Introduction to Algebraic Geometry

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 500

Subsection 1

Plane Curves

Plane Curves

- The ground field k is assumed to be algebraically closed with characteristic not equal to 2 or 3.
- A nonzero homogeneous polynomial $f \in k^d[x_0, x_1, x_2]$ of degree d defines a one-dimensional projective variety

$$V(f) = \{(x_0 : x_1 : x_2) : f(x_0, x_1, x_2) = 0\} \subseteq \mathbb{P}_k^2,$$

which we call a **plane curve**.
- For $c \in k^*$, the polynomials f and cf define the same curve.
- We distinguish between curves defined by f and by powers of f.

Irreducible Components

- Suppose that f has a decomposition into irreducible factors

$$f = f_1^{d_1} \cdots f_r^{d_r}$$

(unique up to permutation and constant factors).

- We write C as a formal sum

$$C = d_1 C_1 + \cdots + d_r C_r,$$

where $C_i := V(f_i)$.

- The curves C_i are called the **irreducible components** of C.
- If f has no repeated components, then this corresponds to the decomposition of $V(f)$ into irreducible components.

Plane Curves of Degree d

- Suppose
$$C = d_1 C_1 + \cdots + d_r C_r.$$
- We use the notation
$$C = \{f = 0\}$$
to denote the set of curves C_i counted with multiplicity.
- We call C a **plane curve of degree** d.
- We have a bijection
$$\{\text{plane curves of degree } d\} \quad \overset{1:1}{\longleftrightarrow} \quad \mathbb{P}(k^d[x_0, x_1, x_2]).$$
- Thus, the set of plane curves of degree d forms a projective space of dimension $\binom{d+2}{2} - 1$.
- The space of lines, i.e., the space of projective curves of degree 1, is the dual projective plane $(\mathbb{P}_k^2)^*$.

Smooth and Singular Points

- If f has no repeated factors, then a point P is a **singular point** of C if and only if
$$f(P)=0 \quad \text{and} \quad \frac{\partial f}{\partial x_i}(P)=0,\ i=0,1,2.$$
- The following relation is known as **Euler's Formula**,
$$d\cdot f=\sum x_i\frac{\partial f}{\partial x_i}.$$
- It implies that, if $\operatorname{char} k=0$ or $\operatorname{char} k>d$, then P is a singular point if and only if
$$\frac{\partial f}{\partial x_i}(P)=0, \quad i=0,1,2.$$
- We will also use this as a definition of **smooth** and **singular points** when f has repeated factors.

Example

- In contrast to the previous chapters, C may have no smooth points.

  Example: Consider
  $$f = x_0^2.$$
  Then we have
  $$C = \{f = 0\}.$$
  So C has no smooth points.

Tangent Space

- For $P \in C$, we define the **tangent space to C at P** by

$$T_P C := \left\{ \sum_{i=0}^{2} \frac{\partial f}{\partial x_i}(P) x_i = 0 \right\} \subseteq \mathbb{P}_k^2.$$

- This is the projective closure of the affine tangent space.
- A point P on C is a smooth point of C if and only if $T_P C$ is a line.

Projective Equivalence

- A map $A \in \mathrm{Gl}(3,k)$ takes lines through the origin to lines through the origin.
- Thus, it induces a **projective transformation**
$$\varphi = \varphi_A : \mathbb{P}_k^2 \to \mathbb{P}_k^2.$$
- Let C and C' be two curves, with defining equations f and f', respectively.
- We say C and C' are **projectively equivalent** if there is a transformation of coordinates $A \in \mathrm{Gl}(3,k)$, such that the corresponding transformation of $\mathbb{P}(k[x_0,x_1,x_2])$ maps f to f'.
- Our goal is to classify plane curves of degree 3 (cubics) up to projective equivalence.

Curves Decomposing Into Three Lines

Lemma

Let C be a plane cubic curve which decomposes into three lines. Then C is projectively equivalent to one of the following curves:

(1) $C = \{x_0x_1x_2 = 0\}$;

(2) $C = \{x_0x_1(x_0 + x_1) = 0\}$;

(3) $C = \{x_0^2x_1 = 0\}$;

(4) $C = \{x_0^3 = 0\}$.

- By assumption,
$$C = \ell_1 \cup \ell_2 \cup \ell_3,$$
where the lines ℓ_i are considered as points of the dual space $(\mathbb{P}_k^2)^*$. We distinguish various cases.

Curves Decomposing Into Three Lines (Cont'd)

(1) The lines ℓ_i are all different and do not all intersect in a common point.

Equivalently, the points $\ell_1, \ell_2, \ell_3 \in (\mathbb{P}_k^2)^*$ do not lie on a line.

Now we use the transitivity of the action of $\mathrm{Gl}(3,k)$ on $(\mathbb{P}_k^2)^*$.

Explicitly, the coordinates of these points define the rows of a matrix in $\mathrm{Gl}(3,k)$ which maps x_0, x_1, x_2 to ℓ_1, ℓ_2, ℓ_3.

Curves Decomposing Into Three Lines (Cont'd)

(2) The three lines are different, and there is a point that lies on all three lines.

Thus, there is a line through the distinct points $\ell_i \in (\mathbb{P}^2_k)^*$.

Now note the following:

- $\mathrm{Gl}(3,k)$ acts transitively on $(\mathbb{P}^2_k)^*$;
- $\mathrm{Gl}(2,k)$ acts 3-transitively on $(\mathbb{P}^1_k)^*$.
  That is, any two triples of distinct points are equivalent under the action of $\mathrm{Gl}(2,k)$.

So all such sets of points are equivalent under the action of $(\mathbb{P}^2_k)^*$.

(3) $\ell_1 = \ell_2 \neq \ell_3$: A similar argument gives Case (3).

(4) $\ell_1 = \ell_2 = \ell_3$: This gives Case (4).

Subsection 2

Intersection Multiplicity

Intersection Multiplicity

- Consider two plane curves.
  - C defined by a polynomial f.
  - C' defined by a polynomial g.
- Assume, first, that C and C' have no common component.

Definition (Intersection Multiplicity)

The **intersection multiplicity** $I_P(C,C')$ **of** C **and** C' **at a point** $P \in \mathbb{P}_k^2$ is defined by

$$I_P(C,C') := \dim_k \mathscr{O}_{\mathbb{P}_k^2,P}/(f,g).$$

Property of Intersection Multiplicity

Lemma

Let C and C' be two plane curves. Then

$$I_P(C, C') \geq 1 \quad \text{iff} \quad P \in C \cap C'.$$

- Assume, first, that $P \notin C \cap C'$. Then $P \notin C$ or $P \notin C'$.
  In the first case, f is a unit in $\mathscr{O}_{\mathbb{P}_k^2, P}$.
  In the second, g is a unit in $\mathscr{O}_{\mathbb{P}_k^2, P}$.
  Thus, $(f, g) = \mathscr{O}_{\mathbb{P}_k^2, P}$. I.e., $\dim_k \mathscr{O}_{\mathbb{P}_k^2, P}/(f, g) = 0$.
  Conversely, suppose $P \in C \cap C'$.
  Then $f, g \in m_P$. So $(f, g) \subseteq m_P$.
  Thus,
  $$I_P(C, C') = \dim_k \mathscr{O}_{\mathbb{P}_k^2, P}/(f, g) \geq \dim_k \mathscr{O}_{\mathbb{P}_k^2, P}/m_P = 1.$$

Line and Point on a Plane Curve

Claim: For a line $L \subseteq \mathbb{P}_k^2$ through a point P on C, we have

$$I_P(C,L) = \text{mult}_P(f|_L).$$

To see this, first make a linear transformation so that:

- $P = (0:0:1)$;
- L is given by $x_1 = 0$.

Consider the affine coordinates

$$x = \frac{x_0}{x_2}, \quad y = \frac{x_1}{x_2}$$

for the affine subspace $\mathbb{A}_k^2$ of $\mathbb{P}_k^2$, where $x_2 \neq 0$.
In terms of x, y, the line $L \cap \mathbb{A}_k^2$ is given by $y = 0$.
We have

$$\begin{aligned} I_P(C,L) &= \dim_k \mathscr{O}_{\mathbb{P}_k^2,P}/(f,x_1) \\ &= \dim_k \mathscr{O}_{\mathbb{A}_k^2,0}/(f(x,y,1),y) \\ &= \text{mult}_P(f|_L). \end{aligned}$$

Transverse Intersection

- Let C be a plane curve defined by f.
- Let $L \subseteq \mathbb{P}_k^2$ be a line through a point P on C.
- We showed

$$I_P(C,L) = \text{mult}_P(f|_L).$$

- It follows that

$$I_P(C,L) \geq 2 \quad \text{iff} \quad L \subseteq T_P C.$$

Definition (Transverse Intersection)

The curves C and C' intersect **transversely** at a point $P \in C \cap C'$ if C and C' are both smooth at P and the tangent lines at P are distinct, i.e.,

$$T_P C \cap T_P C' = \{P\}.$$

Another Version of Nakayama's Lemma

Lemma

Let V be a quasi-projective variety and let $P \in V$. If the elements $f_1, \ldots, f_r \in m_P$ generate the k-vector space m_P/m_P^2, then they also generate the ideal m_P.

- Note that this is a local statement.

  Further, any quasi-projective variety is covered by affine sets.

  So we may assume that V is an affine variety in $\mathbb{A}_k^n$.

  Set

  $$B := \mathscr{O}_{V,P}/(f_1, \ldots, f_r) \quad \text{and} \quad A := m_P/(f_1, \ldots, f_r).$$

  Suppose $P = (a_1, \ldots, a_n)$.

Another Version of Nakayama's Lemma (Cont'd)

- Then the residue classes of

$$x_1 - a_1, \ldots, x_n - a_n$$

generate the module A over B.

Thus, A is a finite B-algebra.

So, by Nakayama's Lemma, either $A = 0$ or $m_P A \neq A$.

However,

$$m_P A = ((f_1, \ldots, f_r) + m_P^2)/(f_1, \ldots, f_r) = m_P/(f_1, \ldots, f_r) = A.$$

So we must have $A = 0$.

I.e., $m_P = (f_1, \ldots, f_r)$.

Criterion for Transverse Intersection

Lemma

Two curves C, C' intersect transversally at P if and only if $I_P(C, C') = 1$.

- We assume that $P = (0:0:1)$ and work with affine coordinates x, y.

  Since $P \in C \cap C'$, we have $(f, g) \subseteq m_P$.

  So $I_P(C, C') = 1$ is equivalent to the statement $(f, g) = m_P$.

  Suppose, first, $I_P(C, C') = 1$.

  Then the linear parts of f and g must span the two-dimensional vector space m_P/m_P^2.

  In particular they are linearly independent.

  So the tangent lines of C and C' at P are distinct.

Criterion for Transverse Intersection (Cont'd)

- Assume, conversely, that C and C' intersect transversally at P.

  Perform a suitable transformation of coordinates so that the linear parts of f and g are given, respectively, by

  $$f_P^{(\ell)} = x \quad \text{and} \quad g_P^{(\ell)} = y.$$

  Then, either by the lemma or using an elementary argument, we may show that f and g generate the ideal m_P.

Intersection Multiplicity in General

- If we omit the condition that C and C' have no common component, we can still define the **local intersection multiplicity** $I_P(C, C')$.
- If P lies on a common component of C and C', we set

$$I_P(C, C') := \infty.$$

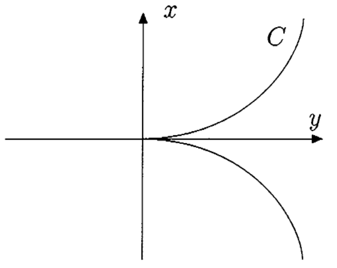
Example

- Consider the semicubical parabola C, given in projective coordinates by

$$z_0^2 z_2 - z_1^3 = 0.$$

In local coordinates, in a neighborhood of $P = (0:0:1)$, it is given by

$$x^2 - y^3 = 0.$$



Let L_1 and L_2 be the lines given by $z_0 = 0$ and $z_1 = 0$, respectively. Then

$$I_P(C, L_1) = \dim_k \mathscr{O}_{\mathbb{P}_k^2, P} / (x^2 - y^3, x) = \dim_k \mathscr{O}_{\mathbb{P}_k^2, P} / (x, y^3) = 3;$$
$$I_P(C, L_2) = \dim_k \mathscr{O}_{\mathbb{P}_k^2, P} / (x^2 - y^3, y) = \dim_k \mathscr{O}_{\mathbb{P}_k^2, P} / (x^2, y) = 2.$$

Bézout's Theorem

Theorem (Bézout)

Let C and C' be two plane curves of degree d and d', respectively, which have no common components. Then C and C' intersect in dd' points, i.e.,

$$C.C' := \sum_P I_P(C, C') = dd'.$$

- We will give a proof of Bézout's Theorem later, in the case that one of the curves is smooth.

Bézout's Theorem: Special Case

- We only need the following special case of Bézout's Theorem.
- It is essentially a reinterpretation of the Fundamental Theorem of Algebra.

Proposition

Let $C \in \mathbb{P}_k^2$ be a curve of degree d. Let L be a line not contained in C. Then C and L intersect in d points (counted with multiplicity), i.e.,

$$\sum_P I_P(C,L) = d.$$

Proof of the Special Case

- Suppose
$$C = \{f = 0\},$$
where $f = f(x_0, x_1, x_2)$ is a homogeneous polynomial of degree d.
Let $L = \{x_2 = 0\}$.
Then
$$f\mid_L = f(x_0, x_1, 0) = a_0 x_0^d + a_1 x_0^{d-1} x_1 + \cdots + a_d x_1^d$$
is a homogeneous polynomial of degree d in two variables.
We may assume that the coordinates have been chosen so that $(1:0) \in L$ is not a zero of $f\mid_L$.
Then the number of points where C and L intersect is obtained by counting, with multiplicity, the zeros of the polynomial
$$f(x) = a_0 x^d + a_1 x^{d-1} + \cdots + a_d.$$
By the Fundamental Theorem of Algebra, there are exactly d zeros.

Projective Classification of Conics

- Every homogeneous quadratic equation can be written in the form

$$q(x_0,x_1,x_2) = q_A(x_0,x_1,x_2) = xA^t x,$$

with $x = (x_0 : x_1 : x_2)$ and $A = {}^tA \in \mathrm{Mat}(3x3,k)$.
- Since we are assuming throughout that k is algebraically closed, the only invariant under a change of coordinates is the rank of the matrix.
- Up to equivalence we have the following cases:
  (1) $q(x_0,x_1,x_2) = x_0^2 + x_1^2 + x_2^2$ (smooth conic);
  (2) $q(x_0,x_1,x_2) = x_0^2 + x_1^2$ (pair of lines);
  (3) $q(x_0,x_1,x_2) = x_0^2$ (double line).
- Note that the conic $x_0^2 + x_1^2 + x_2^2$ is projectively equivalent to $x_0x_2 - x_1^2 = 0$.

Decomposable Plane Cubic

Proposition

Let C be a plane cubic which decomposes into an irreducible conic (a curve of degree 2) and a line. Then C is projectively equivalent to one of the following two curves:

(1) $C_1 = \{(x_0x_2 - x_1^2)x_1 = 0\}$;

(2) $C_2 = \{(x_0x_2 - x_1^2)x_0 = 0\}$.

- By assumption
$$C = C_0 + L,$$
where:
  - C_0 is an irreducible conic;
  - L is a line.

Decomposable Plane Cubic

- The conic C_0 is given by an irreducible quadric

$$\{q(x_0, x_1, x_2) = 0\}.$$

By the Principal Axes Theorem, given by the above classification of conics, C_0 is projectively equivalent to the conic

$$\{x_0x_2 - x_1^2 = 0\}.$$

By the proposition, the line L intersects the conic in two points. Since C_0 is smooth, we have the following possibilities:

(1) L intersects C_0 transversally in two points;
(2) L is tangent to C_0 at a point P_0.

Decomposable Plane Cubic (Cont'd)

- C_0 is the image of

$$\begin{array}{rccl}\varphi: & \mathbb{P}_k^1 & \to & C_0 \subseteq \mathbb{P}_k^2; \\ & (t_0 : t_1) & \mapsto & (t_0^2 : t_0 t_1 : t_1^2).\end{array}$$

A transformation $t_0 \mapsto at_0 + bt_1$, $t_1 \mapsto ct_0 + dt_1$, induces a map

$$\begin{array}{rcl} t_0^2 & \mapsto & a^2 t_0^2 + 2ab t_0 t_1 + b^2 t_1^2, \\ t_0 t_1 & \mapsto & ac t_0^2 + (ad+bc) t_0 t_1 + bd t_1^2, \\ t_1^2 & \mapsto & c^2 t_0^2 + 2cd t_0 t_1 + d^2 t_1^2.\end{array}$$

Thus, the matrix

$$\left(\begin{array}{ccc} a^2 & 2ab & b^2 \\ ac & (ad+bc) & bd \\ c^2 & 2cd & d^2 \end{array}\right)$$

defines a transformation of $\mathbb{P}_k^2$ which maps the conic C_0 to itself.

Decomposable Plane Cubic (Cont'd)

- We use a suitable matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

  - In Case (1), we can map the points of intersection to the points $(1:0:0)$ and $(0:0:1)$.
  - In Case (2), we can map P_0 to the point $(0:0:1)$.

The line through $(1:0:0)$ and $(0:0:1)$ is

$$\{x_1 = 0\}.$$

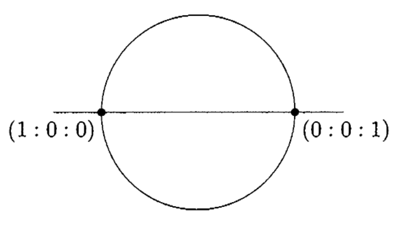
The tangent to the conic C_0 at $(0:0:1)$ is

$$\{x_0 = 0\}.$$

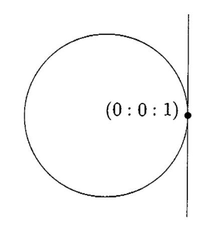
This completes the proof.

Illustration

- The curve $C_1 = \{(x_0x_2 - x_1^2)x_1 = 0\}$.



- The curve $C_1 = \{(x_0x_2 - x_1^2)x_0 = 0\}$.



Irreducible Singular Cubics

Proposition

Let C be an irreducible singular cubic. Then C has exactly one singularity and is projectively equivalent to one of the following curves:

(a) $C = \{x_1^2 x_2 - x_0^3 - x_0^2 x_2 = 0\}$;

(b) $C = \{x_2 x_1^2 - x_0^3 = 0\}$.

- Suppose C has at least two singularities.

  Consider the line through the two singularities.

  It would intersect the curve C in at least 4 points, with multiplicity.

  This is a contradiction.

Irreducible Singular Cubics (Cont'd)

- We may assume that the singularity of C is at $P = (0:0:1)$.

  f does not factor into three linear forms.

  This means that the equation f for C has the form

  $$f = x_2 q(x_0, x_1) + b x_0^3 + c x_0^2 x_1 + d x_0 x_1^2 + e x_1^3,$$

  where

  $$q(x_0, x_1) \not\equiv 0.$$

  The quadratic q factors into linear forms

  $$q(x_0, x_1) = \ell_0(x_0, x_1) \ell_1(x_0, x_1).$$

  Accordingly, there are two cases to consider.

Irreducible Singular Cubics: Case $\ell_0(x_0,x_1) \neq c\ell_1(x_0,x_1)$

(a) The case $\ell_0(x_0,x_1) \neq c\ell_1(x_0,x_1)$.
By applying a transformation of coordinates we may assume that $\ell_0(x_0,x_1) = x_0$ and $\ell_1(x_0,x_1) = x_1$.
Then f has the form

$$f = x_2x_0x_1 + b'x_0^3 + c'x_0^2x_1 + d'x_0x_1^2 + e'x_1^3.$$

Since f is irreducible, $b'e' \neq 0$. Let $\beta^3 = b'$ and $\gamma^3 = e'$.
Then set

$$\begin{array}{rcl} x_2 & = & \beta\gamma(x_2' - 6x_0') + \frac{c'}{\beta}(x_0' + x_1') + \frac{d'}{\gamma}(x_0' - x_1'), \\ x_0 & = & -\frac{1}{\beta}(x_0' + x_1'), \\ x_1 & = & -\frac{1}{\gamma}(x_0' - x_1'), \end{array}$$

Then we obtain

$$f = x_2'(x_0'^2 - x_1'^2) - 8x_0'^3.$$

This is projectively equivalent to $x_1^2x_2 - x_0^2x_2 - x_0^3$.

Irreducible Singular Cubics: Case $\ell_0(x_0,x_1) = c\ell_1(x_0,x_1)$

(b) The case $\ell_0(x_0,x_1) = c\ell_1(x_0,x_1)$.

We may assume that $\ell_0(x_0,x_1) = \ell_1(x_0,x_1) = x_1$.

Then,

$$f = x_2x_1^2 + b'x_0^3 + c'x_0^2x_1 + d'x_0x_1^2 + e'x_1^3.$$

We have $b' \neq 0$, since otherwise x_1 divides f.

Change variables $x_0 = x_0' - \frac{c'}{3b'}x_1$, to get

$$f = x_2x_1^2 + b'x_0'^3 + d''x_0'x_1^2 + e''x_1^3, \quad b' \neq 0.$$

The change variables $x_2 = -b'x_2' - d''x_0' - e''x_1$, to get

$$f = -b'(x_2'x_1^2 - x_0'^3).$$

This is projectively equivalent to $x_2x_1^2 - x_0^3$.

Subsection 3

Classification of Smooth Cubics

Flex Points

Definition (Flex Point)

Let C be a plane curve of arbitrary degree. A smooth point $P \in C$ is called a **point of inflection**, or a **flex point**, of C if $I_P(C, T_P C) \geq 3$.
The tangent $T_P C$ to C at a flex point P is called an **inflectional tangent line**, or an **inflectional tangent**, of C.

- We will show that the inflection points of C are determined by the intersection of C with its Hessian.

The Hessian

Definition (Hessian)

For a plane curve $C = \{f = 0\}$, the **Hessian** of C is given by

$$H_f := \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{0 \le i,j \le 2}.$$

- Suppose H_f does not vanish identically.
- Then H_f is a homogeneous polynomial of degree $3(d-2)$.
- Let
$$H = \{H_f = 0\} \subseteq \mathbb{P}_k^2.$$
- If $d = 2$ then H may be empty.
- Otherwise, either $H = \mathbb{P}_k^2$ or H is a plane curve of degree $3(d-2)$, called the **Hessian curve** of C.

Characterization of Flex Points

Proposition

Let C be a smooth plane curve of degree $d \geq 3$ with Hessian curve H. Assume that $(\text{char}(k), d-1) = 1$. Then $H \cap C$ is equal to the set of inflection points of C.

- We show the statement is invariant under change of coordinates.

  Consider the transformation $A : \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix} \mapsto A \begin{pmatrix} x_0 \\ x_1 \\ x_2 \end{pmatrix}$, $A \in \text{Gl}(3,k)$.

  Given a polynomial $f \in k[x_0, x_1, x_2]$, denote by f^* the transform $f \circ A$.

  Using the chain rule, one can show that $H_{f^*} = (\det A)^2 (H_f)^*$.

  Thus, the statement is invariant under change of coordinates.

  So we may assume that $P = (0:0:1)$ and that $T_p C = \{x_1 = 0\}$.

Characterization of Flex Points (Cont'd)

- Consider the affine coordinates $x = \frac{x_0}{x_2}$ and $y = \frac{x_1}{x_2}$.

  We have, for some $a, b, c, e \in k$, $a \neq 0$,

$$f(x,y) = y(a + bx + cy + g(x,y)) + ex^2 + h(x),$$

  where :
  - All terms of $g(x,y)$ have order ≥ 2;
  - All terms of h have order ≥ 3.

  In homogeneous coordinates we have

$$\begin{aligned} f(x_0,x_1,x_2) &= ax_2^{d-1}x_1 + bx_2^{d-2}x_0x_1 + cx_2^{d-2}x_1^2 + ex_2^{d-2}x_0^2 \\ &\quad + \text{terms of order} \geq 3 \text{ in } x_0, x_1. \end{aligned}$$

  To evaluate $H_f(0:0:1)$ note that the derivatives of all "terms of order ≥ 3 in x_0, x_1" evaluated at $(0:0:1)$ vanish.

Characterization of Flex Points (Cont'd)

- Thus, we have

$$H_f(0:0:1) = \det\begin{pmatrix} 2e & b & 0 \\ b & 2c & (d-1)a \\ 0 & (d-1)a & 0 \end{pmatrix} = -2ea^2(d-1)^2.$$

By hypothesis, $(\text{char}(k), d-1) = 1$ and $a \neq 0$.

So we get

$$H_f(0:0:1) = 0 \quad \text{iff} \quad e = 0.$$

Now we have

$$\mathscr{O}_{\mathbb{P}^2_k, P}/(f, x_1) = \mathscr{O}_{\mathbb{A}^2_k, 0}/(ex^2 + h(x), y).$$

Hence,

$$\begin{aligned} I_P(C, T_P C) \geq 3 \quad & \text{iff} \quad e = 0 \\ & \text{iff} \quad P \in H. \end{aligned}$$

The Veronese Map

- Let d be a positive integer.
- Define
$$\Lambda_d := \{(i_0, i_1, i_2) \in \mathbb{N}_0^3 : i_0 + i_1 + i_2 = d\}.$$
- A combinatorial argument shows that
$$|\Lambda_d| = \binom{d+2}{2}.$$
- For $I := (i_0, i_1, i_2) \in \Lambda_d$, define
$$x_I = x_{(i_0, i_1, i_2)} := x_0^{i_0} x_1^{i_1} x_2^{i_2}.$$
- The **Veronese map** $v_d : \mathbb{P}_k^2 \to \mathbb{P}_k^N$, with $N = \binom{d+2}{2} - 1$, is given by
$$v_d(x_0 : x_1 : x_2) = (x_0^d : x_0^{d-1} x_1 : \ldots : x_2^d) = (\ldots : x_I : \ldots)_{I \in \Lambda_d}.$$

Complement of a Plane Curve in $\mathbb{P}_k^2$

Lemma

The complement $\mathbb{P}_k^2 \setminus C$ of a plane curve C is affine.

- Consider the Veronese map

$$v_d : \mathbb{P}_k^2 \to \mathbb{P}_k^N, \quad N = \binom{d+2}{2} - 1.$$

We denote the coordinates of $\mathbb{P}_k^N$ by

$$z_I, \quad I \in \Lambda_d.$$

As for the Segre map, it can be checked that v_d is an embedding. Its image is given by the equations

$$z_I z_J = z_K z_L, \quad 0 \le |I|, |J|, |K|, |L| \le d,$$

such that $|I| + |J| = |K| + |L|$.

Complement of a Plane Curve in $\mathbb{P}^2_k$ (Cont'd)

- Let f be an equation for C of degree d.

  We can write this in the form:

  $$f = \sum_{I \in \Lambda_d} a_I x_I.$$

  Consider, in $\mathbb{P}^N_k$, the hyperplane

  $$H = \left\{\sum a_I z_I = 0\right\}.$$

  Then we have

  $$v_d(C) = v_d(\mathbb{P}^2_k) \cap H.$$

  This argument shows that $\mathbb{P}^2_k \setminus C$ is affine.

Inflection Points on a Smooth Curve of Degree ≥ 3

- We show that there is at least one inflection point on a smooth curve of degree $d \geq 3$.
- Equivalently, the intersection of a smooth curve of degree $d \geq 3$ with its Hessian is nonempty.
- This is trivial if the Hessian is all of $\mathbb{P}_k^2$.
- If not, we resort to a Weak Form of Bezout's Theorem.

A Weak Form of Bézout's Theorem

Lemma

The intersection of two plane curves C and C' is nonempty.

- Let C be a curve of degree d.

  By the lemma, $\mathbb{P}_k^2 \setminus C$ is affine.

  Assume, to the contrary, that $C \cap C' = \emptyset$.

  Then we have $C' \subseteq \mathbb{P}_k^2 \setminus C \subseteq \mathbb{A}_k^N$.

  But C' does not consist of a point.

  So there is a coordinate function w on $\mathbb{A}_k^N$.

  By restriction, w gives a nonconstant function on C'.

  This contradicts a result, proven previously, to the effect that, if V is an irreducible projective variety, defined over an algebraically close field k, then every regular function on V is constant.

Inflection Points on a Smooth Curve (Cont'd)

Corollary

Every smooth curve C of degree $d \geq 3$, with $d-1$ coprime to char(k), has at least one point of inflection.

- From the lemma and the proposition.
- The statement of the corollary holds in general, for every smooth curve of degree $d \geq 3$ over an algebraically closed field k.
- In the following we will only use this corollary for cubics.
- Since we take char$(k) \neq 2,3$, the hypothesis of the corollary is then always satisfied.
- By using Bézout's Theorem in the proof of the corollary, we obtain the more precise result that, provided $H \neq \mathbb{P}_k^2$,

$$1 \leq \text{number of points of inflection} \leq 3d(d-2).$$

Weierstraß Form of a Cubic

- The **Weierstraß form** of a cubic is given in affine coordinates by

$$y^2 = 4x^3 - g_2 x - g_3.$$

- The corresponding projective curve is

$$C_{g_2,g_3}: \quad x_0 x_2^2 - 4x_1^3 + g_2 x_1 x_0^2 + g_3 x_0^3 = 0.$$

The Discriminant

Definition (The Discriminant)

Consider a polynomial

$$f = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0 = a_n \prod_{i=1}^{n} (x - \alpha_i), \quad a_n \neq 0.$$

The **discriminant** $\mathrm{Disc}(f)$ of f is defined by

$$\mathrm{Disc}(f) := a_n^{2n-2} \prod_{i \neq j} (\alpha_i - \alpha_j).$$

It can be expressed as a polynomial in the coefficients of f.

The Discriminant of the Weierstraß Cubic

Definition (Discriminant of the Weierstraß Cubic)

We define the **discriminant of C_{g_2,g_3}** to be the discriminant of

$$4x^3 - g_2x - g_3$$

divided by 16. A calculation shows that this is given by

$$\Delta := g_2^3 - 27g_3^2.$$

- A way to show this is to:
  - Write $4x^3 - g_2x - g_3 = 4(x-\alpha_1)(x-\alpha_2)(x-\alpha_3)$;
  - Calculate
  $$\frac{1}{16}[-4^4(\alpha_1-\alpha_2)^2(\alpha_1-\alpha_3)^2(\alpha_2-\alpha_3)^2];$$
  - Calculate $g_2^3 - 27g_3^2$.

Smoothness of C_{g_2,g_3}

Proposition

C_{g_2,g_3} is smooth if and only if $\Delta \neq 0$.

- Consider
$$f(x_0,x_1,x_2) = x_0x_2^2 - 4x_1^3 + g_2x_1x_0^2 + g_3x_0^3.$$
Compute the partial derivatives:
$$\begin{aligned} \frac{\partial f}{\partial x_0} &= x_2^2 + 2g_2x_1x_0 + 3g_3x_0^2; \\ \frac{\partial f}{\partial x_1} &= -12x_1^2 + g_2x_0^2; \\ \frac{\partial f}{\partial x_2} &= 2x_0x_2. \end{aligned}$$
We may now verify Euler's identity,
$$\begin{aligned} \sum_{i=0}^2 \frac{\partial f}{\partial x_i}x_i &= x_0(x_2^2 + 2g_2x_1x_0 + 3g_3x_0^2) + x_1(-12x_1^2 + g_2x_0^2) + x_2 2x_0x_2 \\ &= 3x_0x_2^2 - 12x_1^3 + 3g_2x_1x_0^2 + 3g_3x_0^3 \\ &= 3f. \end{aligned}$$

Smoothness of C_{g_2,g_3} (Cont'd)

- It follows that a point P is a singularity of C_{g_2,g_3} if and only if the three equations all evaluate to 0 at P.

  From the last equation, we get $x_0 = 0$ or $x_2 = 0$.

  Suppose $x_0 = 0$.

  Then the second equation implies that $x_1 = 0$.

  Hence, $P = (0:0:1)$.

  But this is not a singular point, since $\frac{\partial f}{\partial x_0}(P) = 1 \neq 0$.

  We conclude $x_2 = 0$.

  Then the first two equations give

$$2g_2x_1x_0 + 3g_3x_0^2 = 0, \quad -12x_1^2 + g_2x_0^2 = 0.$$

Smoothness of C_{g_2,g_3} (Cont'd)

- Suppose $g_2 = g_3 = 0$.
  Then $P = (1:0:0)$ is a singular point.
- Suppose $g_3 = 0, g_2 \neq 0$.
  We have already shown that $x_0 \neq 0$.
  From $2g_2x_1x_0 + 3g_3x_0^2 = 0$, we deduce that $x_1 = 0$.
  But, by $-12x_1^2 + g_2x_0^2 = 0$, the point $P = (1:0:0)$ is a smooth point.
- Suppose $g_2 = 0$, $g_3 \neq 0$.
  Then, since $2g_2x_1x_0 + 3g_3x_0^2 = 0$, $x_0 = 0$.
  We have already dealt with this case.

Smoothness of C_{g_2,g_3} (Cont'd)

- Finally suppose $g_2 g_3 \neq 0$.
  We have seen that $x_0 \neq 0$.
  By $-12x_1^2 + g_2 x_0^2 = 0$, we also have $x_1 \neq 0$.
  From $2g_2 x_1 x_0 + 3g_3 x_0^2 = 0$, we have
  $$x_0 = -\frac{2}{3}\frac{g_2}{g_3}x_1.$$
  Substituting in $-12x_1^2 + g_2 x_0^2 = 0$, we obtain
  $$-12x_1^2 + \frac{4}{9}\frac{g_2^3}{g_3^2}x_1^2 = 0.$$
  This equation has nontrivial solutions if and only if
  $$-12 + \frac{4}{9}\frac{g_2^3}{g_3^2} = 0.$$
  Equivalently, if and only if $\Delta = 0$.

Characterization of Smooth Cubics

Proposition

Let C be a smooth cubic. Then C is projectively equivalent to some curve C_{g_2,g_3}.

- By the corollary C has a point of inflection P.

  We may assume that $P = (0:0:1)$, and that the inflectional line to P is given by $\{x_0 = 0\}$.

  This means that the equation f for C restricted to $\{x_0 = 0\}$ has a zero of order three at $(0:0:1)$.

  So after possible multiplication by a scalar,

$$f = -x_1^3 + x_0(ax_0^2 + bx_1^2 + cx_2^2 + dx_0x_1 + ex_0x_2 + gx_1x_2).$$

  Since $(0:0:1)$ is a smooth point of C, $c \neq 0$.

Characterization of Smooth Cubics (Cont'd)

- We have $f = -x_1^3 + x_0(ax_0^2 + bx_1^2 + cx_2^2 + dx_0x_1 + ex_0x_2 + gx_1x_2),\ c \neq 0.$
  Set $x_2' = (\sqrt{c}x_2 + \frac{1}{2\sqrt{c}}(ex_0 + gx_1))$.
  This transforms f to
  $$f' = x_0x_2'^2 - (x_1^3 + b'x_1^2x_0 + d'x_1x_0^2 + a'x_0^3),$$
  with $b' = \frac{g^2}{4c} - b,\ d' = \frac{eg}{2c} - d,\ a' = \frac{e^2}{4c} - a$.
  Set $x_1' = x_1 + \frac{1}{3}b'x_0$.
  We get
  $$f'' = x_0x_2'^2 - (x_1'^3 + d''x_1'x_0^2 + a''x_0^3),$$
  with $a'' = a' - \frac{1}{27}b'^3 - \frac{1}{3}b'd'',\ d'' = d' - b'^2$.
  Set $x_1'' = \frac{1}{\sqrt[3]{4}}x_1'$.
  We get
  $$f''' = x_0x_2'^2 - (4x_1''^3 + d'''x_1''x_0^2 + a'''x_0^3),$$
  with $a''' = a'',\ d''' = \sqrt[3]{4}d''$.

Characterization of Rational Irreducible Cubics

Proposition

An irreducible cubic is rational if and only if it is singular.

- We classified irreducible singular cubics.

  We showed they re projectively equivalent to one of the curves defined by

$$x_1^2 x_2 - x_0^3 - x_0^2 x_2 = 0 \quad \text{or} \quad x_2 x_1^2 - x_0^3 = 0.$$

  We have seen that both possible cases are rational curves.

  Suppose C is a smooth irreducible cubic.

  Then we can first bring C to Weierstraß normal form

$$y^2 = 4x^3 - g_2 x - g_3 = 4(x - \lambda_1)(x - \lambda_2)(x - \lambda_3).$$

Characterization of Rational Irreducible Cubics (Cont'd)

- By its definition, the discriminant $\Delta = g_2^3 - 27g_3^2$ of the cubic equation $4x^3 - g_2x - g_3 = 0$ is zero if and only if the roots $\lambda_1, \lambda_2, \lambda_3$ are pairwise distinct.

  After a transformation of the line $y = 0$, we can assume that $\lambda_1 = 0$ and $\lambda_2 = 1$.

  Thus C is projectively equivalent to the curve

  $$y^2 = x(x-1)(x-\lambda), \quad \lambda \neq 0, 1.$$

  The result then follows from a preceding result to the effect that, for $\lambda \neq 0, 1$, there is no rational parametrization of

  $$C_\lambda : \quad y^2 = x(x-1)(x-\lambda).$$

Affine Transformations of Weierstraß Curves

- Let C and C' be two smooth cubics.
- Let φ be a projective transformation mapping C to C', such that φ maps a point of inflection P on C to a point of inflection P' on C'.
- By the proposition, we may assume that C and C' are curves in Weierstraß form with $P = P' = (0:0:1)$.
- Then φ must also map the inflectional line $\{x_0 = 0\}$ to itself.
- So in considering transformations between cubics in Weierstraß normal form we can restrict attention to affine transformations.

Form of Affine Transformations of Weierstraß Curves

Lemma

An affine transformation φ which maps a Weierstraß curve

$$y^2 = 4x^3 - g_2x - g_3$$

to another Weierstraß curve has the form

$$x \mapsto u^2x, \quad y \mapsto u^3y, \quad \text{for some } u \in k^*.$$

- A general transformation φ has the form

$$x \mapsto \alpha_1 x + \alpha_2 y + \alpha_3, \quad y \mapsto \beta_1 x + \beta_2 y + \beta_3.$$

  No cubic terms occur on the left side of the Weierstraß equation.

  It follows that $\alpha_2 = 0$.

Affine Transformations of Weierstraß Curves (Cont'd)

- The transformation is invertible.

  It follows that $\alpha_1 \beta_2 \neq 0$.

  The term xy does not occur in the Weierstraß equation.

  We conclude that $\beta_1 = 0$.

  No further linear terms in y occur.

  It follows that $\beta_3 = 0$.

  Similarly, no quadratic terms appear in x.

  Hence, $\alpha_3 = 0$.

  Thus, φ has the form

  $$x \mapsto \alpha_1 x, \quad y \mapsto \beta_2 y.$$

  We must have $\alpha_1^3 = \beta_2^2$.

  This means that $\alpha_1 = u^2, \beta_2 = u^3$, for some $u \in k^*$.

Admissibility and the J-Invariant

Definition (Admissible Transformation)

An **admissible transformation** is a projective transformation φ which:

- Maps a Weierstraß cubic C_{g_2,g_3} to another Weierstraß cubic $C_{g_2',g_3'}$;
- Satisfies $\varphi(0:0:1)=(0:0:1)$.

Definition (J-Invariant)

The ***J*-invariant** of a smooth Weierstraß cubic is defined by

$$J(g_2,g_3) := \frac{g_2^3}{\Delta} = \frac{g_2^3}{g_2^3-27g_3^2}.$$

Admissible Equivalence and the J-Invariant

Proposition

Two smooth Weierstraß cubic curves C_{g_2,g_3} and $C_{g_2',g_3'}$ are equivalent under an admissible transformation if and only if

$$J(g_2,g_3) = J(g_2',g_3').$$

- Let $\varphi : C_{g_2,g_3} \to C_{g_2',g_3'}$ be an admissible transformation.
  Then, since $x \mapsto u^2 x$, and $y \mapsto u^3 y$,
  $$g_2' = \frac{g_2}{u^4}, \quad g_3' = \frac{g_3}{u^6}.$$
  Thus,
  $$J(g_2,g_3) = \frac{g_2'^3}{g_2'^3 - 27g_3'^2} = \frac{\frac{g_2^3}{u^{12}}}{\frac{g_2^3}{u^{12}} - 27\frac{g_3^2}{u^{12}}} = \frac{g_2^3}{g_2^3 - 27g_3^2} = J(g_2',g_3').$$

Admissible Equivalence and the J-Invariant (Cont'd)

- Suppose $J(g_2,g_3)=J(g_2',g_3')$. We consider three cases:
  (1) $J(g_2,g_3)=0$: In this case $g_2=0\neq g_3$.
  Take u, such that $g_3'=\frac{g_3}{u^6}$.
  Then the admissible transformation
  $$x\mapsto u^2x,\quad y\mapsto u^3y$$
  takes C_{g_2,g_3} to $C_{g_2',g_3'}$.
  (2) $J(g_2,g_3)=1$: This is equivalent to $g_3=0\neq g_2$.
  In this case we choose u, such that $g_2'=\frac{g_2}{u^4}$.
  (3) $J(g_2,g_3)\neq 0,1$: Then $g_2,g_3\neq 0$.
  The condition $J(g_2,g_3)=J(g_2',g_3')$ is equivalent to $\frac{g_2^3}{g_3^2}=\frac{g_2'^3}{g_3'^2}$.
  If $g_2=\alpha g_2'$, $g_3=\beta g_3'$, then $\alpha^3=\beta^2$.
  That is, $\alpha=v^2,\beta=v^3$, for $v=\frac{\beta}{\alpha}$.
  We now choose u, with $u^2=v$. So $g_2'=\frac{g_2}{u^4}$, $g_3'=\frac{g_3}{u^6}$.
  Then, the transformation $x\mapsto u^2x,y\mapsto u^3y$ is as required.