Introduction to Algebraic Geometry

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1. Smooth Points and Dimension
   - Smooth and Singular Points
   - Algebraic Characterization of the Dimension of a Variety

Subsection 1

Smooth and Singular Points

The Tangent Space to a Hypersurface at a Point

- Let V be an irreducible affine hypersurface,

$$V = V(f) = \{(x_1,\ldots,x_n) \in \mathbb{A}_k^n : f(x_1,\ldots,x_n) = 0\},$$

where $f \in k[x_1,\ldots,x_n]$ is an irreducible nonconstant polynomial.

Definition (Tangent Space)

Let $P = (a_1,\ldots,a_n) \in V$. The **tangent space to the hypersurface** V **at the point** P is defined by

$$T_P(V) := \left\{(x_1,\ldots,x_n) \in \mathbb{A}_k^n : \sum_{i=1}^n \frac{\partial f}{\partial x_i}(P)(x_i - a_i) = 0\right\},$$

where $\frac{\partial f}{\partial x_i}$ denotes the (formal) derivative of f with respect to x_i.

- The space $T_P(V)$ is an affine subspace of $\mathbb{A}_k^n$, and $P \in T_P(V)$.
- $T_P(V)$ is often given the structure of a vector space, with origin at P.

Tangent Space is "Space of Tangents"

Lemma

Let $L \subseteq \mathbb{A}_k^n$ be an affine line through the point P. Then P is a multiple root of $f\mid_L$ if and only if $L \subseteq T_P(V)$.

- The line L has a parametrization

$$L: \quad x_i = a_i + b_i t,$$

where:

  - $P = (a_1, \ldots, a_n)$;
  - $(b_1, \ldots, b_n)$ is a vector in the direction of L.

Let

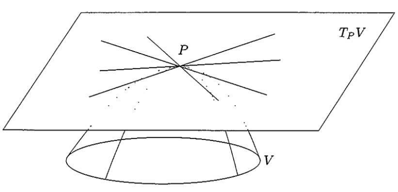
$$g := f\mid_L.$$

Then

$$g(t) = f(a_1 + b_1 t, \ldots, a_n + b_n t).$$

Tangent Space is "Space of Tangents" (Cont'd)

- By hypothesis, $P=(a_1,\ldots,a_n)\in V$.
  So
  $$g(0)=f(P)=0.$$
  So g has a multiple root at 0 if and only if
  $$\frac{\partial g}{\partial t}(0)=0 \quad \text{iff} \quad \sum_{i=1}^n b_i \frac{\partial f}{\partial x_i}(P)=0$$
  $$\text{iff} \quad L\subseteq T_P(V),$$
  by the equation for $T_P(V)$.



Smooth (Regular) Points

Definition (Smooth (Regular) Point)

P is called a **smooth** (or **regular**) point of the hypersurface V if there exists some i, such that

$$\frac{\partial f}{\partial x_i}(P) \neq 0.$$

Otherwise P is called a **singular point** (or a **singularity**) of V.

- This definition gives us the following characterizations of smooth and singular points on affine hypersurfaces.

$$\begin{array}{lll} P \text{ is a smooth point of } V & \text{iff} & T_P V \text{ is an affine hyperplane.} \\ P \text{ is a singular point of } V & \text{iff} & T_P V = \mathbb{A}_k^n. \end{array}$$

Smoothness of General Points

Proposition

For an irreducible affine hypersurface $V \subseteq \mathbb{A}_k^n$, the set

$$V_{\text{smooth}} := \{P \in V : P \text{ is a smooth point of } V\}$$

is open and dense in V.

- The set $V_{\text{sing}} := V \setminus V_{\text{smooth}}$ is given by

$$V_{\text{sing}} = V\left(f, \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) \subseteq \mathbb{A}_k^n.$$

  Since this is obviously a closed subset, V_{smooth} is open.

  We are assuming that V is irreducible.

  So show that V_{smooth} is dense, we must show that $V_{\text{sing}} \neq V$.

Smoothness of General Points (Cont'd)

Claim: $V_{\text{sing}} \neq V$.

Suppose, to the contrary, that $V_{\text{sing}} = V$.

Then all derivatives $\frac{\partial f}{\partial x_i}$ vanish on V.

That is, $\frac{\partial f}{\partial x_i} \in (f)$.

But the degree of $\frac{\partial f}{\partial x_i}$ is less than the degree of f.

This implies that $\frac{\partial f}{\partial x_i} = 0$.

- Assume, first, $\text{char}(k) = 0$.
  It already follows that f is constant, a contradiction.
- Assume, next, that $\text{char}(k) = p > 0$.
  Then it follows that f is constant or is a polynomial in x_i^p.
  But we assume that the ground field k is algebraically closed.
  So this implies that $f = g^p$.
  This contradicts the irreducibility of f.

Real and Complex Manifolds

- Let $k = \mathbb{R}$ or $\mathbb{C}$.
- Let P be a smooth point of a hypersurface $V \subseteq \mathbb{A}_k^n$.
- Assume that $\frac{\partial f}{\partial x_1}(P) \neq 0$.
- Consider the map

$$\begin{array}{rccc} p: & \mathbb{A}_k^n & \to & \mathbb{A}_k^n; \\ & (x_1,\ldots,x_n) & \mapsto & (f(x_1,\ldots,x_n), x_2,\ldots,x_n). \end{array}$$

- At the point P, p has an invertible Jacobian matrix $\left(\frac{\partial p_i}{\partial x_j}\right)_{ij}$.
- By the Inverse Function Theorem, there are neighborhoods $U \subseteq \mathbb{A}_k^n$ of P and $W \subseteq \mathbb{A}_k^n$ of $p(P)$ (in the usual analytic topology), such that $p: U \to W$ is diffeomorphic (or biholomorphic if $k = \mathbb{C}$).

Real and Complex Manifolds (Cont'd)

- In terms of the coordinates $y_1,\ldots,y_n$ of the image space $\mathbb{A}_k^n$,

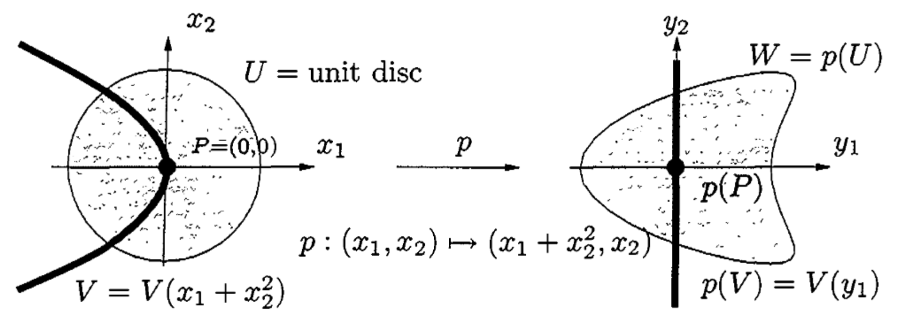
$$p(V)\cap W = V(y_1)\cap W.$$

- Thus, V is:
  - Locally diffeomorphic to an open subset of $\mathbb{R}^{n-1}$; or
  - Locally biholomorphic to an open subset of $\mathbb{C}^{n-1}$.
- This defines the structure of a manifold on the neighborhood

$$V\cap U$$

  of P in V.
- The functions $x_2,\ldots,x_n$ can be used as local coordinates for V in the neighborhood $V\cap U$ of P.

Example


x_2
U = unit disc
$P=(0,0)$
x_1
p
y_2
$W = p(U)$
y_1
$p(P)$
$p : (x_1, x_2) \mapsto (x_1 + x_2^2, x_2)$
$V = V(x_1 + x_2^2)$
$p(V) = V(y_1)$

Linear Part of a Polynomial at a Point

- Consider an arbitrary irreducible affine variety $V \subseteq \mathbb{A}_k^n$.
- Let $f \in k[x_1,\ldots,x_n]$ be a polynomial.
- Let $P = (a_1,\ldots,a_n)$ be a point.
- The **linear part of** f **at** $P = (a_1,\ldots,a_n)$ is defined by

$$f_P^{(\ell)} := \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(P)(x_i - a_i).$$

Tangent Space to an Irreducible Variety at a Point

Definition (Tangent Space)

The **tangent space to an irreducible variety** V **at a point** $P \in V$ is defined by the following intersection of hyperplanes:

$$T_P V := \bigcap_{f \in I(V)} V(f_P^{(\ell)}) \subseteq \mathbb{A}_k^n.$$

- This is an affine subspace of $\mathbb{A}_k^n$ containing P.
- If V is a hypersurface, then this is the same as the tangent space considered previously.

Dimension

Definition (Dimension)

For an irreducible affine variety V, we define the **dimension** of V by

$$\dim V = \min\{\dim T_P V : P \in V\}.$$

- By definition, we have, for all $P \in V$,

$$\dim T_P V \geq \dim V.$$

- We show that $\dim V$ is equal to the dimension of the tangent space $\dim T_P V$, for a general point P on V.

Upper Semicontinuity of Dimension

Proposition

The function $V \to \mathbb{N}$; $P \mapsto \dim T_P V$, is upper semicontinuous in the Zariski topology, i.e., for all $r \in \mathbb{N}$, the set

$$S_r(V) := \{P \in V : \dim T_P V \geq r\}$$

is closed.

- Suppose that $g_1, \ldots, g_m$ generate the ideal $I(V)$.
  For every $f \in I(V)$, the linear part $f_P^{(\ell)}$ can be written as a linear combination of $g_{i,P}^{(\ell)}$.
  It follows that
  $$T_P V = \bigcap_{i=1}^{m} V(g_{i,P}^{(\ell)}) \subseteq \mathbb{A}_k^n.$$

Upper Semicontinuity of Dimension (Cont'd)

- Now we have
$$\dim T_P V = n - \text{rank}\left(\frac{\partial g_i}{\partial x_j}(P)\right)_{ij}.$$
Hence,
$$P \in S_r(V) \quad \text{iff} \quad \text{rank}\left(\frac{\partial g_i}{\partial x_j}(P)\right)_{ij} \leq n - r.$$
The latter is satisfied if and only if all $(n-r+1) \times (n-r+1)$ minors of $\left(\frac{\partial g_i}{\partial x_j}\right)_{ij}$ vanish.
But these minors are polynomial functions.
Therefore, $S_r(V)$ is closed.

Dimension of V and Dimension of $T_P V$

Proposition

Let $r = \dim(V)$ be an irreducible affine variety. Then there is an open dense subset $V_0 \subseteq V$, such that

$$\dim T_P V = \dim V, \quad \text{for all } P \in V_0.$$

- Let $r = \dim(V)$.
  By definition of dimension,
  $$S_r(V) = V \quad \text{and} \quad S_{r+1}(V) \neq V.$$
  By the proposition, $S_{r+1}(V)$ is closed.
  So we can take
  $$V_0 := V \setminus S_{r+1}(V).$$
  V_0 is open and nonempty.

Smooth (Regular) Points of a Variety

Definition (Smooth (Regular) Point)

For an irreducible affine variety V, a point $P \in V$ is a **smooth (regular) point** of V if

$$\dim T_P V = \dim V.$$

Otherwise, P is a **singular point**.

- For a hypersurface $V(f)$ in $\mathbb{A}_k^n$, we have

$$\frac{\partial f}{\partial x_i}(P) \neq 0, \text{ for some } i, \quad \text{iff} \quad \text{rank}\left(\frac{\partial f}{\partial x_i}(P)\right)_i = 1.$$

- Thus, a hypersurface V in $\mathbb{A}_k^n$ has dimension $\dim V = n - 1$.

Codimension

Definition (Codimension)

The **codimension** of an irreducible affine variety V in $\mathbb{A}_k^n$ is defined by

$$\operatorname{codim} V = n - \dim V.$$

- Suppose that $V \subseteq \mathbb{A}_k^n$ is defined by r polynomials $f_1, \ldots, f_r$.
- By the proposition, the codimension is given by the rank of the matrix

$$\left(\frac{\partial f_i}{\partial x_j}\right)_{ij}$$

  at a smooth point P.
- It follows that $\operatorname{codim} V \leq r$.
- Thus, to define an irreducible affine variety of codimension r requires at least r equations.
- In fact this is also the case if V is reducible.

Complex Varieties as Manifolds

- In the case $k = \mathbb{C}$, an irreducible affine variety V of dimension $n - r$ can be considered as a complex manifold of the same dimension.
- Let P be a smooth point of V.
- By definition, there are functions

$$f_1, \ldots, f_r \in I(V),$$

  with linearly independent linear parts at P.
- So the matrix

$$\left(\frac{\partial f_i}{\partial x_j}(P) \right)_{i,j=1,\ldots r}$$

  is nonsingular.

Complex Varieties as Manifolds (Cont'd)

- By the Inverse Function Theorem, the map

$$\begin{array}{rccl} p: & \mathbb{A}_k^n & \to & \mathbb{A}_k^n; \\ & (x_1,\ldots,x_n) & \mapsto & (f_1(x_1,\ldots,x_n),\ldots,f_r(x_1,\ldots,x_n),x_{r+1},\ldots,x_n) \end{array}$$

  induces a diffeomorphism between a neighborhood of P and a neighborhood of $p(P)$.
- We can take

$$x_{r+1},\ldots,x_n$$

  to be local coordinates for V in a neighborhood of P.

Arbitrary Affine Varieties

- Let V be an arbitrary affine variety.
- Suppose V has a decomposition into irreducible components,

$$V = V_1 \cup \cdots \cup V_\ell.$$

Definition (Smooth (Regular) Point)

A point $P \in V$ is a **smooth (regular) point** of V if:

(1) P lies on exactly one irreducible component V_i of V;

(2) P is a smooth point of V_i.

Definition (Dimension)

The **dimension** of V is defined to be the maximum of the dimensions of the irreducible components V.

Subsection 2

Algebraic Characterization of the Dimension of a Variety

Transcendence Basis and Transcendence Degree

- Let V be an irreducible affine variety defined over k.
- Let K be the quotient field of the coordinate ring of V.
- By Noether normalization, there is a (not necessarily unique) field K_t, such that:
  - $k \subseteq K_t \subseteq K$;
  - K_t/k is purely transcendental;
  - K/K_t is algebraic.
- Thus for some integer n, the field K_t is isomorphic to a field of rational functions in n variables over k, i.e., $K_t \cong k(x_1, \ldots, x_n)$.

Transcendence Basis and Transcendence Degree (Cont'd)

- If $K/k(\alpha_1,\ldots,\alpha_m)$ is an algebraic extension, we say that $\alpha_1,\ldots,\alpha_m$ **span** the transcendental part of K.
- If additionally $\alpha_1,\ldots,\alpha_m$ are algebraically independent, then we call the set $\{\alpha_1,\ldots,\alpha_m\}$ a **transcendence basis** of K over k.
- There always exists a transcendence basis.
- Moreover, any two transcendence bases have the same number of elements.
- This number is called the **transcendence degree**, $\text{trdeg}_k K$ of K over the field k.

Dimension and Transcendence Degree

- We will show that the dimension of a variety is equal to the transcendence degree of its function field.
- Consider the case of an irreducible hypersurface $V \subseteq \mathbb{A}_k^n$.
- Let f be an irreducible equation for V.
- The coordinate ring of V is given by

$$k[V] = k[x_1, \ldots, x_n]/(f).$$

- After possible renumbering, we can assume that f contains the variable x_1.
- Then we have

$$k(V) = k(x_2, \ldots, x_n)[x_1]/(f).$$

- So $x_2, \ldots, x_n$ form a transcendence basis of $k(V)$.
- So we have $\text{trdeg}_k k(V) = n - 1 = \dim V$.
- We will show that this equality is true for algebraic varieties in general.

The Maximal Ideal m_P of $\mathscr{O}_{V,P}$

- In the following the tangent space T_PV is considered as a vector space with origin at P.
- Recall the construction of the maximal ideal m_P in the local ring $\mathscr{O}_{V,P}$.
- The maximal ideal of P in $\mathbb{A}_k^n$ is given by

$$M_P = \{f \in k[x_1,\ldots,x_n] : f(P) = 0\} \subseteq k[x_1,\ldots,x_n].$$

- This gives us a maximal ideal in the quotient,

$$\overline{M}_P = M_P/I(V) \subseteq k[V].$$

- Localization gives us the maximal ideal m_P in $\mathscr{O}_{V,P}$,

$$m_P = \left\{\frac{f}{g} \in k(V) : f(P) = 0, g(P) \neq 0\right\} \subseteq \mathscr{O}_{V,P} \subseteq k(V).$$

Algebraic Form of $T_P V$

Theorem

There is an isomorphism of vector spaces

$$T_P V \cong (m_P/m_P^2)^* := \text{Hom}_k(m_P/m_P^2, k),$$

which is natural in the sense of being independent of choice of basis.

- For simplicity we assume that P is the origin, $(0,\ldots,0)$.

  This can always be achieved by a translation.

  Under this hypothesis, $M_P = (x_1,\ldots,x_n)$.

  Let $(k^n)^*$ be the dual space of k^n.

  The coordinates $x_1,\ldots,x_n$ are linear forms on k^n.

  So we can consider them as a basis of $(k^n)^*$.

Algebraic Form of $T_P V$ (Cont'd)

- For all $f \in k[x_1,\ldots,x_n]$, we have a linear function

$$f_P^{(\ell)} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(0) x_i \in (k^n)^*.$$

So we obtain a linear map

$$\begin{array}{rrcl} d: & M_P & \to & (k^n)^*; \\ & f & \mapsto & f_P^{(\ell)}. \end{array}$$

Algebraic Form of $T_P V$ (Cont'd)

Claim: $d : M_P \to (k^n)^*$ is surjective and $\ker d = M_P^2$.

- We have $d(x_i) = x_i$.
  Moreover, the x_i form a basis for $(k^n)^*$.
  So d is surjective.
- For $f(P) \in M_P$, we have

$$f_P^{(\ell)} = 0 \text{ iff all terms of } f \text{ have order at least 2.}$$

  That is, $f_P^{(\ell)} = 0$ iff $f \in M_P^2$.
  So the kernel is M_P^2.

Thus, d induces an isomorphism

$$M_P / M_P^2 \cong (k^n)^*.$$

Algebraic Form of $T_P V$ (Cont'd)

- Now let $V \subseteq \mathbb{A}_k^n$ be an affine variety.
  The inclusion $T_P V \subseteq k^n$ corresponds to a surjection $(k^n)^* \to (T_P V)^*$, given by restriction of linear forms on k^n to $T_P V$.
  Thus, the composition

$$D : M_P / M_P^2 \to (k^n)^* \to (T_P V)^*$$

  is also surjective.
  Claim: $\ker(D) = M_P^2 + I(V)$.
  This can be seen by the following sequence of equivalences.

$$\begin{array}{lll} f \in \ker D & \text{iff} & f_P^{(\ell)}|_{T_P V} = 0 \\ & \text{iff} & f_P^{(\ell)} = \sum a_i g_{i,P}^{(\ell)}, \text{ for some } g_i \in I(V), a_i \in k, \\ & \text{iff} & f - \sum a_i g_i \in M_P^2, \text{ for some } g_i \in I(V), a_i \in k, \\ & \text{iff} & f \in M_P^2 + I(V). \end{array}$$

Algebraic Form of $T_P V$ (Cont'd)

- The last claim gives

$$\overline{M}_P/\overline{M}_P^2 \cong M_P/(M_P^2 + I(V)) \cong (T_P V)^*.$$

It now suffices to show that

$$\overline{M}_P/\overline{M}_P^2 \cong m_P/m_P^2.$$

Then the result will follow by duality.
The inclusion $\overline{M}_P \subseteq m_P$ induces an injection

$$\varphi : \overline{M}_P/\overline{M}_P^2 \to m_P/m_P^2.$$

We show that φ is surjective.
Take $\frac{f}{g} \in m_P$. Then $c := g(0) \neq 0$. Moreover,

$$\frac{f}{c} - \frac{f}{g} = f\left(\frac{1}{c} - \frac{1}{g}\right) \in m_P^2.$$

So $\varphi(\frac{f}{c}) = \overline{\frac{f}{g}} \in m_P/m_P^2$.

Local Dependence of $T_P V$

Corollary

The tangent space $T_P V$ depends only on a neighborhood of P in V. That is, if $f : V \dashrightarrow W$ is a birational map which maps a neighborhood V_0 of P isomorphically to a neighborhood W_0 of $Q = f(P)$, then there is an isomorphism

$$T_P V = T_Q W.$$

- Let $f : V \dashrightarrow W$ be birational and $f(P) = Q$.

  Then we have an isomorphism

  $$f^* : k(W) = k(W_0) \to k(V) = k(V_0),$$

  which maps functions regular at Q to functions regular at P.

Local Dependence of $T_P V$ (Cont'd)

- We consider the isomorphism

$$f^* : k(W) = k(W_0) \to k(V) = k(V_0).$$

Clearly, functions vanishing at P map to functions vanishing at Q.

So m_Q is mapped to m_P.

Thus, f^* induces an isomorphism

$$\begin{array}{cccc} \overline{f^*}: & m_Q/m_Q^2 & \to & m_P/m_P^2 \\ & \| & & \| \\ & (T_Q W)^* & & (T_P V)^* \end{array}$$

Corollary

If V and W are birationally equivalent varieties, then $\dim V = \dim W$.

Differentials

Definition (Differential)

Let $f : V \to W$ be a morphism with $f(P) = Q$.
Dualizing the map

$$\overline{f^*} : m_Q / m_Q^2 \to m_P / m_P^2$$

gives a homomorphism

$$df(P) : T_P V \to T_Q W.$$

It is called the **differential of f at P**.

- In the real and complex cases, this coincides with the usual differentials defined in analysis.

Dimension and Transcendence Degree

Theorem

If V is an irreducible affine variety, then

$$\dim V = \operatorname{trdeg}_k k(V).$$

- V is birationally equivalent to an affine hypersurface.

  This hypersurface has:
  - The same dimension as V;
  - Isomorphic function field.

  So the result follows from the proof for hypersurfaces, given previously.

Set of Smooth Points

Corollary

For every affine variety V, the set of smooth points is an open dense subset of V.

- This can be inferred directly using the definition of regular points.
- Alternatively, one can reduce to the case of a hypersurface.

Smooth Points of Quasi-Projective Varieties

Definition (Smooth or Regular Point)

Let V be a quasi-projective variety. A point $P \in V$ is a **smooth** (or **regular**) **point** of V if there is an affine neighborhood $U \subseteq V$ of P, such that P is a smooth point of U.

- This holds for all affine neighborhoods containing P.
- Thus, the set of smooth points is again an open dense subset.

Dimension of Quasi-Projective Varieties

- We define the *dimension* of a quasi-projective variety.
- Suppose that V is an *irreducible* projective variety in $\mathbb{P}_k^n$.
- Consider its standard affine covering

$$V = V_0 \cup \cdots \cup V_n.$$

- We may assume that V is not contained in any hyperplane $H_i \subseteq \mathbb{P}_k^n$.
- Then, it follows easily that all the V_i have the same dimension.
- We call this the **dimension of** V.
- If V is *reducible*, then, we define the **dimension of** V to be the maximum of the dimensions of the components.

Krull Dimension of a Variety

- Every variety is birationally equivalent to an affine variety.
- So in the following V is always an irreducible affine variety.
- V is a Noetherian topological space.
- That is, every chain

$$V = V_0 \supsetneqq V_1 \supsetneqq V_2 \supsetneqq \cdots \supsetneqq V_\ell \neq \emptyset$$

of irreducible closed sets has finite length ℓ.

Definition (Krull Dimension)

The **Krull dimension** of V, denoted $\text{krdim}(V)$, is the supremum of the lengths of all descending chains of the form

$$V = V_0 \supsetneqq V_1 \supsetneqq V_2 \supsetneqq \cdots \supsetneqq V_\ell \neq \emptyset.$$

Krull Dimension of a Ring

Definition (Krull Dimension)

Let A be a ring.

(1) The **height**, $\text{ht}(I)$, of a prime ideal I is the supremum over the lengths ℓ of all chains of prime ideals of the form

$$I_0 \subsetneqq I_1 \subsetneqq I_2 \subsetneqq \cdots \subsetneqq I_\ell = I.$$

(2) The **Krull dimension** of A, denoted $\dim A$, is the supremum over the heights $\text{ht}(I)$ of all prime ideals $I \neq A$.

Krull Dimension of Varieties and Rings

- Consider a descending chain of irreducible closed sets

$$V = V_0 \supsetneqq V_1 \supsetneqq V_2 \supsetneqq \cdots \supsetneqq V_\ell \neq \emptyset.$$

- Set

$$I_i = I(V_i).$$

- Then

$$\{0\} \subsetneqq I_1 \subsetneqq I_2 \subsetneqq \cdots \subsetneqq I_\ell$$

  is an ascending chain of prime ideals in the coordinate ring $k[V] = k[x_1, \ldots, x_n]/I(V)$.
- Conversely, a chain of prime ideals gives rise to a chain of irreducible closed sets.
- Thus, we immediately have

$$\text{krdim}(V) = \dim k[V].$$

Dimension of a Variety and Krull Dimension

- The fact that the Krull dimension is the same as the dimension defined before follows from a theorem in commutative algebra.

Theorem

Let A be an integral domain which is a finitely generated k-algebra. Let K be the field of fractions of A. Then

$$\dim A = \operatorname{trdeg}_k A.$$

Corollary

If V is an irreducible affine variety, then

$$\operatorname{krdim}(V) = \operatorname{trdeg}_k k(V) = \dim V.$$

- Combining previously obtained results.

Dimension of $\mathscr{O}_{V,P}$ and Dimension of m/m^2

- Concerning the Krull dimension of the local ring $\mathscr{O}_{V,P}$ at any point $P \in V$, we mention, without proof,

Proposition

If V is an irreducible affine variety, then for all points $P \in V$,

$$\dim \mathscr{O}_{V,P} = \dim k[V].$$

- Also, from commutative algebra, we have

Proposition

Let A be a Noetherian local ring with maximal ideal m and residue class field $k = A/m$. Then

$$\dim_k(m/m^2) \geq \dim A.$$

- For $A = \mathscr{O}_{V,P}$, the statement follows from previous results.

Smoothness and Regularity

Definition (Regular Local Ring)

A Noetherian local ring A, with maximal ideal m, is called a **regular local ring** if

$$\dim_k(m/m^2) = \dim A.$$

Corollary

A point $P \in V$ is smooth if and only if $\mathcal{O}_{V,P}$ is a regular ring.

- By a previous theorem, a point P is smooth if and only if

$$\dim_k(m_P/m_P^2) = \dim V.$$

We also know

$$\dim V = \dim k[V] = \dim \mathcal{O}_{V,P}.$$

It follows that P is smooth if and only if $\mathcal{O}_{V,P}$ is a regular ring.

Resolution of Singularities (Curves and Blow-up)

- Consider the curves C and C' in $\mathbb{A}_k^2$, given by

$$\begin{array}{rcl} C & := & \{(x,y) \in \mathbb{A}_k^2 : y^2 = x^3 + x^2\}, \\ C' & := & \{(x,y) \in \mathbb{A}_k^2 : y^2 = x^3\}. \end{array}$$

- Recall that the blow-up of $\mathbb{A}_k^2$ at the origin is given by:
  - The variety

$$\widetilde{\mathbb{A}}_k^2 := \{((x,y),(t_0 : t_1)) \in \mathbb{A}_k^2 \times \mathbb{P}_k^1 : xt_1 - yt_0 = 0\};$$

  - The projection map $\pi : \widetilde{\mathbb{A}}_k^2 \to \mathbb{A}_k^2$ which is a bijection away from the exceptional line $E := \pi^{-1}\{(0,0)\}$.
- Recall that $\widetilde{\mathbb{A}}_k^2$ has a covering by affine sets V_0 and V_1.
- We again identify V_0 with $\mathbb{A}_k^2$, and take coordinates $x, u = \frac{t_1}{t_0}$.
- Set $\rho := \pi \mid_{V_0}$.

Resolution of Singularities (Pullbacks)

- Now $\rho^* : k(\mathbb{A}_k^2) \to k(\widetilde{\mathbb{A}}_k^2)$ is given by
$$x \mapsto x, \quad y \mapsto xu.$$
- So the **pullbacks** of the equations for C and C' are given by
$$\begin{aligned} \rho^*(x^3 + x^2 - y^2) &= x^2(x + 1 - u^2), \\ \rho^*(x^3 - y^2) &= x^2(x - u^2). \end{aligned}$$
- These are the equations for $\pi^{-1}(C)$ and $\pi^{-1}(C')$, respectively.
- Let $\widetilde{C}$ and $\widetilde{C}'$ be the curves given by
$$\begin{aligned} \widetilde{C} \cap V_0 &= \{(x, u) \in \mathbb{A}_k^2 : x + 1 - u^2 = 0\}, \\ \widetilde{C}' \cap V_0 &= \{(x, u) \in \mathbb{A}_k^2 : x - u^2 = 0\}. \end{aligned}$$

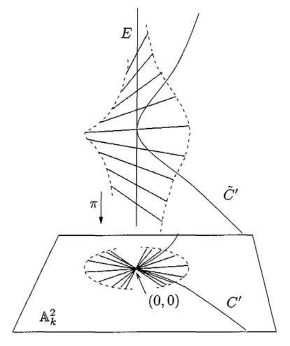
Resolution of Singularities (Pullbacks Cont'd)

- The pullback of the equation for C vanish on $\widetilde{C}$.
- The pullback of the equation for C' vanishes on $\widetilde{C}'$.
- The exceptional line E is given by $x = 0$.
- So the pulled-back equations both vanish with multiplicity 2 along E.
- We express this by writing

$$\begin{aligned} \pi^{-1}(C) &= \widetilde{C} + 2E, \\ \pi^{-1}(C') &= \widetilde{C}' + 2E. \end{aligned}$$

Resolution of Singularities (Resolution)

- Restricted to E, the equations for C and C' are given by u^2-1 and u^2.
- One has simple zeros at $u=\pm 1$, and the other has a double zero at $u=0$.
- E is a tangent line to the curve $\widetilde{C}'$, which meets E at one point with multiplicity two.
- E meets $\widetilde{C}$ transversally at two points.



- In general, the **strict transform** of a curve C is given by the closure of $\pi^{-1}(C\setminus\{(0,0)\})$.
- In this example, the strict transforms $\widetilde{C}$ and $\widetilde{C}'$ are smooth curves, birational to C and C', respectively, via π.
- For each curve the singularity at the origin has been "resolved".