Advanced Calculus

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Subsection 1

Trigonometric Series

Trigonometric Series

- A **trigonometric series** is a series of form
$$\frac{1}{2}a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx + \cdots ,$$
where the coefficients a_n and b_n are constants.
- If these constants satisfy certain conditions, to be specified in the next section, then the series is called a **Fourier series**.
- Each term in the series has the property of repeating itself in intervals of 2π:
$$\begin{gathered} \cos(x + 2\pi) = \cos x, \quad \sin(x + 2\pi) = \sin x, \ldots, \\ \cos[n(x + 2\pi)] = \cos(nx + 2n\pi) = \cos nx, \ldots . \end{gathered}$$
- It follows that if the series converges for all x, then its sum $f(x)$ must also have this property:
$$f(x + 2\pi) = f(x).$$
We say $f(x)$ has **period** 2π.

Periodic Functions

- A function $f(x)$ such that, for some $p > 0$,

$$f(x+p) = f(x), \text{ for all } x,$$

  is said to be **periodic** and have **period** p.
- Note that $\cos 2x$ has, in addition to the period 2π, the period π.
- In general, $\cos nx$ and $\sin nx$ have the periods $\frac{2\pi}{n}$.
- However, 2π is the smallest period shared by all terms of the trigonometric series.
- If $f(x)$ has period p, then the substitution

$$x = p\frac{t}{2\pi}$$

  converts $f(x)$ into a function of t having period 2π.

  Indeed, note that when t increases by 2π, x increases by p.

Periodic Functions as Trigonometric Series

- It can be shown that every periodic function of x satisfying certain very general conditions can be represented as a trigonometric series.
- This theorem reflects physical experience.
- In the case of sound, for example that of a violin string:
  - The term $\frac{1}{2}a_0$ represents the neutral position;
  - The terms $a_1 \cos x + b_1 \sin x$ the fundamental tone;
  - The terms $a_2 \cos 2x + b_2 \sin 2x$ t the first overtone (octave);
  - The other terms represent higher overtones.
- The variable x represents time and the function $f(x)$ the displacement of a point on the string.
- The musical tone heard is a combination of simple harmonic vibrations given by the terms $(a_n \cos nx + b_n \sin nx)$.

Rewriting the Simple Harmonic Vibrations

- Each simple harmonic vibration pair $(a_n \cos nx + b_n \sin nx)$ can be written in the form
$$A_n \sin(nx + \alpha),$$
where
$$A_n = \sqrt{a_n^2 + b_n^2}, \quad \sin\alpha = \frac{a_n}{A_n}, \quad \cos\alpha = \frac{b_n}{A_n}.$$
- The "amplitude" A_{n+1} is a measure of the importance of the n-th overtone in the whole sound.
- The differences in the tones of different musical instruments can be ascribed mainly to the differences in the weights A_n of the overtones.

Subsection 2

Fourier Series

Coefficients of Trigonometric Series

- Suppose now that a periodic function $f(x)$ is the sum of a trigonometric series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Multiply $f(x)$ by $\cos mx$ and integrate from $-\pi$ to π:

$$\int_{-\pi}^{\pi} f(x) \cos mx dx \\ = \int_{-\pi}^{\pi} [\frac{a_0}{2} \cos mx + \sum_{n=1}^{\infty}(a_n \cos nx \cos mx + b_n \sin nx \cos mx)] dx.$$

If term-by-term integration of the series is allowed, then we find

$$\int_{-\pi}^{\pi} f(x) \cos mx dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mx dx \\ + \sum_{n=1}^{\infty} \{a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx\}.$$

Coefficients of Trigonometric Series (Cont'd)

- The integrals on the right are evaluated with the help of the identities

$$\begin{aligned} \cos x \cos y &= \tfrac{1}{2}[\cos(x+y) + \cos(x-y)], \\ \sin x \cos y &= \tfrac{1}{2}[\sin(x+y) + \sin(x-y)], \\ \sin x \sin y &= -\tfrac{1}{2}[\cos(x+y) - \cos(x-y)]. \end{aligned}$$

- $\int_{-\pi}^{\pi} \cos mx dx =$
  - If $m = 0$,
  $$\int_{-\pi}^{\pi} \cos 0 dx = \int_{-\pi}^{\pi} dx = 2\pi.$$
  - If $m \neq 0$,
  $$\int_{-\pi}^{\pi} \cos mx dx = \frac{1}{m} \sin mx \,\Big|_{-\pi}^{\pi} = 0.$$

Coefficients of Trigonometric Series (Cont'd)

- $\int_{-\pi}^{\pi} \cos nx \cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi}(\cos(n+m)x + \cos(n-m)x)dx =$
  - If $m = n \neq 0$,
  $$\frac{1}{2}\int_{-\pi}^{\pi}(\cos 2nx + 1)dx = \frac{1}{2}(\frac{1}{2n}\sin 2nx + x)\mid_{-\pi}^{\pi} = \pi.$$
  - If $m \neq n$,
  $$\frac{1}{2}[\frac{1}{n+m}\sin(n+m)x + \frac{1}{n-m}\sin(n-m)x]_{-\pi}^{\pi} = 0.$$
- $\int_{-\pi}^{\pi} \sin nx \cos mx dx = \frac{1}{2}\int_{-\pi}^{\pi}(\sin(n+m)x + \sin(n-m)x)dx =$
  - If $m = n \neq 0$,
  $$\frac{1}{2}\int_{-\pi}^{\pi}\sin 2nx dx = \frac{1}{2}(-\frac{1}{2n}\cos 2nx)\mid_{-\pi}^{\pi} = 0.$$
  - If $m \neq n$,
  $$\frac{1}{2}[-\frac{1}{n+m}\cos(n+m)x - \frac{1}{n-m}\cos(n-m)x]_{-\pi}^{\pi} = 0.$$

Coefficients of Trigonometric Series (Cont'd)

- We found

$$\begin{aligned} \int_{-\pi}^{\pi} \cos mx dx &= \begin{cases} 2\pi, & \text{if } m = 0 \\ 0, & \text{if } m \neq 0 \end{cases}, \\ \int_{-\pi}^{\pi} \cos nx \cos mx dx &= \begin{cases} 0, & n \neq m \\ \pi, & n = m \neq 0 \end{cases}, \\ \int_{-\pi}^{\pi} \sin nx \cos mx dx &= 0. \end{aligned}$$

Thus, we get:

$$\begin{aligned} & \int_{-\pi}^{\pi} f(x) \cos mx dx = \frac{a_0}{2} \int_{-\pi}^{\pi} \cos mx dx \\ & + \sum_{n=1}^{\infty} \{a_n \int_{-\pi}^{\pi} \cos nx \cos mx dx + b_n \int_{-\pi}^{\pi} \sin nx \cos mx dx\} \\ & = \begin{cases} \frac{a_0}{2} \cdot 2\pi = a_0 \pi, & \text{if } m = 0 \\ a_m \pi, & \text{if } m \neq 0 \end{cases} \end{aligned}$$

Coefficients of Trigonometric Series (Conclusion)

- Multiplying $f(x)$ by $\sin mx$ and proceeding in the same way, we find $\int_{-\pi}^{\pi} f(x) \sin mx dx = \pi b_m$, $m = 1, 2, \ldots$.
  We therefore obtain the following formulas:

$$a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx dx, \quad n = 0, 1, 2, \ldots,$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx dx, \quad n = 1, 2, \ldots.$$

Fourier Series

- Let $f(x)$ be a function such that the integrals

$$\begin{aligned} a_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad n = 0, 1, 2, \ldots, \\ b_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 1, 2, \ldots, \end{aligned}$$

  exist.
- The **Fourier series** of $f(x)$ is the trigonometric series

$$\frac{1}{2}a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx + \cdots$$

  in which the coefficients a_n, b_n are computed from the function $f(x)$ by the integrals above.
- For the integrals defining a_n, b_n to exist it is sufficient that $f(x)$ be continuous except for a finite number of jumps between $-\pi$ and π
- No parentheses are used in the general definition of a Fourier series. If the series converges, then insertion of parentheses is permissible.

Uniformly Convergent Trigonometric Series

Theorem

Every uniformly convergent trigonometric series is a Fourier series. More precisely, if the series

$$\frac{1}{2}a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx + \cdots$$

converges uniformly for all x to $f(x)$, then $f(x)$ is continuous for all x, $f(x)$ has period 2π, and the series is the Fourier series of $f(x)$.

- Since the series converges uniformly for all x, its sum $f(x)$ is continuous, for all x.

  The series remains uniformly convergent if all terms are multiplied by $\cos mx$ or by $\sin mx$.

  Therefore, the term-by-term integration of the series is justified.

Uniformly Convergent Trigonometric Series (Cont'd)

- The formulas

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad n = 0, 1, 2, \dots, \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx, \quad n = 1, 2, \dots, \end{aligned}$$

now follow as previously so that the series is the Fourier series of $f(x)$. The periodicity of $f(x)$ is a consequence of the periodicity of the terms of the series.

Uniform Convergence and Uniqueness

Corollary

If two trigonometric series converge uniformly for all x and have the same sum for all x:

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty} x(a_n \cos nx + b_n \sin nx) \equiv \frac{1}{2}a'_0 + \sum_{n=1}^{\infty}(a'_n \cos nx + b'_n \sin nx),$$

then the series are identical: $a_0 = a'_0$, $a_n = a'_n$, $b_n = b'_n$, for $n = 1, 2, \ldots$. In particular, if a trigonometric series converges uniformly to 0 for all x, then all coefficients are 0.

- Let $f(x)$ denote the sum of both series. Then by the preceding theorem, $a_n = a'_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx dx$, $n = 0, 1, 2, \ldots$, and similarly $b_n = b'_n$, for all n. If $f(x) \equiv 0$, then all coefficients are 0.

Subsection 3

Convergence of Fourier Series

Continuity and Smoothness

- We term a function $f(x)$, defined for $a \leq x \leq b$, **piecewise continuous** in this interval if the interval can be subdivided into a finite number of subintervals, inside each of which $f(x)$ is continuous and has finite limits at the left and right ends of the interval.
- Accordingly, inside the i-th subinterval the function $f(x)$ coincides with a function $f_i(x)$ that is continuous in the closed subinterval.
- If, in addition, the functions $f_i(x)$ have continuous first derivatives, we term $f(x)$ **piecewise smooth**.
- If, in addition, the functions $f_i(x)$ have continuous second derivatives, we term $f(x)$ **piecewise very smooth**.

The Fundamental Theorem

Fundamental Theorem

Let $f(x)$ be piecewise very smooth in the interval $-\pi \leq x \leq \pi$. Then the Fourier series of $f(x)$:

$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx, \quad b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx.$$

converges to $f(x)$ wherever $f(x)$ is continuous inside the interval.
The series converges to $\frac{1}{2}[\lim_{x \to x_1^-} f(x) + \lim_{x \to x_1^+} f(x)]$ at each point of discontinuity x_1 inside the interval, and to $\frac{1}{2}[\lim_{x \to \pi^-} f(x) + \lim_{x \to -\pi^+} f(x)]$ at $x = \pm\pi$.
The convergence is uniform in each closed interval containing no discontinuity.

- The proof of the fundamental theorem will be given later.

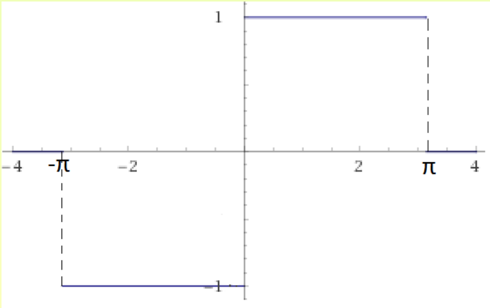
Subsection 4

Examples. Minimizing of Square Error

Example

- Consider $f(x) = \begin{cases} -1, & \text{if } -\pi \leq x < 0 \\ 1, & \text{if } 0 \leq x \leq \pi \end{cases}$.

  The periodic extension of $f(x)$ gives a "square wave".



Example (Cont'd)

- If $n = 0$, then
  $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}[\int_{-\pi}^{0} -dx + \int_0^{\pi} dx] = \frac{1}{\pi}[-\pi + \pi] = 0.$
- If $n \neq 0$:

$$\begin{array}{rcl} a_n & = & \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\ & = & -\frac{1}{\pi}[\int_{-\pi}^{0} -\cos nx dx + \int_0^{\pi} \cos nx dx] \\ & = & \frac{1}{\pi}[-\frac{1}{n}\sin nx \mid_{-\pi}^{0} + \frac{1}{n}\sin nx \mid_0^{\pi}] \\ & = & \frac{1}{\pi}\cdot 0 = 0; \\ b_n & = & \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\ & = & \frac{1}{\pi}[\int_{-\pi}^{0} -\sin nx dx + \int_0^{\pi} \sin nx dx] \\ & = & \frac{1}{\pi}[\frac{1}{n}\cos nx \mid_{-\pi}^{0} - \frac{1}{n}\cos nx \mid_0^{\pi}] \\ & = & \frac{1}{\pi}[\frac{1}{n} - \frac{1}{n}\cos(n\pi) - \frac{1}{n}\cos(n\pi) + \frac{1}{n}] \\ & = & \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4}{n\pi}, & \text{if } n \text{ is odd} \end{cases} \end{array}$$

Example (Cont'd)

- We computed

$$\begin{aligned} a_n &= 0, \quad n = 0, 1, 2, \ldots, \\ b_n &= \begin{cases} 0, & \text{if } n = 2, 4, \ldots \\ \frac{4}{n\pi}, & \text{if } n = 1, 3, 5, \ldots \end{cases} \end{aligned}$$

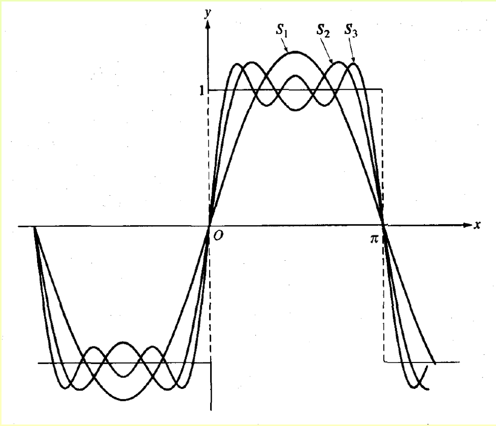
Hence for $-\pi < x < \pi$,

$$f(x) = \frac{4}{\pi} \sin x + \frac{4}{3\pi} \sin 3x + \cdots = \frac{4}{\pi} \sum_{n=1}^{\infty} \frac{\sin (2n-1)x}{2n-1}.$$

Example (Cont'd): Illustration of Partial Sums

- We have

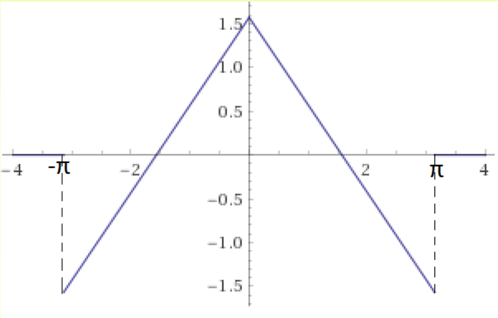
$$S_1 = \frac{4}{\pi}\sin x, \quad S_2 = S_1 + \frac{4}{3\pi}\sin 3x, \quad S_3 = S_2 + \frac{4}{5\pi}\sin 5x.$$



Example

- Let $f(x) = \begin{cases} \frac{1}{2}\pi + x, & \text{if } -\pi \le x \le 0 \\ \frac{1}{2}\pi - x, & \text{if } 0 \le x \le \pi \end{cases}$.

  The periodic extension of $f(x)$ is a triangular wave.



  The extension is continuous for all x.

Example (Cont'd)

- For $n = 0$:

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{0}\left(\frac{1}{2}\pi + x\right)dx + \frac{1}{\pi}\int_{0}^{\pi}\left(\frac{1}{2}\pi - x\right)dx \\ &= \frac{1}{\pi}\left[\frac{1}{2}x^2 + \frac{1}{2}\pi x\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-\frac{1}{2}x^2 + \frac{1}{2}\pi x\right]_{0}^{\pi} \\ &= \frac{1}{\pi}\left(-\frac{1}{2}\pi^2 + \frac{1}{2}\pi^2\right) + \frac{1}{\pi}\left(-\frac{1}{2}\pi^2 + \frac{1}{2}\pi^2\right) \\ &= 0. \end{aligned}$$

Example (Cont'd)

- If $n \neq 0$:

$$
\begin{aligned}
a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0}(\frac{\pi}{2}+2)\cos nx dx + \frac{1}{\pi}\int_{0}^{\pi}(\frac{\pi}{2}-x)\cos nx dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0}(\frac{1}{2}\pi+x)(\frac{1}{n}\sin nx)' dx + \frac{1}{\pi}\int_{0}^{\pi}(\frac{1}{2}\pi-x)(\frac{1}{n}\sin nx)' dx \\
&= \frac{1}{\pi}[[(\frac{1}{2}\pi+x)(\frac{1}{n}\sin nx)]_{-\pi}^{0} - \int_{-\pi}^{0}\frac{1}{n}\sin nx dx] \\
&\quad + \frac{1}{\pi}[[(\frac{1}{2}\pi-x)(\frac{1}{n}\sin nx)]_{0}^{\pi} + \int_{0}^{\pi}\frac{1}{n}\sin nx dx] \\
&= \frac{1}{\pi}[[(\frac{1}{2}\pi+x)(\frac{1}{n}\sin nx)]_{-\pi}^{0} + \frac{1}{n^2}\cos nx \,|_{0}^{\pi}] \\
&\quad + \frac{1}{\pi}[[(\frac{1}{2}\pi-x)(\frac{1}{n}\sin nx)]_{0}^{\pi} - \frac{1}{n^2}\cos nx \,|_{0}^{\pi}] \\
&= \frac{1}{\pi}(\frac{1}{n^2}-\frac{1}{n^2}\cos(n\pi)) - \frac{1}{\pi}(\frac{1}{n^2}\cos(n\pi)-\frac{1}{n^2}) \\
&= \frac{2}{\pi n^2} - \frac{2}{\pi n^2}\cos(n\pi) \\
&= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

Example (Cont'd)

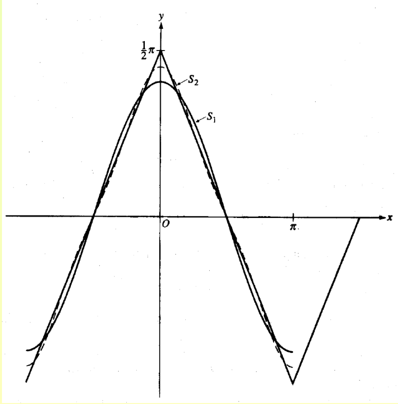
- If $n \neq 0$:

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} (\frac{\pi}{2}+2)\sin nx dx + \frac{1}{\pi}\int_{0}^{\pi} (\frac{\pi}{2}-x)\sin nx dx \\
&= \frac{1}{\pi}\int_{-\pi}^{0} (\frac{1}{2}\pi + x)(-\frac{1}{n}\cos nx)' dx \\
&\quad + \frac{1}{\pi}\int_{0}^{\pi} (\frac{1}{2}\pi - x)(-\frac{1}{n}\cos nx)' dx \\
&= \frac{1}{\pi}[[(\frac{1}{2}\pi + x)(-\frac{1}{n}\cos nx)]_{-\pi}^{0} + \int_{-\pi}^{0}\frac{1}{n}\cos nx dx] \\
&\quad + \frac{1}{\pi}[[(\frac{1}{2}\pi - x)(-\frac{1}{n}\cos nx)]_{0}^{\pi} - \int_{0}^{\pi}\frac{1}{n}\cos nx dx] \\
&= \frac{1}{\pi}[[(\frac{1}{2}\pi + x)(-\frac{1}{n}\cos nx)]_{-\pi}^{0} + \frac{1}{n^2}\sin nx \,|_{-\pi}^{0}] \\
&\quad + \frac{1}{\pi}[[(\frac{1}{2}\pi - x)(-\frac{1}{n}\cos nx)]_{0}^{\pi} - \frac{1}{n^2}\sin nx \,|_{0}^{\pi}] \\
&= \frac{1}{\pi}(-\frac{\pi}{2n} - \frac{\pi}{2n}\cos(n\pi)) + \frac{1}{\pi}(\frac{\pi}{2n}\cos(n\pi) + \frac{\pi}{2n}) \\
&= 0.
\end{aligned}$$

Example (Cont'd): Illustration of Partial Sums

- We conclude

$$f(x) = \frac{4}{\pi}\cos x + \frac{4}{9\pi}\cos 3x + \cdots = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}.$$



The Value $a_0/2$

- The constant term $\frac{a_0}{2}$ of the series is given by the formula
$$\frac{a_0}{2} = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx.$$
- The right-hand member is simply the average or arithmetic mean of $f(x)$ over the interval $-\pi \le x \le \pi$.
- So the line $y = \frac{a_0}{2}$ must be such that the area between the line and the curve $y = f(x)$ lying above the line equals the area between the line and the curve $y = f(x)$ lying below the line.
- The line $y = \frac{a_0}{2}$ is a sort of symmetry line for the graph of $y = f(x)$.
- Taking either of these points of view in the two examples considered, one must have $\frac{a_0}{2} = 0$. The average of $f(x)$ is 0, and there is as much area above the x-axis as below.

Minimizing Total Square Error

- We define the **total square error** of a function $g(x)$ relative to $f(x)$ as the integral
$$E = \int_{-\pi}^{\pi} [f(x) - g(x)]^2 dx.$$
- This error is 0 when $g = f$ (or when $g = f$ except for a finite number of points), and is otherwise positive.
- We seek a constant function $y = g_0$ that minimizes this error.
- The error is
$$\begin{aligned} E(g_0) &= \int_{-\pi}^{\pi} [f(x) - g_0]^2 dx = \int_{-\pi}^{\pi} [f(x)]^2 dx \\ &\qquad - 2g_0 \int_{-\pi}^{\pi} f(x) dx + g_0^2 \cdot 2\pi \\ &= A - 2Bg_0 + 2\pi g_0^2, \end{aligned}$$
where A and B are constants. Thus $E(g_0)$ is a quadratic function of g_0, having a minimum when $\frac{dE}{dg_0} = 0$: $-2B + 4\pi g_0 = 0$. Hence the error is minimized when $g_0 = \frac{B}{2\pi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{a_0}{2}$.

Lemma for Minimization of Square Error

Lemma

The following hold, for all $n, m \neq 0$:

(a) $\int_{-\pi}^{\pi} \sin nx dx = \int_{-\pi}^{\pi} \cos nx dx = 0$;

(b) $\int_{-\pi}^{\pi} \sin^2(nx) dx = \int_{-\pi}^{\pi} \cos^2(nx) dx = \pi$;

(c) $\int_{-\pi}^{\pi} \sin nx \sin mx dx = \int_{-\pi}^{\pi} \sin nx \cos mx dx = \int_{-\pi}^{\pi} \cos nx \cos mx dx = 0$.

- We prove one of each. The rest are handled similarly.
  (a) $\int_{-\pi}^{\pi} \sin nx dx = -\frac{1}{n} \cos nx \,|_{-\pi}^{\pi} = -\frac{1}{n}(\cos(n\pi) - \cos(n\pi)) = 0$.
  (b) $\int_{-\pi}^{\pi} \sin^2(nx) dx = \frac{1}{2} \int_{-\pi}^{\pi} (1 - \cos(2nx)) dx = \frac{1}{2}(x - \frac{1}{2n} \sin(2nx)) \,|_{-\pi}^{\pi} = \frac{1}{2} \cdot 2\pi = \pi$.
  (c) $\int_{-\pi}^{\pi} \sin nx \sin mx dx = -\frac{1}{2} \int_{-\pi}^{\pi} (\cos(n+m)x - \cos(n-m)x) dx =$

$$\begin{cases} -\frac{1}{2} \int_{-\pi}^{\pi} (\cos 2nx - 1) dx = -\frac{1}{2}[\frac{1}{2n} \sin 2nx - x]_{-\pi}^{\pi} = 0, & \text{if } n = m \\ -\frac{1}{2}[\frac{1}{n+m} \sin(n+m)x - \frac{1}{n-m} \sin(n-m)x]_{-\pi}^{\pi} = 0, & \text{if } n \neq m \end{cases}$$

Generalization of the Minimization of Square Error

Theorem

Let $f(x)$ be piecewise continuous for $-\pi \leq x \leq \pi$. The coefficients of the partial sum

$$\frac{1}{2}a_0 + a_1 \cos x + b_1 \sin x + \cdots + a_n \cos nx + b_n \sin nx$$

of the Fourier series of $f(x)$ are precisely those among all coefficients of the function $g_n(x) = p_0 + p_1 \cos x + q_1 \sin x + \cdots + p_n \cos nx + q_n \sin nx$ that minimize the square error

$$\int_{-\pi}^{\pi} [f(x) - g_n(x)]^2 dx.$$

Furthermore, the minimum square error E_n satisfies the equation:

$$E_n = \int_{-\pi}^{\pi} [f(x)]^2 dx - \pi \left[\frac{1}{2}a_0^2 + \sum_{k=1}^{n} (a_k^2 + b_k^2) \right].$$

Proof of the Theorem

- Suppose $g_n(x) = p_0 + p_1 \cos x + q_1 \sin x + \cdots + p_n \cos nx + q_n \sin nx$. Compute the square error of approximating f by g_n:

$$\begin{aligned}
&\int_{-\pi}^{\pi} (f - g_n)^2 dx \\
&= \int_{-\pi}^{\pi} (f - p_0 - p_1 \cos x - q_1 \sin x - \cdots - p_n \cos nx - q_n \sin nx)^2 dx \\
&= \int_{-\pi}^{\pi} [f^2 + p_0^2 + p_1^2 \cos^2 x + q_1^2 \sin^2 x + \cdots + p_n^2 \cos^2 nx + q_n^2 \sin^2 nx \\
&- 2fp_0 - 2fp_1 \cos x - 2fq_1 \sin x - \cdots - 2fp_n \cos nx - 2fq_n \sin nx \\
&+ 2p_0 p_1 \cos x + 2p_0 q_1 \sin x + \cdots + 2p_0 p_n \cos nx + 2p_0 q_n \sin nx \\
&+ \sum_{n,m} 2p_n p_m \cos nx \cos mx + \sum_{n,m} 2p_n q_m \cos nx \sin mx \\
&+ \sum_{n,m} 2q_n q_m \sin nx \sin mx] dx \\
&\overset{\text{Lemma}}{=} \int_{-\pi}^{\pi} f^2 dx + 2\pi p_0^2 + \pi p_1^2 + \pi q_1^2 + \cdots + \pi p_n^2 + \pi q_n^2 \\
&- 2p_0 \int_{-\pi}^{\pi} f dx - 2p_1 \int f \cos x dx - 2q_1 \int_{-\pi}^{\pi} f \sin x dx - \cdots \\
&- 2p_n \int_{-\pi}^{\pi} f \cos nx dx - 2q_n \int_{-\pi}^{\pi} f \sin nx dx \\
&= \int_{-\pi}^{\pi} f^2 dx + (2\pi p_0^2 - 2p_0 \int_{-\pi}^{\pi} f dx) \\
&+ (\pi p_1^2 - 2p_1 \int_{-\pi}^{\pi} f \cos x dx) + \cdots + (\pi q_n^2 - 2q_n \int_{-\pi}^{\pi} f \sin nx dx)
\end{aligned}$$

Proof of the Theorem (Cont'd)

- We found that the square error is given by:

  $$\int_{-\pi}^{\pi} f^2 dx + (2\pi p_0^2 - 2p_0 \int_{-\pi}^{\pi} f dx)$$
  $$+(\pi p_1^2 - 2p_1 \int_{-\pi}^{\pi} f \cos x dx) + \cdots + (\pi q_n^2 - 2q_n \int_{-\pi}^{\pi} f \sin nx dx)$$

  To minimize it, we minimize each of the parentheses:
  - $4\pi_0 - 2\int_{-\pi}^{\pi} f dx = 0 \Rightarrow \; p_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f dx \Rightarrow \; p_0 = \frac{a_0}{2}$.
  - $2\pi p_1 - 2\int_{-\pi}^{\pi} f \cos x dx = 0 \Rightarrow \; p_1 = \frac{1}{\pi}\int_{\pi}^{\pi} f \cos x dx \Rightarrow \; p_1 = a_1$.

    $\vdots$
  - $2\pi q_n - 2\int_{-\pi}^{\pi} f \sin nx dx = 0 \Rightarrow \; q_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f \sin nx dx \Rightarrow \; q_n = b_n$.

  Now for the minimun square error we get

  $$\int_{-\pi}^{\pi} f^2 dx + (2\pi \frac{a_0^2}{4} - 2\frac{a_0}{2}\pi a_0) +$$
  $$+(\pi a_1^2 - 2a_1\pi a_1) + \cdots + (\pi b_n^2 - 2b_n \pi b_n)$$
  $$= \int_{-\pi}^{\pi} f^2 dx - \pi\frac{a_0^2}{2} - \pi a_1^2 - \pi b_1^2 - \cdots - \pi a_n^2 - \pi b_n^2$$
  $$= \int_{-\pi}^{\pi} f^2 dx - \pi[\frac{a_0^2}{2} + \sum_{k=1}^{n}(a_k^2 + b_k^2)].$$

An Inequality Based on the Square-Error Minimization

Corollary

If $f(x)$ is piecewise continuous for $-\pi \leq x \leq \pi$ and $a_0, a_1, \ldots, b_1, b_2, \ldots$ are the Fourier coefficients of $f(x)$, then

$$\frac{1}{2}a_0^2 + \sum_{k=1}^{n}(a_k^2 + b_k^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi}[f(x)]^2 dx.$$

So the series $\sum_{n=1}^{\infty}(a_n^2 + b_n^2)$ converges. Furthermore, $\lim_{n\to\infty} a_n = 0$, $\lim_{n\to\infty} b_n = 0$.

- Since the square error $\int (f-g)^2 dx$ is always positive or 0, the minimum square error E_n is always positive or 0. So the inequality follows from the preceding theorem.

  By the inequality established, the series converges. It then follows that the n-th term of the series converges to 0.

Subsection 5

Generalizations. Fourier Sine and Cosine Series

Using a Nonstandard Interval

- If $f(x)$ is a function of period 2π, one can use as basic interval any interval $c \leq x \leq c + 2\pi$, i.e., any interval of length 2π.
- For such an interval the same reasoning as previously leads to a Fourier series
$$\frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx),$$
where
$$\begin{aligned} a_n &= \frac{1}{\pi}\int_c^{c+2\pi} f(x)\cos nx dx, \\ b_n &= \frac{1}{\pi}\int_c^{2\pi} f(x)\sin nx dx. \end{aligned}$$
- If $f(x)$ is given for all x, with period 2π, this is merely another way of computing the coefficients a_n, b_n.
- If $f(x)$ is given only for $c \leq x \leq c + 2\pi$, the series can be used to represent f in this interval. It will then (if convergent) represent the periodic extension of f outside this interval.

Even and Odd Functions

- Let $f(x)$ be defined in $-\pi \leq x \leq \pi$.
- f is called an **even function** if $f(-x) = f(x)$, for all $-\pi \leq x \leq \pi$.
- f is called an **odd function** if $f(-x) = -f(x)$, for all $-\pi \leq x \leq \pi$.
  Note that:
  - The product of two even functions or of two odd functions is even;
  - The product of an odd function and an even function is odd.
- Furthermore,

$$\int_{-a}^{a} f(x)dx = \begin{cases} 0, & \text{if } f \text{ is odd} \\ 2\int_0^a f(x)dx, & \text{if } f \text{ is even} \end{cases}.$$

The Fourier Cosine Series of an Even Function

- Let f be even in the interval $-\pi \leq x \leq \pi$.
- Then $f(x)\cos nx$ is even (product of two even functions).
- Moreover, $f(x)\sin nx$ is odd (product of odd function and even function).
- Hence
$$\begin{aligned} a_n &= \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx, \quad n = 0, 1, 2, \ldots, \\ b_n &= 0, \quad n = 1, 2, \ldots. \end{aligned}$$
- We have thus the expansion (for a function piecewise very smooth):
$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^\infty a_n \cos nx, \quad f \text{ even}, \\ a_n &= \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx. \end{aligned}$$
- This is called the **Fourier cosine series** of $f(x)$.
- It follows from the fundamental theorem that the series will converge to $f(x)$ for $0 \leq x \leq \pi$ and outside this interval to the even periodic function that coincides with $f(x)$ for $0 \leq x \leq \pi$.

The Fourier Sine Series of an Odd Function

- Similarly, if f is odd,

$$a_n = 0, \quad b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx.$$

- So we have the expansion

$$\begin{array}{rcl} f(x) & = & \sum_{n=1}^\infty b_n \sin nx, \quad f \text{ odd}, \\ b_n & = & \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx. \end{array}$$

- This defines the **Fourier sine series** of a function $f(x)$ defined only between 0 and π.
- The series represents an odd periodic function that coincides with $f(x)$ for $0 \leq x \leq \pi$.

Example

- Let $f(x) = \pi - x$.

  Then one can represent $f(x)$ by a Fourier series over the interval $-\pi < x < \pi$.

  We have

$$\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi}(\pi - x)dx \\ &= \frac{1}{\pi}[-\frac{1}{2}x^2 + \pi x]_{-\pi}^{\pi} \\ &= \frac{1}{\pi}[-\frac{1}{2}\pi^2 + \pi^2 + \frac{1}{2}\pi^2 + \pi^2] \\ &= 2\pi. \end{aligned}$$

Example (Cont'd)

- Next we compute a_n for $n \neq 0$.

$$\begin{aligned} a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)\cos nx dx \\ &= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)(\frac{1}{n}\sin nx)' dx \\ &= \frac{1}{\pi}\left[[(\pi - x)(\frac{1}{n}\sin nx)]_{-\pi}^{\pi} + \int_{-\pi}^{\pi}\frac{1}{n}\sin nx dx\right] \\ &= \frac{1}{\pi}\left[[(\pi - x)(\frac{1}{n}\sin nx)]_{-\pi}^{\pi} - \frac{1}{n^2}\cos nx \,|_{-\pi}^{\pi}\right] \\ &= 0. \end{aligned}$$

Example (Cont'd)

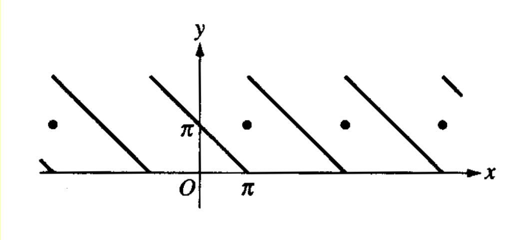
- Finally we compute b_n, $n \neq 0$.

$$\begin{aligned}
b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)\sin nx dx \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} (\pi - x)(-\frac{1}{n}\cos nx)' dx \\
&= \frac{1}{\pi}\left[[(\pi - x)(-\frac{1}{n}\cos nx)]_{-\pi}^{\pi} - \int_{-\pi}^{\pi}\frac{1}{n}\cos nx dx\right] \\
&= \frac{1}{\pi}\left[[(\pi - x)(-\frac{1}{n}\cos nx)]_{-\pi}^{\pi} - \frac{1}{n^2}\sin nx \,\Big|_{-\pi}^{\pi}\right] \\
&= \frac{1}{\pi}(-2\pi)(-\frac{1}{n}\cos n\pi) \\
&= \frac{2}{n}\cos n\pi = \frac{2(-1)^n}{n}.
\end{aligned}$$

Example (Conclusion)

- Hence we have

$$\pi - x = \pi + 2\sum_{n=1}^{\infty} \frac{(-1)^n \sin nx}{n}, \quad -\pi < x < \pi.$$



Example (Cosine Series)

- The same function $f(x) = \pi - x$ can be represented by a Fourier cosine series over the interval $0 \leq x \leq \pi$.

  Now we get

$$\begin{aligned} a_0 &= \frac{2}{\pi}\int_0^\pi f(x)dx \\ &= \frac{2}{\pi}\int_0^\pi (\pi - x)dx \\ &= \frac{2}{\pi}[-\frac{1}{2}x^2 + \pi x]_0^\pi \\ &= \frac{2}{\pi}(-\frac{1}{2}\pi^2 + \pi^2) \\ &= \pi. \end{aligned}$$

Example (Cont'd)

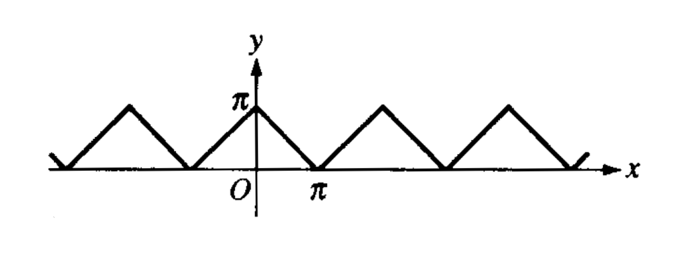
- For $n \neq 0$,

$$
\begin{aligned}
a_n &= \frac{2}{\pi}\int_0^\pi f(x)\cos nx dx \\
&= \frac{2}{\pi}\int_0^\pi (\pi - x)\cos nx dx \\
&= \frac{2}{\pi}\int_0^\pi (\pi - x)(\frac{1}{n}\sin nx)' dx \\
&= \frac{2}{\pi}\left[[(\pi - x)(\frac{1}{n}\sin nx)]_0^\pi + \int_0^\pi \frac{1}{n}\sin nx dx\right] \\
&= \frac{2}{\pi}\left[[(\pi - x)(\frac{1}{n}\sin nx)]_0^\pi - \frac{1}{n^2}\cos nx \,\Big|_0^\pi\right] \\
&= \frac{2}{\pi}(-\frac{1}{n^2}\cos(n\pi) + \frac{1}{n^2}) \\
&= \begin{cases} 0, & \text{if } n \text{ is even} \\ \frac{4}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}
\end{aligned}
$$

Example (Conclusion)

- We have, for $0 \leq x \leq \pi$:

$$\pi - x \quad = \quad \frac{\pi}{2} + \frac{2}{\pi}\left(\frac{2}{1^2}\cos x + \frac{2}{3^2}\cos 3x + \frac{2}{5^2}\cos 5x + \cdots\right).$$



Example (Sine Series)

- Finally, the same function, $f(x) = \pi - x$, can be represented by a Fourier sine series over the interval $0 < x < \pi$.

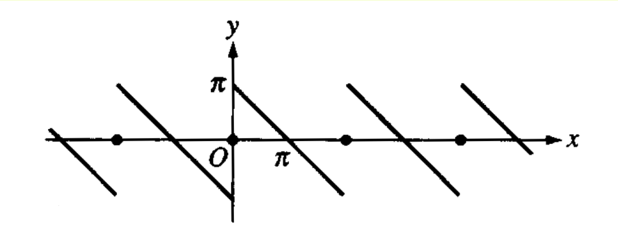
  We have, for $n \geq 1$,

$$\begin{aligned} b_n &= \frac{2}{\pi}\int_0^\pi f(x)\sin nx dx \\ &= \frac{2}{\pi}\int_0^\pi (\pi - x)\sin nx dx \\ &= \frac{2}{\pi}\int_0^\pi (\pi - x)(-\frac{1}{n}\cos nx)' dx \\ &= \frac{2}{\pi}\left[[(\pi - x)(-\frac{1}{n}\cos nx)]_0^\pi - \int_0^\pi \frac{1}{n}\cos nx dx\right] \\ &= \frac{2}{\pi}\left[[(\pi - x)(-\frac{1}{n}\cos nx)]_0^\pi - \frac{1}{n^2}\sin nx \,\Big|_0^\pi\right] \\ &= \frac{2}{\pi}(\frac{\pi}{n}) = \frac{2}{n}. \end{aligned}$$

Example (Conclusion)

- We get, for $0 < x < \pi$,

$$\pi - x = 2 \sum_{n=1}^{\infty} \frac{1}{n} \sin nx.$$



Change of Period

- If $f(x)$ has period p, i.e., $f(x+p) = f(x)$, $p > 0$, then the substitution $x = \frac{p}{2\pi}t$ transforms $f(x)$ into a function $g(t) = f(\frac{p}{2\pi}t)$ that has period 2π.
- We have

$$\begin{array}{rcl} g(t+2\pi) & = & f[\frac{p}{2\pi}(t+2\pi)] \\ & = & f(\frac{p}{2\pi}t+p) = f(\frac{p}{2\pi}t) = g(t). \end{array}$$

- Since g has period 2π, one has a Fourier series for g (assumed piecewise very smooth):

$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos nt + b_n \sin nt),$$

where $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t) \cos nt dt$, $b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(t) \sin nt dt$.

Change of Period (Cont'd)

- We have
$$g(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} (a_n \cos nt + b_n \sin nt),$$
where $a_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \cos nt dt$, $b_n = \frac{1}{\pi} \int_{-\pi}^{\pi} g(t) \sin nt dt$.
- If now t is replaced by $\frac{2\pi}{p} x$, one finds a Fourier series for $f(x)$:
$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos \left(n \cdot \frac{2\pi}{p} x \right) + b_n \sin \left(n \cdot \frac{2\pi}{p} x \right) \right].$$
- The coefficients a_n, b_n can be expressed directly in terms of $f(x)$:
$$\begin{aligned} a_n &= \frac{1}{p/2} \int_{-p/2}^{p/2} f(x) \cos (n \cdot \frac{2\pi}{p} x) dx, \\ b_n &= \frac{1}{p/2} \int_{-p/2}^{p/2} f(x) \sin (n \cdot \frac{2\pi}{p} x) dx. \end{aligned}$$

Change of Period: Cosine and Sine Series

- The Fourier cosine series can also be used in this case:

$$\begin{aligned} f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(n \cdot \tfrac{2\pi}{p} x\right), \quad 0 \leq x \leq \tfrac{p}{2}, \\ a_n &= \frac{2}{p/2} \int_0^{p/2} f(x) \cos\left(n \cdot \tfrac{2\pi}{p} x\right) dx. \end{aligned}$$

- Similarly, $f(x)$ has a Fourier sine series:

$$\begin{aligned} f(x) &= \sum_{n=1}^{\infty} b_n \sin\left(n \cdot \tfrac{2\pi}{p} x\right), \quad 0 < x < \tfrac{p}{2}, \\ b_n &= \frac{2}{p/2} \int_0^{p/2} f(x) \sin\left(n \cdot \tfrac{2\pi}{p} x\right) dx. \end{aligned}$$

Example

- Let $f(x) = 2x + 1$.

  Then $f(x)$ can be represented by a Fourier series over the interval $0 < x < 2$.

  With $p = 2$, we get:

$$\begin{aligned} a_0 &= \int_0^2 f(x)dx \\ &= \int_0^2 (2x+1)dx \\ &= (x^2 + x) \mid_0^2 \\ &= 6. \end{aligned}$$

Example (Cont'd)

- For $n \neq 0$,

$$\begin{aligned} a_n &= \int_0^2 f(x)\cos\left(n\frac{2\pi}{2}x\right)dx \\ &= \int_0^2 (2x+1)\cos(n\pi x)dx \\ &= \int_0^2 (2x+1)\left(\frac{1}{n\pi}\sin(n\pi x)\right)'dx \\ &= \left[(2x+1)\left(\frac{1}{n\pi}\sin(n\pi x)\right)\right]_0^2 - \int_0^2 \frac{2}{n\pi}\sin(n\pi x)dx \\ &= \left[(2x+1)\left(\frac{1}{n\pi}\sin(n\pi x)\right)\right]_0^2 + \frac{2}{n^2\pi^2}\cos(n\pi x)\ \Big|_0^2 \\ &= 0. \end{aligned}$$

Example (Cont'd)

- Finally, for $n \geq 1$,

$$\begin{aligned} b_n &= \int_0^2 f(x)\sin\left(n\tfrac{2\pi}{2}x\right)dx \\ &= \int_0^2 (2x+1)\sin(n\pi x)dx \\ &= \int_0^2 (2x+1)\left(-\tfrac{1}{n\pi}\cos(n\pi x)\right)' dx \\ &= \left[(2x+1)\left(-\tfrac{1}{n\pi}\cos(n\pi x)\right)\right]_0^2 + \int_0^2 \tfrac{2}{n\pi}\cos(n\pi x)dx \\ &= \left[(2x+1)\left(-\tfrac{1}{n\pi}\cos(n\pi x)\right)\right]_0^2 + \tfrac{2}{n^2\pi^2}\sin(n\pi x)\ \Big|_0^2 \\ &= -\tfrac{5}{n\pi} + \tfrac{1}{n\pi} = -\tfrac{4}{n\pi}. \end{aligned}$$

We get

$$f(x) = 3 - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin(n\pi x).$$

Subsection 6

Uniqueness Theorem

Example

- Show that $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$ and $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$.
  We show the first equation (the other can be proved similarly):

$$\begin{array}{rcl}
\sin^3 x & = & [\frac{1}{2i}(e^{ix} - e^{-ix})]^3 \\
 & = & -\frac{1}{8i}(e^{3ix} - 3e^{2ix}e^{-ix} + 3e^{ix}e^{-2ix} - e^{-3ix}) \\
 & = & -\frac{1}{8i}(-3(e^{ix} - e^{-ix}) + (e^{3ix} - e^{-3ix})) \\
 & = & -\frac{1}{4}(-3\frac{e^{ix}-e^{-ix}}{2i} + \frac{e^{i(3x)}-e^{-i(3x)}}{2i}) \\
 & = & -\frac{1}{4}(-3\sin x + \sin 3x) \\
 & = & \frac{3}{4}\sin x - \frac{1}{4}\sin 3x.
\end{array}$$

Preliminary Lemma

Lemma

Both $\sin^n x$ and $\cos^n x$ are expressible as trigonometric polynomials, for all $n \geq 0$.

- We only deal with $\cos^n x$. Moreover, we restrict to n odd.
  For n even, we can then use $\cos x \cos y = \frac{1}{2}[\cos(x+y) + \cos(x-y)]$.
  We have

$$\begin{array}{rcl}
\cos^n x & = & (\frac{1}{2}(e^{ix} + e^{-ix}))^n = \frac{1}{2^n}(e^{ix} + e^{-ix})^n \\
& = & \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}(e^{ix})^k(e^{-ix})^{n-k} \\
& = & \frac{1}{2^n}\sum_{k=0}^{n}\binom{n}{k}e^{ikx}e^{-i(n-k)x} \\
& = & \frac{1}{2^n}\sum_{k=0}^{\frac{n-1}{2}}[\binom{n}{k}e^{ikx}e^{-i(n-k)x} + \binom{n}{n-k}e^{i(n-k)x}e^{-ikx}] \\
& = & \frac{1}{2^{n-1}}\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{k}\frac{1}{2}(e^{i(n-2k)x} + e^{-i(n-2k)x}) \\
& = & \frac{1}{2^{n-1}}\sum_{k=0}^{\frac{n-1}{2}}\binom{n}{k}\cos(n-2k)x.
\end{array}$$

Uniqueness Theorem

Theorem (Uniqueness Theorem)

Let $f(x)$ and $f_1(x)$ be piecewise continuous in the interval $-\pi \leq x \leq \pi$ and have the same Fourier coefficients

$$\begin{aligned} \int_{-\pi}^{\pi} f(x) \cos nx dx &= \int_{-\pi}^{\pi} f_1(x) \cos nx dx, \quad n = 0, 1, 2, \ldots, \\ \int_{-\pi}^{\pi} f(x) \sin nx dx &= \int_{-\pi}^{\pi} f_1(x) \sin nx dx, \quad n = 1, 2, \ldots. \end{aligned}$$

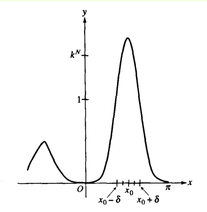
Then $f(x) = f_1(x)$ except perhaps at points of discontinuity.

- Let $h(x) = f(x) - f_1(x)$. Then $h(x)$ is piecewise continuous, and from hypothesis it follows that all Fourier coefficients of $h(x)$ are 0. We then show that $h(x) = 0$ except perhaps at discontinuity points.

  Suppose $h(x_0) \neq 0$ at a point of continuity x_0, for example, $h(x_0) = 2c > 0$. Then, by continuity, $h(x) > c$ for $|x - x_0| < \delta$ and δ sufficiently small. We can assume $-\pi < x_0 < \pi$.

Uniqueness Theorem (Idea)

- We now achieve a contradiction by showing that there exists a "trigonometric polynomial"



$$\begin{aligned} P(x) &= p_0 + p_1 \cos x + p_2 \sin x + \cdots \\ &\quad + p_{2k-1} \cos kx + p_{2k} \sin kx \end{aligned}$$

that represents a "pulse" at x_0 of arbitrarily large amplitude and arbitrarily small width.

If such a pulse can be constructed, then one has a contradiction: On one hand,

$$\int_{-\pi}^{\pi} h(x)P(x)dx = p_0 \int_{-\pi}^{\pi} h(x)dx + p_1 \int_{-\pi}^{\pi} h(x)\cos x dx + \cdots = 0.$$

On the other hand, the major portion of the integral $\int h(x)P(x)dx$ is concentrated in the interval in which the pulse occurs, where $h(x)$ is positive, and $P(x)$ is large and positive. Hence the integral is positive and cannot be 0.

Uniqueness Theorem (Argument)

- Take $P(x) = [\psi(x)]^N$, $\psi(x) = 1 + \cos(x - x_0) - \cos\delta$ for an appropriate positive integer N.

  Since the functions $\sin^n x$ and $\cos^n x$ are expressible as trigonometric polynomials, the function $P(x)$ is a trigonometric polynomial.

  Let $k = \psi(x_0 + \frac{\delta}{2}) = 1 + \cos\frac{\delta}{2} - \cos\delta$.

  Note $\cos\frac{\delta}{2} > \cos\delta$. So, $k > 1$.

  We estimate P:

  - If $x_0 - \frac{1}{2}\delta \leq x \leq x_0 + \frac{1}{2}\delta$, $|x - x_0| \leq \frac{1}{2}\delta$, whence $\cos(x - x_0) \geq \cos\frac{\delta}{2}$, and $\psi(x) \geq k > 1$ giving $P \geq k^N$.
  - If $-\pi \leq x < x_0 - \delta$ or $x_0 + \delta < x \leq \pi$, then $x - x_0 < -\delta$ or $x - x_0 > \delta$, whence $\cos(x - x_0) < \cos\delta$, and $-1 < \psi(x) < 1$, giving $|P| < 1$.

  Since $h(x)$, being piecewise continuous, is bounded by a constant M for $-\pi \leq x \leq \pi$: $|h(x)| \leq M$.

Uniqueness Theorem (Argument Cont'd)

- It follows from the properties of $P(x)$ of the preceding slide that

$$P(x)h(x) > -M, \quad -\pi \le x \le x_0 - \frac{1}{2}\delta \text{ and } x_0 + \frac{1}{2}\delta \le x \le \pi,$$
$$P(x)h(x) \ge ck^N, \quad x_0 - \frac{1}{2}\delta \le x \le x_0 + \frac{1}{2}\delta.$$

Accordingly, we get

$$\begin{aligned}\int_{-\pi}^{\pi} p(x)h(x)dx &= \int_{-\pi}^{x_0-\frac{1}{2}\delta} p(x)h(x)dx + \int_{x_0+\frac{1}{2}\delta}^{\pi} P(x)h(x)dx \\ &+ \int_{x_0-\frac{1}{2}\delta}^{x_0+\frac{1}{2}\delta} P(x)h(x)dx > -M(2\pi-\delta) + ck^N\delta.\end{aligned}$$

Since $k^N \to +\infty$ as $N \to \infty$, the right-hand member of the inequality is surely positive when N is sufficiently large. Accordingly, the left-hand member is positive for appropriate choice of N.
This contradicts the fact that the left-hand member is 0.
Thus, $h(x) = f(x) - f_1(x) = 0$ wherever $f(x)$ and $f_1(x)$ are continuous.

Remarks

- The uniqueness theorem can be looked at as asserting that the system of functions
$$1, \cos x, \sin x, \ldots, \cos nx, \sin nx, \ldots$$
is "large enough", that is, that there are enough functions in this system to construct series for all the periodic functions envisaged.
- It should be noted that omission of any one function of the system would destroy this property.
Thus if $\cos x$ were omitted, one could still form a series
$$\frac{1}{2}a_0 + b_1 \sin x + a_2 \cos 2x + b_2 \sin 2x + \cdots .$$
But there are very smooth periodic functions whose Fourier series in this deficient form could never converge to the function, namely, ail functions $A \cos x$ for $A = \text{const.} \neq 0$. For each such function would have all coefficients 0. So the series reduces to 0 and cannot represent the function.

Convergence of Fourier Series

Theorem

Let the function $f(x)$ be continuous for $-\pi \leq x \leq \pi$ and let the Fourier series of $f(x)$ converge uniformly in this interval. Then the series converges to $f(x)$ for $-\pi \leq x \leq \pi$.

- Let the sum of the Fourier series of $f(x)$ be denoted by $f_1(x)$:

$$f_1(x) = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx).$$

Since the series converges uniformly, it follows from a previous theorem that $f_1(x)$ is continuous and that a_n, b_n are the Fourier coefficients of $f_1(x)$. But the series was given as the Fourier series of $f(x)$. Hence $f(x)$ and $f_1(x)$ have the same Fourier coefficients. By the preceding theorem, $f(x) = f_1(x)$. So $f(x)$ is the sum of its Fourier series for $-\pi \leq x \leq \pi$.

Subsection 7

Fundamental Theorem: A Special Case

Fundamental Theorem: A Special Case

Theorem

Let $f(x)$ be continuous and piecewise very smooth for all x. Let $f(x)$ have period 2π. Then the Fourier series of $f(x)$ converges uniformly to $f(x)$ for all x.

- We only prove the case in which $f(x)$ has continuous first and second derivatives for all x.
  For $n \neq 0$, using integration by parts,
  $$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx dx = \left.\frac{f(x)\sin nx}{n\pi}\right|_{-\pi}^{\pi} - \frac{1}{n\pi}\int_{-\pi}^{\pi} f'(x)\sin nx dx.$$
  The first term on the right is zero.
  A second integration by parts gives
  $$\begin{aligned} a_n &= \frac{f'(x)\cos nx}{n^2\pi}\Big|_{-\pi}^{\pi} - \frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\cos nx dx \\ &\overset{f' \text{ periodic}}{=} -\frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\cos nx dx. \end{aligned}$$

Fundamental Theorem: A Special Case (Cont'd)

- The function $f''(x)$ is continuous in the interval $-\pi \leq x \leq \pi$. Hence $|f''(x)| \leq M$ for an appropriate constant M. One concludes that

$$|a_n| = \left|\frac{1}{n^2\pi}\int_{-\pi}^{\pi} f''(x)\cos nx dx\right| \leq \frac{2M}{n^2}, \quad n = 1, 2, \ldots.$$

  In exactly the same way we prove that $|b_n| \leq \frac{2M}{n^2}$, for all n.

  Hence each term of the Fourier series of $f(x)$ is in absolute value at most equal to the corresponding term of the convergent series $\frac{1}{2}|a_0| + \frac{2M}{1} + \frac{2M}{1} + \frac{2M}{2^2} + \frac{2M}{2^2} + \cdots$.
  Application of the Weierstrass M-test establishes that the Fourier series converges uniformly for all x.

  By the preceding theorem, the sum is $f(x)$.

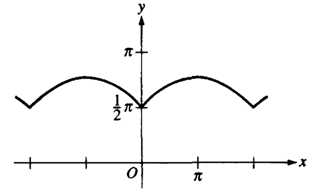
Subsection 8

Proof of Fundamental Theorem

Example

- Consider the function

$$G(x) = \begin{cases} \frac{\pi}{2} - \frac{x}{2} - \frac{x^2}{4\pi}, & \text{if } -\pi \leq x \leq 0 \\ \frac{\pi}{2} + \frac{x}{2} - \frac{x^2}{4\pi}, & \text{if } 0 \leq x \leq \pi \end{cases}$$



  Let G be repeated periodically outside this interval.
- The resulting function $G(x)$ is continuous for all x and is piecewise smooth.
- Its Fourier series is the series

$$\frac{2\pi}{3} - \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{\cos nx}{n^2}.$$

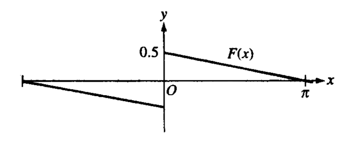
- Hence $|a_n| \leq \frac{M}{n^2}$ as asserted, with $M = \frac{1}{\pi}$. The b_n happens to be 0.
- By the preceding theorem, this series converges uniformly to $G(x)$.

Example (Cont'd)

- Is term-by-term differentiation of the series permissible, in other words, is
$$G'(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n},$$
wherever $G'(x)$ is defined?
- By a theorem on infinite series, this is correct if x lies within an interval within which the differentiated series converges uniformly.
- It turns out that the series $\sum \frac{\sin nx}{n}$ converges uniformly for $a \le |x| \le \pi$, provided that $a > 0$.
- So the formula above for $G'(x)$ is correct for $-\pi \le x \le \pi$, except for $x = 0$.

Example (Conclusion)

- Now let $F(x)$ be the periodic function of period 2π, such that $F(0) = 0$ and



$$F(x) = G'(x) = \begin{cases} -\frac{1}{2} - \frac{x}{2\pi}, & \text{if } -\pi \le x < 0 \\ \frac{1}{2} - \frac{x}{2\pi}, & \text{if } 0 < x \le \pi. \end{cases}$$

- We have stated that $F(x) = \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{\sin nx}{n}$, for all x, the convergence being uniform for $0 < a \le |x| \le \pi$.
- The series on the right was computed as the Fourier series of $F(x)$.
- So $F(x)$ is represented by its Fourier series for all x.
- The remarkable feature of this result is that $F(x)$ has a jump, from $-\frac{1}{2}$ to $\frac{1}{2}$ at $x = 0$.
- The series converges to the average value $F(0) = 0$.

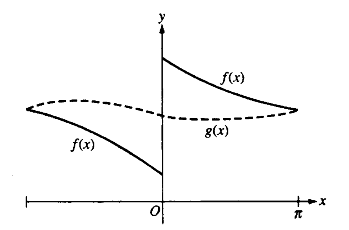
The Fundamental Theorem

Theorem

Let $f(x)$ be defined and piecewise very smooth for $-\pi \leq x \leq \pi$ and let $f(x)$ be defined outside this interval in such a manner that $f(x)$ has period 2π. Then the Fourier series of $f(x)$ converges uniformly to $f(x)$ in each closed interval containing no discontinuity of $f(x)$. At each discontinuity x_0 the series converges to $\frac{1}{2}[\lim_{x \to x_0^+} f(x) + \lim_{x \to x_0^-} f(x)]$.

- For convenience we redefine $f(x)$ at each discontinuity x_0 as the average of left and right limit values.

  Let us suppose, for example, that the only discontinuity is at $x = 0$ (and the points $2k\pi$, $k = \pm 1, \pm 2, \ldots$). Let $\lim_{x \to 0^+} f(x) - \lim_{x \to 0^-} f(x) = s$. So s is precisely the "jump".



The Fundamental Theorem (Cont'd)

- We proceed to eliminate the discontinuity by subtracting from $f(x)$ the function $sF(x)$, where $F(x)$ is the function defined earlier.
  Since $sF(x)$ has also the jump s at $x=0$ (and $x=2k\pi$), $g(x)=f(x)-sF(x)$ has jump 0 at $x=0$ and is continuous for all x: For
  $$\begin{array}{rcl}\lim_{x\to 0^-} g(x) & = & \lim_{x\to 0^-} f(x) - s\lim_{x\to 0^-} F(x) \\ & = & [f(0)-\frac{1}{2}s]+\frac{1}{2}s = f(0) = g(0).\end{array}$$
  A similar statement applies to the right-hand limit.
  Since $F(x)$ is piecewise linear, $g(x)$ is continuous and piecewise very smooth for all x and has period 2π. Hence by the preceding theorem, $g(x)$ is representable by a uniformly convergent Fourier series for all x:
  $$g(x)=\frac{1}{2}A_0+\sum_{n=1}^{\infty}(A_n\cos nx+B_n\sin nx).$$

The Fundamental Theorem (Cont'd)

- Now we have

$$\begin{aligned} f(x) &= g(x) + sF(x) \\ &= \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}(A_n \cos nx + B_n \sin nx) + \tfrac{s}{\pi}\sum_{n=1}^{\infty}\tfrac{\sin nx}{n} \\ &= \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty}[A_n \cos nx + (B_n + \tfrac{s}{n\pi})\sin nx]. \end{aligned}$$

So $f(x)$ is represented by a trigonometric series for all x.

The series is the Fourier series of $f(x)$:

$$\begin{aligned} b_n &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx dx = \tfrac{1}{\pi}\int_{-\pi}^{\pi}[g(x) + sF(x)]\sin nx dx \\ &= \tfrac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sin nx dx + \tfrac{s}{\pi}\int_{-\pi}^{\pi} F(x)\sin nx dx = B_n + \tfrac{s}{n\pi}. \end{aligned}$$

Similarly, $a_n = A_n$.

Therefore the Fourier series of $f(x)$ converges to $f(x)$ for all x.

At $x = 0$ the series converges to $f(0)$, which was defined to be the average of left and right limits at $x = 0$.

The Fundamental Theorem (Conclusion)

- Since the series for $g(x)$ is uniformly convergent for all x, while the series for $F(x)$ converges uniformly in each closed interval not containing $x = 0$ (or $x = 2k\pi$), the Fourier series of $f(x)$ converges uniformly in each such closed interval.

  The theorem has now been proved for the case of just one jump discontinuity. If there are several jumps, at points $x_1, x_2, \ldots$, we simply remove them by subtracting from $f(x)$ the function

$$s_1 F(x - x_1) + s_2 F(x - x_2) + \cdots .$$

  The resulting function $g(x)$ is again continuous and piecewise very smooth, so that the same conclusion holds.

Remark: The Principle of Superposition

- The proof just given uses the **Principle of Superposition**:

  The Fourier series of a linear combination of two functions is the same linear combination of the corresponding two series.
- The idea of subtracting off the series corresponding to a jump discontinuity also has a practical significance:
  - If a function $f(x)$ is defined by its Fourier series and is not otherwise explicitly known, one can, of course, use the series to tabulate the function.
  - If $f(x)$ has a jump discontinuity, the convergence will be poor near the discontinuity; this will reveal itself in the presence of terms having coefficients approaching 0 like $\frac{1}{n}$.
  - If the discontinuity x_1 and jump s_1 are known, one can subtract the corresponding function $s_1 F(x - x_1)$ as before; the new series will converge much more rapidly.

Subsection 9

Orthogonal Functions

Replacing Sine and Cosine by More General Functions

- Let $f(x)$ be given in a fixed interval $a \leq x \leq b$.
- Let $\phi_1(x), \phi_2(x), \ldots, \phi_n(x), \ldots$ be functions all piecewise continuous in this interval intended to replace the system of sines and cosines.
- We then postulate a development, $f(x) = \sum_{n=1}^{\infty} c_n \phi_n(x)$ just as in the case of Fourier series.
- We multiply both sides by $\phi_m(x)$ and integrate term by term:

$$\int_a^b f(x)\phi_m(x)dx = \sum_{n=1}^{\infty} c_n \int_a^b \phi_m(x)\phi_n(x)dx.$$

- In order to achieve a result analogous to that for Fourier series, we must postulate that

$$\int_a^b \phi_m(x)\phi_n(x)dx = 0, \quad m \neq n.$$

Replacing Sine and Cosine (cont'd)

- The series on the right then reduces to one term:

$$\int_a^b f(x)\phi_m(x)dx = c_m \int_a^b [\phi_m(x)]^2 dx.$$

- The integral on the right is a certain constant: $\int_a^b [\phi_m(x)]^2 dx = B_m$.
- B_m will be positive unless $\phi_m(x) \equiv 0$ (except at a finite number of points); to avoid this case, we assume that no B_m is 0. Then

$$c_m = \frac{1}{B_m} \int_a^b f(x)\phi_m(x)dx.$$

- Thus, under the simple conditions

$$\begin{aligned} \int_a^b \phi_m(x)\phi_n(x)dx &= 0, \quad m \neq n, \\ \int_a^b [\phi_m(x)]^2 dx &= B_m \neq 0, \text{ for all } m, \end{aligned}$$

we have a rule for the formation of a series.

Orthogonal System of Functions

Definition

- Two functions $p(x)$, $q(x)$, which are piecewise continuous for $a \leq x \leq b$, are **orthogonal** in this interval if

$$\int_a^b p(x)q(x)dx = 0.$$

- A system of functions $\{\phi_n(x) : n = 1, 2, \ldots\}$ is termed an **orthogonal system** in the interval $a \leq x \leq b$ if ϕ_n and ϕ_m are orthogonal for each pair of distinct indices m, n:

$$\int_a^b \phi_m(x)\phi_n(x)dx = 0, \quad m \neq n,$$

and no $\phi_n(x)$ is identically 0 except at a finite number of points.

Example

- The trigonometric system in the interval $-\pi \leq x \leq \pi$:

$$1,\ \cos x,\ \sin x,\ \ldots, \cos nx,\ \sin nx, \ldots.$$

  - ϕ_1 is the constant 1;
  - ϕ_2 is the function $\cos x$;
  - ϕ_3 is the function $\sin x$;

  $\vdots$

Fourier Series

- If $f(x)$ is piecewise continuous in the interval $a \leq x \leq b$ and $\{\phi_n(x)\}$ is an orthogonal system in this interval, then the series

$$\sum_{n=1}^{\infty} c_n \phi_n(x),$$

where

$$c_n = \frac{1}{B_n} \int_a^b f(x)\phi_n(x)dx, \quad B_n = \int_a^b [\phi_n(x)]^2 dx,$$

is called the **Fourier series of f with respect to the system $\{\phi_n(x)\}$**.

- The numbers $c_1, c_2, \ldots$ are called the **Fourier coefficients of $f(x)$ with respect to the system $\{\phi_n(x)\}$**.

Orthonormal Systems of Functions

- The preceding formulas can be simplified if one assumes that the constant B_n is always 1, i.e., that
$$\int_a^b [\phi_n(x)]^2 dx = 1, \quad n = 1, 2, \ldots.$$
- This can always be achieved by dividing the original $\phi_n(x)$ by appropriate constants.
- When the condition $B_n = 1$ is satisfied for all n, the system of functions $\phi_n(x)$ is called **normalized**.
- A system that is both normalized and orthogonal is called **orthonormal**.

  Example: The following system of functions is orthonormal $\frac{1}{\sqrt{2\pi}}, \frac{\cos x}{\sqrt{\pi}}, \frac{\sin x}{\sqrt{\pi}}, \ldots, \frac{\cos nx}{\sqrt{\pi}}, \frac{\sin nx}{\sqrt{\pi}}, \ldots.$
- The general theory is simpler for normalized systems, but the advantages for the applications are slight.

The Vector Space Formalism

- We can consider the piecewise continuous functions for $a \leq x \leq b$ as a kind of vector space: For $f(x), g(x)$ piecewise continuous,
  - the sum or difference is again piecewise continuous;
  - the product cf by a scalar c is also piecewise continuous.
- We define an **inner product** (or **scalar product**):

$$(f, g) = \int_a^b f(x)g(x)dx.$$

- One can then define a **norm** (or **absolute value**):

$$\|f\| = \sqrt{(f, f)} = \left\{\int_a^b [f(x)]^2 dx\right\}^{\frac{1}{2}}.$$

- The **zero function** 0^* is a function that is 0 except at a finite number of points.
- In general, in this vector theory of functions we consider two functions that differ only at a finite number of points to be the same function.

Orthogonal and Orthonormal Systems in a Vector Space

- In the space V of piece-wise continuous functions on $[a, b]$, two functions $f(x)$ and $g(x)$ are **orthogonal** if
$$(f, g) = 0.$$
- An **orthogonal system** is a system $\{\phi_n(x)\}_{n=1}^{\infty}$ of functions in V, such that
  - $(\phi_m, \phi_n) = 0$, if $m \neq n$;
  - $(\phi_n, \phi_n) = \|\phi_n\|^2 = B_n > 0$, for all $n \geq 1$.
- The system is **orthonornal** if $B_n = 1$, for all $n \geq 1$, i.e., if the ϕ_n are **unit vectors** in V.

The Fourier Formalism in a Vector Space

- The **Fourier serier** of a function $f(x)$ in V **with respect to the orthogonal system** $\{\phi_n(x)\}$ is the series
$$\sum_{n=1}^{\infty} c_n\phi_n(x), \quad c_n = \frac{1}{\|\phi_n\|^2}(f,\phi_n).$$
- If the system $\{\phi_n(x)\}$ is orthonormal, then we get
$$\sum_{n=1}^{\infty} c_n\phi_n(x), \quad c_n = (f,\phi_n).$$
- Compare this with the expressions of vectors in terms of the canonical orthonormal basis $\{\boldsymbol{e}_1,\ldots,\boldsymbol{e}_n\}$ of an n-dimensional vector space:
$$\boldsymbol{v} = v_1\boldsymbol{e}_1 + \cdots + v_n\boldsymbol{e}_n, \quad v_i = \boldsymbol{v}\cdot\boldsymbol{e}_i.$$
- Since the series above is infinite, convergence questions arise.
- Theorems similar to the one proven for the trigonometric Fourier series are proven to address these issues.