Advanced Calculus

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 411

1. Line and Surface Integrals
   - Vector Fields
   - Line Integrals
   - Conservative Vector Fields
   - Parametrized Surfaces and Surface Integrals
   - Surface Integrals of Vector Fields

Subsection 1

Vector Fields

Vector Fields

- A **vector field** $\boldsymbol{F}$ **in** $\mathbb{R}^3$ assigns to each point P in a domain $\mathcal{D}$ a vector $\boldsymbol{F}(P)$.
- A vector field in $\mathbb{R}^3$ is represented by a vector whose components are functions:

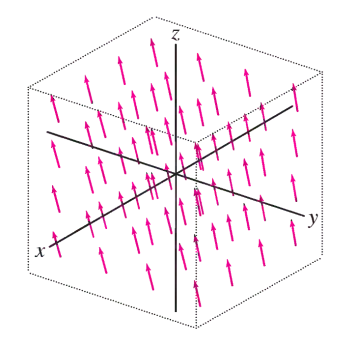
$$\boldsymbol{F}(x,y,z) = \langle F_1(x,y,z), F_2(x,y,z), F_3(x,y,z) \rangle.$$

- To each point $P = (a,b,c)$ is associated the vector $\boldsymbol{F}(a,b,c)$, which is also denoted by $\boldsymbol{F}(P) = F_1(P)\boldsymbol{i} + F_2(P)\boldsymbol{j} + F_3(P)\boldsymbol{k}$.
- When drawing a vector field, we draw $\boldsymbol{F}(P)$ as a vector based at P (rather than the origin).
- The **domain** of $\boldsymbol{F}$ is the set of points P for which $\boldsymbol{F}(P)$ is defined.
- Vector fields in the plane are written in a similar fashion: $\boldsymbol{F}(x,y) = \langle F_1(x,y), F_2(x,y) \rangle = F_1\boldsymbol{i} + F_2\boldsymbol{j}$.
- We will assume that the component functions F_j are smooth, i.e., that they have partial derivatives of all orders on their domains.

Example and Constant Vector Fields

- Which vector is attached to the point $P = (2, 4, 2)$ by the vector field $\boldsymbol{F} = (y - z, x, z - \sqrt{y})$?

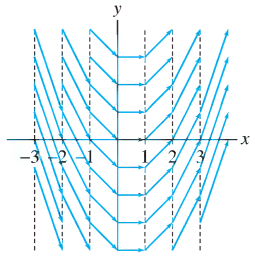
  The vector attached to P is $\boldsymbol{F}(2, 4, 2) = \langle 4 - 2, 2, 2 - \sqrt{4} \rangle = \langle 2, 2, 0 \rangle$.
- A **constant vector field** assigns the same vector to every point in $\mathbb{R}^3$.



Describing a Vector Field I

- Describe the vector field $\boldsymbol{G}(x,y)=\boldsymbol{i}+x\boldsymbol{j}$.

  The vector field assigns the vector $\langle 1,a\rangle$ to the point (a,b). In particular, it assigns the same vector to all points with the same x -coordinate.



$\mathbf{G}=\langle 1,x\rangle$

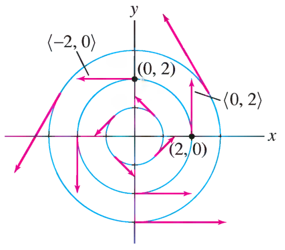
  Notice that $\langle 1,a\rangle$ has slope a and length $\sqrt{1+a^2}$.

  We may describe $\boldsymbol{G}$ as the vector field assigning a vector of slope a and length $\sqrt{1+a^2}$ to all points with $x=a$.

Describing a Vector Field II

- Describe the vector field $\boldsymbol{F}(x,y) = \langle -y, x\rangle$.

  To visualize $\boldsymbol{F}$, observe that $\boldsymbol{F}(a,b) = \langle -b, a\rangle$ has length $r = \sqrt{a^2+b^2}$.
  It is perpendicular to the radial vector $\langle a, b\rangle$ and points counterclockwise.



F = ⟨−y, x⟩

  Thus $\boldsymbol{F}$ is the vector field with vectors along the circle of radius r all having length r and being tangent to the circle, pointing counterclockwise.

Unit and Radial Vector Fields

- A **unit vector field** is a vector field $\boldsymbol{F}$ such that $\|\boldsymbol{F}(P)\| = 1$, for all points P.
- A vector field $\boldsymbol{F}$ is called a **radial vector field** if $\boldsymbol{F}(P) = f(x, y, z)\boldsymbol{r}$, where $f(x, y, z)$ is a scalar function.

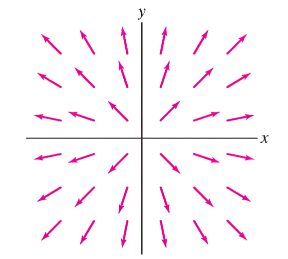
  We use the notation:
  - $\boldsymbol{r} = \langle x, y \rangle$ and $r = (x^2 + y^2)^{1/2}$ for $n = 2$;
  - $\boldsymbol{r} = \langle x, y, z \rangle$ and $r = (x^2 + y^2 + z^2)^{1/2}$ for $n = 3$.

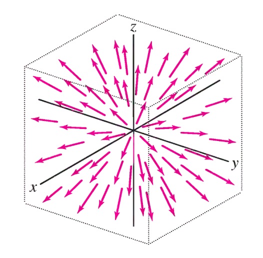
Examples

- Two important examples are the unit radial vector fields in two and three dimensions:

$$\boldsymbol{e}_r = \left\langle \frac{x}{r}, \frac{y}{r} \right\rangle = \left\langle \frac{x}{\sqrt{x^2+y^2}}, \frac{y}{\sqrt{x^2+y^2}} \right\rangle;$$

$$\boldsymbol{e}_r = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle = \left\langle \frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}} \right\rangle.$$





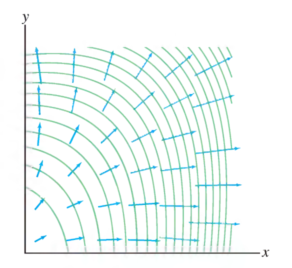
Conservative Vector Fields

- Recall the gradient vector field of a differentiable function $V(x, y, z)$:

$$\boldsymbol{F}(x, y, z) = \nabla V(x, y, z) = \left\langle \frac{\partial V}{\partial x}, \frac{\partial V}{\partial y}, \frac{\partial V}{\partial z} \right\rangle.$$

- A vector field of this type is called a **conservative vector field**.
- The function $V(x, y, z)$ is called a **potential function** (or **scalar potential function**) for $\boldsymbol{F}(x, y, z)$.
- Recall that the gradient vectors are orthogonal to the level curves.

  Thus in a conservative vector field, the vector at every point P is orthogonal to the level curve through P.



Example

- Verify that $V(x,y,z) = xy + yz^2$ is a potential function for the vector field $\boldsymbol{F}(x,y,z) = \langle y, x+z^2, 2yz \rangle$.

  We compute the gradient of V:

  $$\frac{\partial V}{\partial x} = y, \quad \frac{\partial V}{\partial y} = x + z^2, \quad \frac{\partial V}{\partial z} = 2yz.$$

  Thus, $\nabla V = \langle y, x+z^2, 2yz \rangle = \boldsymbol{F}$, i.e., V is a potential function for $\boldsymbol{F}$.

Cross-Partial Property of a Conservative Vector Field

Theorem (Cross-Partial Property of a Conservative Vector Field)

If the vector field $\boldsymbol{F}(x,y,z) = \langle F_1, F_2, F_3 \rangle$ is conservative, then

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}.$$

- If $\boldsymbol{F} = \nabla V$, then $F_1 = \frac{\partial V}{\partial x}$, $F_2 = \frac{\partial V}{\partial y}$ and $F_3 = \frac{\partial V}{\partial z}$. Now compute the "cross"-partial derivatives:

$$\begin{aligned} \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y}\left(\frac{\partial V}{\partial x}\right) = \frac{\partial^2 V}{\partial y \partial x}; \\ \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x}\left(\frac{\partial V}{\partial y}\right) = \frac{\partial^2 V}{\partial x \partial y}. \end{aligned}$$

  Clairaut's Theorem tells us that $\frac{\partial^2 V}{\partial y \partial x} = \frac{\partial^2 V}{\partial x \partial y}$. Thus, $\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}$.

  The other two equalities are proven similarly.

Example: A Non Conservative Function

- Show that $\boldsymbol{F}(x,y,z) = \langle y, 0, 0 \rangle$ is not conservative.

  We have
  $$\frac{\partial F_1}{\partial y} = \frac{\partial}{\partial y} y = 1, \quad \frac{\partial F_2}{\partial x} = \frac{\partial}{\partial x} 0 = 0.$$
  Thus, $\frac{\partial F_1}{\partial y} \neq \frac{\partial F_2}{\partial x}$. By the theorem, $\boldsymbol{F}$ is not conservative, even though the other cross-partials agree:
  $$\frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z} = 0 \quad \text{and} \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y} = 0.$$

Example

(a) Find by inspection a potential function for $\boldsymbol{F}(x,y) = \langle x, 0\rangle$.
(b) Prove that $\boldsymbol{G}(x,y) = \langle y, 0\rangle$ is not conservative.

(a) Suppose $V(x,y)$ is a potential function for $\boldsymbol{F}(x,y)$.
Then,

$$\frac{\partial V}{\partial x} = x, \quad \frac{\partial V}{\partial y} = 0.$$

Thus, we can take $V(x,y) = \frac{1}{2}x^2$.

(b) We have

$$\frac{\partial G_1}{\partial y} = 1, \quad \frac{\partial G_2}{\partial x} = 0.$$

Since $\frac{\partial G_1}{\partial y} \neq \frac{\partial G_2}{\partial x}$, $\boldsymbol{G}$ is not conservative.

Example

- Find a potential function for $\boldsymbol{F}(x,y) = \langle ye^{xy}, xe^{xy} \rangle$ by inspection.

  Suppose that $V(x,y)$ is a potential function for $\boldsymbol{F}$.

  Then we have
  $$\frac{\partial V}{\partial x} = ye^{xy}, \quad \frac{\partial V}{\partial y} = xe^{xy}.$$

  Therefore, we may take
  $$V(x,y) = e^{xy}.$$

Constant Vector Fields

- Show that any constant vector function $\boldsymbol{F}(x,y,z) = \langle a,b,c\rangle$ is conservative.

  Suppose that $V(x,y,z)$ is a potential function for $\boldsymbol{F}$.

  Then we have

  $$\frac{\partial V}{\partial x} = a, \quad \frac{\partial V}{\partial y} = b, \quad \frac{\partial V}{\partial z} = c.$$

  By integration,

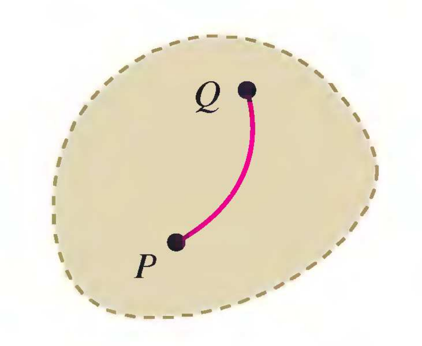
  $$V = ax + f_1(y,z), \quad V = by + f_2(x,z), \quad V = cz + f_3(x,y).$$

  Therefore, we can take

  $$V(x,y,z) = ax + by + cz.$$

Connected Domains

- A domain is "connected" if any two points can be joined by a path within the domain.



Uniqueness of Potential Functions

Theorem (Uniqueness of Potential Functions)

If $\boldsymbol{F}$ is conservative on an open connected domain, then any two potential functions of $\boldsymbol{F}$ differ by a constant.

- If both V_1 and V_2 are potential functions of $\boldsymbol{F}$, then

$$\nabla(V_1 - V_2) = \nabla V_1 - \nabla V_2 = \boldsymbol{F} - \boldsymbol{F} = \mathbf{0}.$$

However, a function whose gradient is zero on an open connected domain is a constant function (this generalizes the fact from single-variable calculus that a function on an interval with zero derivative is a constant function). Thus $V_1 - V_2 = C$, for some constant C. Hence $V_1 = V_2 + C$.

Unit Radial Vector Fields Revisited

- Show that
$$V(x,y,z) = r = \sqrt{x^2+y^2+z^2}$$
is a potential function for $\boldsymbol{e}_r$. I.e., $\boldsymbol{e}_r = \nabla r$.
We have
$$\frac{\partial r}{\partial x} = \frac{\partial}{\partial x}\sqrt{x^2+y^2+z^2} = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{r}.$$
Similarly, $\frac{\partial r}{\partial y} = \frac{y}{r}$ and $\frac{\partial r}{\partial z} = \frac{z}{r}$. Therefore, $\nabla r = \left\langle \frac{x}{r}, \frac{y}{r}, \frac{z}{r} \right\rangle = \boldsymbol{e}_r$.

Inverse-Square Vector Field

- Show that

$$\frac{\boldsymbol{e}_r}{r^2} = \nabla\left(\frac{-1}{r}\right).$$

Recall the Chain Rule for Gradients

$$\nabla F(r) = F'(r)\nabla r.$$

Recall, also, from the preceding example that $\nabla r = \boldsymbol{e}_r$.

Thus, we get

$$\nabla\left(-\frac{1}{r}\right) = \frac{1}{r^2}\nabla r = \frac{1}{r^2}\boldsymbol{e}_r.$$

Example

- Let $\phi(x,y) = \ln r$, where $r = \sqrt{x^2+y^2}$.
  Express $\nabla\phi$ in terms of $\boldsymbol{e}_r$ in $\mathbb{R}^2$.
  Recall again that

$$\nabla F(r) = F'(r)\nabla r \quad \text{and} \quad \nabla r = \boldsymbol{e}_r.$$

  Thus, we have

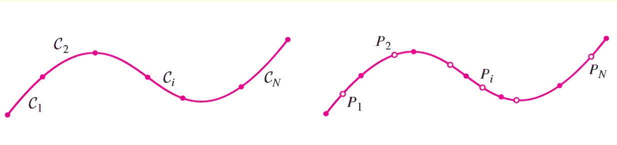
$$\nabla\phi = \nabla(\ln r) = (\ln r)'\nabla r = \frac{1}{r}\boldsymbol{e}_r.$$

Subsection 2

Line Integrals

Scalar Line Integrals

- We begin by defining the **scalar line integral** $\int_{\mathcal{C}} f(x, y, z) ds$ of a function f over a curve $\mathcal{C}$.
- We divide $\mathcal{C}$ into N consecutive arcs $\mathcal{C}_1, \ldots, \mathcal{C}_N$, and choose a sample point P_i in each arc $\mathcal{C}_i$.



- We form the Riemann sum $\sum_{i=1}^N f(P_i) \text{length}(\mathcal{C}_i) = \sum_{i=1}^N f(P_i) \Delta s_i$, where Δs_i is the length of $\mathcal{C}_i$.
- The **line integral** of f over $\mathcal{C}$ is the limit (if it exists) of these Riemann sums as the maximum of the lengths Δs_i approaches zero:

$$\int_{\mathcal{C}} f(x, y, z) ds = \lim_{\{\Delta s_i\} \to 0} \sum_{i=1}^{N} f(P_i) \Delta s_i.$$

Line Integrals and Length of a Curve

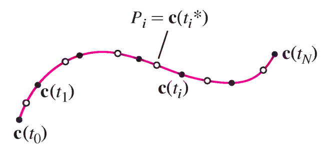
- The scalar line integral of the function $f(x, y, z) = 1$ is simply the length of $\mathcal{C}$.

  In this case, all the Riemann sums have the same value:

$$\int_{\mathcal{C}} 1 ds = \text{length}(\mathcal{C}).$$

Line Integrals Using Parametrizations

- Suppose that $\mathcal{C}$ has a parametrization $\boldsymbol{c}(t)$ for $a \leq t \leq b$ with continuous derivative $\boldsymbol{c}'(t)$. Recall that the derivative is the tangent vector $\boldsymbol{c}'(t) = \langle x'(t), y'(t), z'(t) \rangle$.
- We divide $\mathcal{C}$ into N consecutive arcs $\mathcal{C}_1, \ldots, \mathcal{C}_n$ corresponding to a partition of the interval $[a, b]$: $a = t_0 < t_1 < \cdots < t_{N-1} < t_N = b$ so that d is parametrized by $\boldsymbol{c}(t)$ for $t_{i-1} < t < t_i$.



- Choose sample points $P_i = \boldsymbol{c}(t_i^*)$ with t_i^* in $[t_{i-1}, t_i]$.
- According to the arc length formula

$$\text{length}(\mathcal{C}_i) = \Delta s_i = \int_{t_{i-1}}^{t_i} \|\boldsymbol{c}'(t)\| dt.$$

- Because $\boldsymbol{c}'(t)$ is continuous, the function $\|\boldsymbol{c}'(t)\|$ is nearly constant on $[t_{i-1}, t_i]$ if the length $\Delta t_i = t_i - t_{i-1}$ is small.
- Thus, $\int_{t_{i-1}}^{t_i} \|\boldsymbol{c}'(t)\| dt \approx \|\boldsymbol{c}'(t_i^*)\| \Delta t_i$.

Line Integrals Using Parametrizations (Cont'd)

- This gives us the approximation

$$\sum_{i=1}^{N} f(P_i)\Delta s_i \approx \sum_{i=1}^{N} f(\boldsymbol{c}(t_i^*))\|\boldsymbol{c}'(t_i^*)\|\Delta t_i.$$

- By definition, the sum on the left converges to $\int_{\mathcal{C}} f(x,y,z)ds$ when the maximum of the lengths Δt_i tends to zero.
- The sum on the right is a Riemann sum that converges to the integral $\int_a^b f(\boldsymbol{c}(t))\|\boldsymbol{c}'(t)\|dt$ as the maximum of the lengths Δt_i tends to zero.
- By estimating the errors in this approximation, we can show that the two sums approach the same value.

Computing a Scalar Line Integral

- Our work in the preceding two slides gives:

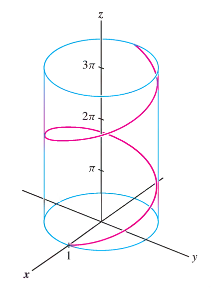
Theorem (Computing a Scalar Line Integral)

Let $\boldsymbol{c}(t)$ be a parametrization of a curve $\mathcal{C}$ for $a \leq t << b$. If $f(x,y,z)$ and $\boldsymbol{c}'(t)$ are continuous, then

$$\int_{\mathcal{C}} f(x,y,z)ds = \int_a^b f(\boldsymbol{c}(t))\|\boldsymbol{c}'(t)\|dt.$$

- The symbol ds is intended to suggest arc length s and is often referred to as the **line element** or **arc length differential**.
- In terms of a parametrization, we have the symbolic equation $ds = \|\boldsymbol{c}'(t)\|dt$, where $\|\boldsymbol{c}'(t)\| = \sqrt{x'(t)^2 + y'(t)^2 + z'(t)^2}$.

Example: Integrating Along the Helix



- Calculate $\int_{\mathcal{C}}(x+y+z)ds$ where $\mathcal{C}$ is the helix $\boldsymbol{c}(t)=\langle\cos t,\sin t,t\rangle$, for $0\leq t\leq\pi$.
  We compute ds:

$$\begin{array}{rcl}\boldsymbol{c}'(t) & = & \langle-\sin t,\cos t,1\rangle;\\ \|\boldsymbol{c}'(t)\| & = & \sqrt{(-\sin t)^2+\cos^2 t+1}=\sqrt{2};\\ ds & = & \|\boldsymbol{c}'(t)\|dt=\sqrt{2}dt.\end{array}$$

$$\begin{array}{rcl}\int_{\mathcal{C}}f(x,y,z)ds & = & \int_0^\pi f(\boldsymbol{c}(t))\|\boldsymbol{c}'(t)\|dt\\ & = & \int_0^\pi(\cos t+\sin t+t)\sqrt{2}dt\\ & = & \sqrt{2}(\sin t-\cos t+\frac{1}{2}t^2)\mid_0^\pi\\ & = & \sqrt{2}(0+1+\frac{1}{2}\pi^2)-\sqrt{2}(0-1+0)\\ & = & 2\sqrt{2}+\frac{\sqrt{2}}{2}\pi^2.\end{array}$$

Example: Arc Length

- Calculate $\int_{\mathcal{C}} 1ds$ for the helix $\boldsymbol{c}(t) = \langle \cos t, \sin t, t \rangle$, for $0 \leq t \leq \pi$. What does the integral represent?

  We found $ds = \sqrt{2}dt$.

  It follows
  $$\int_{\mathcal{C}} 1ds = \int_0^{\pi} \sqrt{2}dt = \pi\sqrt{2}.$$

  This is the length of the helix for $0 \leq t \leq \pi$.

Example: Arc Length

- Calculate $\int_{\mathcal{C}} 1ds$, where $\mathcal{C}$ is parameterized by $\boldsymbol{c}(t) = \langle 4t, -3t, 12t \rangle$, for $2 \leq t \leq 5$.

  What does the integral represent?

  We have

$$\begin{array}{rcl} \boldsymbol{c}'(t) & = & \langle 4, -3, 12 \rangle; \\ \|\boldsymbol{c}'(t)\| & = & \sqrt{4^2 + (-3)^2 + 12^2} = \sqrt{169} = 13; \\ ds & = & \|\boldsymbol{c}'(t)\| dt = 13dt; \\ \int_{\mathcal{C}} 1ds & = & \int_2^5 1 \cdot 13dt \\ & = & 13t \mid_2^5 \\ & = & 39. \end{array}$$

  This is the length of the line segment from the point $\boldsymbol{c}(2) = (8, -6, 24)$ to the point $\boldsymbol{c}(5) = (20, -15, 60)$.

Calculating Mass

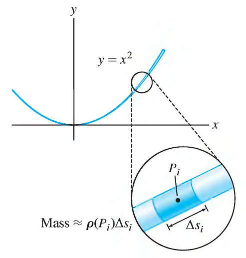
- The general principle that "the integral of a density is the total quantity" applies to scalar line integrals.
- For example, we can view the curve $\mathcal{C}$ as a wire with continuous mass density $\rho(x, y, z)$, given in units of mass per unit length.
- The total mass is defined as the integral of mass density:

$$\text{Total mass of } \mathcal{C} = \int_{\mathcal{C}} \rho(x, y, z) ds.$$

Justification of the Total Mass Formula

- We justify the formulas for the total mass by dividing $\mathcal{C}$ into N arcs $\mathcal{C}_i$ of length Δs_i with N large.

  The mass density is nearly constant on $\mathcal{C}_i$. Therefore, the mass of $\mathcal{C}_i$ is approximately $\rho(P_i)\Delta s_i$, where P_i is any sample point on $\mathcal{C}_i$.



  The total mass is the sum

$$\text{Total mass of } \mathcal{C} = \sum_{i=1}^{N} \text{mass of } \mathcal{C}_i \approx \sum_{i=1}^{N} \rho(P_i)\Delta s_i.$$

  As the maximum of the lengths Δs_i tends to zero, the sums on the right approach the line integral.

Example: Scalar Line Integral as Total Mass

- Find the total mass of a wire in the shape of the parabola $y = x^2$, for $1 \leq x \leq 4$ (in cm), with mass density given by $\rho(x, y) = \frac{y}{x}$ g/cm.

  The arc of the parabola is parametrized by $\mathbf{c}(t) = \langle t, t^2 \rangle$ for $1 \leq t \leq 4$.

  We compute ds:

$$\begin{array}{rcl} \mathbf{c}'(t) & = & \langle 1, 2t \rangle; \\ ds & = & \|\mathbf{c}'(t)\| dt = \sqrt{1 + 4t^2} dt. \end{array}$$

  We write out the integrand and evaluate:

$$\begin{array}{rcl} \int_{\mathcal{C}} \rho(x, y) ds & = & \int_1^4 \rho(\mathbf{c}(t)) \|\mathbf{c}'(t)\| dt \\ & = & \int_1^4 \frac{t^2}{t} \sqrt{1 + 4t^2} dt \\ & \overset{u=1+4t^2}{=} & \frac{1}{8} \int_5^{65} \sqrt{u} du \\ & = & \frac{1}{12} u^{3/2} \mid_5^{65} \\ & = & \frac{1}{12} (65^{3/2} - 5^{3/2}) \text{ g}. \end{array}$$

Calculating Electric Potential

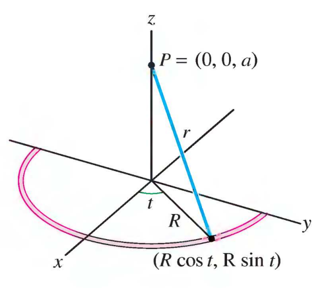
- Scalar line integrals are also used to compute electric potentials.
- When an electric charge is distributed continuously along a curve $\mathcal{C}$, with charge density $\rho(x, y, z)$, the charge distribution sets up an electrostatic field $\boldsymbol{E}$ that is a conservative vector field.
- Coulomb's Law tells us that $\boldsymbol{E} = \nabla V$, where

$$V(P) = k \int_{\mathcal{C}} \frac{\rho(x, y, z) ds}{r_P(x, y, x)}.$$

  In this integral,
  - $r_P(x, y, z)$ denotes the distance from (x, y, z) to P;
  - The constant k has the value $k = 8.99x10^9$ N·m^2/C^2.
- The function V is called the **electric potential**. It is defined for all points P that do not lie on $\mathcal{C}$ and has units of volts (one volt is one N·m/C).

Example: Electric Potential

- A charged semicircle of radius R centered at the origin in the xy-plane has charge density $\rho(x,y,0) = 10^{-8}(2 - \frac{x}{R})$ C/m.
  Find the electric potential at a point $P = (0,0,a)$ if $R = 0.1$ m.
  Parametrize the semicircle by $\boldsymbol{c}(t) = \langle R\cos t, R\sin t, 0\rangle$, $-\frac{\pi}{2} \leq t \leq \frac{\pi}{2}$.



$$\begin{aligned} \|\boldsymbol{c}'(t)\| &= \|\langle -R\sin t, R\cos t, 0\rangle\| = R; \\ ds &= \|\boldsymbol{c}'(t)\| dt = R dt; \\ \rho(\boldsymbol{c}(t)) &= 10^{-8}(2 - \frac{R\cos t}{R}) = 10^{-8}(2 - \cos t). \end{aligned}$$

In our case, the distance r_P from P to a point $(x,y,0)$ on the semicircle has the constant value $r_P = \sqrt{R^2 + a^2}$.

Example: Electric Potential (Cont'd)

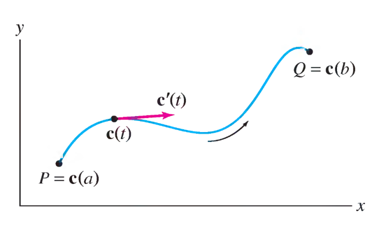
- Thus, we obtain

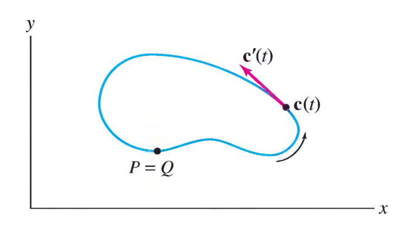
$$\begin{aligned} V(P) &= k \int_{\mathcal{C}} \frac{\rho(x,y,z)ds}{r_P(x,y,z)} = k \int_{\mathcal{C}} \frac{10^{-8}(2-\cos t)Rdt}{\sqrt{R^2+a^2}} \\ &= \frac{10^{-8}kR}{\sqrt{R^2+a^2}} \int_{-\pi/2}^{\pi/2} (2-\cos t)dt \\ &= \frac{10^{-8}kR}{\sqrt{R^2+a^2}} (2t - \sin t) \Big|_{-\pi/2}^{\pi/2} \\ &= \frac{10^{-8}kR}{\sqrt{R^2+a^2}} (2\pi - 2). \end{aligned}$$

With $R = 0.1$ m and $k \approx 9 \times 10^9$, we then obtain $10^{-8}kR(2\pi-2) \approx 9(2\pi-2)$. Hence $V(P) \approx \frac{9(2\pi-2)}{\sqrt{0.01+a^2}}$ volts.

Oriented Curves

- A specified direction along a path curve $\mathcal{C}$ is called an **orientation**.
- We refer to this direction as the **positive direction** along $\mathcal{C}$.
- The opposite direction is the **negative direction**.
- $\mathcal{C}$ provided with an orientation is called an **oriented curve**.





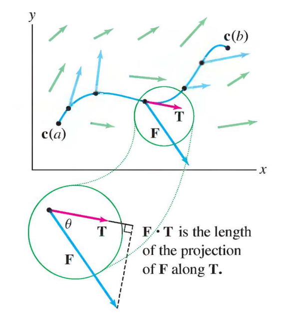
In the left figure, if we reversed the orientation, the positive direction would become the direction from Q to P.

Tangential Component of Vector Field

- Let $\boldsymbol{T} = \boldsymbol{T}(P)$ denote the unit tangent vector at a point P on $\mathcal{C}$ pointing in the positive direction.
- The **tangential component** of $\boldsymbol{F}$ at P is the dot product

$$\begin{aligned} \boldsymbol{F}(P) \cdot \boldsymbol{T}(P) &= \|\boldsymbol{F}(P)\| \|\boldsymbol{T}(P)\| \cos\theta \\ &= \|\boldsymbol{F}(P)\| \cos\theta, \end{aligned}$$

where θ is the angle between $\boldsymbol{F}(P)$ and $\boldsymbol{T}(P)$.



Vector Line Integral

- The vector line integral of $\boldsymbol{F}$ is the scalar line integral of the scalar function $\boldsymbol{F} \cdot \boldsymbol{T}$.
- We make the standing assumption that $\mathcal{C}$ is piece-wise smooth (it consists of finitely many smooth curves joined together with possible corners).

Definition (Vector Line Integral)

The line integral of a vector field $\boldsymbol{F}$ along an oriented curve $\mathcal{C}$ is the integral of the tangential component of $\boldsymbol{F}$:

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\mathcal{C}} (\boldsymbol{F} \cdot \boldsymbol{T}) ds.$$

Parametrizing Line Integrals

- We use parametrizations to evaluate vector line integrals.

  The parametrization $\boldsymbol{c}(t)$ must be:
  - positively oriented, i.e., $\boldsymbol{c}(t)$ must trace $\mathcal{C}$ in the positive direction;
  - regular, i.e., $\boldsymbol{c}'(t) \neq \mathbf{0}$, for $a \leq t \leq b$.

  Then $\boldsymbol{c}'(t)$ is a nonzero tangent vector pointing in the positive direction, and $\boldsymbol{T} = \frac{\boldsymbol{c}'(t)}{\|\boldsymbol{c}'(t)\|}$.
- In terms of the arc length differential $ds = \|\boldsymbol{c}'(t)\| dt$, we have

$$(\boldsymbol{F} \cdot \boldsymbol{T}) ds = \left(\boldsymbol{F}(\boldsymbol{c}(t)) \cdot \frac{\boldsymbol{c}'(t)}{\|\boldsymbol{c}'(t)\|} \right) \|\boldsymbol{c}'(t)\| dt = \boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) dt.$$

Evaluating Line Integrals

- Therefore, the integral $\int_{\mathcal{C}} (\boldsymbol{F} \cdot \boldsymbol{T}) ds$ can be rewritten $\int_a^b \boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) dt$:

Theorem (Computing a Vector Line Integral)

If $\boldsymbol{c}(t)$ is a regular parametrization of an oriented curve $\mathcal{C}$ for $a \leq t \leq b$, then

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_a^b \boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) dt.$$

- It is useful to think of $d\boldsymbol{s}$ as a "vector line element" or "vector differential" that is related to the parametrization by the symbolic equation

$$d\boldsymbol{s} = \boldsymbol{c}'(t) dt.$$

Example

- Evaluate $\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s}$, where $\boldsymbol{F} = \langle z, y^2, x \rangle$ and $\mathcal{C}$ is parametrized (in the positive direction) by $\boldsymbol{c}(t) = \langle t+1, e^t, t^2 \rangle$, for $0 \leq t \leq 2$.

  We calculate the integrand:

$$\begin{array}{rcl} \boldsymbol{c}(t) & = & \langle t+1, e^t, t^2 \rangle; \\ \boldsymbol{F}(\boldsymbol{c}(t)) & = & \langle z, y^2, x \rangle = \langle t^2, e^{2t}, t+1 \rangle; \\ \boldsymbol{c}'(t) & = & \langle 1, e^t, 2t \rangle. \end{array}$$

  The integrand (as a differential) is the dot product:

$$\boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) dt = \langle t^2, e^{2t}, t+1 \rangle \cdot \langle 1, e^t, 2t \rangle dt = (e^{3t} + 3t^2 + 2t) dt.$$

  Finally, we evaluate the integral:

$$\begin{array}{rcl} \int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} & = & \int_0^2 \boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) dt \\ & = & \int_0^2 (e^{3t} + 3t^2 + 2t) dt = (\frac{1}{3} e^{3t} + t^3 + t^2) \mid_0^2 \\ & = & (\frac{1}{3} e^6 + 8 + 4) - \frac{1}{3} = \frac{1}{3}(e^6 + 35). \end{array}$$

Example

- Let $\boldsymbol{F}(x,y,z)=\langle z^2,x,y\rangle$ and $\mathcal{C}$ be the path $\boldsymbol{c}(t)=\langle 3+5t^2,3-t^2,t\rangle$, $0\leq t\leq 2$.
  Evaluate the line integral $\int_{\mathcal{C}}\boldsymbol{F}\cdot d\boldsymbol{s}$.

$$\begin{array}{rcl}
\boldsymbol{c}(t) & = & \langle 3+5t^2,3-t^2,t\rangle,\ 0\leq t\leq 2; \\
\boldsymbol{F}(\boldsymbol{c}(t)) & = & \langle z^2,x,y\rangle=\langle t^2,3+5t^2,3-t^2\rangle; \\
\boldsymbol{c}'(t) & = & \langle 10t,-2t,1\rangle; \\
\boldsymbol{F}(\boldsymbol{c}(t))\cdot\boldsymbol{c}'(t)dt & = & \langle t^2,3+5t^2,3-t^2\rangle\cdot\langle 10t,-2t,1\rangle dt \\
 & = & (10t^3-2t(3+5t^2)+3-t^2)dt \\
 & = & (10t^3-10t^3-6t+3-t^2)dt \\
 & = & (-t^2-6t+3)dt; \\
\int_{\mathcal{C}}\boldsymbol{F}\cdot d\boldsymbol{s} & = & \int_0^2\boldsymbol{F}(\boldsymbol{c}(t))\cdot\boldsymbol{c}'(t)dt \\
 & = & \int_0^2(-t^2-6t+3)dt \\
 & = & (-\frac{1}{3}t^3-3t^2+3t)\ |_0^2 \\
 & = & -\frac{8}{3}-12+6=-\frac{26}{3}.
\end{array}$$

Alternative Notation

- Another standard notation for the line integral $\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s}$ is

$$\int_{\mathcal{C}} F_1 dx + F_2 dy + F_3 dz.$$

- In this notation, we write $d\boldsymbol{s}$ as a vector differential $d\boldsymbol{s} = \langle dx, dy, dz \rangle$ so that $\boldsymbol{F} \cdot d\boldsymbol{s} = \langle F_1, F_2, F_3 \rangle \cdot \langle dx, dy, dz \rangle = F_1 dx + F_2 dy + F_3 dz$.
- In terms of a parametrization $\boldsymbol{c}(t) = \langle x(t), y(t), z(t) \rangle$,

$$\begin{aligned} d\boldsymbol{s} &= \langle dx, dy, dz \rangle = \langle \tfrac{dx}{dt}, \tfrac{dy}{dt}, \tfrac{dz}{dt} \rangle dt; \\ \boldsymbol{F} \cdot d\boldsymbol{s} &= (F_1(\boldsymbol{c}(t)) \tfrac{dx}{dt} + F_2(\boldsymbol{c}(t)) \tfrac{dy}{dt} + F_3(\boldsymbol{c}(t)) \tfrac{dz}{dt}) dt. \end{aligned}$$

So we have the following formula:

$$\begin{aligned} &\int_{\mathcal{C}} F_1 dx + F_2 dy + F_3 dz \\ &\qquad = \int_a^b (F_1(\boldsymbol{c}(t)) \tfrac{dx}{dt} + F_2(\boldsymbol{c}(t)) \tfrac{dy}{dt} + F_3(\boldsymbol{c}(t)) \tfrac{dz}{dt}) dt. \end{aligned}$$

Example

- Consider the ellipse $\mathcal{C}$ with counterclockwise orientation parameterized by $\boldsymbol{c}(\theta) = \langle 5 + 4\cos\theta, 3 + 2\sin\theta \rangle$ for $0 \le \theta \le 2\pi$. Calculate $\int_{\mathcal{C}} 2ydx - 3dy$.

  We have $x(\theta) = 5 + 4\cos\theta$ and $y(\theta) = 3 + 2\sin\theta$. So $\frac{dx}{d\theta} = -4\sin\theta$ and $\frac{dy}{d\theta} = 2\cos\theta$. The integrand of the line integral is

$$\begin{array}{rcl} 2ydx - 3dy & = & (2y\frac{dx}{d\theta} - 3\frac{dy}{d\theta})d\theta \\ & = & (2(3 + 2\sin\theta)(-4\sin\theta) - 3(2\cos\theta))d\theta \\ & = & -(24\sin\theta + 16\sin^2\theta + 6\cos\theta)d\theta. \end{array}$$

  Since the integrals of $\cos\theta$ and $\sin\theta$ over $[0, 2\pi]$ are zero,

$$\begin{array}{rcl} \int_{\mathcal{C}} 2ydx - 3dy & = & -\int_0^{2\pi} (24\sin\theta + 16\sin^2\theta + 6\cos\theta)d\theta \\ & = & -16\int_0^{2\pi} \sin^2\theta d\theta = -16\int_0^{2\pi} (\frac{1}{2} - \frac{1}{2}\cos 2\theta)d\theta \\ & = & -16(\frac{1}{2}\theta - \frac{1}{4}\sin 2\theta)\,|_0^{2\pi} = -16\pi. \end{array}$$

Example

- Evaluate the line integral $\int_{\mathcal{C}} zdx + x^2dy + ydz$, where $\mathcal{C}$ is parameterized by $\boldsymbol{c}(t) = \langle \cos t, \tan t, t\rangle$, with $0 \leq t \leq \frac{\pi}{4}$.
  We have
  $$\begin{array}{lll} x(t) = \cos t, & y(t) = \tan t, & z(t) = t; \\ \frac{dx}{dt} = -\sin t, & \frac{dy}{dt} = \sec^2 t, & \frac{dz}{dt} = 1. \end{array}$$
  Thus, we get
  $$\begin{array}{lll} zdx + x^2dy + ydz & = & (z\frac{dx}{dt} + x^2\frac{dy}{dt} + y\frac{dz}{dt})dt \\ & = & (-t\sin t + \cos^2 t\sec^2 t + \tan t)dt \\ & = & (-t\sin t + 1 + \tan t)dt. \end{array}$$
  Therefore,
  $$\begin{array}{l} \int_{\mathcal{C}} zdx + x^2dy + ydz \\ = \int_0^{\pi/4} (-t\sin t + 1 + \tan t)dt \\ = (t\cos t - \sin t + t - \ln(\cos t)) \mid_0^{\pi/4} \\ = \frac{\sqrt{2}\pi}{8} - \frac{\sqrt{2}}{2} + \frac{\pi}{4} - \ln\frac{\sqrt{2}}{2}. \end{array}$$

Reversing Orientation and Additivity

- Given an oriented curve $\mathcal{C}$, we write $-\mathcal{C}$ to denote the curve $\mathcal{C}$ with the opposite orientation. The unit tangent vector changes sign from $\boldsymbol{T}$ to $-\boldsymbol{T}$ when we change orientation. So the tangential component of $\boldsymbol{F}$ and the line integral also change sign:

$$\int_{-\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} = -\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s}.$$

- If we are given n oriented curves $\mathcal{C}_1, \ldots, \mathcal{C}_n$, we write $\mathcal{C} = \mathcal{C}_1 + \cdots + \mathcal{C}_n$ to indicate the union of the paths.
  We define the line integral over $\mathcal{C}$ as the sum

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{s} + \cdots + \int_{\mathcal{C}_n} \boldsymbol{F} \cdot d\boldsymbol{s}.$$

  We use this formula to define the line integral when $\mathcal{C}$ is piecewise smooth, meaning that $\mathcal{C}$ is a union of smooth curves $\mathcal{C}_1, \ldots, \mathcal{C}_n$.

Properties of Vector Line Integrals

Theorem (Properties of Vector Line Integrals)

Let $\mathcal{C}$ be a smooth oriented curve and let $\boldsymbol{F}$ and $\boldsymbol{G}$ be vector fields.

(i) **Linearity:**

$$\int_{\mathcal{C}} (\boldsymbol{F} + \boldsymbol{G}) \cdot d\boldsymbol{s} = \int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{\mathcal{C}} \boldsymbol{G} \cdot d\boldsymbol{s};$$

$$\int_{\mathcal{C}} k\boldsymbol{F} \cdot d\boldsymbol{s} = k \int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} \quad (k \text{ a constant})$$

(ii) **Reversing Orientation:**

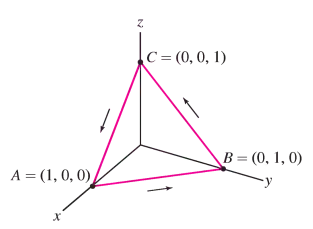
$$\int_{-\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} = -\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s}$$

(iii) **Additivity:** If $\mathcal{C}$ is a union of n smooth curves $\mathcal{C}_1 + \cdots + \mathcal{C}_n$, then

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\mathcal{C}_1} \boldsymbol{F} \cdot d\boldsymbol{s} + \cdots + \int_{\mathcal{C}_n} \boldsymbol{F} \cdot d\boldsymbol{s}.$$

Example

- Compute $\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s}$, where
$$\boldsymbol{F} = \langle e^z, e^y, x + y \rangle$$
and $\mathcal{C}$ is the triangle joining $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ oriented counter-clockwise when viewed from above.



The line integral is the sum of the line integrals over the edges of the triangle:
$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\overline{AB}} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{\overline{BC}} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{\overline{CA}} \boldsymbol{F} \cdot d\boldsymbol{s}.$$

- Segment $\overline{AB}$ is parametrized by $\boldsymbol{c}(t) = \langle 1 - t, t, 0 \rangle$, for $0 \leq t \leq 1$. We have
$$\begin{array}{rcl} \boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) & = & \langle e^0, e^t, 1 \rangle \cdot \langle -1, 1, 0 \rangle = -1 + e^t; \\ \int_{\overline{AB}} \boldsymbol{F} \cdot d\boldsymbol{s} & = & \int_0^1 (e^t - 1) dt = (e^t - t) \mid_0^1 = e - 2. \end{array}$$

Example (Cont'd)

- $\overline{BC}$ is parametrized by $\boldsymbol{c}(t) = \langle 0, 1-t, t \rangle$, for $0 \le t \le 1$. We have

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) &= \langle e^t, e^{1-t}, 1-t \rangle \cdot \langle 0, -1, 1 \rangle = -e^{1-t} + 1 - t; \\ \int_{\overline{BC}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_0^1 (-e^{1-t} + 1 - t) dt \\ &= (e^{1-t} + t - \tfrac{1}{2} t^2) \,|_0^1 = \tfrac{3}{2} - e. \end{aligned}$$

- Finally, $\overline{CA}$ is parametrized by $\boldsymbol{c}(t) = \langle t, 0, 1-t \rangle <$ for $0 \le t \le 1$. We have

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) &= \langle e^{1-t}, 1, t \rangle \cdot \langle 1, 0, -1 \rangle = e^{1-t} - t; \\ \int_{\overline{CA}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_0^1 (e^{1-t} - t) dt \\ &= (-e^{1-t} - \tfrac{1}{2} t^2) \,|_0^1 = -\tfrac{3}{2} + e. \end{aligned}$$

  The total line integral is the sum

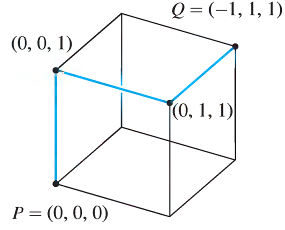
$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} = (e-2) + (\frac{3}{2} - e) + (-\frac{3}{2} + e) = e - 2.$$

Example

- Calculate the line integral of

$$\boldsymbol{F} = \langle e^z, e^{x-y}, e^y \rangle$$

over the blue path from P to Q.



The line integral is the sum of the line integrals over the three edges of the cube:

$$\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\overline{PA}} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{\overline{AB}} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{\overline{BQ}} \boldsymbol{F} \cdot d\boldsymbol{s}.$$

- Segment $\overline{PA}$ is parametrized by $\boldsymbol{c}(t) = \langle 0, 0, t \rangle$, for $0 \leq t \leq 1$. We have

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) &= \langle e^t, 1, 1 \rangle \cdot \langle 0, 0, 1 \rangle = 1; \\ \int_{\overline{PA}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_0^1 1 dt = t \,|_0^1 = 1. \end{aligned}$$

Example (Cont'd)

- $\overline{AB}$ is parametrized by $\mathbf{c}(t) = \langle 0, t, 1 \rangle$, for $0 \leq t \leq 1$. We have

$$\begin{array}{rcl} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) & = & \langle e, e^{-t}, e^t \rangle \cdot \langle 0, 1, 0 \rangle = -e^t; \\ \int_{\overline{AB}} \mathbf{F} \cdot d\mathbf{s} & = & \int_0^1 e^{-t} dt \\ & = & (-e^{-t}) \mid_0^1 = 1 - \frac{1}{e}. \end{array}$$

- Finally, $\overline{BQ}$ is parametrized by $\mathbf{c}(t) = \langle -t, 1, 1 \rangle <$ for $0 \leq t \leq 1$. We have

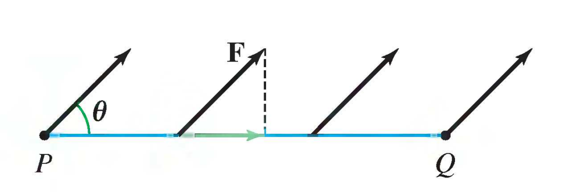
$$\begin{array}{rcl} \mathbf{F}(\mathbf{c}(t)) \cdot \mathbf{c}'(t) & = & \langle e, e^{-t-1}, e \rangle \cdot \langle -1, 0, 0 \rangle = -e; \\ \int_{\overline{BQ}} \mathbf{F} \cdot d\mathbf{s} & = & \int_0^1 -e dt \\ & = & -et \mid_0^1 = -e. \end{array}$$

The total line integral is the sum

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 1 + (1 - \frac{1}{e}) - e = 2 - \frac{1}{e} - e.$$

Work Along a Straight Segment by a Constant Force

- In physics, "work" refers to the energy expended when a force is applied to an object as it moves along a path.
- By definition, the work W performed along the straight segment from P to Q by applying a constant force $\boldsymbol{F}$ at an angle θ



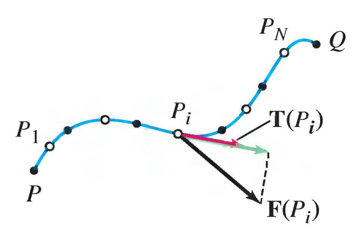
is given by

$$W = (\text{tangential component of } \boldsymbol{F}) \times \text{distance} = (\|\boldsymbol{F}\| \cos\theta) \times PQ.$$

Work Along a Curve by a Force

- When the force acts on the object along a curved path $\mathcal{C}$, it makes sense to define the work W performed as the line integral

$$W = \int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s}.$$



We can divide $\mathcal{C}$ into a large number of short consecutive arcs $\mathcal{C}_1, \ldots, \mathcal{C}_n$, where $\mathcal{C}_i$ has length Δs_i. The work W_i performed along $\mathcal{C}_i$ is approximately equal to the tangential component $\boldsymbol{F}(P_i) \cdot \boldsymbol{T}(P_i)$ times the length Δs_i, where P_i is a sample point in $\mathcal{C}_i$. Thus we have

$$W = \sum_{i=1}^{N} W_i \approx \sum_{i=1}^{N} (\boldsymbol{F}(P_i) \cdot \boldsymbol{T}(P_i)) \Delta s_i.$$

The right side approaches $\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s}$ as the lengths Δs_i tend to zero.

Work Moving an Object in a Force Field

- Often, we are interested in calculating the work required to move an object along a path in the presence of a force field $\boldsymbol{F}$ (such as an electrical or gravitational field).
- In this case, $\boldsymbol{F}$ acts on the object and we must work against the force field to move the object.
- The work required is the negative of the line integral giving the work expended by the field force:

$$(\text{Work performed against } \boldsymbol{F}) = -\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s}.$$

Example: Calculating Work

- Calculate the work performed moving a particle from $P = (0,0,0)$ to $Q = (4,8,1)$ along the path $\boldsymbol{c}(t) = (t^2, t^3, t)$ (in meters), for $1 \leq t \leq 2$, in the presence of a force field $\boldsymbol{F} = \langle x^2, -z, -\frac{y}{z} \rangle$ in newtons.

  We have

$$\begin{array}{rcl} \boldsymbol{F}(\boldsymbol{c}(t)) & = & \boldsymbol{F}(t^2, t^3, t) = \langle t^4, -t, -t^2 \rangle; \\ \boldsymbol{c}'(t) & = & \langle 2t, 3t^2, 1 \rangle; \\ \boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) & = & \langle t^4, -t, -t^2 \rangle \cdot \langle 2t, 3t^2, 1 \rangle = 2t^5 - 3t^3 - t^2. \end{array}$$

  The work performed against the force field in joules is

$$\begin{array}{rcl} W & = & -\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} = -\int_1^2 (2t^5 - 3t^3 - t^2) dt \\ & = & -(\frac{1}{3}t^6 - \frac{3}{4}t^4 - \frac{1}{3}t^3) \mid_1^2 = -(\frac{64}{3} - 12 - \frac{8}{3} - \frac{1}{3} + \frac{3}{4} + \frac{1}{3}) \\ & = & -(\frac{56}{3} + \frac{3}{4} - 12) = -\frac{89}{12}. \end{array}$$

Example

- Calculate the work done by the force field $\boldsymbol{F} = \langle x, y, z \rangle$ along the path $\langle \cos t, \sin t, t \rangle$, for $0 \leq t \leq 3\pi$.

  We have

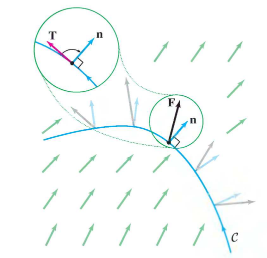
$$\begin{array}{rcl} \boldsymbol{F}(\boldsymbol{c}(t)) & = & \langle \cos t, \sin t, t \rangle; \\ \boldsymbol{c}'(t) & = & \langle -\sin t, \cos t, 1 \rangle; \\ \boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) & = & \langle \cos t, \sin t, t \rangle \cdot \langle -\sin t, \cos t, 1 \rangle = t. \end{array}$$

  The work performed by the force field is

$$\begin{array}{rcl} W & = & \int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} \\ & = & \int_0^{3\pi} \boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) dt \\ & = & \int_0^{3\pi} t dt = \frac{1}{2} t^2 \mid_0^{3\pi} = \frac{9}{2} \pi^2. \end{array}$$

Flux Across a Plane Curve

- Line integrals are also used to compute the "**flux across a plane curve**", defined as the integral of the normal component of a vector field, rather than the tangential component. Suppose that a plane curve $\mathcal{C}$ is parametrized by $\boldsymbol{c}(t)$, for $a \leq t \leq b$. Let $\boldsymbol{n} = \boldsymbol{n}(t) = \langle y'(t), -x'(t) \rangle$, $\boldsymbol{e_n}(t) = \frac{\boldsymbol{n}(t)}{\|\boldsymbol{n}(t)\|}$.



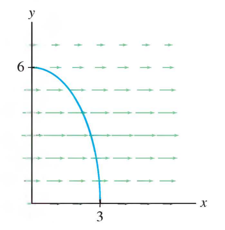
These vectors are normal to $\mathcal{C}$ and point to the right as you follow the curve in the direction of $\boldsymbol{c}$. The flux across $\mathcal{C}$ is the integral of the normal component $\boldsymbol{F} \cdot \boldsymbol{e_n}$, obtained by integrating $\boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{n}(t)$:

$$\text{Flux across } \mathcal{C} = \int_{\mathcal{C}} (\boldsymbol{F} \cdot \boldsymbol{e_n}) ds = \int_a^b \boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{n}(t) dt.$$

- If $\boldsymbol{F}$ is the velocity field of a fluid (a two-dimensional fluid), then the flux is the quantity of water flowing across the curve per unit time.

Example: Flux across a Curve

- Calculate the flux of the velocity vector field $\boldsymbol{v} = \langle 3+2y-\frac{y^2}{3}, 0\rangle$ (in centimeters per second) across the quarter ellipse $\boldsymbol{c}(t) = \langle 3\cos t, 6\sin t\rangle$, for $0 \leq t \leq \frac{\pi}{2}$.



The vector field along the path is

$$\boldsymbol{v}(\boldsymbol{c}(t)) = \langle 3+2(6\sin t) - \frac{(6\sin t)^2}{3}, 0\rangle = \langle 3+12\sin t - 12\sin^2 t, 0\rangle.$$

The tangent vector is

$$\boldsymbol{c}'(t) = \langle -3\sin t, 6\cos t\rangle.$$

Thus

$$\boldsymbol{n}(t) = \langle 6\cos t, 3\sin t\rangle.$$

Example: Flux across a Curve (Cont'd)

- We found

$$\begin{array}{rcl}\boldsymbol{v}(\boldsymbol{c}(t)) & = & \langle 3+12\sin t-12\sin^2 t, 0\rangle;\\ \boldsymbol{n}(t) & = & \langle 6\cos t, 3\sin t\rangle.\end{array}$$

  Compute the dot product

$$\begin{array}{rcl}\boldsymbol{v}(\boldsymbol{c}(t))\cdot\boldsymbol{n}(t) & = & \langle 3+12\sin t-12\sin^2 t, 0\rangle\cdot\langle 6\cos t, 3\sin t\rangle\\ & = & (3+12\sin t-12\sin^2 t)(6\cos t)\\ & = & 18\cos t+72\sin t\cos t-72\sin^2 t\cos t.\end{array}$$

  Integrate to obtain the flux:

$$\begin{array}{l}\int_a^b \boldsymbol{v}(\boldsymbol{c}(t))\cdot\boldsymbol{n}(t)dt\\ =\int_0^{\pi/2}(18\cos t+72\sin t\cos t-72\sin^2 t\cos t)dt\\ =(18\sin t+36\sin^2 t-24\sin^3 t)\mid_0^{\pi/2}\\ =18+36-24=30\ \text{cm}^2/\text{s}.\end{array}$$

Subsection 3

Conservative Vector Fields

Notation

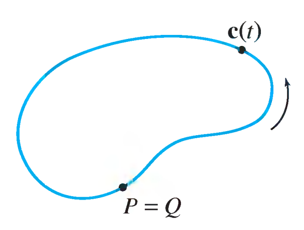
- For convenience, when a particular parametrization $\boldsymbol{c}(t)$ of an oriented curve $\mathcal{C}$ is specified, we will denote the line integral $\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s}$ by

$$\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s}.$$

- When the curve $\mathcal{C}$ is closed, we often refer to the line integral as the **circulation** of $\boldsymbol{F}$ around $\mathcal{C}$.

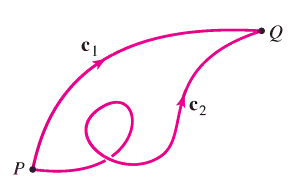
  Then, we denote the integral with the symbol

$$\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s}.$$



Fundamental Theorem for Conservative Vector Fields

- Our first result establishes the fundamental **path independence** of conservative vector fields, which means that the line integral of $\boldsymbol{F}$ along a path from P to Q depends only on the endpoints P and Q, not on the particular path followed.



Theorem (Fundamental Theorem for Conservative Vector Fields)

Assume that $\boldsymbol{F} = \nabla V$ on a domain $\mathcal{D}$.

1. If $\boldsymbol{c}$ is a path from P to Q in $\mathcal{D}$, then
$$\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s} = V(Q) - V(P).$$
In particular, $\boldsymbol{F}$ is path-independent.
2. The circulation around a closed path $\boldsymbol{c}$ $(P = Q)$ is zero: $\oint_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s} = 0$.

Fundamental Theorem (Cont'd)

- Let $\boldsymbol{c}(t)$ be a path in $\mathcal{D}$ for $a \leq t \leq b$, with $\boldsymbol{c}(a) = P$ and $\boldsymbol{c}(b) = Q$. Then

$$\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_{\boldsymbol{c}} \nabla V \cdot d\boldsymbol{s} = \int_a^b \nabla V(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) dt.$$

However, by the Chain Rule for Paths,

$$\frac{d}{dt} V(\boldsymbol{c}(t)) = \nabla V(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t).$$

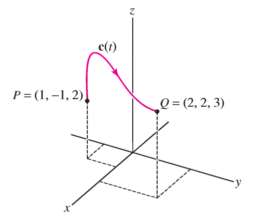
Thus we can apply the Fundamental Theorem of Calculus:

$$\begin{aligned} \int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s} &= \int_a^b \frac{d}{dt} V(\boldsymbol{c}(t)) dt = V(\boldsymbol{c}(t)) \Big|_a^b \\ &= V(\boldsymbol{c}(b)) - V(\boldsymbol{c}(a)) = V(Q) - V(P). \end{aligned}$$

This proves both the equation and path independence, because the quantity $V(Q) - V(P)$ depends only on P, Q, not on the path $\boldsymbol{c}$. If $\boldsymbol{c}$ is a closed path, then $P = Q$ and $V(Q) - V(P) = 0$.

Example I

- Let $\boldsymbol{F} = \langle 2xy + z, x^2, x \rangle$.
  - (a) Verify that $V(x, y, z) = x^2 y + xz$ is a potential function.
  - (b) Evaluate $\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s}$, where $\boldsymbol{c}$ is a path from $P = (1, -1, 2)$ to $Q = (2, 2, 3)$.



(a) The partial derivatives of $V(x, y, z) = x^2 y + xz$ are the components of $\boldsymbol{F}$:

$$\frac{\partial V}{\partial x} = 2xy + z, \quad \frac{\partial V}{\partial y} = x^2, \quad \frac{\partial V}{\partial z} = x.$$

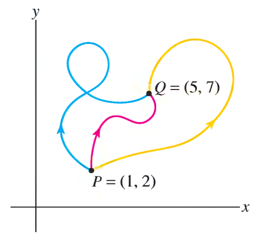
Therefore, $\nabla V = \langle 2xy + z, x^2, x \rangle = \boldsymbol{F}$.

(b) By the theorem, the line integral over any path $\boldsymbol{c}(t)$ from $P = (1, -1, 2)$ to $Q = (2, 2, 3)$ has the value

$$\begin{aligned} \int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s} &= V(Q) - V(P) = V(2, 2, 3) - V(1, -1, 2) \\ &= (2^2(2) + 2(3)) - (1^2(-1) + 1(2)) = 13. \end{aligned}$$

Example II

- Find a potential function for $\boldsymbol{F} = \langle 2x + y, x \rangle$ and use it to evaluate $\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s}$, where $\boldsymbol{c}$ is any path from $(1, 2)$ to $(5, 7)$.
  We will develop later a general method for finding potential functions.



At this point we can see by inspection that $V(x, y) = x^2 + xy$ satisfies $\nabla V = \boldsymbol{F}$:

$$\frac{\partial V}{\partial x} = 2x + y, \quad \frac{\partial V}{\partial y} = x.$$

Therefore, for any path $\boldsymbol{c}$ from $(1, 2)$ to $(5, 7)$,

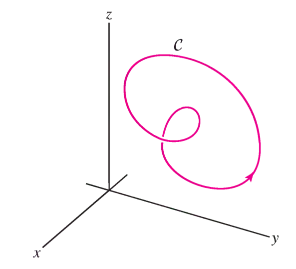
$$\begin{aligned} \int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s} &= V(5, 7) - V(1, 2) \\ &= (5^2 + 5(7)) - (1^2 + 1(2)) = 57. \end{aligned}$$

Example III: Integral around a Closed Path

- Let $V(x, y, z) = xy \sin(yz)$. Evaluate

$$\oint_{\mathcal{C}} \nabla V \cdot d\boldsymbol{s},$$

where $\mathcal{C}$ is the closed curve shown.



By the theorem, the integral of a gradient vector around any closed path is zero. So we have

$$\oint_{\mathcal{C}} \nabla V \cdot d\boldsymbol{s} = 0.$$

Example

- Consider the vector field $\boldsymbol{F} = \frac{z}{x}\boldsymbol{i} + \boldsymbol{j} + \ln x\boldsymbol{k}$ and the function $V(x,y,z) = y + z\ln x$.

  Verify that V is a potential function for $\boldsymbol{F}$ and evaluate the line integral of $\boldsymbol{F}$ over the circle $(x-4)^2 + y^2 = 1$ in the clockwise direction.

  We have

  $$\frac{\partial V}{\partial x} = \frac{z}{x}, \quad \frac{\partial V}{\partial y} = 1, \quad \frac{\partial V}{\partial z} = \ln x.$$

  Therefore $\nabla V = \boldsymbol{F}$.

  Since $\mathcal{C}$ is a closed curve and $\boldsymbol{F}$ is a conservative vector field, we get

  $$\oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} = 0.$$

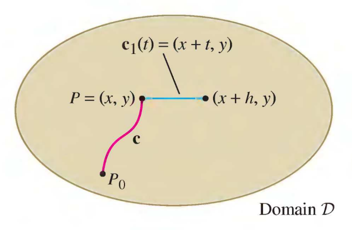
Characterization of Conservativeness

Theorem

A vector field $\boldsymbol{F}$ on an open connected domain $\mathcal{D}$ is path-independent if and only if it is conservative.

- We have already shown that conservative vector fields are path-independent. So we assume that $\boldsymbol{F}$ is path-independent and prove that $\boldsymbol{F}$ has a potential function. To simplify the notation, we treat the case of a planar vector field $\boldsymbol{F}$

  Choose a point P_0 in $\mathcal{D}$. For any point $P = (x, y) \in \mathcal{D}$, define $V(P) = V(x, y) = \int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s}$, where $\boldsymbol{c}$ is any path in V from P_0 to P.



  Note that this definition of $V(P)$ is meaningful only because we are assuming that the line integral does not depend on the path $\boldsymbol{c}$.

Characterization of Conservativeness (Cont'd)

- We prove that $\boldsymbol{F} = \nabla V$, which involves showing that $\frac{\partial V}{\partial x} = F_1$ and $\frac{\partial V}{\partial y} = F_2$. We will only verify the first equation, as the second can be checked in a similar manner.

  Let $\boldsymbol{c}_1$ be the horizontal path $\boldsymbol{c}_1(t) = \langle x+t, y \rangle$, for $0 \le t \le h$. For $|h|$ small enough, $\boldsymbol{c}_1$ lies inside $\mathcal{D}$. Let $\boldsymbol{c} + \boldsymbol{c}_1$ denote the path $\boldsymbol{c}$ followed by $\boldsymbol{c}_1$. It begins at P_0 and ends at $(x+h, y)$. So

$$\begin{aligned} V(x+h,y) - V(x,y) &= \int_{\boldsymbol{c}+\boldsymbol{c}_1} \boldsymbol{F} \cdot d\boldsymbol{s} - \int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s} \\ &= \left(\int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s} + \int_{\boldsymbol{c}_1} \boldsymbol{F} \cdot d\boldsymbol{s}\right) - \int_{\boldsymbol{c}} \boldsymbol{F} \cdot d\boldsymbol{s} \\ &= \int_{\boldsymbol{c}_1} \boldsymbol{F} \cdot d\boldsymbol{s}. \end{aligned}$$

  The path $\boldsymbol{c}_1$ has tangent vector $\boldsymbol{c}_1'(t) = \langle 1, 0 \rangle$. So

$$\begin{aligned} \boldsymbol{F}(\boldsymbol{c}_1(t)) \cdot \boldsymbol{c}_1'(t) &= \langle F_1(x+t,y), F_2(x+t,y) \rangle \cdot \langle 1, 0 \rangle \\ &= F_1(x+t,y); \\ V(x+h,y) - V(x,y) &= \int_{\boldsymbol{c}_1} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_0^h F_1(x+t,y)dt. \end{aligned}$$

Characterization of Conservativeness (Conclusion)

- Using the substitution $u = x + t$, we have

$$\frac{V(x+h,y) - V(x,y)}{h} = \frac{1}{h}\int_0^h F_1(x+h,y)dt = \frac{1}{h}\int_x^{x+h} F_1(u,y)du.$$

The integral on the right is the average value of $F_1(u,y)$ over the interval $[x, x+h]$. It converges to the value $F_1(x,y)$ as $h \to 0$. This yields the desired result:

$$\begin{aligned}\frac{\partial V}{\partial x} &= \lim_{h\to 0}\frac{V(x+h,y) - V(x,y)}{h} \\ &= \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} F_1(u,y)du \\ &= F_1(x,y).\end{aligned}$$

Total Energy

- The Conservation of Energy principle says that the sum $KE + PE$ of kinetic and potential energy remains constant in an isolated system.
- We show now that conservation of energy is valid for the motion of a particle of mass m under a force field $\boldsymbol{F}$ if $\boldsymbol{F}$ has a potential function. This explains why the term "conservative" is used to describe vector fields that have a potential function.
- We follow the convention in physics of writing the potential function with a minus sign: $\boldsymbol{F} = -\nabla V$.
- When the particle is located at $P = (x, y, z)$, it is said to have **potential energy** $V(P)$.
- Suppose that the particle moves along a path $\boldsymbol{c}(t)$.
  The particle's velocity is $\boldsymbol{v} = \boldsymbol{c}'(t)$, and its kinetic energy is
  $KE = \frac{1}{2}m\|\boldsymbol{v}\|^2 = \frac{1}{2}m\boldsymbol{v} \cdot \boldsymbol{v}$.
- By definition, the **total energy** at time t is the sum
  $E = KE + PE = \frac{1}{2}m\boldsymbol{v} \cdot \boldsymbol{v} + V(\boldsymbol{c}(t))$.

Conservation of Energy

Theorem (Conservation of Energy)

The total energy E of a particle moving under the influence of a conservative force field $\boldsymbol{F} = -\nabla V$ is constant in time. That is, $\frac{dE}{dt} = 0$.

- Let $\boldsymbol{a} = \boldsymbol{v}'(t)$ be the particle's acceleration and let m be its mass. According to Newton's Second Law of Motion, $\boldsymbol{F}(\boldsymbol{c}(t)) = m\boldsymbol{a}(t)$. Thus,

$$\begin{array}{rcl}\frac{dE}{dt} & = & \frac{d}{dt}(\frac{1}{2}m\boldsymbol{v}\cdot\boldsymbol{v} + V(\boldsymbol{c}(t))) \\ & = & m\boldsymbol{v}\cdot\boldsymbol{a} + \nabla V(\boldsymbol{c}(t))\cdot\boldsymbol{c}'(t) \quad \text{(Product and Chain Rules)} \\ & = & \boldsymbol{v}\cdot m\boldsymbol{a} - \boldsymbol{F}\cdot\boldsymbol{v} \quad (\text{since } \boldsymbol{F} = -\nabla V \text{ and } \boldsymbol{c}'(t) = \boldsymbol{v}) \\ & = & \boldsymbol{v}\cdot(m\boldsymbol{a} - \boldsymbol{F}) \\ & = & 0. \quad (\text{since } \boldsymbol{F} = m\boldsymbol{a})\end{array}$$

Conservativeness and Cross-Partials

- We showed that every conservative vector field satisfies the cross-partials condition:

$$\frac{\partial F_1}{\partial y} = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \frac{\partial F_3}{\partial y}, \quad \frac{\partial F_3}{\partial x} = \frac{\partial F_1}{\partial z}.$$

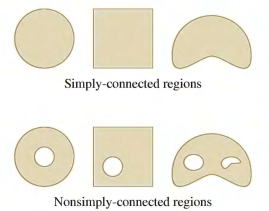
- Does this condition guarantee that $\boldsymbol{F}$ is conservative?

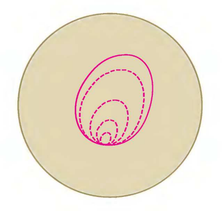
  The answer is a qualified yes:

  The cross-partials condition does guarantee that $\boldsymbol{F}$ is conservative, but only on domains $\mathcal{D}$ with a property called simple-connectedness.

Simple-Connectedness

- Roughly speaking, a domain $\mathcal{D}$ in the plane is **simply-connected** if it does not have any "holes".





- More precisely, $\mathcal{D}$ is simply-connected if every loop in $\mathcal{D}$ can be drawn down, or "contracted", to a point while staying within $\mathcal{D}$.

  Example: Disks, rectangles and the entire plane are simply-connected regions in $\mathbb{R}^2$. The disk with a point removed as in the third figure is not simply-connected. In $\mathbb{R}^3$, the interiors of balls and boxes are simply-connected, as is the entire space $\mathbb{R}^3$.

Existence of a Potential Function

Theorem (Existence of a Potential Function)

Let $\boldsymbol{F}$ be a vector field on a simply-connected domain $\mathcal{D}$. If $\boldsymbol{F}$ satisfies the cross-partials condition, then $\boldsymbol{F}$ is conservative.

Example (Finding a Potential Function): Show that $\boldsymbol{F} = \langle 2xy + y^3, x^2 + 3xy^2 + 2y \rangle$ is conservative and find a potential function.

First we observe that the cross-partial derivatives are equal:

$$\begin{aligned} \frac{\partial F_1}{\partial y} &= \frac{\partial}{\partial y}(2xy + y^3) = 2x + 3y^2; \\ \frac{\partial F_2}{\partial x} &= \frac{\partial}{\partial x}(x^2 + 3xy^2 + 2y) = 2x + 3y^2. \end{aligned}$$

Furthermore, $\boldsymbol{F}$ is defined on all of $\mathbb{R}^2$, which is a simply-connected domain. Therefore, a potential function exists.

Finding a Potential Function (Cont'd)

- The potential function V satisfies $\frac{\partial V}{\partial x} = F_1(x,y) = 2xy + y^3$. This tells us that V is an antiderivative of $F_1(x,y)$, regarded as a function of x alone:

$$V(x,y) = \int F_1(x,y)dx = \int (2xy + y^3)dx = x^2y - xy^3 + g(y).$$

(To obtain the general antiderivative of $F_1(x,y)$ with respect to x, we must add on an arbitrary function $g(y)$ depending on y alone.)
Similarly,

$$\begin{aligned} V(x,y) &= \int F_2(x,y)dy = \int (x^2 + 3xy^2 + 2y)dy \\ &= x^2y + xy^3 + y^2 + h(x). \end{aligned}$$

The two expressions for $V(x,y)$ must be equal:

$$x^2y + xy^3 + g(y) = x^2y + xy^3 + y^2 + h(x).$$

This tells us that $g(y) = y^2$ and $h(x) = 0$, up to the addition of an arbitrary numerical constant C. Thus we obtain the general potential function $V(x,y) = x^2y + xy^3 + y^2 + C$.

Example (Finding a Potential Function)

- Find a potential function for $\boldsymbol{F} = \langle \frac{2xy}{z}, z + \frac{x^2}{z}, y - \frac{x^2 y}{z^2} \rangle$.

  If a potential function V exists, then it satisfies

$$\begin{array}{rcl} V(x,y,z) & = & \int \frac{2xy}{z} dx = \frac{x^2 y}{z} + f(y,z); \\ V(x,y,z) & = & \int (z + \frac{x^2}{z}) dy = zy + \frac{x^2 y}{z} + g(x,z); \\ V(x,y,z) & = & \int (y - \frac{x^2 y}{z^2}) dz = yz + \frac{x^2 y}{z} + h(x,y). \end{array}$$

  These three ways of writing $V(x,y,z)$ must be equal:

$$\frac{x^2 y}{z} + f(y,z) = zy + \frac{x^2 y}{z} + g(x,z) = yz + \frac{x^2 y}{z} + h(x,y).$$

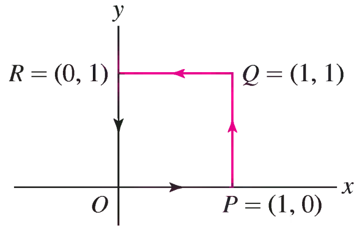
  These equalities hold if $f(y,z) = yz$, $g(x,z) = 0$, and $h(x,y) = 0$. Thus $\boldsymbol{F}$ is conservative and, for any constant C, a potential function is $V(x,y,z) = \frac{x^2 y}{z} + yz + C$.

Example

- Evaluate the circulation $\oint_{\mathcal{C}} \sin x dx + z \cos y dy + \sin y dz$, where $\mathcal{C}$ is the ellipse $4x^2 + 9y^2 = 36$ oriented clockwise.
  We have
  $$\oint_{\mathcal{C}} \sin x dx + z \cos y dy + \sin y dz = \oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s},$$
  where $\boldsymbol{F}(x, y, z) = \langle \sin x, z \cos y, \sin y \rangle$.
  Since
  $$\frac{\partial F_1}{\partial y} = 0 = \frac{\partial F_2}{\partial x}, \quad \frac{\partial F_1}{\partial z} = 0 = \frac{\partial F_3}{\partial x}, \quad \frac{\partial F_2}{\partial z} = \cos y = \frac{\partial F_3}{\partial y},$$
  and $\boldsymbol{F}$ is defined on $\mathbb{R}^3$, which is simply connected, we conclude by the theorem that $\boldsymbol{F}$ is conservative.
  Thus, since $\mathcal{C}$ is a closed curve, we have
  $$\oint_{\mathcal{C}} \sin x dx + z \cos y dy + \sin y dz = \oint_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} = 0.$$

Example

- Calculate the work expedited when a particle is moved from O to Q along $\overline{OP}$ and $\overline{PQ}$ in the presence of the force field $\boldsymbol{F}(x,y) = \langle x^2, y^2 \rangle$.



Note that $\frac{\partial F_1}{\partial y} = 0 = \frac{\partial F_2}{\partial x}$.

Moreover $\boldsymbol{F}$ is defined on $\mathbb{R}^2$, which is simply connected.

Thus, $\boldsymbol{F}$ is conservative.

It is easy to see that a potential function for $\boldsymbol{F}$ is $V(x,y) = \frac{x^3}{3} + \frac{y^3}{3}$.

Hence we have

$$W = -\int_{\mathcal{C}} \boldsymbol{F} \cdot d\boldsymbol{s} = -\int_{\mathcal{C}} \nabla V \cdot d\boldsymbol{s} = V(Q) - V(O) = \frac{1}{3} + \frac{1}{3} = \frac{2}{3}.$$

Assumptions Matter

- We cannot expect the method for finding a potential function to work if $\boldsymbol{F}$ does not satisfy the cross-partials condition (because in this case, no potential function exists).

  Example: Consider $F = \langle y, 0 \rangle$. If we attempted to find a potential function, we would calculate

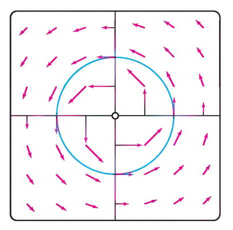
  $$\begin{array}{rcl} V(x,y) & = & \int y dx = xy + g(y); \\ V(x,y) & = & \int 0 dy = 0 + h(x). \end{array}$$

  There is no choice of $g(y)$ and $h(x)$ for which $xy + g(y) = h(x)$. If there were, we could differentiate this equation twice, once with respect to x and once with respect to y. This would yield $1 = 0$, which is a contradiction.

The Vortex Field

- Consider the vortex field

$$\boldsymbol{F} = \langle \frac{-y}{x^2+y^2}, \frac{x}{x^2+y^2} \rangle.$$



Claim: The vortex field satisfies the cross-partials condition but is not conservative.

We check the cross-partials condition directly:

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{x}{x^2+y^2}\right) &= \frac{(x^2+y^2)-x\frac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}; \\ \frac{\partial}{\partial y}\left(\frac{-y}{(x^2+y^2)^2}\right) &= \frac{-(x^2+y^2)+y\frac{\partial}{\partial y}(x^2+y^2)}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}. \end{aligned}$$

The Vortex Field (Cont'd)

- Now consider the line integral of $\boldsymbol{F}$ around the unit circle $\mathcal{C}$ parametrized by $\boldsymbol{c}(t) = \langle \cos t, \sin t \rangle$. We have $\boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) = \langle -\sin t, \cos t \rangle \cdot \langle -\sin t, \cos t \rangle = \sin^2 t + \cos^2 t = 1$. So, we get

$$\oint_{\boldsymbol{C}} \boldsymbol{F} \cdot d\boldsymbol{s} = \int_0^{2\pi} \boldsymbol{F}(\boldsymbol{c}(t)) \cdot \boldsymbol{c}'(t) dt = \int_0^{2\pi} dt = 2\pi \neq 0.$$

If $\boldsymbol{F}$ were conservative, its circulation around every closed curve would be zero.

Note that the domain $\mathcal{D} = \{(x, y) \neq (0, 0)\}$ of $\boldsymbol{F}$ does not satisfy the simply-connected condition of the theorem.

Subsection 4

Parametrized Surfaces and Surface Integrals

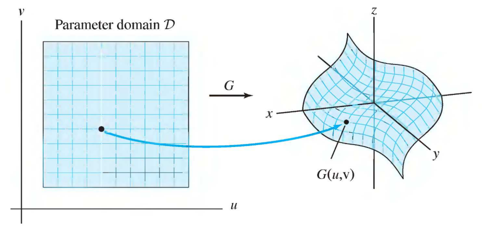
Parametrized Surfaces

- Just as parametrized curves are a key ingredient in the discussion of line integrals, surface integrals require the notion of a parametrized surface.
- A **parametrized surface** is a surface $\mathcal{S}$ whose points are described in the form
$$G(u,v) = (x(u,v), y(u,v), z(u,v)).$$
- The variables u, v (called **parameters**) vary in a region $\mathcal{D}$ called the **parameter domain**.
- Two parameters u and v are needed to parametrize a surface because the surface is two-dimensional.

Example

- The figure below shows a plot of the surface $\mathcal{S}$ with the parametrization

$$G(u,v) = (u+v, u^3 - v, v^3 - u).$$



This surface consists of all points (x, y, z) in $\mathbb{R}^3$, such that

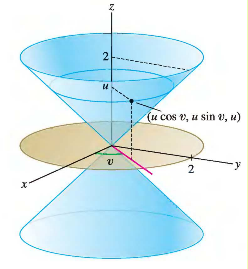
$$x = u + v, \quad y = u^3 - v, \quad z = v^3 - u,$$

for (u, v) in $\mathcal{D} = \mathbb{R}^2$.

Parametrization of a Cone

- Find a parametrization of the portion $\mathcal{S}$ of the cone with equation $x^2 + y^2 = z^2$ lying above or below the disk $x^2 + y^2 \leq 4$. Specify the domain $\mathcal{D}$ of the parametrization.

  This surface $x^2 + y^2 = z^2$ is a cone whose horizonal cross section at height $z = u$ is the circle $x^2 + y^2 = u^2$ of radius u.



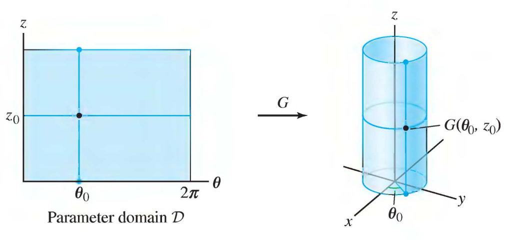
  So a point on the cone at height u has coordinates $(u\cos v, u\sin v, u)$ for some angle v. Thus, the cone has the parametrization

$$G(u,v) = (u\cos v, u\sin v, u).$$

  Since we are interested in the portion of the cone where $x^2 + y^2 = u^2 \leq 4$, the height variable u satisfies $-2 \leq u \leq 2$. The angular variable v varies in the interval $[0, 2\pi)$. Therefore, the parameter domain is $\mathcal{D} = [-2, 2] \times [0, 2\pi)$.

Parametrization of a Cylinder

- The **cylinder of radius** R with equation $x^2 + y^2 = R^2$ is conveniently parametrized in cylindrical coordinates.



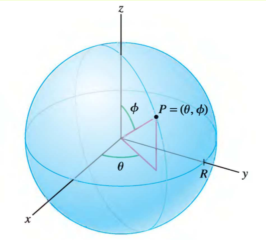
Points on the cylinder have cylindrical coordinates (R, θ, z).
So we use θ and z as parameters (with fixed R). We obtain the **Parametrization of a Cylinder**:

$$G(\theta, z) = (R\cos\theta, R\sin\theta, z),$$

$0 \leq \theta < 2\pi$, $-\infty < z < \infty$.

Parametrization of a Sphere

- The sphere of radius R with center at the origin is parametrized conveniently using spherical coordinates (ρ, θ, ϕ), with $\rho = R$.



Parametrization of a Sphere:

$$G(\theta, \phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\theta),$$

$0 \leq \theta < 2\pi$, $0 \leq \phi \leq \pi$.

- The North and South Poles correspond to $\phi = 0$ and $\phi = \pi$ with any value of θ (the map G fails to be one-to-one at the poles).

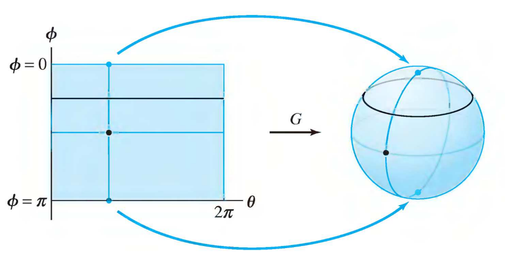
Parametrization of a Sphere (Cont'd)

- We gave the parametrization

$$G(\theta,\phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\theta),$$

$0 \leq \theta < 2\pi$, $0 \leq \phi \leq \pi$.

  - G maps each horizontal segment $\phi = c$ $(0 < c < \pi)$ to a latitude (a circle parallel to the equator);
  - G maps each vertical segment $\theta = c$ to a longitudinal arc from the the North Pole to the South Pole.



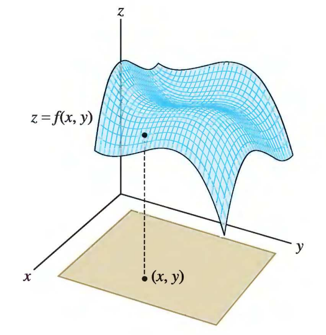
Parametrization of a Graph

- The **graph of a function** $z = f(x, y)$ always has the following simple parametrization:

  Parametrization of a Graph:

  $$G(x, y) = (x, y, f(x, y)).$$

  In this case the parameters are x, y.

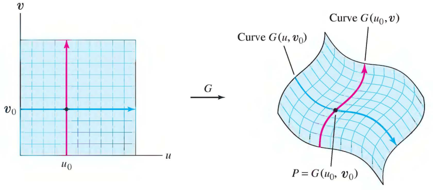


Grid Curves on a Surface

- Suppose that a surface $\mathcal{S}$ has a parametrization

$$G(u,v) = (x(u,v), y(u,v), z(u,v))$$

  that is one-to-one on a domain $\mathcal{D}$. We shall always assume that G is **continuously differentiable**, meaning that the functions $x(u,v)$, $y(u,v)$ and $z(u,v)$ have continuous partial derivatives.
- In the uv-plane, we can form a grid of lines parallel to the coordinates axes. These grid lines correspond under G to a system of **grid curves** on the surface.



  More precisely, the horizontal and vertical lines through (u_0, v_0) in the domain correspond to the grid curves $G(u, v_0)$ and $G(u_0, v)$ that intersect at the point $P = G(u_0, v_0)$.

Tangent and Normal Vectors to the Surface

- Consider the tangent vectors to these grid curves:

$$\begin{array}{rcl}\boldsymbol{T}_u(P) & = & \frac{\partial G}{\partial u}(u_0,v_0) = \langle \frac{\partial x}{\partial u}(u_0,v_0), \frac{\partial y}{\partial u}(u_0,v_0), \frac{\partial z}{\partial u}(u_0,v_0)\rangle; \\ \boldsymbol{T}_v(P) & = & \frac{\partial G}{\partial v}(u_0,v_0) = \langle \frac{\partial x}{\partial v}(u_0,v_0), \frac{\partial y}{\partial v}(u_0,v_0), \frac{\partial z}{\partial v}(u_0,v_0)\rangle.\end{array}$$

- The parametrization G is called **regular at** P if the following cross product is nonzero:

$$\boldsymbol{n}(P) = \boldsymbol{n}(u_0,v_0) = \boldsymbol{T}_u(P) \times \boldsymbol{T}_v(P).$$

- In this case, $\boldsymbol{T}_u$ and $\boldsymbol{T}_v$ span the tangent plane to $\mathcal{S}$ at P and $\boldsymbol{n}(P)$ is a normal vector to the tangent plane. We call $\boldsymbol{n}(P)$ a **normal to the surface** $\mathcal{S}$.
- We often write $\boldsymbol{n}$ instead of $\boldsymbol{n}(P)$ or $\boldsymbol{n}(u,v)$, but it is understood that the vector $\boldsymbol{n}$ varies from point to point on the surface.
  Similarly, we often denote the tangent vectors by $\boldsymbol{T}_u$ and $\boldsymbol{T}_v$.
- Note that $\boldsymbol{T}_u$, $\boldsymbol{T}_v$ and $\boldsymbol{n}$ need not be unit vectors.

Example

- Consider the parametrization

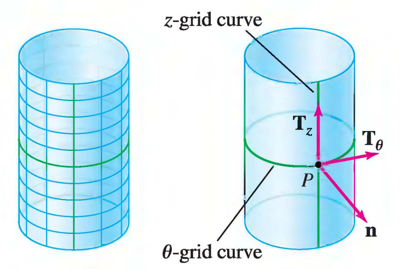
$$G(\theta, z) = (2\cos\theta, 2\sin\theta, z)$$

of the cylinder $x^2 + y^2 = 4$:

(a) Describe the grid curves.

(b) Compute $\boldsymbol{T}_\theta$, $\boldsymbol{T}_z$, and $\boldsymbol{n}(\theta, z)$.

(c) Find an equation of the tangent plane at $P = G(\frac{\pi}{4}, 5)$.



(a) The grid curves on the cylinder through $P = (\theta_0, z_0)$ are

$$G(\theta, z_0) = (2\cos\theta, 2\sin\theta, z_0) \quad \text{(circle of radius 2 at height } z = z_0\text{)}$$
$$G(\theta_0, z) = (2\cos\theta_0, 2\sin\theta_0, z) \quad \text{(vertical line through } P \text{ with } \theta = \theta_0\text{)}$$

Example (Part (b))

(b) The partial derivatives of $G(\theta, z) = (2\cos\theta, 2\sin\theta, z)$ give us the tangent vectors

$$\begin{aligned} \boldsymbol{T}_\theta &= \frac{\partial G}{\partial \theta} = \frac{\partial}{\partial \theta}(2\cos\theta, 2\sin\theta, z) = \langle -2\sin\theta, 2\cos\theta, 0\rangle; \\ \boldsymbol{T}_z &= \frac{\partial G}{\partial z} = \frac{\partial}{\partial z}(2\cos\theta, 2\sin\theta, z) = \langle 0, 0, 1\rangle. \end{aligned}$$

Observe that $\boldsymbol{T}_\theta$ is tangent to the θ-grid curve and $\boldsymbol{T}_z$ is tangent to the z-grid curve.

The normal vector is

$$\boldsymbol{n}(\theta, z) = \boldsymbol{T}_\theta \times \boldsymbol{T}_z = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = 2\cos\theta\boldsymbol{i} + 2\sin\theta\boldsymbol{j}.$$

The coefficient of $\boldsymbol{k}$ is zero, so $\boldsymbol{n}$ points directly out of the cylinder.

Example (Part (c))

(c) We have $G(\theta, z) = (2\cos\theta, 2\sin\theta, z)$ and $\boldsymbol{n}(\theta, z) = \langle 2\cos\theta, 2\sin\theta, 0\rangle$.

For $\theta = \frac{\pi}{4}$, $z = 5$,

$$P = G(\frac{\pi}{4}, 5) = \langle\sqrt{2}, \sqrt{2}, 5\rangle, \quad \boldsymbol{n} = \boldsymbol{n}(\frac{\pi}{4}, 5) = \langle\sqrt{2}, \sqrt{2}, 0\rangle.$$

The tangent plane through P has normal vector $\boldsymbol{n}$. Thus it has equation

$$\sqrt{2}(x - \sqrt{2}) + \sqrt{2}(y - \sqrt{2}) = 0.$$

Equivalently,

$$x + y = 2\sqrt{2}.$$

The tangent plane is vertical (because z does not appear in the equation).

Example

- Calculate the tangent vectors and the normal to the surface

$$G(\theta,\phi) = (\cos\theta\sin\phi, \sin\theta\sin\phi, \cos\phi)$$

at $\theta = \frac{\pi}{2}$ and $\phi = \frac{\pi}{4}$.
We have

$$\begin{aligned}\boldsymbol{T}_\theta &= \frac{\partial G}{\partial\theta} = \langle -\sin\theta\sin\phi, \cos\theta\sin\phi, 0\rangle;\\ \boldsymbol{T}_\phi &= \frac{\partial G}{\partial\phi} = \langle \cos\theta\cos\phi, \sin\theta\cos\phi, -\sin\phi\rangle.\end{aligned}$$

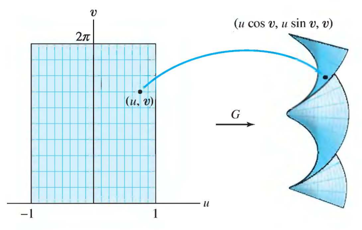
Therefore,

$$\begin{aligned}\boldsymbol{T}_\theta(\tfrac{\pi}{2},\tfrac{\pi}{4}) &= \langle -\sin\tfrac{\pi}{2}\sin\tfrac{\pi}{4}, \cos\tfrac{\pi}{2}\sin\tfrac{\pi}{4}, 0\rangle = \langle -\tfrac{\sqrt{2}}{2}, 0, 0\rangle;\\ \boldsymbol{T}_\phi &= \langle \cos\tfrac{\pi}{2}\cos\tfrac{\pi}{4}, \sin\tfrac{\pi}{2}\cos\tfrac{\pi}{4}, -\sin\tfrac{\pi}{4}\rangle = \langle 0, \tfrac{\sqrt{2}}{2}, -\tfrac{\sqrt{2}}{2}\rangle;\\ \boldsymbol{n}(\tfrac{\pi}{2},\tfrac{\pi}{4}) &= \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ -\frac{\sqrt{2}}{2} & 0 & 0\\ 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2}\end{vmatrix} = -\tfrac{1}{2}\boldsymbol{j} - \tfrac{1}{2}\boldsymbol{k}.\end{aligned}$$

Example: Helicoid Surface

- Describe the surface $\mathcal{S}$ with parametrization $G(u,v) = (u\cos v, u\sin v, v)$, $-1 \le u \le 1$, $0 \le v < 2\pi$. Compute $\boldsymbol{n}(u,v)$ at $u = \frac{1}{2}$, $v = \frac{\pi}{2}$.

  For each fixed value $u = a$, the curve $G(a,v) = (a\cos v, a\sin v, v)$ is a helix of radius a. Therefore, as u varies from -1 to 1, $G(u,v)$ describes a family of helices of radius u. The resulting surface is a "helical ramp".



  The tangent and normal vectors are

$$\boldsymbol{T}_u = \frac{\partial G}{\partial u} = \langle \cos v, \sin v, 0\rangle; \quad \boldsymbol{T}_v = \frac{\partial G}{\partial v} = \langle -u\sin v, u\cos v, 1\rangle;$$

$$\boldsymbol{n}(u,v) = \boldsymbol{T}_u \times \boldsymbol{T}_v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos v & \sin v & 0 \\ -u\sin v & u\cos v & 1 \end{vmatrix} = (\sin v)\boldsymbol{i} - (\cos v)\boldsymbol{j} + u\boldsymbol{k}.$$

  At $u = \frac{1}{2}$, $v = \frac{\pi}{2}$, we have $\boldsymbol{n} = \boldsymbol{i} + \frac{1}{2}\boldsymbol{k}$.

Normal Vector of the Parametrization of the Sphere

- Consider the standard parametrization of the sphere of radius R centered at the origin $C(\theta,\phi)=(R\cos\theta\sin\phi,R\sin\theta\sin\phi,R\cos\phi)$. Since the distance from $G(\theta,\phi)$ to the origin is R, the unit radial vector at $G(\theta,\phi)$ is obtained by dividing by R:

$$\boldsymbol{e}_r=\langle\cos\theta\sin\phi,\sin\theta\sin\phi,\cos\phi\rangle.$$

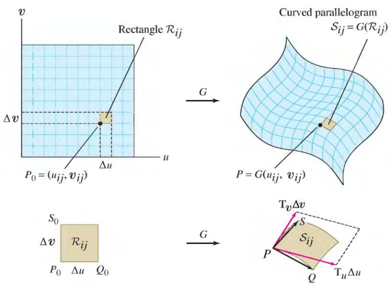
Furthermore,

$$\begin{aligned}
\boldsymbol{T}_\theta &= \langle -R\sin\theta\sin\phi, R\cos\theta\sin\phi, 0\rangle;\\
\boldsymbol{T}_\phi &= \langle R\cos\theta\cos\phi, R\sin\theta\cos\phi, -R\sin\phi\rangle;\\
\boldsymbol{n} &= \boldsymbol{T}_\theta\times\boldsymbol{T}_\phi=\begin{vmatrix}\boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k}\\ -R\sin\theta\sin\phi & R\cos\theta\sin\phi & 0\\ R\cos\theta\cos\phi & R\sin\theta\cos\phi & -R\sin\phi\end{vmatrix}\\
&= -R^2\cos\theta\sin^2\phi\boldsymbol{i}-R^2\sin\theta\sin^2\phi\boldsymbol{j}-R^2\cos\phi\sin\phi\boldsymbol{k}\\
&= -R^2\sin\phi\langle\cos\theta\sin\phi,\sin\theta\sin\phi,\cos\phi\rangle\\
&= -(R^2\sin\phi)\boldsymbol{e}_r.
\end{aligned}$$

The outward-pointing normal vector is $\boldsymbol{n}=\boldsymbol{T}_\phi\times\boldsymbol{T}_\theta=(R^2\sin\phi)\boldsymbol{e}_r$.

Area of a Surface Element

- Assume, for simplicity, that $\mathcal{D}$ is a rectangle (the argument also applies to more general domains). Divide $\mathcal{D}$ into a grid of small rectangles $\mathcal{R}_{ij}$ of size $\Delta u \times \Delta v$. Compare the area of $\mathcal{R}_{ij}$ with the area of its image under G. This image is a "curved" parallelogram $\mathcal{S}_{ij} = G(\mathcal{R}_{ij})$.



First, we note that if Δu and Δv are small, then the curved parallelogram $\mathcal{S}_{ij}$ has approximately the same area as the "genuine" parallelogram with sides $\overrightarrow{PQ}$ and $\overrightarrow{PS}$.

Recall that the area of the parallelogram spanned by two vectors is the length of their cross product $\text{Area}(\mathcal{S}_{ij}) \approx \|\overrightarrow{PQ} \times \overrightarrow{PS}\|$.

Area of a Surface Element (Cont'd)

- Use linear approximation to estimate the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PS}$:

$$\begin{aligned}\overrightarrow{PQ} &= G(u_{ij}+\Delta u, v_{ij}) - G(u_{ij}, v_{ij}) \approx \frac{\partial G}{\partial u}(u_{ij}, v_{ij})\Delta u = \boldsymbol{T}_u \Delta u;\\ \overrightarrow{PS} &= G(u_{ij}, v_{ij}+\Delta v) - G(u_{ij}, v_{ij}) \approx \frac{\partial G}{\partial v}(u_{ij}, v_{ij})\Delta v = \boldsymbol{T}_v \Delta v.\end{aligned}$$

Thus we have

$$\text{Area}(\mathcal{S}_{ij}) \approx \|\boldsymbol{T}_u \Delta u \times \boldsymbol{T}_v \Delta v\| = \|\boldsymbol{T}_u \times \boldsymbol{T}_v\| \Delta u \Delta v.$$

Since $\boldsymbol{n}(u_{ij}, v_{ij}) = \boldsymbol{T}_u \times \boldsymbol{T}_v$ and $\text{Area}(\mathcal{R}_{ij}) = \Delta u \Delta v$, we obtain

$$\text{Area}(\mathcal{S}_{ij}) \approx \|\boldsymbol{n}(u_{ij}, v_{ij})\| \text{Area}(\mathcal{R}_{ij}).$$

Conclusion: $\|\boldsymbol{n}\|$ is a distortion factor that measures how the area of a small rectangle $\mathcal{R}_{ij}$ is altered under the map G.

Area of a Surface

- To compute the surface area of $\mathcal{S}$, we assume:
  - G is one-to-one, except possibly on the boundary of $\mathcal{D}$;
  - G is regular, except possibly on the boundary of $\mathcal{D}$.
    Recall that "regular" means that $\boldsymbol{n}(u, v)$ is nonzero.
- The entire surface $\mathcal{S}$ is the union of the small patches $\mathcal{S}_{ij}$. So we can apply the approximation on each patch to obtain

$$\text{Area}(\mathcal{S}) = \sum_{i,j} \text{Area}(\mathcal{S}_{ij}) \approx \sum_{i,j} \|\boldsymbol{n}(u_{ij}, v_{ij})\| \Delta u \Delta v.$$

The sum on the right is a Riemann sum for the double integral of $\|\boldsymbol{n}(u, v)\|$ over the parameter domain $\mathcal{D}$. As Δu and Δv tend to zero, these Riemann sums converge to a double integral, which we take as the definition of **surface area**:

$$\text{Area}(\mathcal{S}) = \iint_{\mathcal{D}} \|\boldsymbol{n}(u, v)\| du dv.$$

Example

- Use spherical coordinates to compute the surface area of a sphere of radius R.

  The parametrization using spherical coordinates is

  $$G(\theta,\phi) = (R\cos\theta\sin\phi, R\sin\theta\sin\phi, R\cos\phi).$$

  So we have

  $$\begin{array}{rcl} \boldsymbol{T}_\theta & = & \langle -R\sin\theta\sin\phi, R\cos\theta\sin\phi, 0\rangle; \\ \boldsymbol{T}_\phi & = & \langle R\cos\theta\cos\phi, R\sin\theta\cos\phi, -R\sin\phi\rangle; \\ \boldsymbol{n} & = & \left|\begin{array}{ccc} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -R\sin\theta\sin\phi & R\cos\theta\sin\phi & 0 \\ R\cos\theta\cos\phi & R\sin\theta\cos\phi & -R\sin\phi \end{array}\right| \\ & = & \langle -R^2\cos\theta\sin^2\phi, -R\sin\theta\sin^2\phi, -R^2\sin\phi\cos\phi\rangle; \end{array}$$

Example (Cont'd)

- Now we get

$$\begin{aligned} &\|\boldsymbol{n}\| \\ &= \sqrt{(-R^2\cos\theta\sin^2\phi)^2+(-R\sin\theta\sin^2\phi)^2+(-R^2\sin\phi\cos\phi\rangle)^2} \\ &= \sqrt{R^4[(\cos^2\theta+\sin^2\theta)\sin^4\phi+\sin^2\phi\cos^2\phi]} \\ &= R^2\sqrt{\sin^2\phi(\sin^2\phi+\cos^2\phi)} \\ &= R^2|\sin\phi|; \end{aligned}$$

Therefore

$$\begin{aligned} \text{Area} &= \int_0^{2\pi}\int_0^{\pi}\|\boldsymbol{n}\|d\phi d\theta \\ &= R^2\int_0^{2\pi}\int_0^{\pi}|\sin\phi|d\phi d\theta \\ &= R^2\int_0^{2\pi}\int_0^{\pi}\sin\phi d\phi d\theta \\ &= R^2\int_0^{2\pi}-\cos\phi\ |_0^{\pi}\ d\theta \\ &= R^2\int_0^{2\pi}2d\theta = R^2 2\cdot 2\pi = 4\pi R^2. \end{aligned}$$

Surface Integral

- We define the surface integral of a function $f(x, y, z)$:

$$\iint_{\mathcal{S}} f(x, y, z) dS.$$

- Choose a sample point $P_{ij} = G(u_{ij}, v_{ij})$ in each small patch $\mathcal{S}_{ij}$ and form the sum: $\sum_{i,j} f(P_{ij})\text{Area}(\mathcal{S}_{ij})$.
- The limit of these sums as Δu and Δv tend to zero (if it exists) is the **surface integral**:

$$\iint_{\mathcal{S}} f(x, y, z) dS = \lim_{\Delta u, \Delta v \to 0} \sum_{i,j} f(P_{ij})\text{Area}(\mathcal{S}_{ij}).$$

Evaluating Surface Integrals

- To evaluate the surface integral $\iint_{\mathcal{S}} f(x,y,z)dS$, we write
$$\sum_{i,j} f(P_{ij})\text{Area}(\mathcal{S}_{ij}) \approx \sum_{i,j} f(G(u_{ij},v_{ij}))\|\boldsymbol{n}(u_{ij},v_{ij})\|\Delta u\Delta v.$$
On the right we have a Riemann sum for the double integral of $f(G(u,v))\|\boldsymbol{n}(u,v)\|$ over the parameter domain $\mathcal{D}$.
If G is continuously differentiable, we can show the two sums in the displayed equation approach the same limit:

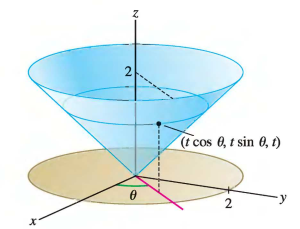
Theorem (Surface Integrals and Surface Area)

Let $G(u,v)$ be a parametrization of a surface $\mathcal{S}$ with parameter domain $\mathcal{D}$. Assume that G is continuously differentiable, one-to-one, and regular (except possibly at the boundary of $\mathcal{D}$). Then
$$\iint_{\mathcal{S}} f(x,y,z)dS = \iint_{\mathcal{D}} f(G(u,v))\|\boldsymbol{n}(u,v)\|dudv.$$
For $f(x,y,z)=1$, we get $\text{Area}(\mathcal{S}) = \iint_{\mathcal{D}} \|\boldsymbol{n}(u,v)\|dudv$.

Example

- Calculate the surface area of the portion $\mathcal{S}$ of the cone $x^2 + y^2 = z^2$ lying above the disk $x^2+y^2 \leq 4$. Then calculate $\iint_{\mathcal{S}} x^2 z dS$.

  A parametrization of the cone is $G(\theta, t) = (t\cos\theta, t\sin\theta, t)$, $0 \leq t \leq 2$, $0 \leq \theta < 2\pi$.



  Compute the tangent and normal vectors:

$$\boldsymbol{T}_\theta = \frac{\partial G}{\partial\theta} = \langle -t\sin\theta, t\cos\theta, 0\rangle, \quad \boldsymbol{T}_t = \frac{\partial G}{\partial t} = \langle \cos\theta, \sin\theta, 1\rangle,$$

$$\boldsymbol{n} = \boldsymbol{T}_\theta \times \boldsymbol{T}_t = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -t\sin\theta & t\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = t\cos\theta\boldsymbol{i} + t\sin\theta\boldsymbol{j} - t\boldsymbol{k}.$$

  The normal vector has length

$$\|\boldsymbol{n}\| = \sqrt{t^2\cos^2\theta + t^2\sin^2\theta + (-t)^2} = \sqrt{2t^2} = \sqrt{2}|t|.$$

  Thus, $dS = \|\boldsymbol{n}\| d\theta dt = \sqrt{2}|t| d\theta dt \stackrel{t\geq 0}{=} \sqrt{2} t d\theta dt$.

Example (Cont'd)

- Calculate the surface area:

$$\begin{array}{rcl}\text{Area}(\mathcal{S}) & = & \iint_{\mathcal{D}} \|\boldsymbol{n}\| dudv = \int_0^2 \int_0^{2\pi} \sqrt{2} t d\theta dt \\ & = & \int_0^2 2\sqrt{2}\pi t dt = \sqrt{2}\pi t^2 \mid_0^2 = 4\sqrt{2}\pi.\end{array}$$

Calculate the surface integral. We express $f(x,y,z) = x^2 z$ in terms of the parameters t and θ:
$f(G(\theta,t)) = f(t\cos\theta, t\sin\theta, t) = (t\cos\theta)^2 t = t^3\cos^2\theta$. Now we have

$$\begin{array}{rcl}\iint_{\mathcal{S}} f(x,y,z)dS & = & \int_0^2 \int_0^{2\pi} f(G(\theta,t))\|\boldsymbol{n}(\theta,t)\| d\theta dt \\ & = & \int_0^2 \int_0^{2\pi} (t^3\cos^2\theta)(\sqrt{2}t) d\theta dt \\ & = & \sqrt{2}(\int_0^2 t^4 dt)(\int_0^{2\pi} \cos^2\theta d\theta) \\ & = & \sqrt{2}(\int_0^2 t^4 dt)(\int_0^{2\pi} (\frac{1}{2} + \frac{1}{2}\cos 2\theta) d\theta) \\ & = & \sqrt{2}(\frac{32}{5})(\pi) = \frac{32\sqrt{2}\pi}{5}.\end{array}$$

Example

- Let $\mathcal{S} = G(\mathcal{D})$, where $\mathcal{D} = \{(u,v) : u^2 + v^2 \leq 1, u \geq 0, v \geq 0\}$ and $G(u,v) = (2u+1, u-v, 3u+v)$.
  Calculate the surface area of $\mathcal{S}$.
  We have
  $$\boldsymbol{T}_u = \frac{\partial G}{\partial u} = \langle 2,1,3\rangle; \quad \boldsymbol{T}_v = \frac{\partial G}{\partial v} = \langle 0,-1,1\rangle;$$
  $$\boldsymbol{n} = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2 & 1 & 3 \\ 0 & -1 & 1 \end{vmatrix} = \langle 4,-2,-2\rangle; \quad \|\boldsymbol{n}\| = \sqrt{24} = 2\sqrt{6}.$$
  So we get
  $$\begin{aligned} \text{Area} &= \iint_{\mathcal{D}} \|\boldsymbol{n}\| du dv = \int_0^1 \int_0^{\sqrt{1-v^2}} 2\sqrt{6} du dv \\ &= 2\sqrt{6}\int_0^1 \sqrt{1-v^2} dv \overset{v=\sin\theta}{=} 2\sqrt{6}\int_0^{\pi/2} \cos^2\theta d\theta \\ &= 2\sqrt{6}\int_0^{\pi/2} \frac{1}{2}(1+\cos 2\theta) d\theta \\ &= \sqrt{6}(\theta + \frac{1}{2}\sin 2\theta) \Big|_0^{\pi/2} = \frac{\sqrt{6}\pi}{2}. \end{aligned}$$

Total Mass of and Total Charge on a Surface

- A surface with mass density $\rho(x, y, z)$ (in units of mass per area) is the surface integral of the mass density:

$$(\text{Mass of } \mathcal{S}) = \iint_{\mathcal{S}} \rho(x, y, z) dS.$$

- Similarly, if an electric charge is distributed over $\mathcal{S}$ with charge density $\rho(x, y, z)$, then the surface integral of $\rho(x, y, z)$ is the total charge on $\mathcal{S}$,

$$(\text{Total Charge on } \mathcal{S}) = \iint_{\mathcal{S}} \rho(x, y, z) dS.$$

Computing Total Charge on a Surface

- Find the total charge (in coulombs) on a sphere $\mathcal{S}$ of radius 5 cm whose charge density in spherical coordinates is $\rho(\theta,\phi) = 0.003\cos^2\phi$ C/cm^2.

  We parametrize $\mathcal{S}$ in spherical coordinates:

$$G(\theta,\phi) = (5\cos\theta\sin\phi, 5\sin\theta\sin\phi, 5\cos\phi).$$

  We have shown that $\|\boldsymbol{n}\| = 5^2\sin\phi$. Now we have

$$\begin{aligned}
\text{Total Charge} &= \iint_{\mathcal{S}} \rho(\theta,\phi)dS = \int_0^{2\pi}\int_0^{\pi} \rho(\theta,\phi)\|\boldsymbol{n}\|d\phi d\theta \\
&= \int_0^{2\pi}\int_0^{\pi} (0.003\cos^2\phi)(25\sin\phi)d\phi d\theta \\
&= (0.075)(2\pi)\int_0^{\pi} \cos^2\phi\sin\phi d\phi \\
&= 0.15\pi(-\tfrac{\cos^3\phi}{3})\ |_0^{\pi} = 0.15\pi(\tfrac{2}{3}) = 0.1\pi \text{ C}.
\end{aligned}$$

Surface Integral Over a Graph

- When a graph $z = g(x,y)$ is parametrized by $G(x,y) = (x,y,g(x,y))$, the tangent and normal vectors are

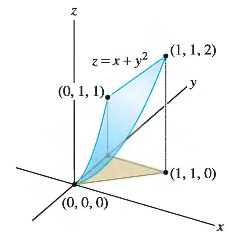
$$\boldsymbol{T}_x = (1,0,g_x), \quad \boldsymbol{T}_y = (0,1,g_y),$$
$$\boldsymbol{n} = \boldsymbol{T}_x \times \boldsymbol{T}_y = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & g_x \\ 0 & 1 & g_y \end{vmatrix} = -g_x\boldsymbol{i} - g_y\boldsymbol{j} + \boldsymbol{k}.$$
$$\|\boldsymbol{n}\| = \sqrt{1 + g_x^2 + g_y^2}.$$

The surface integral over the portion of a graph lying over a domain $\mathcal{D}$ in the xy-plane is

$$\text{Surface integral over a graph} = \iint_{\mathcal{D}} f(x,y,g(x,y))\sqrt{1+g_x^2+g_y^2}\,dxdy.$$

Example

- Calculate $\int_{\mathcal{S}}(z-x)dS$, where $\mathcal{S}$ is the portion of the graph of $z=x+y^2$ where $0\leq x\leq y$, $0\leq y\leq 1$.
  Let $z=g(x,y)=x+y^2$. Then $g_x=1$ and $g_y=2y$. We get $dS=\sqrt{1+g_x^2+g_y^2}dxdy=\sqrt{1+1+4y^2}dxdy=\sqrt{2+4y^2}dxdy$.
  On the surface $\mathcal{S}$, we have $z=x+y^2$. Thus $f(x,y,z)=z-x=(x+y^2)-x=y^2$. Now we get



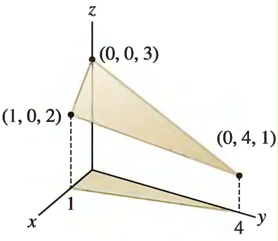
$$\begin{aligned}\int\int_{\mathcal{S}}f(x,y,z)dS &= \int_0^1\int_0^y y^2\sqrt{2+4y^2}dxdy\\ &= \int_0^1(y^2\sqrt{2+4y^2})x\,|_0^y\,dy=\int_0^1 y^3\sqrt{2+4y^2}dy.\end{aligned}$$

Substitute $u=2+4y^2$, $du=8ydy$. Then $y^2=\frac{1}{4}(u-2)$. We get

$$\begin{aligned}\int_0^1 y^3\sqrt{2+4y^2}dy &= \frac{1}{8}\int_2^6\frac{1}{4}(u-2)\sqrt{u}du=\frac{1}{32}\int_2^6(u^{3/2}-2u^{1/2})du\\ &= \frac{1}{32}(\frac{2}{5}u^{5/2}-\frac{4}{3}u^{3/2})\,|_2^6\,=\frac{1}{30}(6\sqrt{6}+\sqrt{2}).\end{aligned}$$

Example

- Calculate $\int_{\mathcal{S}}(xy+e^z)dS$, where $\mathcal{S}$ is the triangle with vertices $(0,0,3)$, $(1,0,2)$ and $(0,4,1)$.

  The plane contains the vectors $\langle 1,0,-1\rangle$ and $\langle 0,4,-2\rangle$.

  Therefore a normal to the plane is given by

$$\begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 1 & 0 & -1 \\ 0 & 4 & -2 \end{vmatrix} = \langle 4,2,4\rangle.$$

  Thus, the plane has equation $4x+2y+4(z-3)=0$ or $z=g(x,y)=3-x-\frac{1}{2}y$. So $g_x=-1$, $g_y=-\frac{1}{2}$,

$$dS = \sqrt{1+g_x^2+g_y^2}dxdy = \sqrt{1+(-1)^2+(-\frac{1}{2})^2}dxdy$$
$$= \sqrt{\frac{9}{4}}dxdy = \frac{3}{2}dxdy; \quad f(x,y,z) = xy+e^z = xy+e^{3-x-\frac{1}{2}y}.$$

Example (Cont'd)

- Finally, we get

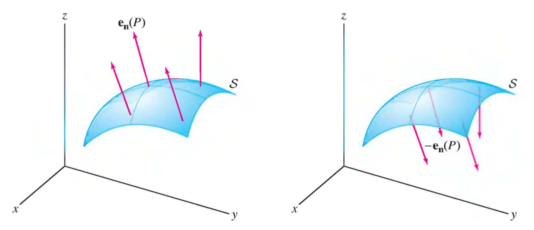
$$\begin{aligned}
&\iint_{\mathcal{S}} f(x,y,z)dS \\
&= \int_0^4 \int_0^{1-\frac{1}{4}y} (xy + e^{3-x-\frac{y}{2}})\frac{3}{2}dxdy \\
&= \frac{3}{2}\int_0^4 [\frac{1}{2}x^2y - e^{3-x-\frac{y}{2}}]_0^{1-\frac{y}{4}}dy \\
&= \frac{3}{2}\int_0^4 [\frac{1}{2}(1-\frac{y}{4})^2y - e^{3-(1-\frac{y}{4})-\frac{y}{2}} + e^{3-\frac{y}{2}}]dy \\
&= \frac{3}{2}\int_0^4 (\frac{y}{2} - \frac{y^2}{4} + \frac{y^3}{32} - e^{2-\frac{y}{4}} + e^{3-\frac{y}{2}})dy \\
&= \frac{3}{2}[\frac{y^2}{4} - \frac{y^3}{12} + \frac{y^4}{128} + 4e^{2-\frac{y}{4}} - 2e^{3-\frac{y}{2}}]_0^4 \\
&= \frac{3}{2}[4 - \frac{16}{3} + 2 + 4e - 2e - 4e^2 + 2e^3] \\
&= \frac{3}{2}[\frac{2}{3} + 2e - 4e^2 + 2e^3] \\
&= 1 + 3e - 6e^2 + 3e^3.
\end{aligned}$$

Subsection 5

Surface Integrals of Vector Fields

Orientation of a Surface

- Flux through a surface goes from one side of the surface to the other.
- To compute flux we need to specify a **positive direction** of flow.
- This is done by means of an **orientation**, which is a choice of unit normal vector $\boldsymbol{e_n}(P)$ at each point P of $\mathcal{S}$, chosen in a continuously varying manner.



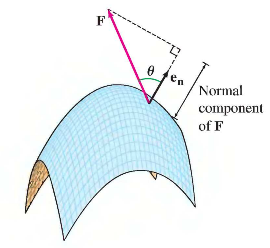
- The unit vectors $-\boldsymbol{e_n}(P)$ define the opposite orientation.
- If $\boldsymbol{e_n}$ are outward-pointing unit normal vectors on a sphere, then a flow from the inside of the sphere to the outside is a positive flux.

Vector Surface Integrals

- The **normal component** of a vector field $\boldsymbol{F}$ at a point P on an oriented surface $\mathcal{S}$ is the dot product

  $$\text{Normal component at } P = \boldsymbol{F}(P)\cdot \boldsymbol{e_n}(P) = \|\boldsymbol{F}(P)\|\cos\theta,$$

  where θ is the angle between $\boldsymbol{F}(P)$ and $\boldsymbol{e_n}(P)$.



- Often, we write $\boldsymbol{e_n}$ instead of $\boldsymbol{e_n}(P)$, but it is understood that $\boldsymbol{e_n}$ varies from point to point on the surface.
- The **vector surface integral**, denoted $\iint_{\mathcal{S}} \boldsymbol{F}\cdot d\boldsymbol{S}$ is defined as the integral of the normal component:

  $$\text{Vector surface integral: } \iint_{\mathcal{S}} \boldsymbol{F}\cdot d\boldsymbol{S} = \iint_{\mathcal{S}} (\boldsymbol{F}\cdot \boldsymbol{e_n})dS.$$

- This quantity is also called the **flux** of $\boldsymbol{F}$ across or through $\mathcal{S}$.

Computing Vector Surface Integrals

- An **oriented parametrization** $G(u,v)$ is a regular parametrization (meaning that $\boldsymbol{n}(u,v)$ is nonzero for all u,v) whose unit normal vector defines the orientation: $\boldsymbol{e_n} = \boldsymbol{e_n}(u,v) = \frac{\boldsymbol{n}(u,v)}{\|\boldsymbol{n}(u,v)\|}$.
- Applying the formula for the scalar surface integral to the function $\boldsymbol{F} \cdot \boldsymbol{e_n}$, we obtain

$$\begin{aligned} \iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S} &= \iint_{\mathcal{D}} (\boldsymbol{F} \cdot \boldsymbol{e_n}) \|\boldsymbol{n}(u,v)\| dudv \\ &= \iint_{\mathcal{D}} \boldsymbol{F}(G(u,v)) \cdot \left(\frac{\boldsymbol{n}(u,v)}{\|\boldsymbol{n}(u,v)\|}\right) \|\boldsymbol{n}(u,v)\| dudv \\ &= \iint_{\mathcal{D}} \boldsymbol{F}(G(u,v)) \cdot \boldsymbol{n}(u,v) dudv. \end{aligned}$$

- This formula remains valid even if $\boldsymbol{n}(u,v)$ is zero at points on the boundary of the parameter domain $\mathcal{D}$.
- If we reverse the orientation of $\mathcal{S}$ in a vector surface integral, $\boldsymbol{n}(u,v)$ is replaced by $-\boldsymbol{n}(u,v)$ and the integral changes sign.
- We can think of $d\boldsymbol{S}$ as a "vector surface element" that is related to a parametrization by the symbolic equation $d\boldsymbol{S} = \boldsymbol{n}(u,v) dudv$.

The Vector Surface Integral Theorem

- Summarizing the work on the previous slide:

Theorem (Vector Surface Integral)

Let $G(u,v)$ be an oriented parametrization of an oriented surface $\mathcal{S}$ with parameter domain $\mathcal{D}$. Assume that G is one-to-one and regular, except possibly at points on the boundary of $\mathcal{D}$. Then

$$\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S} = \iint_{\mathcal{D}} \boldsymbol{F}(G(u,v)) \cdot \boldsymbol{n}(u,v) du dv.$$

If the orientation of $\mathcal{S}$ is reversed, the surface integral changes sign.

Example

- Calculate $\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F} = \langle 0, 0, x \rangle$ and $\mathcal{S}$ is the surface with parametrization $G(u, v) = (u^2, v, u^3 - v^2)$, for $0 \leq u \leq 1$, $0 \leq v \leq 1$ and oriented by upward-pointing normal vectors.

  Compute the tangent and normal vectors.

$$\boldsymbol{T}_u = \langle 2u, 0, 3u^2 \rangle, \quad \boldsymbol{T}_v = \langle 0, 1, -2v \rangle,$$

$$\boldsymbol{n}(u, v) = \boldsymbol{T}_u \times \boldsymbol{T}_v = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ 2u & 0 & 3u^2 \\ 0 & 1 & -2v \end{vmatrix}$$

$$= -3u^2 \boldsymbol{i} + 4uv \boldsymbol{j} + 2u \boldsymbol{k} = \langle -3u^2, 4uv, 2u \rangle.$$

  The z-component of $\boldsymbol{n}$ is positive on the domain $0 \leq u \leq 1$. So $\boldsymbol{n}$ is the upward-pointing normal.

Example (Cont'd)

- We found $\boldsymbol{n}(u,v) = \langle -3u^2, 4uv, 2u\rangle$.

  We now evaluate $\boldsymbol{F} \cdot \boldsymbol{n}$.

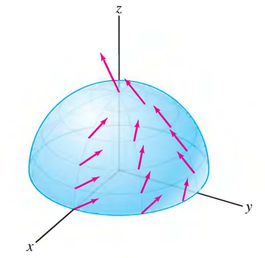
$$\begin{gathered}\boldsymbol{F}(G(u,v)) = \langle 0,0,x\rangle = \langle 0,0,u^2\rangle, \\ \boldsymbol{F}(G(u,v)) \cdot \boldsymbol{n}(u,v) = \langle 0,0,u^2\rangle \cdot \langle -3u^2, 4uv, 2u\rangle = 2u^3.\end{gathered}$$

  Finally, we evaluate the surface integral.

$$\begin{array}{rcl}\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S} & = & \int_0^1 \int_0^1 \boldsymbol{F}(G(u,v)) \cdot \boldsymbol{n}(u,v) dv du \\ & = & \int_0^1 \int_0^1 2u^3 dv du \\ & = & \int_0^1 2u^3 du \\ & = & \frac{1}{2}u^4 \mid_0^1 = \frac{1}{2}.\end{array}$$

Example: Integral over a Hemisphere

- Calculate the flux of $\boldsymbol{F} = \langle z, x, 1 \rangle$ across the upper hemisphere $\mathcal{S}$ of the sphere $x^2 + y^2 + z^2 = 1$, oriented with outward-pointing normal vectors.

  Parametrize the hemisphere by $G(\theta, \phi) = (\cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi)$, $0 \leq \phi \leq \frac{\pi}{2}$, $0 \leq \theta \leq 2\pi$.

  We have computed the outward-pointing normal vector $\boldsymbol{n} = \boldsymbol{T}_\phi \times \boldsymbol{T}_\phi = (R^2 \sin\phi)\boldsymbol{e}_r = \sin\phi \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle$.

  We now evaluate $\boldsymbol{F} \cdot \boldsymbol{n}$:

$$\boldsymbol{F}(G(\theta, \phi)) = \langle z, x, 1 \rangle = \langle \cos\phi, \cos\theta \sin\phi, 1 \rangle;$$

$$\begin{aligned} &\boldsymbol{F}(G(\theta, \phi)) \cdot \boldsymbol{n}(\theta, \phi) \\ &= \langle \cos\phi, \cos\theta \sin\phi, 1 \rangle \cdot \langle \cos\theta \sin^2\phi, \sin\theta \sin^2\phi, \cos\phi \sin\phi \rangle \\ &= \cos\theta \sin^2\phi \cos\phi + \cos\theta \sin\theta \sin^3\phi + \cos\phi \sin\phi. \end{aligned}$$

Example: Integral over a Hemisphere (Cont'd)

- Finally, we evaluate the surface integral.

$$\begin{aligned}\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S} &= \int_0^{\pi/2} \int_0^{2\pi} \boldsymbol{F}(G(\theta, \phi)) \cdot \boldsymbol{n}(\theta, \phi) d\theta d\phi \\ &= \int_0^{\pi/2} \int_0^{2\pi} (\underbrace{\cos\theta \sin^2\phi \cos\phi + \cos\theta \sin\theta \sin^3\phi}_{\text{Integral over } \theta \text{ is zero}} \\ &\quad + \cos\phi \sin\phi) d\theta d\phi.\end{aligned}$$

The integrals of $\cos\theta$ and $\cos\theta \sin\theta$ over $[0, 2n]$ are both zero. So we are left with

$$\begin{aligned}\int_0^{\pi/2} \int_0^{2\pi} \cos\phi \sin\phi d\theta d\phi &= 2\pi \int_0^{\pi/2} \cos\phi \sin\phi d\phi \\ &= 2\pi \frac{\sin^2\phi}{2} \Big|_0^{\pi/2} \\ &= \pi.\end{aligned}$$

Example

- Compute the integral $\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F} = \langle x, y, e^z \rangle$ and $\mathcal{S}$ is the cylinder $x^2 + y^2 = 4$, $1 \leq z \leq 5$, with the outward-pointing normal.
  We use cylindrical coordinates $x = r\cos\theta,\ y = r\sin\theta$ and $z = z$.
  The cylinder in expressed as $G(\theta, z) = (2\cos\theta, 2\sin\theta, z)$, $0 \leq \theta \leq 2\pi$, $1 \leq z \leq 5$. So we have:

$$\boldsymbol{T}_\theta = \frac{\partial G}{\partial \theta} = \langle -2\sin\theta, 2\cos\theta, 0\rangle, \quad \boldsymbol{T}_z = \frac{\partial G}{\partial \theta} = \langle 0, 0, 1\rangle;$$

$$\boldsymbol{n} = \boldsymbol{T}_\theta \times \boldsymbol{T}_z = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -2\sin\theta & 2\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = \langle 2\cos\theta, 2\sin\theta, 0\rangle;$$

$$\boldsymbol{F} = \langle x, y, e^z\rangle = \langle 2\cos\theta, 2\sin\theta, e^z\rangle;$$

$$\boldsymbol{F} \cdot \boldsymbol{n} = \langle 2\cos\theta, 2\sin\theta, e^z\rangle \cdot \langle 2\cos\theta, 2\sin\theta, 0\rangle = 4.$$

  Now we get

$$\begin{aligned} \iint_{\mathcal{D}} \boldsymbol{F} \cdot d\boldsymbol{S} &= \int_0^{2\pi} \int_1^5 \boldsymbol{F} \cdot \boldsymbol{n} dz d\theta \\ &= \int_0^{2\pi} \int_1^5 4 dz d\theta = \int_0^{2\pi} 16 d\theta = 32\pi. \end{aligned}$$

Example

- Compute the integral $\iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S}$, where $\boldsymbol{F} = \langle xy, y, 0\rangle$ and $\mathcal{S}$ is the cone $z^2 = x^2 + y^2$, $x^2 + y^2 \leq 4$, $z \geq 0$, with the downward-pointing normal.

  We use cylindrical coordinates $x = r\cos\theta,\ y = r\sin\theta$ and $z = z$.

  The cone in expressed as

$$G(r,\theta) = (r\cos\theta, r\sin\theta, r), \quad 0 \leq r \leq 2, 0 \leq \theta \leq 2\pi.$$

  So we have:

$$\boldsymbol{T}_\theta = \frac{\partial G}{\partial \theta} = \langle -r\sin\theta, r\cos\theta, 0\rangle, \quad \boldsymbol{T}_r = \frac{\partial G}{\partial r} = \langle \cos\theta, \sin\theta, 1\rangle;$$

$$\boldsymbol{n} = \boldsymbol{T}_\theta \times \boldsymbol{T}_r = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ -r\sin\theta & r\cos\theta & 0 \\ \cos\theta & \sin\theta & 1 \end{vmatrix} = \langle r\cos\theta, r\sin\theta, -r\rangle;$$

$$\boldsymbol{F} = \langle xy, y, 0\rangle = \langle r^2\sin\theta\cos\theta, r\sin\theta, 0\rangle;$$

$$\boldsymbol{F} \cdot \boldsymbol{n} = \langle r^2\sin\theta\cos\theta, r\sin\theta, 0\rangle \cdot \langle r\cos\theta, r\sin\theta, -r\rangle = r^3\sin\theta\cos^2\theta + r^2\sin^2\theta.$$

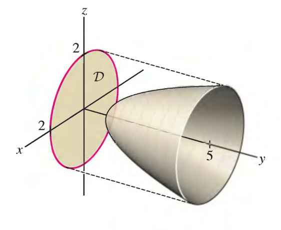
Example (Cont'd)

- Now we get

$$\begin{aligned}\iint_{\mathcal{D}} \boldsymbol{F} \cdot d\boldsymbol{S} &= \int_0^{2\pi} \int_0^2 \boldsymbol{F} \cdot \boldsymbol{n} dr d\theta \\ &= \int_0^{2\pi} \int_0^2 (r^3 \sin\theta \cos^2\theta + r^2 \sin^2\theta) dz d\theta \\ &= \int_0^{2\pi} (\tfrac{1}{4} r^4 \sin\theta \cos^2\theta + \tfrac{1}{3} r^3 \sin^2\theta) \mid_0^2 d\theta \\ &= \int_0^{2\pi} (4 \sin\theta \cos^2\theta + \tfrac{8}{3} \sin^2\theta) d\theta \\ &= -\tfrac{4}{3} \cos^3\theta \mid_0^{2\pi} + \tfrac{8}{3} \tfrac{1}{2} (\theta - \tfrac{1}{2} \sin 2\theta) \mid_0^{2\pi} \\ &= \tfrac{8\pi}{3}.\end{aligned}$$

Example: Integral over a Graph

- Calculate the flux of $\boldsymbol{F} = x^2\boldsymbol{j}$ through the surface $\mathcal{S}$ defined by $y = 1 + x^2 + z^2$ for $1 \leq y \leq 5$. Orient $\mathcal{S}$ with normal pointing in the negative y-direction.



  This surface is the graph of $y = 1 + x^2 + z^2$, where x and z are the independent variables.

  We find a parametrization. Using x and z, because y is given explicitly as a function of x and z, $G(x, z) = (x, 1 + x^2 + z^2, z)$. The condition $1 \leq y \leq 5$ is equivalent to $1 \leq 1 + x^2 + z^2 \leq 5$ or $0 \leq x^2 + z^2 \leq 4$. Therefore, the parameter domain is the disk of radius 2 in the xz-plane. I.e., we have $\mathcal{D} = \{(x, z) : x^2 + z^2 \leq 4\}$.

  Because the parameter domain is a disk, it makes sense to use the polar variables r and θ in the xz-plane. So we write $x = r\cos\theta$, $z = r\sin\theta$. Then $y = 1 + x^2 + z^2 = 1 + r^2$ and
  $G(r, \theta) = (r\cos\theta, 1 + r^2, r\sin\theta)$, $0 \leq \theta \leq 2\pi$, $0 \leq r \leq 2$.

Example: Integral over a Graph (Cont'd)

- We compute the tangent and normal vectors.

$$\boldsymbol{T}_r = \langle \cos\theta, 2r, \sin\theta \rangle, \quad \boldsymbol{T}_\theta = \langle -r\sin\theta, 0, r\cos\theta \rangle,$$

$$\boldsymbol{n} = \boldsymbol{T}_r \times \boldsymbol{T}_\theta = \begin{vmatrix} \boldsymbol{i} & \boldsymbol{j} & \boldsymbol{k} \\ \cos\theta & 2r & \sin\theta \\ -r\sin\theta & 0 & r\cos\theta \end{vmatrix} = 2r^2\cos\theta\boldsymbol{i} - r\boldsymbol{j} + 2r^2\sin\theta\boldsymbol{k}.$$

  The coefficient of $\boldsymbol{j}$ is $-r$. Because it is negative, $\boldsymbol{n}$ points in the negative y-direction, as required.

  We now evaluate $\boldsymbol{F} \cdot \boldsymbol{n}$.

$$\begin{aligned} &\boldsymbol{F}(G(r,\theta)) = x^2\boldsymbol{i} = r^2\cos^2\theta\boldsymbol{j} = \langle 0, r^2\cos^2\theta, 0 \rangle, \\ &\boldsymbol{F}(G(r,\theta)) \cdot \boldsymbol{n} = \langle 0, r^2\cos^2\theta, 0 \rangle \cdot \langle 2r^2\cos\theta, -r, 2r^2\sin\theta \rangle \\ &= -r^3\cos^2\theta. \end{aligned}$$

  For the integral

$$\begin{aligned} \iint_{\mathcal{S}} \boldsymbol{F} \cdot d\boldsymbol{S} &= \iint_{\mathcal{D}} \boldsymbol{F}(G(r,\theta)) \cdot \boldsymbol{n} dr d\theta = \int_0^{2\pi} \int_0^2 (-r^3\cos^2\theta) dr d\theta \\ &= -(\int_0^{2\pi} \cos^2\theta d\theta)(\int_0^2 r^3 dr) = -(\pi)(\tfrac{2^4}{4}) = -4\pi. \end{aligned}$$

The Flow Rate Through a Surface

- Imagine dipping a net into a stream of flowing water.

  The **flow rate** is the volume of water that flows through the net per unit time.

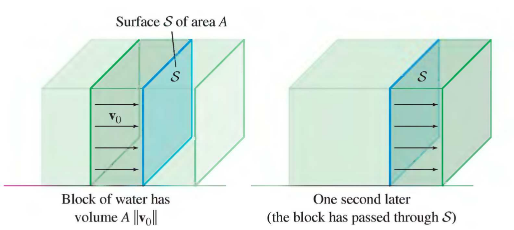
  To compute the flow rate, let $\mathbf{v}$ be the velocity vector field. At each point P, $\mathbf{v}(P)$ is the velocity vector of the fluid particle located at the point P.

  Claim: The flow rate through a surface $\mathcal{S}$ is equal to the surface integral of $\mathbf{v}$ over $\mathcal{S}$.

Perpendicular Flow Through a Rectangular Surface

- Suppose first that $\mathcal{S}$ is a rectangle of area A and that $\boldsymbol{v}$ is a constant vector field with value $\boldsymbol{v}_0$ perpendicular to the rectangle.

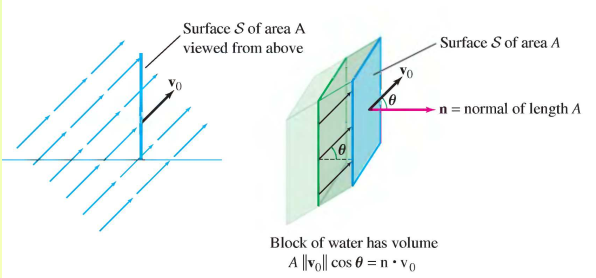
  The particles travel at speed $\|\boldsymbol{v}_0\|$, say in meters per second. So a given particle flows through $\mathcal{S}$ within a one-second time interval if its distance to $\mathcal{S}$ is at most $\|\boldsymbol{v}_0\|$ meters, i.e., if its velocity vector passes through $\mathcal{S}$.



  Thus the block of fluid passing through $\mathcal{S}$ in a one-second interval is a box of volume $\|\boldsymbol{v}_0\|A$: Flow rate $=$ (velocity)(area) $= \|\boldsymbol{v}_0\|A$.

Flow Through a Rectangular Surface

- If the fluid flows at an angle θ relative to $\mathcal{S}$, then the block of water is a parallelepiped (rather than a box) of volume $A\|\boldsymbol{v}_0\|\cos\theta$.

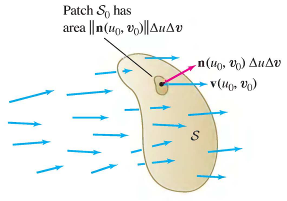


- If $\boldsymbol{n}$ is a vector normal to $\mathcal{S}$ of length equal to the area A, then we can write the flow rate as a dot product:

$$\text{Flow rate} = A\|\boldsymbol{v}_0\|\cos\theta = \boldsymbol{v}_0 \cdot \boldsymbol{n}.$$

Flow: The General Case

- In the general case, the velocity field $\boldsymbol{v}$ is not constant, and the surface $\mathcal{S}$ may be curved. Choose a parametrization $G(u, v)$.
  Consider a small rectangle of size $\Delta u \times \Delta v$ mapped by G to a small patch $\mathcal{S}_0$ of $\mathcal{S}$.
  For any sample point $G(u_0, v_0)$ in $\mathcal{S}_0$, the vector $\boldsymbol{n}(u_0, v_0)\Delta u \Delta v$ is a normal vector of length approximately equal to the area of $\mathcal{S}_0$.



  This patch is nearly rectangular, so we have the approximation

$$\text{Flow rate through } \mathcal{S}_0 \approx \boldsymbol{v}(u_0, v_0) \cdot \boldsymbol{n}(u_0, v_0)\Delta u \Delta v.$$

  The total flow per second is the sum of the flows through the small patches. The limit of the sums as Δu and Δv tend to zero is the integral of $\boldsymbol{v}(u, v) \cdot \boldsymbol{n}(u, v)$, which is the surface integral of $\boldsymbol{v}$ over $\mathcal{S}$:

$$\text{Flow Rate across } \mathcal{S} = \iint_{\mathcal{S}} \boldsymbol{v} \cdot d\boldsymbol{S}.$$

Example

- Let $\boldsymbol{v} = \langle x^2 + y^2, 0, z^2 \rangle$ be the velocity field (in centimeters per second) of a fluid in $\mathbb{R}^3$. Compute the flow rate through the upper hemisphere $\mathcal{S}$ of the unit sphere centered at the origin.

  We use spherical coordinates: $x = \cos\theta \sin\phi$, $y = \sin\theta \sin\phi$, $z = \cos\phi$. The upper hemisphere corresponds to the ranges $0 \leq \phi \leq \frac{\pi}{2}$ and $0 \leq \theta \leq 2\pi$.

  We know that the upward-pointing normal is

$$\boldsymbol{n} = (R^2 \sin\phi)\boldsymbol{e}_r = \sin\phi \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle.$$

  Now we compute:

$$\begin{array}{rcl} \boldsymbol{v} & = & \langle x^2 + y^2, 0, z^2 \rangle = \langle \sin^2\phi, 0, \cos^2\phi \rangle; \\ \boldsymbol{v} \cdot \boldsymbol{n} & = & \sin\phi \langle \sin^2\phi, 0, \cos^2\phi \rangle \cdot \langle \cos\theta \sin\phi, \sin\theta \sin\phi, \cos\phi \rangle \\ & = & \sin^4\phi \cos\theta + \sin\phi \cos^3\phi. \end{array}$$

Example (Cont'd)

- Finally, for the integral, we have

$$\begin{aligned}\iint_{\mathcal{S}} \boldsymbol{v} \cdot d\boldsymbol{S} &= \int_0^{\pi/2} \int_0^{2\pi} (\sin^4 \phi \cos \theta + \sin \phi \cos^3 \phi) d\theta d\phi \\ &= \int_0^{\pi/2} \int_0^{2\pi} \sin \phi \cos^3 \phi d\theta d\phi \\ &= 2\pi \int_0^{\pi/2} \cos^3 \phi \sin \phi d\phi \\ &= 2\pi(-\frac{\cos^4 \phi}{4}) \Big|_0^{\pi/2} \\ &= \frac{\pi}{2} \text{ cm}^3/\text{s}.\end{aligned}$$

Since $\boldsymbol{n}$ is an upward-pointing normal, this is the rate at which fluid flows across the hemisphere from below to above.