Introduction to Algebraic Geometry

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LSSU Math 500

Subsection 1

The Existence of Lines on a Cubic

The Framework

- The ground field $k = \overline{k}$ has characteristic not equal to 2 or 3.
- Consider a homogeneous polynomial of degree 3

$$f = f(x_0, x_1, x_2, x_3) \in k^3[x_0, x_1, x_2, x_3].$$

- Consider, also, the corresponding **cubic surface**:

$$S = \{(x_0 : x_1 : x_2 : x_3) \in \mathbb{P}_k^3 : f(x_0, x_1, x_2, x_3) = 0\}.$$

- We assume that S is smooth, which is the case for general cubics f.
- This means that there is a Zariski open subset of cubic polynomials f, such that S is smooth.

The Main Theorem

- The main result of this section is the following.

Theorem

Every smooth cubic surface $S \subseteq \mathbb{P}_k^3$ contains a line.

- First we show that, the exists a point $P \in S$, such that, for the intersection
$$C_P := S \cap T_P S,$$
one of the following holds:
  - C_P contains a line;
  - C_P is a plane cubic with a cusp.

  As a consequence:
  - In the first case, we obtain the line required.
  - In the second case, we show that there is a line in S passing through a point on C_P.

Two Surfaces in $\mathbb{P}_k^3$

Lemma

Two surfaces in $\mathbb{P}_k^3$ have nonempty intersection.

- The proof is analogous to the proof of the corresponding lemma for plane cubics.

The Intersection $C_P = S \cap T_P S$

Lemma

Let S be a smooth cubic surface. Then, for every point $P \in S$, the intersection

$$C_P := S \cap T_P S$$

is a singular plane cubic curve. There is a point P in S, such that C_P is either reducible or a plane cubic with a cusp (a semicubical parabola).

- A reducible cubic surface is singular.

  Since S is smooth, S is irreducible.

  So S does not contain $T_P S$.

  Therefore, C_P is indeed a curve.

  S is defined by a cubic.

  Its restriction to a linear subspace is also given by a cubic.

  So C_P is a plane cubic curve.

The Intersection $C_P = S \cap T_P S$ (Cont'd)

- We show, next, that P is a singularity of C_P.
  Choose coordinates so that:
  - $P = (0:0:0:1) \in S$;
  - $T_P S = \{x_2 = 0\}$.

  In affine coordinates x, y, z, the equation f of S is then given by

  $$f = z + q(x,y,z) + h(x,y,z),$$

  where:
  - q is a homogeneous quadratic polynomial;
  - h is a homogeneous cubic polynomial.

  In homogeneous coordinates we have

  $$f = x_2 x_3^2 + q(x_0, x_1, x_2) x_3 + h(x_0, x_1, x_2).$$

  Hence, $f \mid_{\{x_2=0\}}$ vanishes quadratically at P.
  So C_P is singular at P.

The Intersection $C_P = S \cap T_P S$ (Cont'd)

- Now suppose that C_P is irreducible for all points $P \in S$.

  We must show that, for some P, the curve C_P is a cuspidal cubic.

  As in a preceding proof, the tangents of C_P at the point P are given by the factors of the quadratic form $\widetilde{q}(x_0, x_1) = q(x_0, x_1, 0)$.

  C_P has a cusp if and only if the two tangents lines are equal.

  Equivalently, iff q is the square of a linear form.

  Write

  $$q(x_0, x_1, x_2) = \sum_{i,j=0}^{2} a_{ij} x_i x_j, \quad \text{with } a_{ij} = a_{ji}.$$

  Then q is a square iff the matrix $\widetilde{A} = (a_{ij})_{i,j=0,1}$ has rank 1.

  But $\widetilde{q}$ is not identically zero.

  So this is equivalent to $\det \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = 0$.

The Intersection $C_P = S \cap T_P S$ (Cont'd)

- Consider, next, the Hessian of f,

$$H_f := \det\left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_{i,j}.$$

Similar to the Hessian of a cubic curve, a transformation of coordinates, given by a matrix $M \in \mathrm{Gl}(4,k)$, results in

$$H_{f^*} = (\det M)^2 (H_f)^*,$$

where:
  - $f^* = f \circ M$;
  - $(H_f)^* = H_f \circ M$.

Recall that now f has the form

$$f = x_2 x_3^2 + q(x_0,x_1,x_2)x_3 + h(x_0,x_1,x_2),$$

with

$$q(x_0,x_1,x_2) = \sum_{i,j=0}^{2} a_{ij} x_i x_j, \quad a_{ij} = a_{ji}.$$

The Intersection $C_P = S \cap T_P S$ (Conclusion)

- So we get

$$H_f(P) = \det \begin{pmatrix} 2a_{00} & 2a_{01} & 2a_{02} & 0 \\ 2a_{10} & 2a_{11} & 2a_{12} & 0 \\ 2a_{20} & 2a_{21} & 2a_{22} & 2 \\ 0 & 0 & 2 & 0 \end{pmatrix}.$$

  From the form of this matrix we have

$$H_f(P) = 0 \quad \text{iff} \quad \det \begin{pmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{pmatrix} = 0.$$

  Thus, C_P is a cubic with a cusp iff $P \in S \cap H$, where $H = \{H_f = 0\}$.

  It remains to show that $S \cap H$ is nonempty.

  But f is a cubic.

  So H must be one of the following:
  - $\mathbb{P}^3_k$, in which case $S \cap H = S \neq \emptyset$;
  - A cubic surface, in which case we use the preceding lemma.

Polar of a Homogeneous Cubic Polynomial

- We want to know when a line is contained in a cubic surface.
- Any line can be given as the line $\overline{PQ}$ through two points P and Q.
- So we introduce the following concept, which will help us determine when the line $\overline{PQ}$ is contained in the surface $\{f = 0\}$.

Definition (Polar)

For a homogeneous cubic polynomial f in $x_0, \ldots, x_3$, the **polar** of f is defined by

$$f_1(x_0, \ldots, x_3; y_0, \ldots, y_3) := \sum_{i=0}^{3} \frac{\partial f}{\partial x_i} y_i.$$

Geometric Meaning of the Polar

- For $P \in S$, by definition of the tangent space, we have

$$\overline{PQ} \subseteq T_P S \quad \text{iff} \quad f_1(P;Q) = 0.$$

- For arbitrary points $P \neq Q$, an elementary computation shows that

$$f(\lambda P + \mu Q) = \lambda^3 f(P) + \lambda^2 \mu f_1(P;Q) + \lambda \mu^2 f_1(Q;P) + \mu^3 f(Q).$$

- So we have

$$\overline{PQ} \subseteq S \quad \text{iff} \quad f(P) = f_1(P;Q) = f_1(Q;P) = f(Q) = 0.$$

- Now we determine when two or more polynomials have a common zero.

The Resultant

Definition (The Resultant)

Let r and s be homogeneous polynomials in variables u and v, given by

$$\begin{array}{rcl} r(u,v) & = & a_0u^2+a_1uv+a_2v^2, \\ s(u,v) & = & b_0u^3+b_1u^2v+b_2uv^2+b_3v^3. \end{array}$$

The **resultant** of r and s is given by

$$R(r,s) := \det\begin{pmatrix} a_0 & a_1 & a_2 & & \\ & a_0 & a_1 & a_2 & \\ & & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & b_3 & \\ & b_0 & b_1 & b_2 & b_3 \end{pmatrix}.$$

The Sylvester Matrix

- The matrix

$$\begin{pmatrix} a_0 & a_1 & a_2 & & \\ & a_0 & a_1 & a_2 & \\ & & a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 & b_3 & \\ & b_0 & b_1 & b_2 & b_3 \end{pmatrix}$$

  is known as the **Sylvester matrix**.
- The resultant $R(r,s)$ is also known as the **Sylvester resultant**.

Importance of the Resultant

Lemma

Two homogeneous polynomials r and s of degree 2 and 3, respectively, have a common zero in $\mathbb{P}_k^1$ if and only if $R(r,s)=0$.

- Consider the 5-dimensional vector space V of all homogeneous polynomials of degree 4 in u and v.

  The rows of the Sylvester matrix are the coefficients of

  $$u^2r,\ uvr,\ v^2r,\ us,\ vs,$$

  in terms of the standard basis of monomials for this space.

  The determinant, i.e., the Sylvester resultant, is zero if and only if there is a linear relation between these polynomials.

Importance of the Resultant (Cont'd)

- A linear relationship between u^2r, uvr, v^2r, us, vs can be written as

$$qr = \ell s,$$

where:

  - $q = q(u,v)$ is homogeneous of degree 2;
  - $\ell = \ell(u,v)$ is linear.

This equality implies that qr and ℓs have the same zeros.

This is only possible if r and s have a common zero.

On the other hand, suppose r and s have a common zero.

Then so do the five polynomials.

This means that they certainly cannot span V.

So they must be linearly dependent.

Proof of the Main Theorem

Main Theorem

Every smooth cubic surface $S \subseteq \mathbb{P}_k^3$ contains a line.

- Suppose, first, $C_P = S \cap T_P S$ is reducible for some point $P \in S$.

  Then C_P, and thus S, contains a line.

  Suppose, next, that C_P is irreducible.

  By a preceding lemma, we can assume that there is a point P, such that C_P has a cusp.

  By a transformation of coordinates, we can assume that:
  - $P = (0:0:1:0)$;
  - $T_P S = \{x_3 = 0\}$.

Proof of the Main Theorem (Cont'd)

- By the proposition classifying irreducible singular cubics, we can further assume that

$$C_P = \{x_0^2 x_2 - x_1^3 = x_3 = 0\}.$$

Thus, the equation f of S has the form

$$f = x_0^2 x_2 - x_1^3 + x_3 g,$$

for some homogeneous polynomial $g = g(x_0, \ldots, x_3)$ of degree 2.

But S is smooth at P.

So we have $g(0,0,1,0) \neq 0$.

By scaling, we can assume that $g(0,0,1,0) = 1$.

Proof of the Main Theorem (Cont'd)

- We show that S contains a line through a point on C_P.
  Points on C_P are parametrized by $P_\alpha := (1 : \alpha : \alpha^3 : 0)$, for $\alpha \in k$.
  We can represent any line through P_α by a line $\overline{P_\alpha Q}$ through P_α and $Q = (0 : x_1 : x_2 : x_3)$, for some point Q in the plane $\{x_0 = 0\}$.
  We have the equivalence
  $$\overline{PQ} \subseteq S \quad \text{iff} \quad f(P) = f_1(P;Q) = f_1(Q;P) = f(Q) = 0.$$
  Since $f(P_\alpha) = 0$, we obtain
  $$\overline{P_\alpha Q} \subseteq S \quad \text{iff} \quad f_1(P_\alpha;Q) = f_1(Q;P_\alpha) = f(Q) = 0.$$
  Let A_α, B_α and C_α denote, respectively,
  $$f_1(P_\alpha;Q), f_1(Q;P_\alpha), f(Q) \in k[\alpha][x_1,x_2,x_3].$$
  These are homogeneous polynomials in x_1,x_2,x_3 of degree 1, 2 and 3, respectively, with coefficients given by polynomials in α.

Proof of the Main Theorem (Cont'd)

- We now use resultants to determine whether two polynomials have a common root.

  We define a polynomial $R_{27}(\alpha)$ to be the resultant

  $$R_{27}(\alpha) := R(B_\alpha(x_1, \widetilde{A}_\alpha(x_1,x_3), x_3), C_\alpha(x_1, \widetilde{A}_\alpha(x_1,x_3), x_3)),$$

  where $x_2 = \widetilde{A}_\alpha(x_1,x_3)$ is determined by the linear polynomial A_α.

  The preceding lemma implies that $R_{27}(\alpha)$ satisfies

  $$R_{27}(\alpha) = 0 \quad \text{iff} \quad A_\alpha, B_\alpha, C_\alpha \text{ have a common zero } (\eta_\alpha : \xi_\alpha : \tau_\alpha).$$

  Suppose α_0 is a root of $R_{27}(\alpha)$.

  Then the line $\overline{P_{\alpha_0} Q}$ with $Q = (0 : \eta_{\alpha_0} : \xi_{\alpha_0} : \tau_{\alpha_0})$ lies on S.

  Consequently, it suffices to prove that R_{27} has a root.

  It suffices, in turn, to show that R_{27} is nonconstant.

  We will do this by explicitly computing R_{27}.

Proof of the Main Theorem (Cont'd)

- Consider the equation

$$f = f(x_0,\dots,x_3) = x_0^2 x_2 - x_1^3 + x_3 g.$$

Its polar $f_1(x_0,\dots;\dots,y_3)$ is given by

$$f_1 = 2x_0x_2y_0 - 3x_1^2y_1 + x_0^2y_2 + g(x_0,\dots,x_3)y_3 + x_3g_1(x_0,\dots;\dots,y_3),$$

where g_1 is the polar of the quadratic equation g.

The polynomials $A_\alpha, B_\alpha, C_\alpha$ are then given by

$$\begin{array}{rcl} A_\alpha & = & -3\alpha^2 x_1 + x_2 + g(1,\alpha,\alpha^3,0)x_3, \\ B_\alpha & = & -3\alpha x_1^2 + x_3 g_1(1,\alpha,\alpha^3,0;0,x_1,x_2,x_3), \\ C_\alpha & = & -x_1^3 + x_3 g(0,x_1,x_2,x_3). \end{array}$$

Now g is quadratic and $g(0,0,1,0) = 1$.

So we define $a^{(6)} := g(1,\alpha,\alpha^3,0) = \alpha^6 + \text{terms of lower order}$.

Proof of the Main Theorem (Cont'd)

- From $A_\alpha = 0$, we have

$$x_2 = 3\alpha^3 x_1 - a^{(6)} x_3.$$

Substituting in B_α gives the expression

$$B_\alpha = -3\alpha x_1^2 + x_3 g_1(1, \alpha, \alpha^3, 0; 0, x_1, 3\alpha^2 x_1 - a^{(6)} x_3, x_3).$$

Now $g_1(1, \alpha, \alpha^3, 0; 0, x_1, x_2, x_3)$ is linear in x_1, x_2 and x_3.
So we have

$$B_\alpha = b_0 x_1^2 + b_1 x_1 x_3 + b_2 x_3^2,$$

with

$$\begin{array}{rcl} b_0 & = & -3\alpha, \\ b_1 & = & g_1(1, \alpha, \alpha^3, 0; 0, 1, 3\alpha^2, 0) = 6\alpha^5 + \cdots, \\ b_2 & = & g_1(1, \alpha, \alpha^3, 0; 0, 0, -a^{(6)}, 1) = -2\alpha^9 + \cdots, \end{array}$$

where the dots denote terms of lower order.

Proof of the Main Theorem (Cont'd)

- Similarly, substituting

$$x_2 = 3\alpha^3 x_1 - a^{(6)} x_3$$

in C_α gives

$$C_\alpha = -x_1^3 + x_3 g(0, x_1, 3\alpha^2 x_1 - a^{(6)} x_3, x_3).$$

We obtain

$$C_\alpha = c_0 x_1^3 + c_1 x_1^2 x_3 + c_2 x_1 x_3^2 + c_3 x_3^3,$$

where

$$\begin{aligned} c_0 &= -1, \\ c_1 &= g(0,1,3\alpha^2,0) = 9\alpha^4 + \cdots, \\ c_2 &= g_1(0,1,3\alpha^2,0;0,0,-a^{(6)},1) = -6\alpha^8 + \cdots, \\ c_3 &= g(0,0,-a^{(6)},1) = \alpha^{12} + \cdots. \end{aligned}$$

Proof of the Main Theorem (Cont'd)

- By definition we have $R_{27}(\alpha) = \det \begin{pmatrix} b_0 & b_1 & b_2 & & \\ & b_0 & b_1 & b_2 & \\ & & b_0 & b_1 & b_2 \\ c_0 & c_1 & c_2 & c_3 & \\ & c_0 & c_1 & c_2 & c_3 \end{pmatrix}.$

  The leading term of R_{27} is given by the leading term of the determinant of the matrix of leading terms.

  This is the matrix obtained by replacing b_i and c_j by their leading terms, assuming this determinant is nonzero.

  Thus the leading term of R_{27} is given by

$$\det \begin{pmatrix} -3\alpha & 6\alpha^5 & -2\alpha^9 & & \\ & -3\alpha & 6\alpha^5 & -2\alpha^9 & \\ & & -3\alpha & 6\alpha^5 & -2\alpha^9 \\ -1 & 9\alpha^4 & -6\alpha^8 & \alpha^{12} & \\ & -1 & 9\alpha^4 & -6\alpha^8 & \alpha^{12} \end{pmatrix} = \alpha^{27} \begin{pmatrix} -3 & 6 & -2 & & \\ & -3 & 6 & -2 & \\ & & -3 & 6 & -2 \\ -1 & 9 & -6 & 1 & \\ & -1 & 9 & -6 & 1 \end{pmatrix} = \alpha^{27}.$$

  It follows that $R_{27}(\alpha)$ is a polynomial of degree 27.

  In particular it is nonconstant.

The 27 Lines

- Let S be a cubic surface.
- Suppose S contains a point P, such that $S \cap T_P S$ is a cuspidal cubic.
- The preceding proof shows that there is a polynomial R_α of degree 27, such that every root of R_α gives rise to at least one line on S.
- It follows that if the roots of R_α are distinct, we should expect S to contain at least 27 lines.

Subsection 2

The Configuration of the 27 Lines

Singular Locus of a Quadratic

- We show that:
  - There are 27 lines on a smooth cubic, and determine their configuration;
  - Every smooth cubic surface is rational.
- Consider the quadric

$$Q = \left\{ \sum_{i,j=0}^{n} a_{ij} x_i x_j = 0 \right\} \subseteq \mathbb{P}_k^n, \quad a_{ij} = a_{ji}.$$

Lemma

The quadric Q has singular locus given by the linear subspace

$$\operatorname{Sing} Q = \mathbb{P}(\ker(A)) = \{x \in \mathbb{P}_k^n : Ax = 0\},$$

where the symmetric matrix $A = (a_{ij})$ is the matrix of the quadratic form corresponding to Q. In particular, Q is smooth if and only if A has maximal rank $n+1$.

Proof of the Lemma

- By the Principal Axes Theorem, there is an invertible matrix M, such that
$$MA^tM = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & & & \\ & & & 0 & & \\ & & & & \ddots & \\ & & & & & 0 \end{pmatrix} = E_{r+1}.$$
The matrix M induces a transformation of coordinates.
This transformation maps Q to the quadric
$$Q_{r+1} = \{x_0^2 + \cdots + x_r^2 = 0\},$$
where $r = \text{rk}(A) - 1$.
It is then easy to see that
$$\text{Sing}\, Q_{r+1} = \{x_0 = \cdots = x_r = 0\} = \mathbb{P}(\ker E_{r+1}).$$

The Rank of a Quadric

Definition (Rank)

Consider the quadric

$$Q = \left\{ \sum_{i,j=0}^{n} a_{ij} x_i x_j = 0 \right\} \subseteq \mathbb{P}_k^n, \quad a_{ij} = a_{ji}.$$

The **rank** of Q is defined to be the rank of the corresponding matrix

$$A = (a_{ij}).$$

The rank is well defined, even though A is only determined up to multiplication by a nonzero scalar.

Smooth Cubic Sufaces and Planes

Proposition

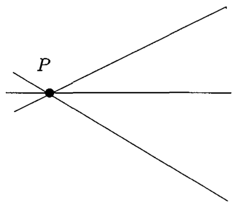
Let S be a smooth cubic surface.

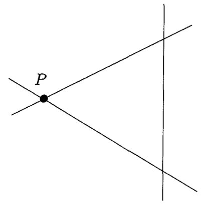
(1) If E is a plane, then $E \cap S$ is either an irreducible cubic curve, or a conic and a line, or three distinct lines.

(2) The surface S contains at most 3 lines through any given point $P \in S$.

If S contains two or three lines passing through a point $P \in S$, then these lie in a plane E.

Moreover, $E \cap S$ has one of the possible configurations shown below.





Proof of the Proposition

(1) As in a preceding proof the intersection $E \cap S$ is a plane cubic curve.

So we must show that $E \cap S$ does not contain a multiple line.

By a multiple line, we mean a line L, such that

$$E \cap S = 2L + M,$$

for some line M, possibly with $M = L$.

Suppose, to the contrary, that $E \cap S$ has this form.

Choose coordinates so that $E = \{x_3 = 0\}$ and so that the multiple line is given by

$$\ell = \{x_2 = x_3 = 0\} \subseteq E \cap S.$$

Proof of the Proposition (Cont'd)

- These assumptions imply that the defining equation of S has the form

$$f = x_2^2 g(x_0, x_1, x_2) + x_3 h(x_0, x_1, x_2, x_3),$$

where $\deg g = 1$ and $\deg h = 2$.

Consider the set

$$\Delta := \ell \cap \{h = 0\} = \{(x_0 : x_1 : 0 : 0) : h(x_0, x_1, 0, 0) = 0\}.$$

Since k is algebraically closed, $\Delta \neq \emptyset$.

However, the points in Δ are singular points of S.

This contradicts the fact that S is smooth.

(2) Note that if $\ell \subseteq S$ is a line, $\ell = T_P \ell \subseteq T_P S$.

So any line $\ell \subseteq S$, passing through P, lies in the tangent space $T_P S$.

Thus, it suffices to take $E = T_P S$.

Since S is smooth, $T_P S$ is a plane.

So the result follows from Part (1).

Line on a Smooth Cubic and Points of Intersection

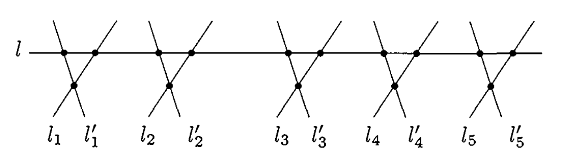
Proposition

Let S be a smooth cubic and $\ell \subseteq S$ a line. Then there are exactly 10 lines in S which intersect ℓ. These lines are given by five pairs of lines (ℓ_i, ℓ'_i), $1 \leq i \leq 5$, which have the following properties:

(1) $\ell \cup \ell_i \cup \ell'_i$ lie in a plane. In particular, the lines ℓ_i and ℓ'_i intersect the line ℓ for $i = 1, \ldots, 5$.

(2) $(\ell_i \cup \ell'_i) \cap (\ell_j \cup \ell'_j) = \emptyset$, for $i \neq j$.

Such a configuration is shown below:



Note that it is also possible that ℓ, ℓ_i, ℓ'_i meet at a point.

Line on a Smooth Cubic (Cont'd)

- Let E be a plane containing the line ℓ.

  Then $E \cap S = \ell \cup q$, where $q \subseteq E$ is a conic.

  Either q is irreducible, or else, by the proposition, S has one of the configurations shown previously.

  So we must show that there are exactly five planes $E \supseteq \ell$, such that the conic q decomposes.

  Each of these planes would give a pair of lines.

  By the proposition, they would have the given configuration.

Line on a Smooth Cubic (Cont'd)

- We can assume that $\ell = \{x_2 = x_3 = 0\}$.
  So the set of planes containing the line ℓ is given by the pencil of planes
  $$E_{\lambda,\mu}: \quad \{\lambda x_3 - \mu x_2 = 0\}.$$
  We want to determine for which values of $(\lambda : \mu) \in \mathbb{P}_k^1$ the intersection $E_{\lambda,\mu} \cap S$ consists of three lines.
  Now S contains ℓ.
  So the defining equation of S has the form
  $$f = Ax_0^2 + Bx_0x_1 + Cx_1^2 + Dx_0 + Ex_1 + F,$$
  where $A, \ldots, F \in k[x_2, x_3]$ are homogeneous polynomials, with:
  - A, B, C linear;
  - D, E quadratic;
  - F cubic.

  Points on $E_{\lambda,\mu}$ have the form $(x_0 : x_1 : \lambda t : \mu t)$, for some $t \in k$.
  So we can take homogeneous coordinates $(x_0 : x_1 : t)$ on $E_{\lambda,\mu}$.

Line on a Smooth Cubic (Cont'd)

- In terms of these coordinates:
  - $\ell \mid_{E_{\lambda,\mu}} = \{t = 0\}$;
  - $f \mid_{E_{\lambda,\mu}} = tq(x_0, x_1, t)$, where

$$\begin{array}{rcl} q(x_0,x_1,t) & = & A(\lambda,\mu)x_0^2 + B(\lambda,\mu)x_0x_1 + C(\lambda,\mu)x_1^2 \\ & & +D(\lambda,\mu)x_0t + E(\lambda,\mu)x_1t + F(\lambda,\mu)t^2. \end{array}$$

By a preceding lemma, this conic is singular if and only if $\Delta(\lambda,\mu) = 0$, where $\Delta(\lambda,\mu)$ is the homogeneous polynomial of degree 5 given by

$$\begin{array}{rcl} \Delta(\lambda,\mu) & = & 4\det\begin{pmatrix} A(\lambda,\mu) & \frac{1}{2}B(\lambda,\mu) & \frac{1}{2}D(\lambda,\mu) \\ \frac{1}{2}B(\lambda,\mu) & C(\lambda,\mu) & \frac{1}{2}E(\lambda,\mu) \\ \frac{1}{2}D(\lambda,\mu) & \frac{1}{2}E(\lambda,\mu) & F(\lambda,\mu) \end{pmatrix} \\ & = & 4A(\lambda,\mu)C(\lambda,\mu)F(\lambda,\mu) + B(\lambda,\mu)E(\lambda,\mu)D(\lambda,\mu) \\ & & -C(\lambda,\mu)D^2(\lambda,\mu) - A(\lambda,\mu)E^2(\lambda,\mu) - F(\lambda,\mu)B^2(\lambda,\mu). \end{array}$$

To prove the theorem we need to show that:

- $\Delta(\lambda,\mu) \not\equiv 0$;
- Δ has no multiple roots.

Line on a Smooth Cubic (Cont'd)

- By a coordinate transformation, take $\lambda = 0$ to be a zero of $\Delta(\lambda,\mu)$.

  We must show that λ^2 does not divide $\Delta(\lambda,\mu)$.

  The preceding proposition gives the possible configurations for the corresponding cubic $E_{0,1} \cap S$.

  We can choose coordinates so that $E_{0,1} \cap S$ is given by either

  $$\begin{array}{l} \ell = \{x_3 = 0\},\ \ell_1 = \{x_0 = 0\},\ \ell_1' = \{x_1 = 0\} \quad \text{or} \\ \ell = \{x_3 = 0\},\ \ell_1 = \{x_0 = 0\},\ \ell_1' = \{x_0 - x_3 = 0\}. \end{array}$$

  We only treat the second case.

  We have

  $$f = x_0 x_3 (x_0 - x_3) + x_2 g,$$

  where g is quadratic.

Line on a Smooth Cubic (Conclusion)

- We have
$$f = x_0x_3(x_0 - x_3) + x_2g.$$
Compare with
$$f = Ax_0^2 + Bx_0x_1 + Cx_1^2 + Dx_0 + Ex_1 + F.$$
We see that:
  - $D(x_2,x_3) = -x_3^2 + x_2\beta(x_2,x_3)$, for some $\beta \in k^1[x_2,x_3]$;
  - B, C, E and $F \in k[x_2,x_3]$ must all be divisible by x_2.

  Thus,
$$\Delta \equiv -C(\lambda,\mu)D^2(\lambda,\mu) \mod \lambda^2.$$
Since C is linear,
$$C(x_2,x_3) = cx_2,$$
for some $c \in k$.
Since S is smooth at $P = (0:1:0:0)$, $c \neq 0$.
Thus, $\Delta(\lambda,\mu)$ is not divisible by λ^2.

Meeting Mode of Lines on a Cubic Surface

- To say two lines **meet transversally** means that they intersect in exactly one point.

Lemma

Suppose the lines $\ell_1,\ldots,\ell_4 \in \mathbb{P}_k^3$ are disjoint.
Then one of the following holds:

(1) $\ell_1,\ldots,\ell_4$ lie on a smooth quadric Q. In this case there are infinitely many lines which intersect $\ell_1,\ldots,\ell_4$ transversally.

(2) $\ell_1,\ldots,\ell_4$ do not lie on any quadric. Then there are one or two common transversal lines.

- First we show that ℓ_1, ℓ_2 and ℓ_3 are contained in a quadric surface Q.
  The dimension of the space of homogeneous polynomials of degree 2 in 4 variables is $\dim k^2[x_0,x_1,x_2,x_3] = 10$.
  So the space of quadrics in $\mathbb{P}_k^3$ may be considered as a 9-dimensional projective space.

Three Lines are Contained in a Quadric Surface

- The space of quadrics in $\mathbb{P}_k^3$ may be considered as a 9-dimensional projective space.

  The requirement that a quadric $Q = \{f = 0\}$ passes through a given point is a linear condition on the coefficients of f.

  For a line ℓ to lie on a quadric Q, it is sufficient to show that three distinct points on ℓ lie on Q.

  Otherwise $\ell \cap Q$ consists of at most 2 points.

  Hence, the requirement that three lines lie on a quadric can be expressed by giving 9 linear conditions.

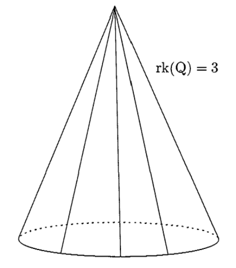
  But the space of quadrics is projectively 9-dimensional.

  Therefore, there is at least one quadric Q containing all three lines.

Smoothness of the Quadric Q

- We now show that Q is smooth.
  The lines ℓ_1, ℓ_2, ℓ_3 are disjoint.
  Any two lines in a plane intersect.
  It follows that Q is not the union of two planes.

  Similarly, Q cannot be a quadric of rank 3.

  On any such quadric, every line passes through the unique singularity of Q, i.e., through the vertex of the cone.

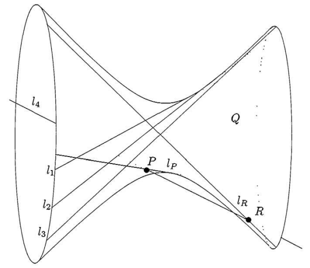
  This can be seen by considering a projection from the vertex.
  Lines either pass through the vertex, and are mapped to points, or are mapped to lines. But the image is a smooth quadric curve.
  So it does not contain any lines.
  Thus, Q is a smooth quadric containing ℓ_1, ℓ_2, ℓ_3.
  Being disjoint, these lines lie in the same ruling on Q.

Relative Position of Q and ℓ_4

- We now have the following possibilities:
  (1) $\ell_4 \subseteq Q$: Then ℓ_4 must lie in the same ruling as ℓ_1, ℓ_2, ℓ_3.
  There are infinitely many transversal lines.
  They are given by the lines in the other ruling on Q.

  (2) $\ell_4 \not\subseteq Q$: Then $\ell_4 \cap Q$ consists of one or two points.

  Suppose ℓ_4 meets Q in distinct points P and R. Then the lines ℓ_P and ℓ_R in Q passing through P and R, which are not in the same ruling as ℓ_1, ℓ_2 and ℓ_3, meet all four lines transversally.

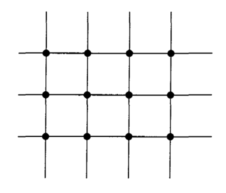


  Suppose ℓ_4 only meets Q in one point.
  Then we obtain one line in Q transversal to all four lines $\ell_1, \ldots, \ell_4$.
  Note that any line meeting ℓ_1, ℓ_2 and ℓ_3 meets Q in at least 3 points.
  Thus, it is contained in Q.

Configuration of Lines on a Smooth Cubic Surface

Lemma

If S is a smooth cubic, then S cannot contain four skew lines m_1, m_2, m_3, m_4 which are all intersected by three lines n_1, n_2, n_3. That is, S cannot contain 7 lines configured as in the figure.



- Suppose we have such a configuration on S.

  By hypothesis, $m_1, \ldots, m_4$ are intersected by three lines.

  The lemma implies that they lie on a smooth quadric Q.

  The m_i are in one ruling on Q, and the n_i on the other ruling.

Configuration of Lines on a Smooth Cubic Surface (Cont'd)

- Every point $P \in Q$ is contained in a line L in the same ruling as the n_i.
  The line L intersects m_1, m_2, m_3 and m_4.
  This gives 4 points in $L \subseteq S$.
  But a cubic and line intersect in at most 3 points, unless $L \subseteq S$.
  Thus, $P \in S$, for all $P \in Q$.
  So $Q \subseteq S$.
  Hence, there is a decomposition $S = Q \cup H$, for some hyperplane H.
  But $Q \cup H$ is singular along the intersection $Q \cap H$.
  This contradicts the hypothesis that S is smooth.

Configuration of Lines intersecting a Line in a Cubic

- Recall that given a smooth cubic S and a line $\ell \subseteq S$, there are exactly 10 lines in S, intersecting ℓ, and we know their configuration.

Lemma

If ℓ, ℓ_i and ℓ'_i are lines on a smooth cubic surface S, as before, then any other line m on S must intersect exactly one of ℓ, ℓ_i and ℓ'_i.

- Let E_i be the plane with $E_i \cap S = \ell \cup \ell_i \cup \ell'_i$.

  A line and a plane in $\mathbb{P}^3_k$ always intersect.

  So m meets the plane E_i.

  Thus,

$$m \cap (\ell \cup \ell_i \cup \ell'_i) = m \cap (E_i \cap S) = (m \cap S) \cap E_i = m \cap E_i \neq \emptyset.$$

  So m meets at least one of these three lines.

Lines intersecting a Line in a Cubic (Cont'd)

- Suppose m meets more than one of the lines ℓ, ℓ_i and ℓ'_i.

  Then oe of the following holds:
  - m meets E_i in two points;
  - m meets two of the lines at a single point in S.

  In the first case m is contained in E_i.

  But this contradicts the configuration lemma.

  In the second case, by the second part of the configuration lemma, m is contained in E_i.

  But again, this contradicts the first part of the configuration lemma.

The 27 Lines on a Smooth Cubic

Theorem

A smooth cubic surface S contains exactly 27 lines.

- We saw S contains a line ℓ.

  We then obtained five pairs of lines (ℓ_i, ℓ_i'), $i = 1, \ldots, 5$, intersecting ℓ.

  Now apply the same proposition to line ℓ_1.

  We obtain a skew line $m \subseteq S$ to ℓ.

  By the preceding lemma, m meets exactly one of ℓ_i and ℓ_i', $i = 1, \ldots, 5$.

  By relabeling, if necessary, we can assume that m intersects each of the lines $\ell_1, \ldots, \ell_5$, and does not intersect any of the lines $\ell_1', \ldots, \ell_5'$.

  Now there are 5 pairs of lines meeting m.

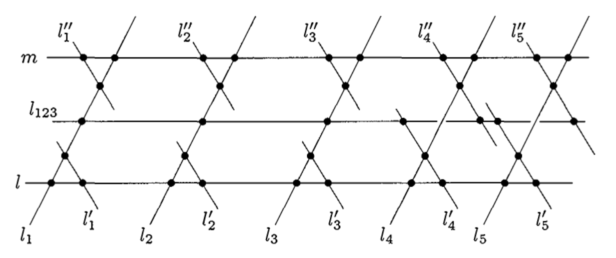
  They must be of the form (ℓ_i, ℓ_i''), for $i = 1, \ldots, 5$.

The 27 Lines on a Smooth Cubic (Cont'd)

- By a previous proposition, no four distinct lines on S can be coplanar.
  So the four lines ℓ_i'', ℓ_i, ℓ and m cannot be coplanar.
  Thus, ℓ_i'' cannot intersect ℓ.
  So the ℓ_i'' are all different from the ℓ_j'.
  We have now found 17 lines $\ell, m, \ell_i, \ell_i', \ell_i''; i = 1, \ldots, 5$.
  By the preceding lemma, each ℓ'' must meet one of the lines ℓ, ℓ_j, ℓ_j'.
  But the proposition implies that $\ell_i'' \cap \ell_j = \emptyset$ for $i \neq j$.
  Moreover, as already noted, $\ell_i'' \cap \ell = \emptyset$.
  So $\ell_i'' \cap \ell_j' \neq \emptyset$, for $i \neq j$.

The 27 Lines on a Smooth Cubic (Cont'd)

- We must find 10 more lines.



Consider each of the 10 triples (i,j,k) with $1 \leq i < j < k \leq 5$.

We show that, for each such triple, there is exactly one line ℓ_{ijk} that:

- Is different from, and skew to, ℓ and m;
- Meets ℓ_i, ℓ_j and ℓ_k.

To conclude, we will show that any line in S not equal to one of the 17 lines already found must be equal to one of these 10 lines.

A Line Other than the 17 Intersects Exactly 3 of the ℓ_i

- Suppose n is a line on S not equal to one of the 17 lines already given.

  First note that all lines intersecting ℓ or m are given by the ℓ_i, ℓ'_i, ℓ''_i.

  Thus, n cannot intersect ℓ or m.

  We show that n intersects exactly 3 of the lines ℓ_i.

  (1) n cannot intersect 4 of the ℓ_i.
  Otherwise, the three skew lines ℓ, n and m would all be transverse to these four skew ℓ_i.
  Such a configuration contradicts a preceding lemma.

  (2) n cannot intersect less than 2 of the ℓ_i.
  Otherwise, by the lemma, n intersects at least 4 of the ℓ'_i.
  This leads to the same contradiction.

A Line Other than the 17 and the ℓ_i (Cont'd)

(3) n cannot intersect exactly 2 of the ℓ_i.
Assume, to the contrary, that n:

- Does not intersect ℓ_1, ℓ_2 or ℓ_3;
- Does intersect both ℓ_4 and ℓ_5.

As already noted, n does not intersect ℓ or m.
The lemma implies that n:

- Intersects ℓ'_1, ℓ'_2 and ℓ'_3;
- Does not intersect ℓ''_4 or ℓ''_5.

As discussed above, the $\ell'_i \cap \ell''_j \neq \emptyset$, for $i \neq j$.
So we have four skew lines ℓ'_1, ℓ'_2, ℓ'_3, ℓ'_4 which all intersect the three skew lines n, ℓ''_4 and ℓ''_5.
That is, we end up again with the same forbidden configuration.

Uniqueness of ℓ_{ijk}

- We show that ℓ_{ijk} is unique.

  Suppose there are two lines ℓ_{ijk} and ℓ'_{ijk}, which:
  - Are different from, and skew to, ℓ and m;
  - Meet ℓ_i, ℓ_j and ℓ_k.

  By a previous proposition, the five lines $\ell_i, \ell_j, \ell_k, \ell_{ij}, \ell'_{ijk}$ do not all lie in a common plane.

  Therefore, ℓ_{ijk} and ℓ'_{ijk} cannot intersect.

  So we have three skew lines ℓ_i, ℓ_j and ℓ_k which all intersect the four skew lines ℓ, m, ℓ_{ijk} and ℓ'_{ijk} transversally.

  By a previous lemma, this is also a contradiction.

Existence of ℓ_{ijk}

- We show that the lines ℓ_{ijk} exist.

  By the proposition, ℓ_1 intersects ten distinct lines.

  Four of these lines are given by ℓ, ℓ_1', m and ℓ_1''.

  By the above argument, the remaining six lines must be of the form ℓ_{1jk}, for some triple $(1,j,k)$, with $1 < j < k \leq 5$.

  But there are exactly $\binom{4}{2} = 6$ such triples.

  So all the lines ℓ_{1jk} occur.

  Similarly all the lines ℓ_{ijk} exist.
- We have now constructed 27 lines on S, given by

$$\{\ell, m, \ell_i, \ell_i', \ell_i'', \ell_{ijk} : 1 \leq i < j < k \leq 5\}.$$

Summary

- A smooth cubic surface S contains 27 lines, $\ell, m, \ell_i, \ell'_i, \ell''_i, \ell_{ijk}$, for $i < j < k \leq 5$.
- Each line meets 10 others, as follows:
  - ℓ meets $\ell_1, \ldots, \ell_5$ and $\ell'_1, \ldots, \ell'_5$;
  - m meets $\ell''_1, \ldots, \ell''_5$ and $\ell'_1, \ldots, \ell'_5$;
  - ℓ_1 meets $\ell, m, \ell'_1, \ell''_1$ and ℓ_{1jk}, for $2 \leq j < k \leq 5$;
  - ℓ'_1 meets ℓ, ℓ_1, ℓ''_j, for $2 \leq j \leq 5$, and $\ell_{234}, \ell_{235}, \ell_{245}, \ell_{345}$;
  - ℓ''_1 meets m, ℓ_1, ℓ'_j, for $2 \leq j \leq 5$, and $\ell_{234}, \ell_{235}, \ell_{245}, \ell_{345}$;
  - ℓ_{123} meets $\ell_1, \ell_2, \ell_3, \ell'_4, \ell''_4, \ell'_5, \ell''_5$ and $\ell_{145}, \ell_{245}, \ell_{345}$.
- The intersections of the remaining lines are given by appropriate permutations of the indices.

Subsection 3

Rationality of the Cubics

Rationality of a Smooth Surface

Proposition

Let S be a smooth cubic.

(1) There exist two disjoint lines on S.

(2) S is a rational surface.

(1) This follows from the construction of the 27 lines on a cubic surface.
Alternatively, by our first theorem, S contains a line ℓ.
Then a previous proposition produces disjoint lines ℓ_1 and ℓ_2.

(2) Given two skew lines $\ell, m \subseteq \mathbb{P}_k^3$, we can construct a rational map.
Consider a point $Q \notin \ell \cup m$.
There is exactly one line $n = n(Q)$, through Q, which intersects both ℓ and m.

Rationality of a Smooth Surface (Cont'd)

- Thus, we can define a map

$$\begin{array}{rrcl} \pi_{\ell,m}: & \mathbb{P}_k^3 \backslash (\ell \cup m) & \rightarrow & \ell \times m = \mathbb{P}_k^1 \times \mathbb{P}_k^1; \\ & Q & \mapsto & (n(Q) \cap \ell, n(Q) \cap m). \end{array}$$

  By a transformation of coordinates, we can take:
  - $\ell = \{x_2 = x_3 = 0\}$;
  - $m = \{x_0 = x_1 = 0\}$.

  Then $\pi_{\ell,m}$ is given by

$$\begin{array}{rrcl} \pi_{\ell,m}: & \mathbb{P}_k^3 \backslash (\ell \cup m) & \rightarrow & \ell \times m; \\ & (x_0 : x_1 : x_2 : x_3) & \mapsto & ((x_0 : x_1), (x_2 : x_3)), \end{array}$$

  This shows that $\pi_{\ell,m}$ is a rational map on $\mathbb{P}_k^3$, with domain of definition $\mathbb{P}_k^3 \backslash (\ell \cup m)$.

Rationality of a Smooth Surface (Cont'd)

- We show that $\varphi = \pi_{\ell,m}|_S : S \dashrightarrow \ell \times m$ has a rational inverse.
  For $(P,Q) \in \ell \times m$, we consider the line $\overline{PQ}$ in $\mathbb{P}_k^3$.
  At most 10 of the lines $\overline{PQ}$ are contained in S.
  If $\overline{PQ} \nsubseteq S$, then, by the Fundamental Theorem, $\overline{PQ} \cap S = \{P,Q,R\}$, where the points of intersection are counted with multiplicity.
  We define a map
  $$\begin{array}{rccc} \psi : & \ell \times m & \dashrightarrow & S; \\ & (P,Q) & \mapsto & R. \end{array}$$
  The solutions of $f|_{\overline{PQ}}$ depend algebraically on the points P and Q.
  So ψ is a rational map.
  Clearly, the maps φ and ψ are mutually inverse.
  Note that $\ell \times m \cong \mathbb{P}_k^1 \times \mathbb{P}_k^1$ is a rational surface (i.e. birational to $\mathbb{P}_k^2$).
  This proves the result.