f^C(x_1, \dots, x_{n+m}))) \\ &\quad = \pi_A^*(f^C(x_1, \dots, x_{n+m})). \end{aligned}$$

Similarly, it can be shown that $\pi_B : A \times B \to B$ is a TAN morphism from $A \times B$ into B.

Finally, it is shown that $A \times B$ possesses the universal mapping property of the product of A and B in the category **TAN**. To this end, suppose that D is a TAN with set of agents $D = \{D_i\}_{1 \le i \le p}$, such that $D_i = \langle m_i, R_i \rangle$, $1 \le i \le p$, and dynamics $f^D : k^{\mathbf{p}} \to k^{\mathbf{p}}$. In addition, assume that $h_A : D \to A$ and $h_B : D \to B$ are two TAN morphisms, i.e., that $h_A : \mathbf{n} \to \mathbf{p}$ and $h_B : \mathbf{m} \to \mathbf{p}$ are two set mappings such that

$$\begin{aligned} &m_{h_A(i)} = k_i, \qquad h_A(P_i) \subseteq R_{h_A(i)}, \quad \text{for all } 1 \le i \le n, \\ &m_{h_B(i)} = l_i, \qquad h_B(Q_i) \subseteq R_{h_B(i)}, \quad \text{for all } 1 \le i \le m, \end{aligned}$$

and such that the following two rectangles commute

$$\begin{tikzcd} k^{\mathbf{p}} \arrow[r, "f^D"] \arrow[d, "h_A^*"'] & k^{\mathbf{p}} \arrow[d, "h_A^*"] \\ k^{\mathbf{n}} \arrow[r, "f^A"'] & k^{\mathbf{n}} \end{tikzcd} \qquad \begin{tikzcd} k^{\mathbf{p}} \arrow[r, "f^D"] \arrow[d, "h_B^*"'] & k^{\mathbf{p}} \arrow[d, "h_B^*"] \\ k^{\mathbf{m}} \arrow[r, "f^B"'] & k^{\mathbf{m}} \end{tikzcd}$$

A TAN morphism $h : D \to A \times B$ has to be constructed that makes the following diagram commute and its uniqueness must be shown.

$$\begin{tikzcd} A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B \\ & D \arrow[ul, "h_A"] \arrow[u, "h"] \arrow[ur, "h_B"'] & \end{tikzcd}$$

Let $h : \mathbf{n} + \mathbf{m} \to \mathbf{p}$ be defined by

$$h(i) = \begin{cases} h_A(i), & \text{if } 1 \le i \le n, \\ h_B(i-n), & \text{if } n < i \le n+m. \end{cases}$$

Then, $m_{h(i)} = m_{h_A(i)} = k_i$, if $1 \le i \le n$, and $m_{h(i)} = m_{h_B(i-n)} = l_{i-n}$, if $n < i \le n+m$. Also $h(P_i) = h_A(P_i) \subseteq R_{h_A(i)} = R_{h(i)}$, if $1 \le i \le n$, and $h(Q_i) = h_B(Q_{i-n}) \subseteq R_{h_B(i-n)} = R_{h(i-n)}$, if $n < i \le n+m$. Furthermore

$$\begin{tikzcd} k^{\mathbf{p}} \arrow[r, "f^D"] \arrow[d, "h^*"'] & k^{\mathbf{p}} \arrow[d, "h^*"] \\ k^{\mathbf{n+m}} \arrow[r, "f^C"'] & k^{\mathbf{n+m}} \end{tikzcd}$$

$$\begin{aligned} & h^*(f^D(x_1, \ldots, x_p)) \\ &\quad = h^*(f_1^D(x_1, \ldots, x_p), \ldots, f_p^D(x_1, \ldots, x_p)) \\ &\quad = (f_{h(1)}^D(x_1, \ldots, x_p), \ldots, f_{h(n+m)}^D(x_1, \ldots, x_p)) \\ &\quad = (f_{h_A(1)}^D(x_1, \ldots, x_p), \ldots, f_{h_A(n)}^D(x_1, \ldots, x_p), \\ &\qquad f_{h_B(1)}^D(x_1, \ldots, x_p), \ldots, f_{h_B(m)}^D(x_1, \ldots, x_p)) \\ &\quad = (h_A^*(f^D(x_1, \ldots, x_p)), h_B^*(f^D(x_1, \ldots, x_p))) \\ &\quad = (f^A(h_A^*(x_1, \ldots, x_p)), f^B(h_B^*(x_1, \ldots, x_p))) \\ &\quad = (f^A(x_{h_A(1)}, \ldots, x_{h_A(n)}), f^B(x_{h_B(1)}, \ldots, x_{h_B(m)})) \\ &\quad = f^C(x_{h_A(1)}, \ldots, x_{h_A(n)}, x_{h_B(1)}, \ldots, x_{h_B(m)}) \\ &\quad = f^C(x_{h(1)}, \ldots, x_{h(n+m)}) \\ &\quad = f^C(h^*(x_1, \ldots, x_p)). \end{aligned}$$

Hence, $h : D \to A \times B$, thus defined, is indeed a TAN morphism. It makes the required triangles commute since, by its construction, $h \circ \pi_a = h_a$ and $h \circ \pi_B = h_B$. Finally, it is the unique such morphism, since it is uniquely determined by h_A and h_B.

The following theorem has been established.

THEOREM 2. *The category* **TAN** *has finite products. Moreover, the functor* S : **TAN** $\to$ **Set**$^{\mathrm{op}}$ *that sends a* TAN *to the index set of its underlying set of agents preserves and creates finite products.*

3.2. S : TAN $\to$ SET$^{\mathrm{op}}$ DOES NOT CREATE EQUALIZERS

A simple counterexample is given to the effect that S : **TAN** $\to$ **Set**$^{\mathrm{op}}$ does not create equalizers.

Let A be a TAN with the set of agents $A = \{A_1, A_2, A_3\}$, such that $A_1 = \langle 2, \{2, 3\}\rangle$, $A_2 = \langle 2, \{1, 3\}\rangle$ and $A_3 = \langle 2, \{1, 2\}\rangle$ and dynamics f^A.

$$\begin{tikzcd}
& 1(2) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\
2(2) \arrow[rr, leftrightarrow] & & 3(2)
\end{tikzcd}$$

Consider the two TAN morphisms $h_1, h_2 : A \to A$, defined by $h_1 : \{1, 2, 3\} \to \{1, 2, 3\}$ being the identity and $h_2 : \{1, 2, 3\} \to \{1, 2, 3\}$ being given by $1 \mapsto 2$, $2 \mapsto 3$ and $3 \mapsto 1$. Suppose, for the sake of obtaining a contradiction, that $\langle C, c : C \to A\rangle$ is the equalizer in **TAN** of h_1 and h_2.

$$\begin{tikzcd}
C \arrow[r, "c"] & A \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & A
\end{tikzcd}$$

Then, in **Set** $c : \{1, 2, 3\} \to \{1, \ldots, |C|\}$ has to coequalize h_1 and h_2.

$$\begin{tikzcd}
\{1, \ldots, |C|\} & \{1, 2, 3\} \arrow[l, "c"'] & \{1, 2, 3\} \arrow[l, shift right, "h_1"'] \arrow[l, shift left, "h_2"]
\end{tikzcd}$$

But then C has to be a singleton. Thus, C has a singleton set of agents with its single agent having threshold 2 and output set the singleton containing itself. However, it is not hard to check that there is no morphism $c : C \to A$ in **TAN** from this TAN to A.

It has thus been shown that

PROPOSITION 3. *The functor* S : **TAN** $\to$ **Set**$^{\mathrm{op}}$ *does not create equalizers.*

3.3. **TAN** DOES NOT HAVE COPRODUCTS

A simple counterexample is given to the effect that **TAN** does not have binary coproducts.

Let A be a TAN with the singleton set of agents $A = \{A_1\}$, such that $A_1 = \langle 0, \{1\}\rangle$ and dynamics f^A. Similarly, let B be a TAN with set of agents $B = \{B_1\}$, such that $B_1 = \langle 1, \{1\}\rangle$ and dynamics f^B. Suppose, for the sake of obtaining a contradiction, that the TAN C is the coproduct in **TAN** of A and B with coprojections $i_A : A \to C, i_B : B \to C$.

$$A \xrightarrow{i_A} C \xleftarrow{i_B} B$$

Since the coprojections $i_A : A \to C, i_B : B \to C$ preserve thresholds of agents and the thresholds of A_1 and B_1 are different, the set of agents of C must contain at least two agents, one with threshold 0 and the other with threshold 1. But then $i_A : \{1, \ldots, |C|\} \to \{1\}$ must send the indices of both of these agents to 1, which violates the preservation of thresholds.

It has thus been shown that

PROPOSITION 4. *The category* **TAN** *does not have coproducts.*

4. The Category GTAN of Generalized TANs

4.1. THE CATEGORY **GTAN**

In this section, the notion of a TAN is generalized to obtain the notion of a generalized TAN (GTAN). The notion of mapping between TANs is accordingly modified to obtain a notion of morphism between GTANs. As is the case with TANs, a mapping between GTANs preserves the TAN structure, i.e., the thresholds and the output sets of corresponding agents. The third condition in the definition of a TAN morphism is in effect and, hence, the nice preservation properties with respect to state spaces still apply.

A *generalized TAN* (GTAN) A consists of a collection $A = \{A_i\}_{1\leq i\leq n}$ of agents and a dynamics function $f^A : k^{\mathbf{n}} \to k^{\mathbf{n}}$. Each agent A_i, $1 \leq i \leq n$, is a pair $A_i = \langle k_i, P_i\rangle$, where k_i is an integer, called the *threshold* of agent i, and P_i is a multiset $\{1^{e^i_1}, \ldots, n^{e^i_n}\}$, $e^i_j \in \mathbb{N}$, $1 \leq j \leq n$, i.e., a multiset that contains finitely many copies e^i_j of each index j. The dynamics function $f^A : k^{\mathbf{n}} \to k^{\mathbf{n}}$ is defined as before except that, at each time step, each agent A_i contributes to attainment of another agent's A_j threshold as many times as appearances of j in A_i's output set. More precisely, f^A is defined by

$$f^A(x) = (f^A_1(x), \ldots, f^A_n(x)), \quad \text{for all } x \in k^{\mathbf{n}},$$

where

$$f_i^A(x) = \begin{cases} \chi_{|\{j^{e_i^j} : x_j = 1\}| \geq k_i}(x), & \text{if } k_i \geq 0, \\ \chi_{|\{j^{e_i^j} : x_j = 1\}| < -k_i}(x), & \text{if } k_i < 0, \end{cases} \quad \text{for all } x \in k^{\mathbf{n}},\ 1 \leq i \leq n.$$

Let A, B be two GTANs with sets of agents $A = \{A_i\}_{1 \leq i \leq n}$, $B = \{B_i\}_{1 \leq i \leq m}$, such that $A_i = \langle k_i, P_i \rangle$, $1 \leq i \leq n$, and $B_i = \langle l_i, Q_i \rangle$, $1 \leq i \leq m$, and dynamics $f^A : k^{\mathbf{n}} \to k^{\mathbf{n}}$ and $f^B : k^{\mathbf{m}} \to k^{\mathbf{m}}$, respectively. A *GTAN morphism* $h : A \to B$ *from* the GTAN A *to* the GTAN B is a set mapping $h : \mathbf{m} \to \mathbf{n}$ satisfying, for all $1 \leq i \leq m$,

(1) $k_{h(i)} = l_i$,
(2) $h(Q_i) \subseteq P_{h(i)}$ (as multisets),
(3) the following diagram commutes

$$\begin{tikzcd} k^{\mathbf{n}} \arrow[r, "f^A"] \arrow[d, "h^*"'] & k^{\mathbf{n}} \arrow[d, "h^*"] \\ k^{\mathbf{m}} \arrow[r, "f^B"'] & k^{\mathbf{m}} \end{tikzcd}$$

where $h^* : k^{\mathbf{n}} \to k^{\mathbf{m}}$ is defined, as before, by

$$h^*(x_1, \ldots, x_n) = \langle x_{h(1)}, \ldots, x_{h(m)} \rangle, \quad \text{for all } \langle x_1, \ldots, x_n \rangle \in k^{\mathbf{n}}.$$

Given a GTAN A, with set of agents $A = \{A_i\}_{1 \leq i \leq n}$, the identity morphism $i_n : \mathbf{n} \to \mathbf{n}$ is a GTAN morphism. Also, given two GTAN morphisms $h_1 : A \to B$ and $h_2 : B \to C$, their composition as set functions is also a GTAN morphism $h_1 \circ h_2 : A \to C$. Thus, the collection of all GTANs with GTAN morphisms between them forms a category. This category will be denoted by **GTAN**. It is not difficult to see that the category **TAN** of TANs is a full subcategory of **GTAN**. Also, GTANs are, like TANs, special cases of finite threshold networks. The thresholds are again restricted to be integer-valued but the weight function w in this case is not completely determined by the graph and the threshold function of the network. Rather, it is any function

$$w(i, j) = \begin{cases} k_{ij}, & \text{if } (i, j) \in E \text{ and } t(j) \geq 0, \\ -k_{ij}, & \text{if } (i, j) \in E \text{ and } t(j) < 0, \\ 0, & \text{otherwise}, \end{cases} \quad \text{for all } (i, j) \in I \times I,$$

where k_{ij} is a positive integer, for all $i, j \in I$.

A proposition similar to Proposition 1 holds for the case of GTANs

PROPOSITION 5. *Let A, B be GTANs with sets of agents $A = \{A_i\}_{1 \leq i \leq n}$ and $B = \{B_i\}_{1 \leq i \leq m}$, such that $A_i = \langle k_i, P_i \rangle$, $1 \leq i \leq n$, and $B_i = \langle l_i, Q_i \rangle$, $1 \leq i \leq m$, respectively. $h : A \cong B$ if and only if h is a bijection and*

(1) $k_{h(i)} = l_i$,
(2) $h(Q_i) = P_{h(i)}$ (*as multisets*).

4.2. **GTAN** HAS FINITE LIMITS

As has already been proven, **TAN** has finite products but S : **TAN** $\to$ **Set**$^{\mathrm{op}}$ does not create equalizers and **TAN** does not have coproducts. Since having finite limits is a very desirable property for a category, lack of finite limits in **TAN** was the partial motivation for searching for another category in which **TAN** would be embeddable as a subcategory and which would possess finite limits. **GTAN** is a category that has this property. It will be shown in the following sections that **GTAN** has finite products and equalizers that are preserved and created by the underlying index set functor. So, it has finite limits (see, e.g., [1], Thm. 9.2.10).

4.2.1. **GTAN** *Has Finite Products*

In this section, it is shown that the category **GTAN** has finite products. Since, in **GTAN** the GTAN with the empty set of agents is also a terminal object, it suffices to show that GTAN has binary products.

Let A and B be two GTANs with sets of agents $A = \{A_i\}_{1\le i\le n}$ and $B = \{B_i\}_{1\le i\le m}$, such that $A_i = \langle k_i, P_i\rangle$, $1 \le i \le n$, and $B_i = \langle l_i, Q_i\rangle$, $1 \le i \le m$, and dynamics $f^A : k^{\mathbf{n}} \to k^{\mathbf{n}}$ and $f^B : k^{\mathbf{m}} \to k^{\mathbf{m}}$, respectively. Construct the GTAN $C = A \times B$ with set of agents $C = \{C_i\}_{1\le i\le n+m}$, where

$$C_i = \begin{cases} \langle k_i, P_i\rangle, & \text{if } 1 \le i \le n, \\ \langle l_{i-n}, Q_{i-n}\rangle, & \text{if } n < i \le n+m \end{cases}$$

and dynamics $f^C : k^{\mathbf{n+m}} \to k^{\mathbf{n+m}}$, uniquely determined by

$$\begin{aligned} &\pi_i^{n+m}(f^C(x_1, \ldots, x_{n+m})) \\ &\quad = \begin{cases} \pi_i^n(f^A(x_1, \ldots, x_n)), & \text{if } 1 \le i \le n, \\ \pi_{n-i}^m(f^B(x_{n+1}, \ldots, x_{n+m})), & \text{if } n < i \le n+m, \end{cases} \end{aligned}$$

where, as before, by $\pi_q^p : k^p \to k$, $1 \le q \le p$, is denoted the projection function of p variables, projecting onto the q-th variable. Once more, it is not difficult to see that this is indeed the dynamics that is induced by the output multisets and the thresholds in C.

Now consider the two set mappings $\pi_A : \mathbf{n} \to \mathbf{n} + \mathbf{m}$ and $\pi_B : \mathbf{m} \to \mathbf{n} + \mathbf{m}$, such that $i \overset{\pi_A}{\mapsto} i$ and $i \overset{\pi_B}{\mapsto} n + i$. It is now shown that $\pi_A : A \times B \to A$ and $\pi_B : A \times B \to B$ are GTAN morphisms. First, for π_A, conditions (1) and (2) in the definition of a GTAN morphism are clearly satisfied. For (3), commutativity of the diagram

$$\begin{tikzcd} k^{\mathbf{n}+\mathbf{m}} \arrow[r, "f^C"] \arrow[d, "\pi_A^*"'] & k^{\mathbf{n}+\mathbf{m}} \arrow[d, "\pi_A^*"] \\ k^{\mathbf{n}} \arrow[r, "f^A"'] & k^{\mathbf{n}} \end{tikzcd}$$

may be shown exactly as in the case for TANs. Similarly, it can be shown that $\pi_B : A \times B \to B$ is a GTAN morphism from $A \times B$ into B.

Finally, it can be shown that $A \times B$ possesses the universal mapping property of the product of A and B in the category **GTAN** in a way very similar to the one employed to show the corresponding property for TANs. To this end, suppose that D is a GTAN with set of agents $D = \{D_i\}_{1 \leq i \leq p}$, such that $D_i = \langle m_i, R_i \rangle$, $1 \leq i \leq p$, and dynamics $h^D : k^{\mathbf{p}} \to k^{\mathbf{p}}$. In addition assume that $h_A : D \to A$ and $h_B : D \to B$ are two GTAN morphisms, i.e., that $h_A : \mathbf{n} \to \mathbf{p}$ and $h_B : \mathbf{m} \to \mathbf{p}$ are two set mappings such that

$$\begin{aligned} m_{h_A(i)} &= k_i, & h_A(P_i) &\subseteq R_{h_A(i)}, & &\text{for all } 1 \leq i \leq n, \\ m_{h_B(i)} &= l_i, & h_B(Q_i) &\subseteq R_{h_B(i)}, & &\text{for all } 1 \leq i \leq m, \end{aligned}$$

and such that the following two rectangles commute

$$\begin{tikzcd} k^{\mathbf{p}} \arrow[r, "f^D"] \arrow[d, "h_A^*"'] & k^{\mathbf{p}} \arrow[d, "h_A^*"] \\ k^{\mathbf{n}} \arrow[r, "f^A"'] & k^{\mathbf{n}} \end{tikzcd} \qquad \begin{tikzcd} k^{\mathbf{p}} \arrow[r, "f^D"] \arrow[d, "h_B^*"'] & k^{\mathbf{p}} \arrow[d, "h_B^*"] \\ k^{\mathbf{m}} \arrow[r, "f^B"'] & k^{\mathbf{m}} \end{tikzcd}$$

The unique morphism $h : D \to A \times B$ that makes the following diagram commute

$$\begin{tikzcd} A & A \times B \arrow[l, "\pi_A"'] \arrow[r, "\pi_B"] & B \\ & D \arrow[ul, "h_A"] \arrow[u, "h"] \arrow[ur, "h_B"'] & \end{tikzcd}$$

is the map $h : \mathbf{n} + \mathbf{m} \to \mathbf{p}$, defined by

$$h(i) = \begin{cases} h_A(i), & \text{if } 1 \leq i \leq n, \\ h_B(i-n), & \text{if } n < i \leq n+m. \end{cases}$$

The details will be omitted.

Denote by G : **GTAN** $\to$ **Set** the functor that sends a GTAN to the index set of its underlying set of agents, constructed very similarly to the functor S. The following theorem has been established

THEOREM 6. *The category* **GTAN** *has finite products. Moreover, the functor* $\mathrm{G} : \mathbf{GTAN} \to \mathbf{Set}^{\mathrm{op}}$ *that sends a GTAN to the index set of its underlying set of agents preserves and creates finite products.*

4.2.2. **GTAN** *Has Equalizers*

In this section, it is shown that the category **GTAN** of GTANs has equalisers.

Let A and B be two GTANs with sets of agents $A = \{A_i\}_{1 \le i \le n}$ and $B = \{B_i\}_{1 \le i \le m}$, such that $A_i = \langle k_i, P_i \rangle$, $1 \le i \le n$, and $B_i = \langle l_i, Q_i \rangle$, $1 \le i \le m$, and dynamics $f^A : k^{\mathbf{n}} \to k^{\mathbf{n}}$ and $f^B : k^{\mathbf{m}} \to k^{\mathbf{m}}$, respectively. Assume, in addition, that $h_1 : A \to B$ and $h_2 : A \to B$ are two GTAN morphisms from A to B, i.e., that $h_1, h_2 : \mathbf{m} \to \mathbf{n}$ are set mappings such that

$$\begin{aligned} &k_{h_1(i)} = k_{h_2(i)} = l_i, \quad \text{for all } 1 \le i \le m, \\ &h_1(Q_i) \subseteq P_{h_1(i)}, \qquad h_2(Q_i) \subseteq P_{h_2(i)}, \quad \text{for all } 1 \le i \le m, \end{aligned}$$

and such that the following rectangles commute

$$\begin{tikzcd} k^{\mathbf{n}} \arrow[r, "f^A"] \arrow[d, "h_1^*"'] & k^{\mathbf{n}} \arrow[d, "h_1^*"] \\ k^{\mathbf{m}} \arrow[r, "f^B"'] & k^{\mathbf{m}} \end{tikzcd} \qquad \begin{tikzcd} k^{\mathbf{n}} \arrow[r, "f^A"] \arrow[d, "h_2^*"'] & k^{\mathbf{n}} \arrow[d, "h_2^*"] \\ k^{\mathbf{m}} \arrow[r, "f^B"'] & k^{\mathbf{m}} \end{tikzcd}$$

Let $\langle X, h : \mathbf{n} \to X \rangle$ be the coequalizer of h_1, h_2 in **Set**.

$$\begin{tikzcd} X & \mathbf{n} \arrow[l, "h"'] & \mathbf{m} \arrow[l, shift right, "h_1"'] \arrow[l, shift left, "h_2"] \end{tikzcd}$$

X is constructed by taking the relation $\eta = \{(h_1(i), h_2(i)) : 1 \le i \le m\}$, generating the smallest equivalence relation θ on $\mathbf{n}$ containing η by taking the reflexive, symmetric and transitive closure of η in that order and then setting $X = \mathbf{n}/\theta$. Now construct the generalized TAN $C = \mathrm{eq}(h_1, h_2)$ with set of agents $C = \{C_I\}_{I \in X}$, where

$$C_I = \left\langle k_i, \biguplus_{i \in I} P_i/\theta \right\rangle,$$

where $I = i/\theta$, i.e., i is any representative of I. P_i/θ is a multiset of copies of elements in X. $\biguplus$ takes into account the number of copies of elements in the multisets and keeps the highest multiplicity of each element. This is well defined since $k_i = k_j$, for all $\langle i, j \rangle \in \theta$. Finally $\mathrm{eq}(h_1, h_2)$ is endowed with the dynamics $f^C : k^{|X|} \to k^{|X|}$, uniquely determined by taking $f^C(x_{I_1}, \ldots, x_{I_{|X|}})$ to be the $|X|$-tuple $(y_{I_1}, \ldots, y_{I_{|X|}})$, obtained by the n-tuple $f^A(x_{1/\theta}, \ldots, x_{n/\theta})$ by taking y_I to be the element occupying the i-th position for some $i \in I$. To show that this definition makes sense, it suffices to show that, if $f^A(x_{1/\theta}, \ldots, x_{n/\theta}) = (x'_1, \ldots, x'_n)$

and $\langle i, j\rangle \in \theta$, then $x'_i = x'_j$. This will also establish that the newly defined dynamics is induced by the thresholds and output multisets of $\mathrm{eq}(f_1, f_2)$. In turn, it suffices to show that this is the case for all $\langle i, j\rangle \in \eta$. But this holds because of the commutativity of the two rectangles

$$\begin{tikzcd} k^{\mathbf{n}} \arrow[r, "f^A"] \arrow[d, "h_1^*"'] & k^{\mathbf{n}} \arrow[d, "h_1^*"] \\ k^{\mathbf{m}} \arrow[r, "f^B"'] & k^{\mathbf{m}} \end{tikzcd} \qquad \begin{tikzcd} k^{\mathbf{n}} \arrow[r, "f^A"] \arrow[d, "h_2^*"'] & k^{\mathbf{n}} \arrow[d, "h_2^*"] \\ k^{\mathbf{m}} \arrow[r, "f^B"'] & k^{\mathbf{m}} \end{tikzcd}$$

Indeed, since $\langle h_1(1), h_2(1)\rangle, \ldots, \langle h_1(m), h_2(m)\rangle \in \theta$, we get

$$f^B(x_{h_1(1)/\theta}, \ldots, x_{h_1(m)/\theta}) = f^B(x_{h_2(1)/\theta}, \ldots, x_{h_2(m)/\theta}),$$

whence

$$(x'_{h_1(1)}, \ldots, x'_{h_1(m)}) = (x'_{h_2(1)}, \ldots, x'_{h_2(m)}),$$

which shows that, if $\langle i, j\rangle \in \eta$, then $x'_i = x'_j$, as required.

Now consider the projection $\pi_\theta : \mathbf{n} \to X$. It is shown that $\pi_\theta : \mathrm{eq}(h_1, h_2) \to A$ is a GTAN morphism. For π_θ conditions (1) and (2) in the definition of a GTAN morphism are clearly satisfied. For (3), the following diagram must be shown to commute

$$\begin{tikzcd} k^{|X|} \arrow[r, "f^C"] \arrow[d, "\pi_\theta^*"'] & k^{|X|} \arrow[d, "\pi_\theta^*"] \\ k^{\mathbf{n}} \arrow[r, "f^A"'] & k^{\mathbf{n}} \end{tikzcd}$$

Its commutativity, however, follows directly from the definition of the dynamics f^C of $\mathrm{eq}(h_1, h_2)$.

Finally, it is shown that $\mathrm{eq}(h_1, h_2)$ possesses the universal mapping property of the equaliser of A and B in the category **GTAN**. To this end, suppose that D is a GTAN with set of agents $D = \{D_i\}_{1\leq i\leq p}$, such that $D_i = \langle m_i, R_i\rangle$, $1 \leq i \leq p$, and dynamics $f^D : k^{\mathbf{p}} \to k^{\mathbf{p}}$. In addition, assume that $h : D \to A$ is a GTAN morphism, i.e., that $h : \mathbf{n} \to \mathbf{p}$ is a set mapping such that

$$m_{h(i)} = k_i, \qquad h(P_i) \subseteq R_{h(i)}, \quad \text{for all } 1 \leq i \leq n,$$

and such that the following rectangle commutes

$$\begin{tikzcd} k^{\mathbf{p}} \arrow[r, "f^D"] \arrow[d, "h^*"'] & k^{\mathbf{p}} \arrow[d, "h^*"] \\ k^{\mathbf{n}} \arrow[r, "f^A"'] & k^{\mathbf{n}} \end{tikzcd}$$

and that $h \circ h_1 = h \circ h_2$.

A morphism $d : D \to \mathrm{eq}(h_1, h_2)$ has to be constructed that makes the following diagram commute and its uniqueness must be shown.

$$\begin{tikzcd}
\mathrm{eq}(h_1, h_2) \arrow[r, "\pi_\theta"] & A \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & B \\
D \arrow[u, "d"] \arrow[ur, "h"'] & &
\end{tikzcd}$$

Since $h \circ h_1 = h \circ h_2$ and X is the coequalizer of h_1, h_2 in **Set**, there exists unique $d : \{1, \ldots, |X|\} \to \mathbf{p}$, such that $d \circ \pi_\theta = h$. This d is defined, for all equivalence classes I of θ, by

$$d(I) = h(i), \quad \text{for some } i \in I,$$

and is independent of the representative $i \in I$ chosen.

$d : D \to \mathrm{eq}(h_1, h_2)$ is a GTAN morphism. Indeed, for all $I \in X$, $m_{d(I)} = m_{h(i)} = k_i = k_I$ and

$$\begin{aligned}
d\left(\biguplus_{i\in I} P_i/\theta\right) &= d\left(\biguplus_{i\in I} \bigcup_{j\in P_i} \{j/\theta\}\right) \\
&= \biguplus_{i\in I} \bigcup_{j\in P_i} \{h(j)\} \\
&= \biguplus_{i\in I} h(P_i) \subseteq \biguplus_{i\in I} R_{h(i)} \\
&= R_{h(i)},
\end{aligned}$$

since, for all $\langle i, j\rangle \in \theta$, $h(i) = h(j)$. So it suffices to show that the following rectangle commutes

$$\begin{tikzcd}
k^{\mathbf{p}} \arrow[r, "f^D"] \arrow[d, "d^*"'] & k^{\mathbf{p}} \arrow[d, "d^*"] \\
k^{|X|} \arrow[r, "f^C"'] & k^{|X|}
\end{tikzcd}$$

It suffices, in turn, to show that

$$\pi_\theta^* d^* f^D = \pi_\theta^* f^C d^*.$$

$$\begin{aligned}
\pi_\theta^*(f^C(d^*(x_1, \ldots, x_p))) &= f^A(\pi_\theta^*(d^*(x_1, \ldots, x_p))) \\
&= f^A((d\pi_\theta)^*(x_1, \ldots, x_p)) \\
&= f^A(h^*(x_1, \ldots, x_p)) \\
&= h^*(f^D(x_1, \ldots, x_p)) \\
&= (d\pi_\theta)^*(f^D(x_1, \ldots, x_p)) \\
&= \pi_\theta^*(d^*(f^D(x_1, \ldots, x_p))).
\end{aligned}$$

Hence, $d : D \to \mathrm{eq}(h_1, h_2)$, thus defined, is indeed a GTAN morphism. It makes the required triangle commute since, by its construction, $d \circ \pi_\theta = h$. Finally, it is the unique such morphism, since it is uniquely determined by h.

The following theorem has been established.

THEOREM 7. *The category* **GTAN** *has equalizers. The functor* G : **GTAN** $\to$ $\mathbf{Set}^{\mathrm{op}}$ *that sends a GTAN to the index set of its underlying set of agents preserves and creates equalizers.*

The counterexample on the existence of equalizers for **TAN** is revisited and the equalizer of the same two morphisms, considered as morphisms in **GTAN** is given.

Recall that A was the TAN with the set of agents $A = \{A_1, A_2, A_3\}$, such that $A_1 = \langle 2, \{2, 3\}\rangle$, $A_2 = \langle 2, \{1, 3\}\rangle$ and $A_3 = \langle 2, \{1, 2\}\rangle$ and dynamics f^A.

$$\begin{tikzcd} & 1(2) \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \\ 2(2) \arrow[rr, leftrightarrow] & & 3(2) \end{tikzcd}$$

The two TAN morphisms $h_1, h_2 : A \to A$, were defined by $h_1 : \{1, 2, 3\} \to \{1, 2, 3\}$ being the identity and $h_2 : \{1, 2, 3\} \to \{1, 2, 3\}$ being given by $1 \mapsto 2$, $2 \mapsto 3$ and $3 \mapsto 1$. The equalizer in **GTAN** of h_1 and h_2 $\langle C, c : C \to A\rangle$

$$\begin{tikzcd} C \arrow[r, "c"] & A \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & A \end{tikzcd}$$

consists of the GTAN C with set of agents $C = \{C_1\}$, such that $C_1 = \langle 2, \{1^2\}\rangle$. The GTAN morphism $c : C \to A$ sends all three indices 1, 2 and 3 of A to the single index 1 of C.

Theorems 6 and 7, together with Theorem 9.2.10 of [1] yield the following

THEOREM 8. *The category* **GTAN** *has finite limits. The functor* G : **GTAN** $\to$ $\mathbf{Set}^{\mathrm{op}}$ *that sends a GTAN to the index set of its underlying set of agents preserves and creates finite limits.*

Acknowledgements

The author wishes to thank Professors Laubenbacher of the Virginia Institute of Technology and Pareigis of the University of Munich, Germany, for introducing him to the area of Finite Dynamical Systems and for their guidance and support.

References

1. Barr, M. and Wells, C.: *Category Theory for Computing Science*, 3rd edn, Les Publications CRM, Montréal, 1999.

2. Barrett, C. L. and Reidys, C. M.: Elements of a theory of simulation I: Sequential CA over random graphs, *Appl. Math. Comput.* **98** (1999), 241–259.
3. Barrett, C. L., Mortveit, H. S. and Reidys, C. M.: Elements of a theory of simulation II: Sequential dynamical systems, *Appl. Math. Comput.* **107** (1999), 121–136.
4. Borceux, F.: *Handbook of Categorical Algebra*, Encyclopedia Math. Appl. 50, Cambridge University Press, Cambridge, U.K., 1994.
5. Goles, E. and Martínez, S.: *Neural and Automata Networks, Dynamical Behavior and Applications*, Kluwer Academic Publishers, Dordrecht, 1990.
6. Gumm, H. P.: Elements of the general theory of coalgebras, Preprint, 2000.
7. Laubenbacher, R. and Pareigis, B.: Update schedules of sequential dynamical systems, Preprint, 2001.
8. Laubenbacher, R. and Pareigis, B.: Decomposition and simulation of sequential dynamical systems, *Adv. Appl. Math.* **30** (2003), 655–678.
9. Mac Lane, S.: *Categories for the Working Mathematician*, Springer-Verlag, 1971.
10. McCulloch, W. S. and Pitts, W.: A logical calculus of the ideas immanent in nervous activity, *Bull. Math. Biophys.* **5** (1943), 115–133.
11. Rutten, J. J. M. M.: A calculus of transition systems (towards universal coalgebra), *CSLI Lecture Notes* 53, CSLI Publications, Stanford, CA, 1995, pp. 231–256.
12. Rutten, J. J. M. M.: Universal coalgebra: A theory of systems, *Theoret. Comput. Sci.* **249** (2000), 3–80.
13. Voutsadakis, G.: Threshold agent networks: An approach to modeling and simulation, *Appl. Math. Comput.* **142** (2003), 521–543.