Introduction to Algebraic Geometry

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Subsection 1

Projective Space

Projective Space

- Let V be a finite dimensional vector space over k.
- We consider the following equivalence relation on $V\setminus\{0\}$:

$$u \sim v \quad \text{iff} \quad \text{there exists } \lambda \in k^*, \text{ with } u = \lambda v.$$

Definition (Projective Space)

The **projective space** associated to V is defined by

$$\mathbb{P}(V) := (V\setminus\{0\})/\sim.$$

The **dimension** of $\mathbb{P}(V)$ is defined by

$$\dim \mathbb{P}(V) := \dim V - 1.$$

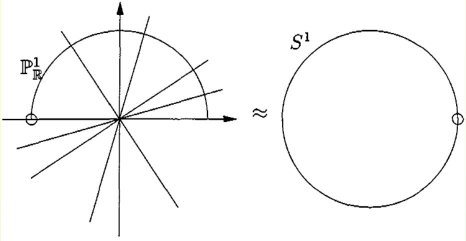
Remarks

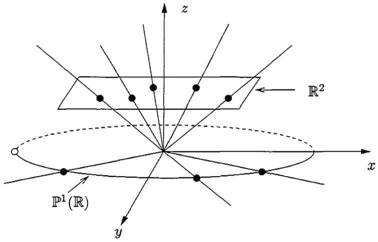
- Two vectors are equivalent if and only if they span the same line in V.
- So geometrically, the projective space associated to V is the set of all lines through the origin in V.
- In particular, taking $V = k^{n+1}$, we define

$$\mathbb{P}^n := \mathbb{P}^n_k := \mathbb{P}(k^{n+1}).$$

Examples

- The space $\mathbb{P}^1_{\mathbb{R}} = \mathbb{P}(\mathbb{R}^2)$ is homeomorphic to S^1.





- The real projective plane has a decomposition

$$\mathbb{P}^2_{\mathbb{R}} = \mathbb{P}(\mathbb{R}^3) = \mathbb{R}^2 \cup \mathbb{P}^1(\mathbb{R}).$$

Under this decomposition:
  - $\mathbb{R}^2$ corresponds to the set of lines that do not lie in the (x, y)-plane;
  - $\mathbb{P}^1(\mathbb{R})$ corresponds to the set of lines in the (x, y)-plane.

The Residue Class and the Homogeneous Coordinates

- We will denote the residue class map

$$\pi : V \setminus \{0\} \to \mathbb{P}(V).$$

- For the special case $\mathbb{P}(V) = \mathbb{P}_k^n$, we use the notation

$$(x_0 : \ldots : x_n) := \pi((x_0, \ldots, x_n)).$$

- We call $(x_0 : \ldots : x_n)$ the **homogeneous coordinates** of the point $P = \pi((x_0, \ldots, x_n)) \in \mathbb{P}_k^n$.
- The homogeneous coordinates are well defined only up to multiplication by a common scalar.
- However, we can "compute" using them.

Decomposition into an Affine and a Projective Space

- Any projective space can be decomposed into an affine subspace and a projective subspace of smaller dimension.
- For $\mathbb{P}_k^n$, such a decomposition is given by setting

$$U_\ell := \{(x_0 : \ldots : x_n) \in \mathbb{P}_k^n : x_\ell \neq 0\},$$
$$H_\ell := \{(x_0 : \ldots : x_n) \in \mathbb{P}_k^n : x_\ell = 0\}.$$

- The space H_ℓ can be identified with $\mathbb{P}_k^{n-1}$.
- U_ℓ can be identified with $\mathbb{A}_k^n$.

Decomposition into Affine and Projective Space (Cont'd)

- The latter identification can be realized, e.g., by the mutually inverse maps

$$\begin{array}{rrcl} i_\ell : & \mathbb{A}_k^n & \to & U_\ell, \\ & (x_1,\ldots,x_n) & \mapsto & (x_1:\ldots:x_{\ell-1}:1:x_\ell:\ldots:x_n) \end{array}$$

$$\begin{array}{rrcl} j_\ell : & U_\ell & \to & \mathbb{A}_k^n; \\ & (x_0:\ldots:x_{\ell-1}:x_\ell:\ldots:x_n) & \mapsto & (\frac{x_0}{x_\ell},\ldots,\frac{x_{\ell-1}}{x_\ell},\frac{x_{\ell+1}}{x_\ell},\ldots,\frac{x_n}{x_\ell}) \end{array}$$

- So we get a decomposition

$$\mathbb{P}_k^n = U_\ell \cup H_\ell = \mathbb{A}_k^n \cup \mathbb{P}_k^{n-1}.$$

Terminology of Decomposition

- Consider again
$$\mathbb{P}_k^n = U_\ell \cup H_\ell = \mathbb{A}_k^n \cup \mathbb{P}_k^{n-1}.$$
- We fix the value of ℓ (usually $\ell = 0$ or n).
- We refer to:
  - U_ℓ as the **affine part** of $\mathbb{P}_k^n$;
  - H_ℓ as the **hyperplane at infinity**.
- Points in H_ℓ are called "points at infinity".
- This particular decomposition into an affine and a projective piece is conventional.
- More generally, any projective hyperplane can be taken in $\mathbb{P}_k^n$, and the complement will always be an affine space.

Projective Subspaces

Definition (Projective Subspace)

A **projective subspace** of $\mathbb{P}(V)$ is a subset of the form $\pi(W \setminus \{0\})$, where $W \subseteq V$ is a linear subspace and π is the residue class map. We write $\mathbb{P}(W) \subseteq \mathbb{P}(V)$.

- A projective subspace is itself naturally a projective space.
- If $\dim W = \dim V - 1$, then we call $\mathbb{P}(W)$ a **hyperplane** in $\mathbb{P}(V)$.
- A **projective line** is a projective space of dimension 1.
- A **projective plane** is a projective space of dimension 2.

The Intersection Lemma

Lemma

Let $\mathbb{P}(W_1)$ and $\mathbb{P}(W_2)$ be projective subspaces of an n-dimensional projective space $\mathbb{P}(V)$. If $\dim\mathbb{P}(W_1)+\dim\mathbb{P}(W_2)\geq n$, then $\mathbb{P}(W_1)$ and $\mathbb{P}(W_2)$ intersect, i.e., $\mathbb{P}(W_1)\cap\mathbb{P}(W_2)\neq\emptyset$.

- We have $\dim W_1+\dim W_2\geq n+2=\dim V+1$.

  So W_1 and W_2 intersect at least in a line.
- In projective space the distinction between the cases of parallel and nonparallel lines no longer exists.
  - Two lines in the projective plane always intersect.
  - In contrast, in the affine plane, two lines may be parallel.

Covering by Affine Spaces

- Any projective space has a covering by affine spaces

$$\mathbb{P}_k^n = U_0 \cup U_1 \cup \cdots \cup U_n,$$

  where

$$U_i := \{(x_0 : \ldots : x_n) \in \mathbb{P}_k^n : x_i \neq 0\}.$$

- In the case of $k = \mathbb{R}$ or $k = \mathbb{C}$ this covering can be used to give $\mathbb{P}_{\mathbb{R}}^n$ or $\mathbb{P}_{\mathbb{C}}^n$ the structure of a **compact** n**-dimensional real** or **complex manifold**, respectively.

  Example: The complex projective line has the structure of a compact Riemann surface, namely the Riemann sphere,

$$\mathbb{P}_{\mathbb{C}}^1 = \mathbb{C} \cup \{\infty\} \approx S^2.$$

Subsection 2

Projective Varieties

Homogeneous Polynomials

- The homogeneous coordinates of a point $P = (x_0 : \ldots : x_n) \in \mathbb{P}_k^n$ are only determined up to multiplication by a common scalar.
- So to consider the zero sets of polynomial equations defined on $\mathbb{P}_k^n$, we must make a restriction to *homogeneous polynomials*.
- A polynomial
$$f(x_0, \ldots, x_n) = \sum a_{\nu_0 \cdots \nu_n} x_0^{\nu_0} \cdots x_n^{\nu_n}$$
is called **homogeneous of degree** d if all the monomials have the same degree $d = \nu_0 + \cdots + \nu_n$.
- We also use the word **form** to refer to homogeneous polynomials.
- So we refer, e.g., to **linear forms**, **quadratic forms**, **cubic forms**, etc.

Projective Varieties

- If f is homogeneous of degree d, then we have

$$f(\lambda x_0,\ldots,\lambda x_n)=\lambda^d f(x_0,\ldots,x_n).$$

- This shows the zero set of f,

$$V(f):=\{(x_0:\ldots:x_n)\in\mathbb{P}_k^n : f(x_0,\ldots,x_n)=0\}\subseteq\mathbb{P}_k^n$$

  is well defined.

Definition (Projective Variety)

A **projective variety** is a subset $V\subseteq\mathbb{P}_k^n$, such that, there exists a set of homogeneous polynomials $T\subseteq k[x_0,\ldots,x_n]$, with

$$V=\left\{P\in\mathbb{P}_k^n : f(P)=0,\ \text{for all } f\in T\right\}.$$

- As in the affine case, we may assume that T has only finitely many elements.

Examples

- We have already seen the projective subvariety of $\mathbb{P}_k^n$ given by the hyperplane at infinity,

$$H_n = \{(x_0 : x_1 : \ldots : x_n) \in \mathbb{P}_k^n : x_n = 0\}.$$

- We discussed the following curves:

$$C_1 = \{(x : y : z) \in \mathbb{P}_{\mathbb{C}}^2 : y^2 z = 4x^3 - g_2 x z^2 - g_3 z^3\}$$

and

$$C_2 = \{(x : y : z) \in \mathbb{P}_{\mathbb{C}}^2 : y^2 z = x(x - z)(x - \lambda z)\}.$$

- We described these curves by giving affine equations in two variables.
- The process of obtaining the projective equations given here is called **homogenization**.

The Projective Rational Normal Curve of Degree 3

- Consider the map

$$\begin{array}{rcl}\varphi: & \mathbb{P}^1_k & \to & \mathbb{P}^3_k; \\ \varphi(t_0:t_1) & = & (t_0^3:t_0^2t_1:t_0t_1^2:t_1^3).\end{array}$$

The image $C := \varphi(\mathbb{P}^1_k)$ is a projective variety, given by

$$C = \left\{(x_0:x_1:x_2:x_3)\in\mathbb{P}^3_k : \operatorname{rank}\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3\end{pmatrix} \le 1\right\}.$$

This means that C is the intersection of three quadrics $C = Q_1 \cap Q_2 \cap Q_3$, where

$$\begin{array}{rcl} Q_1 & := & \{(x_0:x_1:x_2:x_3)\in\mathbb{P}^3_k : x_0x_2 - x_1^2 = 0\}, \\ Q_2 & := & \{(x_0:x_1:x_2:x_3)\in\mathbb{P}^3_k : x_2x_3 - x_1x_2 = 0\}, \\ Q_3 & := & \{(x_0:x_1:x_2:x_3)\in\mathbb{P}^3_k : x_1x_3 - x_2^2 = 0\}. \end{array}$$

The Projective Rational Normal Curve of Degree 3 (Cont'd)

- The curve
$$C := \varphi(\mathbb{P}^1_k)$$
cannot be defined by only two quadratic equations.
On the other hand, we have
$$C = Q_1 \cap F,$$
where
$$F := \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}^3_k : x_0 x_3^2 - 2x_1 x_2 x_3 + x_2^3 = 0\}.$$
That is, the quadric Q_1 and the cubic F meet along the curve C.
C is called the (**projective**) **rational normal curve** of degree 3.

Example

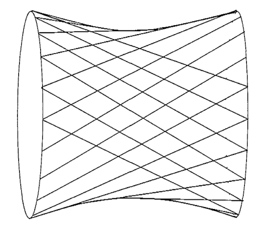
- The image of the map

$$\begin{array}{rccl} \varphi: & \mathbb{P}_k^1 \times \mathbb{P}_k^1 & \to & \mathbb{P}_k^3, \\ & \varphi((x_0 : x_1),(y_0 : y_1)) & = & (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1), \end{array}$$

  is given by the quadric

$$Q := \{(z_0 : z_1 : z_2 : z_3) \in \mathbb{P}_k^3 : z_0 z_3 - z_1 z_2 = 0\}.$$

  There are two families of lines on Q (in each case P runs through the points of $\mathbb{P}_k^1$):

  - The family of lines $\varphi(\mathbb{P}_k^1 \times \{P\})$;
  - The family of lines $\varphi(\{P\} \times \mathbb{P}_k^1)$.

  Each of these families of lines is called a **ruling** of Q.
  Any two lines in the same ruling are disjoint.
  Any two lines in different rulings intersect.

Graded Rings

Definition (Graded Ring)

A **graded ring** is a ring S together with a decomposition into abelian groups

$$S = \bigoplus_{d \geq 0} S_d,$$

such that:

- For $d \neq e$, we have $S_d \cap S_e = \{0\}$;
- Multiplication satisfies $S_d \cdot S_e \subseteq S_{d+e}$.

The elements of S_d are called the **homogeneous elements of degree** d.

- An important example is the polynomial ring

  $$S = k[x_0, \ldots, x_n] = \bigoplus_{d \geq 0} k^d[x_0, \ldots, x_n],$$

  where

  $$k^d[x_0, \ldots, x_n] := \{f \in k[x_0, \ldots, x_n] : f \text{ is homogeneous of degree } d\} \cup \{0\}.$$

Homogeneous Ideals

Definition (Homogeneous Ideals)

A **homogeneous ideal** I in a graded ring S is an ideal which satisfies

$$I = \bigoplus_{d \geq 0} (I \cap S_d).$$

- An ideal I is homogeneous if and only if every element $f \in I$ has a unique decomposition

$$f = f_0 + \cdots + f_N,$$

where $f_i \in I$ is a homogeneous element of degree d_i.

Properties of Homogeneous Ideals

Lemma

For an ideal I in a graded ring S we have:

(1) The ideal I is homogeneous if and only if it can be generated by homogeneous elements.

(2) If I is homogeneous, then I is prime if and only if for any pair of homogeneous elements $f, g \in S$, we have: $fg \in I$ iff $f \in I$ or $g \in I$.

(3) The sum, product, intersection and radical of homogeneous ideals are also homogeneous ideals.

Generators of $I(T)$

- A projective variety was defined as the set of zeros of a system of homogeneous polynomials.
- Equivalently, a projective variety is the set of zeros of a homogeneous ideal, or of finitely many homogeneous polynomials.
- Let T be a set of homogeneous polynomials.
- Let $I(T)$ be the homogeneous ideal generated by T.
- Then we have

$$V(T) = V(I(T)) = V(f_1, \ldots, f_k),$$

for homogeneous generators $f_1, \ldots, f_k$ of $I(T)$.

The Zariski Topology on $\mathbb{P}_k^n$

- As for affine space, the zero sets define a topology on $\mathbb{P}_k^n$.

Lemma

Projective varieties satisfy the axioms of the closed sets of a topology on $\mathbb{P}_k^n$. In other words, we have the following:

(1) The union of finitely many projective varieties is a projective variety.

(2) The intersection of any number of projective varieties is a projective variety.

(3) The empty set and $\mathbb{P}_k^n$ are projective varieties.

- This topology is called the **Zariski topology** on $\mathbb{P}_k^n$.
- As in the affine case, we can decompose projective varieties into irreducible components.

Quasi-Projective Varieties

Definition (Quasi-Projective Variety)

A **quasi-projective variety** is an open subset of a projective variety.

Homogeneous Ideals and Projective Varieties

- We describe the relationship between projective varieties and homogeneous ideals.
- In one direction we have

$$\begin{array}{ccc}\left\{\begin{array}{c}\text{homogeneous ideals}\\ I\subseteq k[x_0,\ldots,x_n]\end{array}\right\} & \to & \left\{\begin{array}{c}\text{projective varieties}\\ V\subseteq \mathbb{P}_k^n\end{array}\right\}\\ I & \mapsto & V(I),\end{array}$$

where

$$V(I) = \left\{(x_0:\ldots:x_n)\in \mathbb{P}_k^n : \begin{array}{l} f(x_0,\ldots,x_n)=0 \\ \text{for } f\in I, f \text{ homogeneous}\end{array}\right\}.$$

Projective Varieties and Homogeneous Ideals

- In the opposite direction

$$\left\{ \begin{array}{c} \text{projective varieties} \\ V \subseteq \mathbb{P}_k^n \end{array} \right\} \rightarrow \left\{ \begin{array}{c} \text{homogeneous ideals} \\ I \subseteq k[x_0, \ldots, x_n] \end{array} \right\}$$

$$V \mapsto I(V),$$

where

$$I(V) = \left\{ \begin{array}{l} \text{ideal generated by} \\ \text{homogeneous polynomials } f, \text{ with } f|_V = 0 \end{array} \right\}.$$

The Irrelevant Ideal

- In the affine case, the corresponding maps I and V are mutually inverse if we make a restriction to radical ideals.
- As in the affine case, we have $V((1)) = \emptyset$.
- On the other hand, there is another homogeneous ideal, namely

$$m = (x_0, \ldots, x_n) = \bigoplus_{d \geq 1} k^d[x_0, \ldots, x_n],$$

for which we also have $V(m) = \emptyset$.

Definition (Irrelevant Ideal)

The ideal m is called the **irrelevant ideal**.

The Affine Cone

- If I is a homogeneous ideal, we can consider
  - The projective zero set $V = V(I) \subseteq \mathbb{P}_k^n$;
  - The affine zero set $V^a = V(I) \subseteq \mathbb{A}_k^{n+1}$.
- Geometrically, if $I \neq k[x_1, \ldots, x_n]$, we have

$$V^a = \pi^{-1}(V) \cup \{0\},$$

where, as before, π is the residue class map $\pi : \mathbb{A}_k^{n+1} \backslash \{0\} \to \mathbb{P}_k^n$.
- In particular,

$$(x_0, \ldots, x_n) \in V^a \quad \text{iff} \quad (\lambda x_0, \ldots, \lambda x_n) \in V^a, \text{ for } \lambda \in k^*.$$

Definition (Affine Cone)

The set V^a is called the **affine cone** over the projective variety $V(I) \subseteq \mathbb{P}_k^n$.

Projective Nullstellensatz

Theorem (Projective Nullstellensatz)

Let k be an algebraically closed field. Then for a homogeneous ideal J, we have the following:

(1) $V(J) = \varnothing$ iff $\sqrt{J} \supseteq (x_0, \ldots, x_n)$.

(2) If $V(J) \neq \varnothing$, then $I(V(J)) = \sqrt{J}$.

(1) We have

$$\begin{aligned} V(J) = \varnothing \quad & \text{iff} \quad V^a(J) \subseteq \{0\} \\ & \text{iff} \quad \sqrt{J} \supseteq (x_0, \ldots, x_n). \\ & \qquad \text{(affine Nullstellensatz)} \end{aligned}$$

Projective Nullstellensatz (Cont'd)

(2) We make use of the following observation.

Suppose $f = \sum f_i$ is a polynomial with homogeneous components f_i.

Then, using the fact that the field k has infinitely many elements,

$$f(\lambda x_0, \ldots, \lambda x_n) = 0, \text{ for all } \lambda, \quad \text{iff} \quad f_i(x_0, \ldots, x_n) = 0, \text{ for all } i.$$

Now, we get

$$\begin{aligned} f \in I(V(J)) \quad & \text{iff} \quad f \in I(V^a(J)) \\ & \text{iff} \quad \exists\, n \geq 1 (f^n \in J) \\ & \qquad\; \text{(affine Nullstellensatz)} \\ & \text{iff} \quad f \in \sqrt{J}. \end{aligned}$$

Projective Varieties and Homogeneous Radical Ideals

Corollary

The maps $J \mapsto V(J)$ and $V \mapsto I(V)$ give bijections

$$\left\{ \begin{array}{c} \text{homogeneous radical ideals} \\ J \subsetneqq k[x_0,\ldots,x_n] \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{projective varieties} \\ V \subseteq \mathbb{P}_k^n \end{array} \right\}$$

$$\left\{ \begin{array}{c} \text{homogeneous prime ideals} \\ J \subsetneqq k[x_0,\ldots,x_n] \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{irreducible projective} \\ \text{varieties } V \subseteq \mathbb{P}_k^n \end{array} \right\}$$

Here the irrelevant ideal m corresponds to the empty set.

- It remains only to show that

  $$V \text{ is irreducible if and only if } I(V) \text{ is prime.}$$

  We stated that a homogeneous ideal I is prime if and only if, for any pair of homogeneous elements $f,g \in S$, $fg \in I$ iff $f \in I$ or $g \in I$.
  So the proof becomes analogous to the proof in the affine case.

Covering of $\mathbb{P}_k^n$ by Affine Sets

- We now return to the covering of $\mathbb{P}_k^n$ by affine sets

$$\mathbb{P}_k^n = U_0 \cup \cdots \cup U_n,$$

where

$$U_i := \{(x_0 : \ldots : x_n) \in \mathbb{P}_k^n : x_i \neq 0\}.$$

- For every open set U_i, we have a bijection

$$\begin{array}{rrcl} j_i : & U_i & \to & \mathbb{A}_k^n, \\ & (x_0 : \ldots : x_i : \ldots : x_n) & \mapsto & (\frac{x_0}{x_i}, \ldots, \frac{x_{i-1}}{x_i}, \frac{x_{i+1}}{x_i}, \ldots, \frac{x_n}{x_i}). \end{array}$$

- The Zariski topology on $\mathbb{P}_k^n$ induces a topology on U_i.
- $\mathbb{A}_k^n$ is also equipped with the Zariski topology.

Homeomorphism $j_i : U_i \to \mathbb{A}_k^n$

Proposition

The map $j_i : U_i \to \mathbb{A}_k^n$ is a homeomorphism.

- For simplicity we take $i = 0$.

  We must show that j_0 and j_0^{-1} are both continuous.

  Equivalently, j_0^{-1} and j_0 both map closed sets to closed sets.

  The closed subsets of U_i and $\mathbb{A}_k^n$ are defined by polynomials in the rings:

$$\begin{aligned} S^h &:= \{f \in k[x_0,\dots,x_n] : f \text{ is homogeneous}\}, \\ A &:= k[x_1,\dots,x_n]. \end{aligned}$$

Homeomorphism $j_i : U_i \to \mathbb{A}_k^n$ (Cont'd)

- We have a map $\alpha : S^h \to A$, defined by

$$\alpha(f) = f(1, x_1, \ldots, x_n).$$

We also have a map $\beta : A \to S^h$, defined by

$$\beta(g) = x_0^{\deg g} g\left(\frac{x_1}{x_0}, \ldots, \frac{x_n}{x_0}\right).$$

These maps satisfy

$$\alpha \circ \beta(g) = g.$$

Any closed subset of U_0 has the form

$$X = \overline{X} \cap U_0,$$

where $\overline{X} \subseteq \mathbb{P}_k^n$ is the closure of X in $\mathbb{P}_k^n$.

Homeomorphism $j_i : U_i \to \mathbb{A}^n_k$ (Cont'd)

- Now $\overline{X}$ is a projective variety.
  So there is a (finite) subset $T \subseteq S^h$, such that:
  - $\overline{X} = V(T)$;
  - $j_0(X) = V(\alpha(T))$.

  Any closed subset of $\mathbb{A}^n_k$ has the form
  $$W = V(T'),$$
  for a (finite) subset $T' \subseteq A$.
  Moreover,
  $$j_0^{-1}(W) = V(\beta(T')) \cap U_0.$$
  Thus j_0 and j_0^{-1} map closed subsets to closed subsets as follows:
  $$\begin{array}{rcl} V(T) \cap U_0 & \xrightarrow{j_0} & V(\alpha(T)), \\ V(\beta(T')) \cap U_0 & \xleftarrow{j_0^{-1}} & V(T'). \end{array}$$
  Hence, j_0 is a homeomorphism.

Standard Affine Covering

- Any projective variety $X \subseteq \mathbb{P}_k^n$ has a covering
$$X = X_0 \cup \cdots \cup X_n,$$
where $X_i := X \cap U_i$.
- By means of j_i we can identify U_i with $\mathbb{A}_k^n$.
- So the X_i may be regarded as affine varieties.
- This covering is called the **standard affine covering** of X.

Irreducible Projective and Affine Varieties

Corollary

The map $X \mapsto X_0 = X \cap U_0$ defines a bijection

$$\left\{ \begin{array}{c} \text{irreducible projective varieties} \\ X \subseteq \mathbb{P}^n_k, \text{ with } X \not\subseteq \{x_0 = 0\} \end{array} \right\} \overset{1:1}{\longleftrightarrow} \left\{ \begin{array}{c} \text{irreducible affine} \\ \text{varieties } X_0 \subseteq \mathbb{A}^n_k \end{array} \right\}$$

The inverse of this map is given by taking the Zariski closure.

- The above corollary implies that we can consider both affine and quasi-affine varieties as quasi-projective varieties.

The Cubic C_0 Revisited

- We return to the cubic curve C_0, given by

$$\{(x_1,x_2)\in\mathbb{A}_k^2 : x_2^2 - x_1(x_1-1)(x_1-\lambda)=0\}\subseteq\mathbb{A}_k^2\cong U_0\subseteq\mathbb{P}_k^2.$$

Set

$$f(x_1,x_2):=x_2^2-x_1(x_1-1)(x_1-\lambda).$$

Note that $\deg(f)=3$.

So, with β the map defined in the proposition, we have

$$\begin{aligned}\beta(f) &= x_0^3 f\left(\frac{x_1}{x_0},\frac{x_2}{x_0}\right)\\ &= x_0^3\left[\left(\frac{x_2}{x_0}\right)^2-\frac{x_1}{x_0}\left(\frac{x_1}{x_0}-1\right)\left(\frac{x_1}{x_0}-\lambda\right)\right]\\ &= x_0^3\left[\left(\frac{x_2}{x_0}\right)^2-\frac{x_1}{x_0}\left(\frac{x_1-x_0}{x_0}\right)\left(\frac{x_1-\lambda x_0}{x_0}\right)\right]\\ &= x_0x_2^2-x_1(x_1-x_0)(x_1-\lambda x_0).\end{aligned}$$

The Cubic C_0 Revisited (Cont'd)

- Thus, the Zariski closure of C_0 in $\mathbb{P}_k^2$ is given by
$$\overline{C}_0 = \{(x_0 : x_1 : x_2) \in \mathbb{P}_k^2 : x_0 x_2^2 - x_1(x_1 - x_0)(x_1 - \lambda x_0) = 0\}.$$
Let
$$H_0 := V(x_0) = \mathbb{P}_k^n \setminus U_0.$$
We have
$$\overline{C}_0 \cap H_0 = \{(0:0:1)\}.$$
This shows that $\overline{C}_0$ is obtained from C_0 by adding a single point at infinity.
- In general, for a plane curve of degree d we must add d points at infinity.

Subsection 3

Rational Functions and Morphisms

Rational Functions

- Let f and g be both homogeneous polynomials of degree d on $\mathbb{P}_k^n$.
- Then
$$\begin{aligned}\frac{f(\lambda x_0,\ldots,\lambda x_n)}{g(\lambda x_0,\ldots,\lambda x_n)} &= \frac{\lambda^d f(x_0,\ldots,x_n)}{\lambda^d g(x_0,\ldots,x_n)}\\ &= \frac{f(x_0,\ldots,x_n)}{g(x_0,\ldots,x_n)}.\end{aligned}$$
- So the quotient is a well defined function on V.

Function Fields

- For an irreducible projective variety V, we define

$$k(V) := \left\{ \frac{f}{g} : \begin{array}{r} f, g \in k[x_0, \ldots, x_n] \text{ homogeneous} \\ \deg f = \deg g,\ g \notin I(V) \end{array} \right\} / \sim,$$

where

$$\frac{f}{g} \sim \frac{f'}{g'} \quad \text{iff} \quad fg' - gf' \in I(V).$$

- It can be checked that, with the natural operations, $k(V)$ is a field.

Definition (Function Field of V)

$k(V)$ is called the **function field** of V. The elements of $k(V)$ are called **rational functions**.

Standard Affine Covering and Function Fields

Lemma

Let V be an irreducible projective variety, with standard affine covering

$$V = V_0 \cup \cdots \cup V_n.$$

If $V \not\subseteq V(x_0)$, then we have an isomorphism of projective and affine function fields,

$$k(V) \cong k(V_0).$$

- The following maps are mutually inverse.

$$\begin{array}{ccc} k(V) & \to & k(V_0); \\ \frac{f(x_0,\ldots,x_n)}{g(x_0,\ldots,x_n)} & \mapsto & \frac{f(1,x_1,\ldots,x_n)}{g(1,x_1,\ldots,x_n)}, \end{array} \qquad \begin{array}{ccc} k(V_0) & \to & k(V); \\ \frac{f(x_1,\ldots,x_n)}{g(x_1,\ldots,x_n)} & \mapsto & \frac{f\left(\frac{x_1}{x_0},\ldots,\frac{x_n}{x_0}\right)}{g\left(\frac{x_1}{x_0},\ldots,\frac{x_n}{x_0}\right)}. \end{array}$$

The Homogeneous Coordinate Ring

- The function field $k(V)$ can also be defined by localization of the *coordinate ring*, defined as a graded ring.
- Let $V \subseteq \mathbb{P}_k^n$ be an irreducible variety with affine cone $V^a \subseteq \mathbb{A}_k^{n+1}$.
- Then the ring

$$S(V) := k[V^a] := k[x_0, \ldots, x_n]/I(V)$$

is equipped with the structure of a *graded ring*,

$$S(V) = \bigoplus_{d \geq 0} S_d(V),$$

where

$$S_d(V) := \{\overline{f} \in S(V) : f \text{ is homogeneous with } \deg f = d\} \cup \{0\}.$$

The Homogeneous Coordinate Ring (Cont'd)

- We must show that

$$S_d(V) \cap S_e(V) = \{0\}, \quad \text{for } e \neq d.$$

Suppose $\overline{f} = \overline{g}$.

Then $f - g \in I(V)$.

So, if $\deg f \neq \deg g$, then, since $I(V)$ is homogeneous, $f, g \in I(V)$.

Hence, $\overline{f} = \overline{g} = 0$.

Definition (Homogeneous Coordinate Ring)

$S(V)$ is the **homogeneous coordinate ring** of V.

The Degree of a Quotient

- Consider an arbitrary graded ring
$$S = \bigoplus_{d \geq 0} S_d.$$
- Let $T \subseteq S$ be a multiplicatively closed system of homogeneous elements.
- We wish to give the local ring S_T the structure of a graded ring.
- For homogeneous elements $f \in S$ and $g \in T$, we define $\frac{f}{g}$ to be **homogeneous of degree**
$$\deg \frac{f}{g} := \deg f - \deg g.$$
- This is well defined.
Suppose $\frac{f}{g} = \frac{f'}{g'}$. Then, by definition, for some $h \in S$,
$$h(fg' - gf') = 0.$$
Therefore, $hfg' = hf'g$.

The Degree of a Quotient (Cont'd)

- By taking the homogeneous components, we may assume h is homogeneous.

  Now deg is multiplicative.

  Thus, we have

$$\deg h + \deg f + \deg g' = \deg h + \deg f' + \deg g.$$

This shows that $\deg\left(\frac{f}{g}\right)$ is well defined.

Definition

For any multiplicatively closed system $T \subseteq S$ of a graded ring S, we define

$$S_{(T)} := \left\{ \frac{f}{g} \in S_T : \frac{f}{g} \text{ is homogeneous of degree } 0 \right\}.$$

The Coordinate Ring as a Local Ring

- Let $\mathfrak{p} \subseteq S$ be a homogeneous prime ideal.
  The set
  $$T_{\mathfrak{p}} := \{f \in S : f \text{ is homogeneous}, \ f \notin \mathfrak{p}\}$$
  is a multiplicatively closed system.
  We define
  $$S_{(\mathfrak{p})} := S_{(T_{\mathfrak{p}})}.$$
- Let S be an integral domain.
  Let $f \in S$ be a nonzero homogeneous element.
  The set
  $$T_f := \{f^n : n \geq 0\}$$
  is multiplicatively closed.
  We define
  $$S(f) := S_{(T_f)}.$$

The Coordinate Ring as a Local Ring (Cont'd)

Lemma

For a projective variety $V \subseteq \mathbb{P}_k^n$, we have an isomorphism of graded rings

$$k(V) \cong S(V)_{((0))}.$$

- This follows immediately from the definition of $k(V)$.

Regular Functions

Definition (Regular Function)

A rational function $f \in k(V)$ is called **regular at a point** P if there is a representation

$$f = \frac{g}{h}, \quad \text{with } h(P) \neq 0.$$

Definition (Domain of Definition)

The **domain of definition** $\text{dom}(f)$ of $f \in k(V)$ is the set of all points where f is regular.

- As in the affine case, $\text{dom}(f)$ is a nonempty open subset of V.

Local Ring of a Variety

Definition (Local Ring of V)

The **local ring of V at a point** $P \in V$ is defined by

$$\mathscr{O}_{V,P} := \{f \in k(V) : f \text{ is regular at } P\}.$$

Definition (The Maximal Ideal of V)

The **maximal ideal of V at a point** $P \in V$ is defined by

$$m_{V,P} := \{f \in \mathscr{O}_{V,P} : f(P) = 0\} \subseteq \mathscr{O}_{V,P}.$$

- Note that every element $g \in \mathscr{O}_{V,P}$, with $g(P) \neq 0$, is a unit.
- So $m_{V,P}$ is the unique maximal ideal of $\mathscr{O}_{V,P}$.
- Thus, $\mathscr{O}_{V,P}$ is indeed a local ring.

Isomorphism of Local Rings

- Let V be an irreducible variety, with $V \not\subseteq V(x_0)$.
- Consider a point $P \in V_0 = V \cap U_0$.
- Depending on whether we interpret P as a point in the projective variety V or in the affine variety V_0, we have defined two local rings,
$$\mathscr{O}_{V_0,P} \quad \text{and} \quad \mathscr{O}_{V,P}.$$
- The isomorphism given in the lemma induces an isomorphism
$$\mathscr{O}_{V,P} \cong \mathscr{O}_{V_0,P}.$$
- For $P \in V$, we can consider the maximal ideal
$$M_P := \{f \in S(V) : f \text{ homogeneous, } f(P) = 0\} \subseteq S(V).$$

Lemma

$\mathscr{O}_{V,P} \cong S(V)_{(M_P)}$.

- Immediately clear from the definitions.

The Ring of Regular Functions

Definition (Ring of Regular Functions)

For a quasi-projective variety, given by an open subset $U \subseteq V$, the **ring of regular functions** on U is defined by

$$\mathscr{O}(U) := \{f \in k(V) : U \subseteq \mathrm{dom}(f)\}.$$

- Considering $\mathscr{O}(U)$ as a subset of $k(V)$, we have

$$\mathscr{O}(U) = \bigcap_{P \in U} \mathscr{O}_{V,P}.$$

The Regular Function Theorem

Theorem

If V is an irreducible projective variety defined over an algebraically close field k, then every regular function on V is constant, i.e.,

$$\mathscr{O}(V) \cong k.$$

- In the complex case, if $V \subseteq \mathbb{P}^n_{\mathbb{C}}$ is a smooth projective variety, this result follows from the fact that any holomorphic function on a connected compact complex manifold is constant.
- We would like to give an algebraic proof.

R-Modules

- Let R be a ring with 1.

Definition (R-Module)

A **module over** R (or an R-**module**) is an abelian group M, together with a multiplication map

$$\begin{array}{rcl} R \times M & \to & M; \\ (r, m) & \mapsto & rm, \end{array}$$

such that the following hold:

(1) $r(m_1 + m_2) = rm_1 + rm_2$;

(2) $(r_1 + r_2)m = r_1 m + r_2 m$;

(3) $(r_1 r_2)m = r_1(r_2 m)$;

(4) $1m = m$.

- A module in which the ring R is a field is a vector space.
- A **submodule** is defined analogously to vector subspaces.
- Similarly for **homomorphisms** of modules.

Finitely Generated and Noetherian Modules

Definition (Finitely Generated Module)

An R-module M is called **finitely generated** if there are finitely many elements $m_1, \ldots, m_k$ with

$$M = Rm_1 + \cdots + Rm_k.$$

Definition (Noetherian Module)

An R-module M is called **Noetherian** if all submodules $U \subseteq M$ are finitely generated.

Finitely Generated Modules over Noetherian Rings

Lemma

Let R is a Noetherian ring. If M is a finitely generated R-module, then M is a Noetherian module.

- Let $M = Rm_1 + \cdots + Rm_k$.
  Let $e_i, 1 \leq i \leq k$, be a basis for R^k, with

$$e_i = (0, \ldots, 0, 1, 0, \ldots, 0),$$

i.e., all components are 0 except a 1 in the i-th place.
For $1 \leq i \leq k$, there is a surjective homomorphism

$$\begin{array}{rccc} \varphi: & R^k & \to & M; \\ & e_i & \mapsto & m_i. \end{array}$$

Suppose U is a submodule of M.
Then $\phi^{-1}(U)$ is a submodule of R^k.
So it is enough to prove the result for R^k.

Finitely Generated and Noetherian Modules (Cont'd)

- We will prove the result by induction on k.

  If $k = 1$, then a submodule of $M \cong R$ is isomorphic to an ideal of R.

  The result follows from the assumption that R is a Noetherian ring.

  Now take $U \subseteq R^k$, with $k \geq 2$.

  The first components of the vectors in U generate an ideal I in R,

  $$I := (u_1 : (u_1, \ldots, u_k) \in U).$$

  Since R is Noetherian, this ideal is finitely generated.

  So there are elements $u^{(i)} \in U$, $1 \leq i \leq \ell$, with first component $u_1^{(i)}$, such that

  $$I = (u_1^{(1)}, \ldots, u_1^{(\ell)}).$$

Finitely Generated and Noetherian Modules (Cont'd)

- Thus, for $u \in U$, there are elements $r_1, \ldots, r_\ell \in R$, such that

$$u - r_1 u^{(1)} - \cdots - r_\ell u^{(\ell)} = (0, u_2^*, \ldots, u_k^*).$$

Let $R^{k-1} \subseteq R^k$ be the submodule of elements of R^k, with first component 0.

Consider the submodule

$$U' := U \cap R^{k-1}.$$

By induction, U' is finitely generated, by some elements $v_1, \ldots, v_m$.

Thus,

$$u^{(1)}, \ldots, u^{(\ell)}, v_1, \ldots, v_m$$

generate U.

The Regular Function Theorem

Theorem

If V is an irreducible projective variety defined over an algebraically close field k, then every regular function on V is constant, i.e.,

$$\mathscr{O}(V) \cong k.$$

- Let V be an irreducible projective variety in $\mathbb{P}_k^n$.
  We assume that V is not contained in any hyperplane $H_i := V(x_i)$.
  Otherwise $V \subseteq H_i \cong \mathbb{P}_k^{n-1}$, and we can replace $\mathbb{P}_k^n$ by $\mathbb{P}_k^{n-1}$.
  Let $f \in \mathscr{O}(V)$.
  Consider the affine covering $V = V_0 \cup \cdots \cup V_n$.
  Since f is regular on V, the restriction $f|_{V_i}$ is regular on V_i.

The Regular Function Theorem (Cont'd)

- We make use of the explicit isomorphism $k(V) \cong k(V_i)$.

  Then $f|_{V_i}$ is a polynomial in $\frac{x_j}{x_i}$, $1 \le j \ne i \le n$.

  Thus, for $1 \le i \le n$, we can write

  $$f|_{V_i} = \frac{g_i}{x_i^{N_i}},$$

  where $g_i \in S(V)$ is homogeneous of degree N_i.

  Since V is irreducible, $I(V)$ is a prime ideal.

  This implies that

  $$S(V) = k[x_0, \ldots, x_n]/I(V)$$

  is an integral domain.

  So we can consider the field of fractions

  $$L := \mathrm{Quot} S(V) = k(V^a),$$

  where V^a is the affine cone over V.

The Regular Function Theorem (Cont'd)

- The rings $\mathscr{O}(V), k(V)$ and $S(V)$ are all contained in L.
  We have
  $$x_i^{N_i} f \in S_{N_i}(V),$$
  where $S_d(V)$ is the homogeneous part of degree d of $S(V)$.
  Now take an integer $N > \sum N_i$.
  Then $S_N(V)$ is a finite dimensional k-vector space, spanned by the monomials of degree N.
  Every monomial m in $S_N(V)$ is divisible by $x_i^{N_i}$, for some i.
  Since $x_i^{N_i} f \in S_{N_i}(V)$, we get $mf \in S_N(V)$.
  Hence, $S_N(V)f \subseteq S_N(V)$.
  This shows that, for $q \geq 1$, we have a sequence of inclusions,
  $$S_N(V)f^q \subseteq S_N(V)f^{q-1} \subseteq \cdots \subseteq S_N(V)f \subseteq S_N(V).$$
  In particular, this implies that $x_0^N f^q \in S_N(V)$, for all $q \geq 1$.
  Hence, $S(V)[f] \subseteq x_0^{-N} S(V) \subseteq L$.

The Regular Function Theorem (Final Steps)

- We have $x_0^{-N}S(V)$ is a finitely generated $S(V)$-module.

  Hence, $x_0^{-N}S(V)$ is Noetherian.

  So the submodule $S(V)[f]$ is also finitely generated over $S(V)$.

  Now f is integral over $S(V)$.

  It, thus, satisfies an equation of the form

  $$f^m + a_{m-1}f^{m-1} + \cdots + a_1 f + a_0 = 0, \quad a_i \in S(V).$$

  Since f is homogeneous of degree 0, we can assume the same for the a_i, since we can take the degree 0 component of the a_i.

  Hence, $a_i \in S_0(V) = k$.

  Thus, f is algebraic over k.

  But, we are assuming that $k = \overline{k}$.

  It follows that $f \in k$.

Affine Rational Maps

Definition (Affine Rational Map)

Let V be an irreducible projective variety.

(1) A **rational map** $f : V \dashrightarrow \mathbb{A}_k^m$ is an m-tuple

$$f = (f_1, \ldots, f_m)$$

of rational functions $f_1, \ldots, f_m \in k(V)$.
The **domain of definition** of f is given by

$$\text{dom}(f) := \bigcap_{i=1}^{m} \text{dom}(f_i).$$

On this set, f is well defined, with $f(P) = (f_1(P), \ldots, f_m(P))$.

(2) A **rational map** $f : V \dashrightarrow W \subseteq \mathbb{A}_k^m$ is given by a rational map $f : V \dashrightarrow \mathbb{A}_k^m$ with $f(\text{dom}(f)) \subseteq W$.

Projective Rational Maps

- Now let V be an irreducible projective or affine variety.

Definition (Projective Rational Map)

A **rational map** $f : V \dashrightarrow \mathbb{P}_k^m$ on V is given by

$$f(P) = (f_0(P) : \ldots : f_m(P))$$

for rational functions $f_0, \ldots, f_m \in k(V)$.

- If $0 \neq g \in k(V)$, then the tuples

$$(f_0, \ldots, f_m) \quad \text{and} \quad (gf_0, \ldots, gf_m)$$

define the same rational map.

Regular Maps

Definition (Regular Map)

A rational map $f : V \dashrightarrow \mathbb{P}_k^m$ is **regular at a point** P if there is a representation

$$f = (f_0 : \ldots : f_m),$$

such that:

(1) For $1 \leq i \leq m$, the function f_i is regular at P.

(2) There is some i, such that $f_i(P) \neq 0$.

Definition (Projective Rational Map)

A **rational map** $f : V \dashrightarrow W \subseteq \mathbb{P}_k^m$ is a rational map $f : V \dashrightarrow \mathbb{P}_k^m$, with

$$f(\mathrm{dom}(f)) \subseteq W.$$

Morphisms of Quasi-Projective Varieties

Definition (Morphism)

Let V_1 and V_2 be irreducible affine or projective varieties containing open subsets U_1 and U_2, respectively.

(1) A **morphism** $f : U_1 \to U_2$ is a rational map $f : V_1 \dashrightarrow V_2$ with

$$U_1 \subseteq \mathrm{dom}(f) \quad \text{and} \quad f(U_1) \subseteq U_2.$$

(2) A morphism $f : U_1 \to U_2$ is an **isomorphism** if there is a morphism $g : U_2 \to U_1$, with

$$g \circ f = \mathrm{id}_{U_1} \quad \text{and} \quad f \circ g = \mathrm{id}_{U_2}.$$

- This definition allows us to speak of:
  - Morphisms on quasi-projective varieties;
  - Isomorphisms between quasi-projective varieties.

Example

- Consider the **rational normal curve of degree** n, parametrized by the map

$$\begin{array}{rrcl} \varphi: & \mathbb{P}^1_k & \to & \mathbb{P}^n_k; \\ & \varphi(t_0:t_1) & = & (t_0^n : t_0^{n-1}t_1 : \ldots : t_1^n). \end{array}$$

Notice that we can write

$$\begin{array}{rcl} \varphi(t_0:t_1) & = & (t_0^n : t_0^{n-1}t_1 : \ldots : t_1^n) \\ & = & \left(\left(\frac{t_0}{t_1}\right)^n : \left(\frac{t_0}{t_1}\right)^{n-1} : \ldots : 1\right) \\ & = & \left(1 : \ldots : \left(\frac{t_1}{t_0}\right)^{n-1} : \left(\frac{t_1}{t_0}\right)^n\right). \end{array}$$

So the map φ is rational and everywhere regular.

The Isomorphism j_ℓ

- We return to the affine covering of $\mathbb{P}_k^n$.
- We have maps

$$\begin{array}{rcl} i_\ell: \quad \mathbb{A}_k^n & \to & U_\ell = \{(x_0 : \ldots : x_n) : x_\ell \neq 0\} \subseteq \mathbb{P}_k^n; \\ (x_1, \ldots, x_n) & \mapsto & (x_1 : \ldots : x_{\ell-1} : 1 : x_\ell : \ldots : x_n), \\ j_\ell: \quad U_\ell & \to & \mathbb{A}_k^n; \\ (x_0 : \ldots : x_{\ell-1} : x_\ell \ldots : x_n) & \mapsto & (\frac{x_0}{x_\ell}, \ldots, \frac{x_{\ell-1}}{x_\ell}, \frac{x_{\ell+1}}{x_\ell}, \ldots, \frac{x_n}{x_\ell}). \end{array}$$

We have already seen that j_ℓ is a homeomorphism.

Proposition

$j_\ell : U_\ell \to \mathbb{A}_k^n$ is an isomorphism.

- The maps i_ℓ and j_ℓ are inverse morphisms of each other.

Birational Equivalences

- Let V and W be irreducible quasi-projective varieties.

Definition (Birational Map)

A rational map $f : V \dashrightarrow W$ is called **birational** (or a **birational equivalence**) if there is a rational map $g : W \dashrightarrow V$, with

$$f \circ g = \mathrm{id}_W \quad \text{and} \quad g \circ f = \mathrm{id}_V.$$

Definition (Birational Equivalence)

Two varieties V and W are said to be **birationally equivalent** if there is a birational equivalence $f : V \dashrightarrow W$.

Characterization of Birational Maps

- Recall that a rational map $f : V \dashrightarrow W$ is called *dominant* if $f(\text{dom}(f))$ is a Zariski dense subset of W.

Theorem

For a rational map $f : V \dashrightarrow W$, the following statements are equivalent:

(1) f is birational;

(2) f is dominant and $f^* : k(W) \to k(V)$ is an isomorphism;

(3) There are open sets $V_0 \subseteq V$ and $W_0 \subseteq W$, such that the restriction $f|_{V_0} : V_0 \to W_0$ is an isomorphism.

Proof of the Equivalence Theorem ((3)⇒(1))

- The equivalence of (1) and (2) is proved as in the affine case.

  We show that (3)⇒(1).

  Under (3), $f|_{V_0}: V_0 \to W_0$ has an inverse $g: W_0 \to V_0$.

  Moreover, by definition, $g: W \dashrightarrow V$ is a rational map.

  Then $g \circ f: V \dashrightarrow V$ and $f \circ g: W \dashrightarrow W$ are rational maps which are the identity maps on V_0 and W_0, respectively.

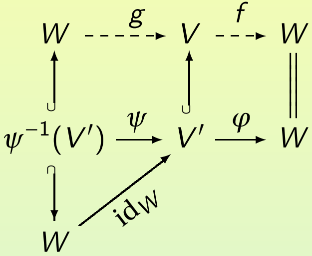
  But V_0 and W_0 are dense.

  It follows that

$$g \circ f = \mathrm{id}_V \quad \text{and} \quad f \circ g = \mathrm{id}_W.$$

Proof of the Equivalence Theorem ((1)⇒(3))

- It remains to show the implication (1)⇒(3).
  Let $g : W \dashrightarrow V$ be a rational map inverse to f.
  We set $V' := \mathrm{dom}(f)$ and $W' = \mathrm{dom}(g)$.

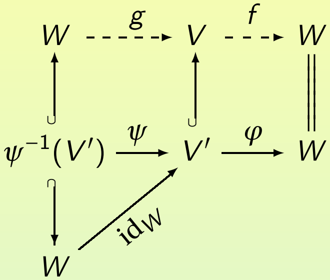


  Then we have morphisms
  $$\varphi := f\,|_{V'} : V' \to W \quad \text{and} \quad \psi := g\,|_{W'} : W' \to V.$$
  We also have an equality of rational maps $f \circ g = \mathrm{id}_W$.
  Thus, we obtain $\varphi(\psi(P)) = P$, for all $P \in \psi^{-1}(V')$.

Proof of the Equivalence Theorem ((1)⇒(3) Cont'd)

- Now let

$$V_0 := \varphi^{-1}(\psi^{-1}(V')),$$
$$W_0 := \psi^{-1}(\varphi^{-1}(W')).$$



Suppose $Q \in V_0$.

Then $\varphi(Q) \in \psi^{-1}(V')$.

So $\varphi(\psi(\varphi(Q))) = \varphi(Q)$.

Hence, $\varphi(Q) \in W'$ and $\varphi(Q) \in \psi^{-1}(\varphi^{-1}(W')) = W_0$.

Thus, the map $\varphi : V_0 \to W_0$ is a morphism.

Similarly, $\psi : W_0 \to V_0$ is a morphism.

Obviously, φ and ψ are inverse to each other.

Categorical Formulation

Corollary

There is a contravariant equivalence between the category of irreducible quasi-projective varieties, with dominant rational maps as morphisms, and the category of finitely generated field extensions of k and k-homomorphisms, given by

$$\begin{array}{rcl} V & \mapsto & k(V), \\ (f : V \dashrightarrow W) & \mapsto & (f^* : k(W) \to k(V)). \end{array}$$

- In the classification problem of algebraic geometry, one tries to classify varieties either up to birational equivalence (coarse classification), or up to isomorphism (fine classification).
- It makes sense to restrict attention to the properties of a variety that are invariant under birational equivalence.

Irreducible Varieties are "Almost" Hypersurfaces

Proposition

Every quasi-projective irreducible variety is birationally equivalent to an affine hypersurface.

- Since every quasi-projective variety is birationally equivalent to an affine variety, we may restrict attention to this case.

  Let $V \subseteq \mathbb{A}_k^n$ be an irreducible affine variety.

  By a previous corollary, there are

  $$y_1, \ldots, y_{m+1} \in k[V],$$

  such that:

  - $y_1, \ldots, y_m$ algebraically independent over k;
  - $k(V)$ is an algebraic extension of $k(y_1, \ldots, y_m)$, generated by y_{m+1}.

Irreducible Varieties are "Almost" Hypersurfaces (Cont'd)

- Consider the minimal polynomial

$$y_{m+1}^N + a_1 y_{m+1}^{N-1} + \cdots + a_n = 0, \quad a_i \in k(y_1, \ldots, y_m),$$

of y_{m+1} over $k(y_1, \ldots, y_m)$.

Multiply by the highest common denominator of the a_i.

We obtain an irreducible polynomial

$$b_0 y_{m+1}^N + b_1 y_{m+1}^{N-1} + \cdots + b_N = 0, \quad b_i \in k[y_1, \ldots, y_m].$$

This equation defines an irreducible hypersurface $W \subseteq \mathbb{A}_k^{m+1}$.

Now the y_i are algebraically independent, for $1 \le i \le m$.

So we have an isomorphism $k(W) \cong k(V)$.

The result then follows from the preceding theorem.

Rational Quasi-Projective Varieties

Definition (Rational Quasi-Projective Variety)

A quasi-projective variety V is called **rational** if V is birationally equivalent to $\mathbb{A}_k^n$ (or equivalently, to $\mathbb{P}_k^n$).

Proposition

The following statements are equivalent:

(1) V is rational;

(2) $k(V) \cong k(x_1, \ldots, x_n)$;

(3) There are isomorphic open sets $V_0 \subseteq V$ and $U_0 \subseteq \mathbb{A}_k^n$.

- This follows immediately from the theorem.

Examples

- We have already visited the curves

$$\begin{array}{llll} C_0: & y^2 & = & x^3 \qquad \text{(semicubical parabola)}, \\ C_1: & y^2 & = & x^3+x^2. \end{array}$$

  They are rational.
- We also visited the elliptic curve

$$C_\lambda: \quad y^2 = x(x-1)(x-\lambda), \quad \lambda \neq 0,1.$$

  We showed that it is not rational.

Example

- Consider the map

$$\begin{array}{rccl} f: & \mathbb{A}_k^1 & \to & C := \{(x,y) \in \mathbb{A}_k^2 : y^2 - x^3 = 0\}; \\ & t & \mapsto & (t^2, t^3), \end{array}$$

  It is a birational map.

  However, f is not an isomorphism.

  The restriction

$$f_0 := f\,|_{\mathbb{A}_k^1 \setminus \{0\}} : \mathbb{A}_k^1 \setminus \{0\} \to C \setminus \{0\}$$

  is an isomorphism of Zariski open sets.

Projections

- Consider the map

$$\pi := (x_1 : \ldots : x_n) : \begin{array}{rcl} \mathbb{P}_k^n & \to & \mathbb{P}_k^{n-1}; \\ (x_0 : \ldots : x_n) & \mapsto & (x_1 : \ldots : x_n). \end{array}$$

It is a rational map.

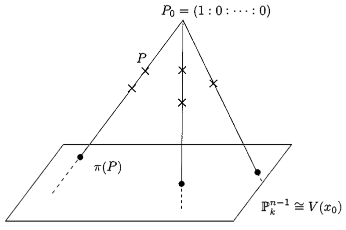
It is defined everywhere on $\mathbb{P}_k^n$ except at the point $P_0 = (1 : 0 : \ldots : 0)$.

The map π is called the **projection** from P_0.

Identify $\mathbb{P}_k^{n-1}$ with $V(x_0)$ in $\mathbb{P}_k^n$.

Consider a point $P \neq P_0$.

The image $\pi(P)$ is given by the intersection of the line $\overline{P_0 P}$ with the plane $\mathbb{P}_k^{n-1}$.



Projections (Cont'd)

- Now set $n = 3$ and consider the quadric

$$Q := \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}_k^3 : x_0x_3 - x_1x_2 = 0\}.$$

The point $P_0 = (1:0:0:0)$ lies on Q.

The restriction of the projection π to Q is a rational map

$$p = \pi\mid_Q : Q \dashrightarrow \mathbb{P}_k^2.$$

This map is birational.

Its inverse $q : \mathbb{P}_k^2 \dashrightarrow Q$ is given by

$$\begin{array}{rcl} q(x_1 : x_2 : x_3) & = & (\frac{x_1x_2}{x_3} : x_1 : x_2 : x_3) \\ & = & (x_1x_2 : x_1x_3 : x_2x_3 : x_3^2). \end{array}$$

q is defined everywhere except at the points $(1:0:0)$ and $(0:1:0)$.

Cremona Transformations

- A birational map from $\mathbb{P}_k^2$ to $\mathbb{P}_k^2$ is called a **Cremona transformation**.
- An example is given by the map $\varphi : \mathbb{P}_k^2 \dashrightarrow \mathbb{P}_k^2$, defined by

$$\begin{aligned} \varphi(x_0 : x_1 : x_2) &= (x_1x_2 : x_0x_2 : x_0x_1) \\ &= \left(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right). \end{aligned}$$

  It is birational, with $\varphi = \varphi^{-1}$.

  The map φ is not defined at the points $(1:0:0)$, $(0:1:0)$, $(0:0:1)$.

  Moreover, it contracts the three lines $V(x_i)$ to points.

The Segre Map and the Segre Variety

Definition (The Segre Map and the Segre Variety)

We define the **Segre map** by

$$\begin{array}{rrcl} s_{n,m}: & \mathbb{P}_k^n \times \mathbb{P}_k^m & \to & \mathbb{P}_k^N; \\ & ((x_0 : \ldots : x_n), (y_0 : \ldots : y_m)) & \mapsto & (x_0 y_0 : \ldots : x_i y_j : \ldots : x_n y_m) \end{array}$$

where $N = (n+1)(m+1) - 1$ and $0 \le i \le n$, $0 \le j \le m$.

This map is well defined.

The image

$$\Sigma_{n,m} := s_{n,m}(\mathbb{P}_k^n \times \mathbb{P}_k^m)$$

is called the **Segre variety**.

The Segre Variety

Lemma

The Segre map $s_{n,m} : \mathbb{P}_k^n \times \mathbb{P}_k^m \to \Sigma_{n,m}$ is bijective.
The image $\Sigma_{n,m}$ is a projective variety in $\mathbb{P}_k^N$.

- We denote the coordinates of $\mathbb{P}_k^N$ by z_{ij}, where $0 \le i \le n$, $0 \le j \le m$.
  These coordinates are ordered to be compatible with $s_{n,m}$.
  Let $\pi_{ij\ell r}$ is the projection

$$\pi_{ij\ell r} : \begin{array}{ccc} \mathbb{P}_k^N & \dashrightarrow & \mathbb{P}_k^1, \\ (z_{00} : \ldots : z_{ij} : \ldots : z_{nm}) & \mapsto & (z_{ij} : z_{\ell r}). \end{array}$$

  Then the composition $\pi_{ij\ell r} \circ s_{n,m}$ is given by the rational map

$$\pi_{ij\ell r} \circ s_{n,m}((x_0 : \ldots : x_n), (y_0 : \ldots : y_m)) = (x_i y_j : x_\ell y_r).$$

The Segre Map and the Segre Variety (Cont'd)

- So the points in $\Sigma_{n,m}$ satisfy the homogeneous equations

$$z_{ir}z_{j\ell} - z_{i\ell}z_{jr} = 0, \quad i,j = 0,\ldots,n;\ \ell,r = 0,\ldots,m.$$

Let Z be the variety described by these equations.

Clearly $\Sigma_{n,m} \subseteq Z$.

Claim: For every point $R \in Z$, there exists a pair $(P,Q) \in \mathbb{P}_k^n \times \mathbb{P}_k^m$, such that

$$s_{n,m}(P,Q) = R.$$

The Claim implies that $\sum_{n,m} = Z$.

Thus, $s_{n,m}$ maps $\mathbb{P}_k^n \times \mathbb{P}_k^m$ bijectively to its image.

Proof of the Claim

Claim: For every point $R \in Z$, there exists a pair $(P,Q) \in \mathbb{P}_k^n \times \mathbb{P}_k^m$, such that

$$s_{n,m}(P,Q) = R.$$

Let $R = (z_{00}^0 : z_{01}^0 : \ldots : z_{nm}^0)$ be a point in Z.
Without loss of generality, we may assume that $z_{00}^0 \neq 0$.
So, by scaling, we can assume that $z_{00}^0 = 1$.
The other cases can be handled analogously.
Set

$$\begin{array}{lcl} Q & := & (1 : z_{01}^0 : \ldots : z_{0m}^0) \in \mathbb{P}_k^m, \\ P & := & (1 : z_{10}^0 : \ldots : z_{n0}^0) \in \mathbb{P}_k^n. \end{array}$$

We have

$$z_{i0}^0 z_{0j}^0 = z_{00}^0 z_{ij}^0 = z_{ij}^0.$$

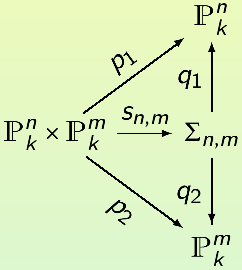
Hence, $s_{n,m}((P,Q)) = R$. It is also clear that (P,Q) is the unique point in $\mathbb{P}_k^n \times \mathbb{P}_k^m$ mapped to R by $s_{n,m}$.

Irreducibility of the Segre Variety

Lemma

The Segre variety $\Sigma_{n,m}$ is irreducible.

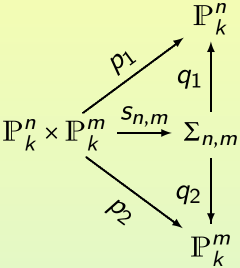
- The projections from $\mathbb{P}_k^n \times \mathbb{P}_k^m$ to $\mathbb{P}_k^n$ and to $\mathbb{P}_k^m$ form a commutative diagram.



The proof of the lemma shows that q_1 and q_2 are morphisms.

Irreducibility of the Segre Variety

- Let $P \in \mathbb{P}_k^n$ and $Q \in \mathbb{P}_k^m$ be any points.



We have the following restrictions of $s_{n,m}$,

$$\begin{array}{rcl} s_{n,m}^Q : \mathbb{P}_k^n \times \{Q\} & \to & \Sigma_{n,m}, \\ s_{n,m}^P : \{P\} \times \mathbb{P}_k^m & \to & \Sigma_{n,m}. \end{array}$$

These maps induce isomorphisms between $\mathbb{P}_k^n$ and $\mathbb{P}_k^m$ and projective subspaces of $\mathbb{P}_k^N$.

Thus, via $s_{n,m}^P$ and $s_{n,m}^Q$, the fibers of q_1 and q_2 are projective varieties, isomorphic to $\mathbb{P}_k^m$ and $\mathbb{P}_k^n$, respectively.

In particular, the fibers of q_1 and q_2 are irreducible.

The irreducibility of $\Sigma_{n,m}$ can now be shown as in the affine case.

Example

- We may identify $\mathbb{P}_k^n \times \mathbb{P}_k^m$ with $\Sigma_{n,m}$ by means of the Segre map.
- In this way, $\mathbb{P}_k^n \times \mathbb{P}_k^m$ is viewed as an irreducible projective variety.

  Example: We have already seen the map

$$s_{1,1}: \begin{array}{rcl} \mathbb{P}_k^1 \times \mathbb{P}_k^1 & \to & \mathbb{P}_k^3; \\ ((x_0 : x_1), (y_0 : y_1)) & \mapsto & (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1). \end{array}$$

  In this case the Segre variety $\Sigma_{1,1}$ is the quadric

$$\Sigma_{1,1} = Q = V(z_{00} z_{11} - z_{01} z_{10}).$$

Product of Projective Varieties

Proposition

Let V and W be projective varieties. Then:

(1) The product $V \times W$ is also a projective variety.

(2) If V, W are irreducible, then so is $V \times W$.

(1) Let $V \subseteq \mathbb{P}_k^n$, $W \subseteq \mathbb{P}_k^m$ be given by homogeneous equations

$$\begin{array}{rrcll} V: & f_i(x_0,\dots,x_n) & = & 0, & i=1,\dots,r; \\ W: & g_j(y_0,\dots,y_m) & = & 0, & j=1,\dots,s. \end{array}$$

Let d_i be the degree of f_i.

Let e_j the degree of g_j.

Product of Projective Varieties (Cont'd)

- The elements of the set $V \times W \subseteq \mathbb{P}_k^n \times \mathbb{P}_k^m$ are given by the zeros of the polynomials

$$\begin{array}{rcll} F_{ik} & := & f_i y_k^{d_i}, & i=1,\ldots,r,\ k=0,\ldots,m, \\ G_{j\ell} & := & g_j x_\ell^{e_j}, & j=1,\ldots,s,\ \ell=0,\ldots,n. \end{array}$$

These may be considered as homogeneous polynomials

$$F_{ik} = F_{ik}(z_{\mu k}) \quad \text{and} \quad G_{j\ell} = G_{j\ell}(z_{\ell\nu}).$$

Append the equations

$$z_{\mu\nu} z_{\rho\sigma} - z_{\mu\sigma} z_{\rho\nu} = 0.$$

We obtain a system of homogeneous equations for the set

$$V \times W \subseteq \mathbb{P}_k^N.$$

(2) Irreducibility can be shown as in the lemma.

Remarks

- In fact $V \times W$ is a product in the category of projective varieties.
- Note that $V \times W$ does not have the product topology.
- The product of quasi-projective varieties is obtained analogously.

The Exceptional Line

- Consider the quasi-projective variety

$$\widetilde{\mathbb{A}}_k^2 := \{((x,y),(t_0:t_1)) \in \mathbb{A}_k^2 \times \mathbb{P}_k^1 : xt_1 - yt_0 = 0\}.$$

- Projection onto the factor $\mathbb{A}_k^2$ induces a surjective morphism

$$\pi : \widetilde{\mathbb{A}}_k^2 \to \mathbb{A}_k^2.$$

- We have

$$\pi^{-1}(x,y) = \begin{cases} \{(0,0)\} \times \mathbb{P}_k^1, & \text{if } (x,y) = (0,0), \\ ((x,y),(x:y)), & \text{otherwise.} \end{cases}$$

- The fiber $E := \pi^{-1}((0,0))$ is a projective line, which is called the **exceptional line**.

The Blow Up

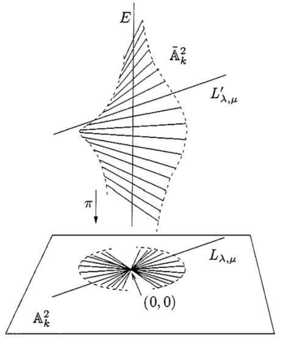
- For $(x,y) \neq (0,0)$, the inverse image is the point $((x,y),(x:y))$.
- So π is birational with inverse

$$\begin{array}{rrcl} \pi^{-1}: & \mathbb{A}_k^2 & \dashrightarrow & \widetilde{\mathbb{A}}_k^2; \\ & \pi^{-1}(x,y) & = & ((x,y),(x:y)). \end{array}$$

- This map is not regular at the origin.
- Away from the origin, π gives an isomorphism between the quasi-projective varieties

$$\mathbb{A}_k^2 \backslash \{0\} \quad \text{and} \quad \widetilde{\mathbb{A}}_k^2 \backslash E.$$

- The surface $\widetilde{\mathbb{A}}_k^2$ is called the **blow-up** of $\mathbb{A}_k^2$ at the origin.



Points of E and Lines through Origin in $\mathbb{A}_k^2$

- The points of E correspond to the lines through the origin in $\mathbb{A}_k^2$.
- Let $L_{\lambda,\mu}$ be the line through the origin in $\mathbb{A}_k^2$, given by

$$L_{\lambda,\mu} := \{(x,y) \in \mathbb{A}_k^2 : \lambda x - \mu y = 0\}.$$

- The inverse image of $\{(0,0)\} \in L_{\lambda,\mu}$ is the exceptional line E.
- Any other point of $L_{\lambda,\mu}$ is of the form $(\mu t, \lambda t)$, for $t \in k \setminus \{0\}$.
- We have

$$\pi^{-1}(\mu t, \lambda t) = ((\mu t, \lambda t), (\mu : \lambda)).$$

- So the inverse image of $L_{\lambda,\mu} \setminus \{0\}$ is given by

$$\pi^{-1}(L_{\lambda,\mu} \setminus \{0\}) = \{((\mu t, \lambda t), (\mu : \lambda)) : t \in k \setminus \{0\}\}.$$

Points of E and Lines in $\mathbb{A}_k^2$ (Cont'd)

- Let the line $L'_{\lambda,\mu} \subseteq \mathbb{A}_k^2 \times \mathbb{P}_k^1$ be the closure of
$$\pi^{-1}(L_{\lambda,\mu} \setminus \{0\}) = \{((\mu t, \lambda t), (\mu : \lambda)) : t \in k \setminus \{0\}\}.$$
- The projection π induces an isomorphism between $L'_{\lambda,\mu}$ and $L_{\lambda,\mu}$.
- We have
$$L'_{\lambda,\mu} = \pi^{-1}(L_{\lambda,\mu} \setminus \{0\}) \cup \{((0,0),(\mu,\lambda))\} \in E.$$
- So
$$\pi^{-1}(L_{\lambda,\mu}) = E \cup L'_{\lambda,\mu}.$$
- Identify E with $\mathbb{P}_k^1$ via $((0,0),(x:y)) \mapsto (x:y) \in \mathbb{P}_k^1$.
- We obtain $L'_{\lambda,\mu} \cap E = (\mu : \lambda) \in \mathbb{P}_k^1$.
- Thus, every point $(\mu : \lambda) \in E \cong \mathbb{P}_k^1$ corresponds to a unique line through the origin in $\mathbb{A}_k^2$.

Affine Covering of $\widetilde{\mathbb{A}}_k^2$

- Recall the affine covering $\mathbb{P}_k^1 = U_0 \cup U_1$, with $U_i = \{t_i \neq 0\}$.
- It induces a covering

$$\widetilde{\mathbb{A}}_k^2 = V_0 \cup V_1, \quad V_i \subseteq \mathbb{A}_k^2 \times \mathbb{A}_k^1,$$

where

$$\begin{aligned} V_0 &:= \left\{((x,y),(t_0:t_1)) \in \mathbb{A}_k^2 \times \mathbb{P}_k^1 : t_0 \neq 0 \text{ and } x\tfrac{t_1}{t_0} - y = 0\right\}, \\ V_1 &:= \left\{((x,y),(t_0:t_1)) \in \mathbb{A}_k^2 \times \mathbb{P}_k^1 : t_1 \neq 0 \text{ and } x - y\tfrac{t_0}{t_1} = 0\right\}. \end{aligned}$$

- Now we use coordinates:
  - $x, u := \frac{t_1}{t_0}$ for V_0;
  - $y, v := \frac{t_0}{t_1}$ for V_1.
- We see that V_0 and V_1 are both isomorphic to $\mathbb{A}_k^2$.

Example

- Consider the curve

$$C: \quad y^2 = x^3 + x^2.$$

- It has a double point at the origin.
- We have

$$\pi^{-1}(C) = \{((x,y),(t_0 : t_1)) : y^2 = x^3 + x^2, t_0 y = t_1 x\}.$$

In terms of the preceding affine covering, we have

$$\begin{aligned} \pi^{-1}(C \backslash \{(0,0)\}) \cap V_0 &= \{(x,u) \in V_0 \cong \mathbb{A}_k^2 : x^2(x+1-u^2) = 0\}, \\ \pi^{-1}(C \backslash \{(0,0)\}) \cap V_1 &= \{(y,v) \in V_1 \cong \mathbb{A}_k^2 : y^2(yv^3 + v^2 - 1) = 0\}. \end{aligned}$$

Example (Cont'd)

- Let $\widetilde{C}$ denote the closure of this curve in $\mathbb{A}_k^2 \times \mathbb{P}_k^1$
- Note $\widetilde{C} \subseteq V_0$.
- Identifying E with $\mathbb{P}_k^1$, we get

$$\widetilde{C} \setminus \pi^{-1}(C \setminus \{(0,0)\}) = \widetilde{C} \cap E = \{(1:1),(1:-1)\} \subseteq E \cong \mathbb{P}_k^1.$$

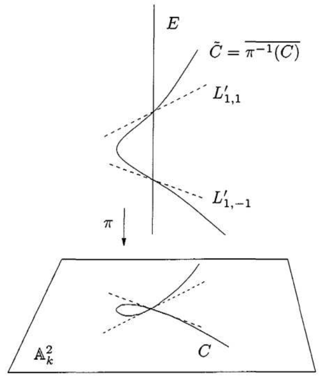
- These points correspond to the two tangents to C at the origin, given by $L_{1,1}$ and $L_{1,-1}$ in the notation of the previous example.

Example (Strict Transform of C)

- The preimage $\pi^{-1}(C)$ also contains the exceptional line (with multiplicity two).
- Moreover, we have

$$\pi^{-1}(C) = E \cup \widetilde{C}.$$

- The curve $\widetilde{C}$ is "smooth".
- We call $\widetilde{C}$ the **strict transform** of C.



- $\widetilde{C}$ is birationally equivalent, but not isomorphic, to C.
- The figure shows C and its preimage, together with the tangent lines to C at $(0,0)$ and their preimages.