Introduction to Algebraic Geometry

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Subsection 1

Divisors on Curves

Smooth Projective Curves

- Let k be an algebraically closed field.
- Let C be a smooth projective curve over k.
- That is, C is a smooth irreducible projective variety of dimension 1.

Divisors and Divisor Groups of Curves

Definition (Divisor of a Curve)

A **divisor** D on C is a finite formal sum of points on C.
For a divisor D given by

$$D = n_1 P_1 + \cdots + n_k P_k, \quad n_i \in \mathbb{Z},\ P_i \in C,$$

the **degree** of D is defined by

$$\deg D := n_1 + \cdots + n_k.$$

The **divisor group** of C is given by

$$\operatorname{Div} C := \{D : D \text{ is a divisor on } C\}.$$

That is, $\operatorname{Div} C$ is the set of all divisors, or equivalently, the free abelian group generated by the points of C.

The Maximal Ideal m_P of the Local Ring $\mathscr{O}_{C,P}$

- Recall the unique maximal ideal of the local ring $\mathscr{O}_{C,P}$, given by
$$m_P = \{g \in \mathscr{O}_{C,P} : g(P) = 0\}.$$
- Since C is smooth,
$$\dim_k m_P / m_P^2 = \dim C = 1.$$
- Suppose $t \in m_P$, such that the residue class $\overline{t}$ spans the k-vector space m_P / m_P^2.
- Then, by Nakayama's Lemma, t generates m_P.

A Strictly Descending Chain of Ideals

- Consider the chain

$$m_P \supsetneqq m_P^2 \supsetneqq \cdots \supsetneqq m_P^k \supsetneqq m_P^{k+1} \supsetneqq \cdots$$

It is a strictly descending chain of ideals.

Suppose, to the contrary, $m_P^k = m_P^{k+1}$, for some k.

Then, for some $g \in \mathscr{O}_{C,P}$,

$$t^k(1-gt)=0.$$

So $t^k = 0$.

However, $\mathscr{O}_{C,P}$ is contained in the function field $k(C)$.

So $t^k \neq 0$.

Intersection of m_P^k, $k = 1, 2, \ldots$

Lemma

$$\bigcap_{k=1}^{\infty} m_P^k = \{0\}.$$

- Let U be an affine neighborhood of P in C.

  The coordinate ring $k[U]$ is Noetherian.

  The local ring of $k[U]$ is obtained by localization at a maximal ideal.

  So the local ring $\mathscr{O}_{C,P}$ is also Noetherian.

  By the Hilbert Basis Theorem, the ring $\mathscr{O}_{C,P}[T]$ is also Noetherian.

Intersection of m_P^k, $k = 1, 2, \ldots$ (Cont'd)

- Suppose we have $\alpha \in \bigcap_{k=1}^{\infty} m_P^k$.

  Then, for all k, we have $\alpha \in m_P^k$.

  So
  $$\alpha = f_k(t),$$
  for some polynomial $f_k \in \mathscr{O}_{C,P}[T]$ of the form
  $$f_k = g_k T^k, \quad g_k \in \mathscr{O}_{C,P}.$$
  Let I be the ideal in $\mathscr{O}_{C,P}[T]$ generated by the polynomials f_k.

  Since $\mathscr{O}_{C,P}[T]$ is Noetherian, there are elements $f_1, \ldots, f_\ell$ which generate I.

Intersection of m_P^k, $k = 1, 2, \ldots$ (Cont'd)

- Now we obtain

$$f_{\ell+1}(T) = \sum_{i=1}^{\ell} h_i(T) f_i(T), \quad h_i(T) \in \mathscr{O}_{C,P}[T],$$

where $h_i(T) = p_i T^{\ell+1-i}$, for some $p_i \in \mathscr{O}_{C,P}$.
By substituting t for T, we obtain $h_i(t) \in m_P^{\ell+1-i} \subseteq m_P$.
Putting $\mu_i := h_i(t)$, the equation above becomes

$$\alpha = \sum_{i=1}^{\ell} \mu_i \alpha = \mu \alpha, \quad \mu = \sum_{i=1}^{\ell} \mu_i \in m.$$

Thus, $\alpha(1-\mu) = 0$.
But $(1-\mu)(P) = 1 \neq 0$.
So $1-\mu$ is a unit in $\mathscr{O}_{C,P}$.
It now follows that $\alpha = 0$.

Multiplicity of a Function in $\mathscr{O}_{C,P}$

Definition

For every function $g \in \mathscr{O}_{C,P}$, regular at P, the **multiplicity** of g at P is given by

$$v_P(g) := \max\{k : g \in m_P^k\}.$$

- A function g vanishes at P if and only if $v_P(g) \geq 1$.
- If g has multiplicity k at P, then

$$g = ht^k,$$

for some $h \in \mathscr{O}_{C,P}$, with $h(P) \neq 0$.

Multiplicity of a Rational Function at a Point

Definition

For every rational function $0 \neq f \in k(C)$, the **multiplicity** of f at P is defined by

$$v_P(f) := v_P(g) - v_P(h),$$

where $f = \frac{g}{h}$, for some $g, h \in \mathscr{O}_{C,P}$.

- If $v_P(f) > 0$, then f has a **zero of order** $v_P(f)$ in P.
- If $v_P(f) < 0$, then f has a **pole of order** $-v_P(f)$ in P.

- This definition is independent of the representation $f = \frac{g}{h}$.

  Suppose $f = \frac{g}{h} = \frac{g'}{h'}$.
  Then $gh' = g'h$ and we have

$$v_P(g) + v_P(h') = v_P(gh') = v_P(g'h) = v_P(g') + v_P(h).$$

Multiplicity as a Discrete Valuation

- For every point $P \in C$, the multiplicity at P defines a map

$$\begin{array}{rrcl} v_P : & k(C)^* & \to & \mathbb{Z}; \\ & f & \mapsto & v_P(f). \end{array}$$

- This map has the following properties:
  (1) $v_P(fg) = v_P(f) + v_P(g)$;
  (2) $v_P(f+g) \geq \min\{v_P(f), v_P(g)\}$.
- A map with these properties is called a **discrete valuation** on the field $k(C)$.
- We also have

$$\begin{array}{rcl} \mathscr{O}_{C,P} & = & \{f \in k^*(C) : v_P(f) \geq 0\} \cup \{0\}, \\ m_P & = & \{f \in k^*(C) : v_P(f) > 0\} \cup \{0\}, \end{array}$$

- We call $\mathscr{O}_{C,P}$ a **valuation ring** of $k(C)$.

Discrete Valuations and Valuation Rings

Definition (Discrete Valuation Ring)

An integral domain R is called a **discrete valuation ring** if the field of fractions K of R has a valuation v, that is, a map

$$v : K^* \to \mathbb{Z},$$

such that, for all $x, y \in K$:

(1) $v(xy) = v(x) + v(y)$;

(2) $v(x+y) \geq \min\{v(x), v(y)\}$,

and such that R is the **valuation ring** of v, that is,

$$R = \{x \in K^* : v(x) \geq 0\} \cup \{0\}.$$

Integral Closure

- We borrow from Commutative Algebra a characterization of discrete valuation rings.
- First we recall the definition of a integral closure.
- Let B be a ring and A a subring of B.
- An element $x \in B$ is **integral over** A if x is a root of a monic polynomial with coefficients in A.
- This means that x satisfies an equation of the form

$$x^n + a_1 x^{n-1} + \cdots + a_n = 0,$$

  where the a_i are elements of A.
- The set C of elements of B which are integral over A is a subring of B containing A, called the **integral closure of** A **in** B.
- If $C = A$, then A is said to be **integrally closed in** B.
- If $C = B$, then B is said to be **integral over** A.

Characterization of Discrete Valuation Rings

- We give, without proof, a characterization of discrete valuation rings.

Proposition

Let A be a Noetherian local integral domain of dimension 1. Let m be its maximal ideal and $k = A/m$ its residue field. Then the following statements are equivalent:

(1) A is a discrete valuation ring;

(2) A is integrally closed;

(3) A is a regular local ring (i.e., $\dim_k(m/m^2) = \dim A = 1$);

(4) m is a principal ideal;

(5) Every non-zero ideal is a power of m;

(6) There exists $x \in A$, such that every non-zero ideal is of the form (x^k), for some $k \geq 0$.

Values of Functions at Points on Smooth Projective Curves

Lemma

If $0 \neq f \in k(C)$, then there are only finitely many $P \in C$, such that

$$v_P(f) \neq 0.$$

- We can write

$$f = \frac{g}{h},$$

for some homogeneous polynomials g and h of the same degree (this depends on the chosen embedding $C \subseteq \mathbb{P}_k^n$).

The sets $\{g = 0\}$ and $\{h = 0\}$ are proper closed subsets of C.

Since C is a curve, both have only finitely many points.

Principal Divisors of C

Definition (Divisor Defined by f)

Let $0 \neq f \in k(C)$ be a rational function. The **divisor defined by** f is given by

$$(f) := \sum_{P \in C} v_P(f) P \in \operatorname{Div} C.$$

Definition (Principal Divisor)

A divisor $D \in \operatorname{Div} C$ is called a **principal divisor** if there is a rational function $0 \neq f \in k(C)$, such that

$$D = (f).$$

Structure of the Sets of Principal Divisors

- We have
$$(fg) = (f) + (g) \quad \text{and} \quad \left(\frac{1}{f}\right) = -(f).$$
- So there is a group homomorphism
$$\begin{array}{rcl} k(C)^* & \to & \operatorname{Div} C; \\ f & \mapsto & (f), \end{array}$$
from the multiplicative group $k(C)^*$ to the additive group $\operatorname{Div} C$.
- It follows that the principal divisors form a subgroup of $\operatorname{Div} C$.

Linear Equivalence

Definition (Linear Equivalence)

Two divisors D and D' are said to be **linearly equivalent** if their difference is a principal divisor, i.e., if

$$D - D' = (f), \text{ for some } f \in k(C)^*.$$

This relation is denoted by $D \sim D'$.

- We have
$$D \sim 0 \text{ if and only if } D \text{ is a principal divisor.}$$
- Linear equivalence is an equivalence relation.

The Divisor Class Group

Definition (The Divisor Class Group)

The **divisor class group** of C is defined by

$$\mathrm{Cl}\, C := \mathrm{Div}\, C / \sim.$$

- The principal divisors form a subgroup of the abelian group $\mathrm{Div}\, C$.
- So the divisor class group $\mathrm{Cl}\, C$ is an abelian group.

Example

- Let $C = \mathbb{P}^1_k$.
  Then, in $\text{Div}\,C$, we have
  $$D \sim 0 \quad \text{iff} \quad \deg D = 0.$$
  Consider a rational function f on $\mathbb{P}^1_k$.
  It can be written as
  $$f = \frac{g}{h},$$
  for homogeneous polynomials $g, h \in k[x_0, x_1]$, with $\deg g = \deg h$.
  So $\deg(f) = 0$.

Example (Cont'd)

- Conversely, suppose $\deg D = 0$.
  Then we have
  $$D = D' - D'',$$
  where:
  - $D' = \sum n_P P$, for some $n_P > 0$;
  - $D'' = \sum m_P P$, for some $m_P > 0$;
  - $\sum n_P = \sum m_P$.

  Then, there are homogeneous polynomials g and h, of degree $N = \sum n_P = \sum m_P$, which vanish precisely on D' and D'', respectively.
  Set
  $$f = \frac{g}{h}.$$
  Then we have $(f) = D' - D'' = D$.
  It follows that the degree function induces an isomorphism
  $$\deg : \mathrm{Cl}(\mathbb{P}_k^1) \cong \mathbb{Z}.$$

Subsection 2

The Degree of a Principal Divisor

The Degree of Map

- We introduce the concept of the degree of a map between projective curves.
- Let $f : C \to C'$ be a surjective map.
- By pull-back of functions, f induces an inclusion $k(C') \subseteq k(C)$.
- Now $k(C)$ and $k(C')$ both have transcendence degree 1.
- So the field extension $k(C)/k(C')$ is finite.

Definition (Degree of a Map)

If $f : C \to C'$ is a surjective map between projective curves, then the degree of f is defined by

$$\deg f := \deg[k(C) : k(C')].$$

Surjectivity of Nonconstant Maps to $\mathbb{P}_k^1$

- We will show that every nonconstant map $f: C \to C'$ between two projective curves is surjective.
- For now, we only need this for $C' = \mathbb{P}_k^1$.
- In this case, we can give an elementary proof.

Lemma

Every nonconstant map $f: C \to \mathbb{P}_k^1$ is surjective.

- Suppose f is not surjective.

  Then we can assume that $\infty := \mathbb{P}_k^1 \backslash \mathbb{A}_k^1$ is not in the image.

  Hence, we may regard f as a map $f: C \to \mathbb{A}_k^1$.

  By hypothesis, f is nonconstant.

  Thus, $f^*(x) := x \circ f$ is a nonconstant regular function on C.

  This is not possible.

Relating the Local Rings

- Let $f : C \to C'$ be a surjective map.
- Suppose the point $Q \in C'$ has preimage

$$f^{-1}(Q) = \{P_1, \ldots, P_m\}.$$

- Associated to Q we have the ring

$$\widetilde{\mathscr{O}} = \bigcap_{i=1}^{m} \mathscr{O}_{C,P_i} \subseteq k(C)$$

  of rational functions on C regular at $P_1, \ldots, P_m$.
- By means of the inclusion

$$\mathscr{O}_{C',Q} \subseteq k(C') \subseteq k(C)$$

  $\mathscr{O}_{C',Q}$ is contained in $\widetilde{\mathscr{O}}$.
- In particular, we may view $\widetilde{\mathscr{O}}$ as an $\mathscr{O}_{C',Q}$-module.

Existence of Local Parameters

Lemma

The ring $\widetilde{\mathscr{O}}$ has the following properties.

(1) There are elements $t_1, \ldots, t_m \in \widetilde{\mathscr{O}}$ with $v_{P_i}(t_j) = \delta_{ij}$.
In particular, t_i is a local parameter at P_i.

(2) If $u \in \widetilde{\mathscr{O}}$, then

$$u = t_1^{\ell_1} \cdots t_m^{\ell_m} v,$$

where:

- $\ell_i = v_{P_i}(u)$;
- v is an invertible element of $\widetilde{\mathscr{O}}$.

Existence of Local Parameters (Cont'd)

(1) Consider a projective embedding $C \subseteq \mathbb{P}_k^n$.

Choose a hyperplane H, not containing any of the points P_i, so that

$$\{P_1, \ldots, P_m\} \subseteq U = C \setminus H,$$

where:

- $U \subseteq \mathbb{A}_k^n$;
- $\mathbb{A}_k^n$ is such that we have a disjoint union $\mathbb{P}_k^n = \mathbb{A}_k^n \cup H$.

Now we can choose affine hyperplanes H_i which:

- Intersect C transversally at P_i (i.e., $T_{P_i} C \nsubseteq H_i$);
- Do not pass through any of the points P_j, for $j \neq i$.

In this argument we use the fact that the field $k = \overline{k}$ has infinitely many elements.

Define t_i to be given by the restriction of the equation of the hyperplane H_i to the curve C.

Then the t_i behave as required.

Existence of Local Parameters (Cont'd)

(2) Let $u \in \widetilde{\mathscr{O}}$.

Set $\ell_i := v_{P_i}(u) \geq 0$.

Define
$$u' := t_1^{-\ell_1} \cdots t_m^{-\ell_m} u.$$

Then $v_{P_i}(u') = 0$, for $i = 1, \ldots, m$.

Thus, $u' \in \widetilde{\mathscr{O}}^*$ is a unit.

The result then follows from the equality
$$u = t_1^{\ell_1} \cdots t_m^{\ell_m} u'.$$

Rational Maps Between Smooth Curves

Lemma

Let C and C' be smooth curves. If C' is projective, then every rational map $f : C \dashrightarrow C'$ is a morphism.

- It is enough to consider rational maps $f : C \dashrightarrow \mathbb{P}_k^n$.
  The statement concerns the local behavior of f at each point P.
  Let $P \in C$ and let t be a **local parameter** at P (generator of m_P).
  Then, there are rational functions f_i, such that

  $$f = (f_0 : \ldots : f_n), \quad f_i \in k(C),$$

  and each f_i can be written in the form

  $$f_i = t^{\ell_i} \widetilde{f}_i,$$

  for some $\ell_i \in \mathbb{Z}$, $\widetilde{f}_i \in \mathcal{O}_{C,P}$, with $\widetilde{f}_i(P) \neq 0$.

Rational Maps Between Smooth Curves (Cont'd)

- We may assume that $\ell_0 \leq \cdots \leq \ell_n$.

  Then

$$\begin{aligned} f &= (t^{\ell_0}\widetilde{f}_0 : t^{\ell_1}\widetilde{f}_1 : \ldots : t^{\ell_n}\widetilde{f}_n) \\ &= (\widetilde{f}_0 : t^{\ell_1-\ell_0}\widetilde{f}_1 : \ldots : t^{\ell_n-\ell_0}\widetilde{f}_n). \end{aligned}$$

  So we have a representation of f with:
  - All components being regular functions;
  - Nonzero first component, since $\widetilde{f}_0(P) \neq 0$.

  Hence, f is regular at P.

Isomorphisms of Smooth Projective Curves

Corollary

Two smooth projective curves C and C' are isomorphic if and only if they are birationally equivalent.

- Consider mutually inverse rational maps

$$\begin{aligned} \varphi : \quad & C \dashrightarrow C'; \\ \varphi^{-1} : \quad & C' \dashrightarrow C. \end{aligned}$$

By the lemma, the maps φ and φ^{-1} are morphisms.

Moreover, we have

$$\varphi^{-1} \circ \varphi = \mathrm{id}_C \quad \text{and} \quad \varphi \circ \varphi^{-1} = \mathrm{id}_{C'}.$$

Freeness of $\widetilde{\mathscr{O}}$ as an $\mathscr{O}_{C',Q}$-Module

- Recall that $\mathscr{O}_{C',Q} \subseteq k(C')$ is contained in $\widetilde{\mathscr{O}} = \bigcap_{i=1}^m \mathscr{O}_{C,P_i} \subseteq k(C)$.
- Moreover, this allows viewing $\widetilde{\mathscr{O}}$ as an $\mathscr{O}_{C',Q}$-module.

Lemma

The module $\widetilde{\mathscr{O}}$ is a free $\mathscr{O}_{C',Q}$-module of rank $d = \deg f$, i.e.,

$$\widetilde{\mathscr{O}} \cong \mathscr{O}_{C',Q}^d.$$

- We started with:
  - A surjective map $f : C \to C'$ between two smooth curves;
  - A point $Q \in C'$.

  We want to show that

$$\widetilde{\mathscr{O}} = \bigcap_{i=1}^m \mathscr{O}_{C,P_i} \cong \mathscr{O}_{C',Q}^d,$$

  where $f^{-1}(Q) = \{P_1, \ldots, P_m\}$ and $d = \deg f$.

Freeness of $\widetilde{\mathscr{O}}$ (Cont'd)

- Let $V \subseteq C'$ be an affine neighborhood of Q.

  Denote the coordinate ring of V by $B = k[V]$.

  We can view B as a subring of $k(C)$.

  Let

  $$A := \text{the algebraic closure of } B \text{ in } k(C).$$

  A result from Commutative Algebra ensures that A is a finitely generated k-algebra.

  Moreover, A has field of fractions $k(C)$.

  Thus, there is an affine curve U, with $k[U] = A$.

  The proof proceeds in four steps.

Freeness of $\widetilde{\mathscr{O}}$: Step (1)

(1) We show that U is smooth.

By a previous corollary, this is equivalent to the statement that every local ring $\mathscr{O}_{U,P}$ is a regular local ring.

Consider a point $P \in U$.

There is a maximal ideal m in A, such that

$$\mathscr{O}_{U,P} \cong A_m.$$

Now A is integrally closed in $k(C)$.

An elementary argument (involving taking common denominators and clearing denominators) shows that this also holds for A_m.

By the characterization of discrete valuation rings, $\mathscr{O}_{U,P}$ is a regular local ring.

We conclude that U is smooth.

Freeness of $\widetilde{\mathscr{O}}$: Step (2)

(2) We show that U is isomorphic to an open subset of C.
The field of fractions of $A = k[U]$ is equal to the field $k(C)$.
So there is a birational map $\varphi : U \dashrightarrow C$.
By a preceding lemma, the map φ is a morphism.
We show that φ maps the affine curve U isomorphically to $\varphi(U) \subseteq C$.
But, first, we show that $\varphi(U)$ is open.
Note that a nonempty subset of C is open if and only if it equals C minus finitely many points.
Now $\varphi : U \to C$ is birational.
So there are nonempty open sets $U' \subseteq U$ and $U'' \subseteq C$, such that

$$\varphi|_{U'} : U' \to U''$$

is an isomorphism.
But U'' is open and nonempty and $U'' \subseteq \varphi(U)$.
So $\varphi(U)$ is also open.

Freeness of $\widetilde{\mathscr{O}}$: Step (2) (Cont'd)

- We will now show that the map $\varphi : U \to \varphi(U)$ is an isomorphism.
  To prove this, it is enough to show that the rational inverse map $\varphi^{-1} : \varphi(U) \dashrightarrow U$ is a morphism.
  Since U is affine, we cannot use the Rational Map Lemma.
  Assume that $U \subseteq \mathbb{A}_k^n$ and that $\varphi^{-1} = (g_1, \ldots, g_n)$, $g_i \in k(C)$.
  Let $S = \varphi(R)$ be a point that is not regular for φ^{-1}.
  Possibly after renumbering, we have $g_1 = \frac{h_1}{h_2}$, $h_1(S) \neq 0$, $h_2(S) = 0$.
  If $z_1, \ldots, z_n$ are the coordinates of $\mathbb{A}_k^n$, then $g_1 = (\varphi^{-1})^*(z_1)$.
  That is, $\varphi^*(g_1) = z_1$. Thus, $\varphi^*(h_1) = z_1 \varphi^*(h_2)$.
  Now we get
  $$\varphi^*(h_2)(R) = h_2(\varphi(R)) = h_2(S) = 0.$$
  But from this it also follows that $\varphi^*(h_1)(R) = 0$.
  So $h_1(S) = 0$, contradicting the hypothesis.
  In the following we identify U with the image $\varphi(U)$ by means of φ, i.e., we view U as an open subset of C.

Freeness of $\widetilde{\mathscr{O}}$: Step (3)

(3) We show that $U = f^{-1}(V)$.
We have an inclusion

$$B = k[V] \stackrel{f^*}{\hookrightarrow} A = k[U].$$

$$\begin{array}{ccc} U & \hookrightarrow & C \\ f \downarrow & & \downarrow f \\ V & \hookrightarrow & C' \end{array}$$

It induces a commutative diagram.
In particular, $U \subseteq f^{-1}(V)$.
Suppose that this is a strict inclusion.
Then there is a point $\widetilde{R} \in C$, such that $\widetilde{R} \notin U$ and $\widetilde{S} := f(\widetilde{R}) \in V$.
Let $f^{-1}(\widetilde{S}) \cap U = \{\widetilde{R}_1, \ldots, \widetilde{R}_\ell\}$.
By the preceding lemma, there is a rational function $g \in k(C)$, which is regular at $\widetilde{R}_1, \ldots, \widetilde{R}_\ell$ but not at $\widetilde{R}$,

$$g \notin \mathscr{O}_{C,\widetilde{R}} \quad \text{and} \quad g \in \mathscr{O}_{C,\widetilde{R}_i}, \quad i = 1, \ldots, \ell.$$

Let $X \subseteq C$ be the set of points where g is not regular (its poles).
Then $\widetilde{S} \notin f(X \cap U)$.

Freeness of $\widetilde{\mathscr{O}}$: Step (3) (Cont'd)

- Similarly, there is a function $h \in k[V] = B$ with

$$hg \in k[U] = A \quad \text{and} \quad hg \notin \mathscr{O}_{C,\widetilde{R}}.$$

Here h is chosen to:
  - Have zeros of sufficiently high order in $f(X \cap U)$;
  - Be nonzero at $\widetilde{S}$.

By definition of A, $g' := hg \in A$ is algebraic over B.

That is, g' satisfies an equation

$$g'^n + b_{n-1}g'^{n-1} + \cdots + b_0 = 0, \quad b_i \in B = k[V].$$

Thus, in the field $k(C)$, we have

$$g' = -b_{n-1} - b_{n-2}g'^{-1} - \cdots - b_0 g'^{-n+1}.$$

This is a contradiction, since $g' \notin \mathscr{O}_{C,\widetilde{R}}$ but $b_i g'^{-1} \in \mathscr{O}_{C,\widetilde{R}}$.

We conclude that $U = \varphi^{-1}(V)$.

Freeness of $\widetilde{\mathscr{O}}$: Step (4)

(4) We show that

$$\widetilde{\mathscr{O}} = A\mathscr{O}_{C',Q}.$$

This, as we will see, leads to the statement of the lemma.

The inclusion $A\mathscr{O}_{C',Q} \subseteq \widetilde{\mathscr{O}}$ is obvious.

Now let $g \in \widetilde{\mathscr{O}}$ and let X be the set of poles of g.

Then we have $Q \notin f(X)$.

By Part (3), we can find a function $h \in k[V]$, with

$$h(Q) \neq 0 \quad \text{and} \quad hg \in A.$$

Since $h(Q) \neq 0$, we have $h^{-1} \in \mathscr{O}_{C',Q}$.

Thus, $g \in A\mathscr{O}_{C',Q}$.

Freeness of $\widetilde{\mathscr{O}}$: Conclusion

- By a previously quoted result of Commutative Algebra, A is finitely generated as a B-module.

  We showed that $\widetilde{\mathscr{O}} = A\mathscr{O}_{C',Q}$.

  It follows that $\widetilde{\mathscr{O}}$ is a finitely generated $\mathscr{O}_{C',Q}$-module.

  The local ring $\mathscr{O}_{C',Q}$ is a principal ideal ring.

  That is, every ideal has the form (t^k).

  From the Structure Theorem of Finitely Generated Modules over principal ideal rings, for some integer m,

  $$\widetilde{\mathscr{O}} = \mathscr{O}^m_{C',Q} \oplus T, \quad T = \text{torsion part}.$$

  But $\mathscr{O}_{C',Q} \subseteq \widetilde{\mathscr{O}} \subseteq k(C)$, i.e., $\widetilde{\mathscr{O}}$ is contained in the field $k(C)$.

  This shows that there can be no torsion.

  So we get $T = 0$.

Freeness of $\widetilde{\mathscr{O}}$: Conclusion (Cont'd)

- It now remains to determine the number m.

  This is the number of independent elements of $\widetilde{\mathscr{O}}$ over $\mathscr{O}_{C',Q}$.

  By multiplying through by the denominators, one sees that this is the same as the number of independent elements of $\widetilde{\mathscr{O}}$ over $k(C')$.

  Now $d = \deg[k(C) : k(C')]$ is the degree of the field extension.

  Therefore, we have $m \leq d$.

  Now let $f_1, \ldots, f_d$ be a basis of $k(C)$ over $k(C')$.

  It is possible that some of $f_1, \ldots, f_d$ could have a pole in the set $f^{-1}(Q)$.

  However, we can multiply with a suitable power t^ℓ, where t is a local parameter in Q.

  So the functions $f_1 t^\ell, \ldots, f_d t^\ell \in \widetilde{\mathscr{O}}$ are independent over $k(C')$.

  This shows that $m \geq d$.

Surjective Map and Map Between Groups of Divisors

- Let C and C' be smooth projective curves.
- Suppose that $f : C \to C'$ is a surjective map.
- Consider $Q \in C'$.
- Choose a local parameter t in Q.
- That is, t is a generator of the maximal ideal m_Q.
- The inverse image $f^{-1}(Q)$ is a proper closed subset of C.
- So $f^{-1}(Q)$ contains only finitely many points.
- We set

$$f^*(Q) := \sum_{P_i \in f^{-1}(Q)} v_{P_i}(f^*(t)) P_i,$$

where $f^*(t) = t \circ f$.

Map Between Groups of Divisors

- The divisor
$$f^*(Q) := \sum_{P_i \in f^{-1}(Q)} v_{P_i}(f^*(t))P_i$$
is independent of the choice of t.
Suppose t' is another local parameter.
Then $t' = ut$, for some unit $u \in \mathscr{O}_{C,Q}$.
In particular $u(Q) \neq 0$.
Hence, also $f^*(u)(P_i) \neq 0$.
So
$$v_{P_i}(f^*(t')) = v_{P_i}(f^*(ut)) = v_{P_i}(f^*(u)) + v_{P_i}(f^*(t)) = v_{P_i}(f^*(t)).$$
By extending linearly we obtain a group homomorphism
$$f^* : \operatorname{Div} C' \to \operatorname{Div} C.$$

Degree of a Map and Degree of a Divisor

Proposition

If $f : C \to C'$ is a surjective map between smooth projective curves, then for all points $Q \in C'$,

$$\deg f^*(Q) = \deg f.$$

- This result gives a geometrical meaning to the degree

$$\deg f := \deg[k(C) : k(C')]$$

  of a map $f : C \to C'$.
- The degree of f is the number of preimages (correctly counted) of any point $Q \in C'$.

Degree of a Map and Degree of a Divisor (Cont'd)

- Let $t \in \mathscr{O}_{C',Q} \subseteq \widetilde{\mathscr{O}}$ be a local parameter.
  By a preceding lemma, we can write
  $$t = t_1^{\ell_1} \cdots t_m^{\ell_m} v, \quad \ell_i = v_{P_i}(t),\ v \in \widetilde{\mathscr{O}}^*.$$
  Hence,
  $$f^*(Q) = \sum_{i=1}^m \ell_i P_i.$$
  Moreover,
  $$\deg f^*(Q) = \sum_{i=1}^m \ell_i.$$
  Since we have $v_{P_i}(t_j) = \delta_{ij}$, the t_i are pairwise coprime.
  By the Chinese Remainder Theorem, we get
  $$\widetilde{\mathscr{O}}/(t) = \bigoplus_{i=1}^m \widetilde{\mathscr{O}}/(t_i^{\ell_i}).$$

Degree of a Map and Degree of a Divisor (Cont'd)

Claim: We have

$$\dim_k \widetilde{\mathscr{O}}/(t_i^{\ell_i}) = \ell_i.$$

The functions $1, t_i, \ldots, t_i^{\ell_i - 1}$ are linearly independent over k.

So we have

$$\dim_k \widetilde{\mathscr{O}}/(t_i^{\ell_i}) \geq \ell_i.$$

Thus, it is enough to show that every element $w \in \widetilde{\mathscr{O}}$ can be written as

$$w \equiv \alpha_0 + \alpha_1 t_i + \cdots + \alpha_{\ell_i - 1} t_i^{\ell_i - 1} \mod t_i^{\ell_i}, \quad \alpha_i \in k.$$

We prove this by induction on $s = \ell_i$.

If $s = 0$, there is nothing to prove.

Assume, next, that the statement is true for $s \geq 0$.

That is, we have

$$w \equiv \alpha_0 + \alpha_1 t_i + \cdots + \alpha_{s-1} t_i^{s-1} \mod t_i^s.$$

Degree of a Map and Degree of a Divisor (Cont'd)

- By a preceding lemma,

$$\widetilde{w} := t_i^{-s}(w - \alpha_0 - \alpha_1 t_1 - \cdots - \alpha_{s-1} t_i^{s-1}) \in \widetilde{\mathscr{O}} \subseteq \mathscr{O}_{C,P_i}.$$

Set $\alpha_s := \widetilde{w}(P_i)$.

So $\widetilde{w} - \alpha_s$ has a zero in P_i.

I.e., $\widetilde{w} - \alpha_s \in (t_i)$.

In other words,

$$\alpha_s \equiv t_i^{-s}(w - \alpha_0 - \cdots - \alpha_{s-1} t_i^{s-1}) \mod t_i.$$

Multiplication by t_i^s gives

$$w \equiv \alpha_0 + \cdots + \alpha_{s-1} t_i^{s-1} + \alpha_s t_i^s \mod t_i^{s+1}.$$

Degree of a Map and Degree of a Divisor (Conclusion)

- By the Claim, we know that $\dim_k \widetilde{\mathscr{O}}/(t_i^{\ell_i}) = \ell_i$.

  We now get

$$\dim \widetilde{\mathscr{O}}/(t) = \sum_{i=1}^{m} \dim \widetilde{\mathscr{O}}/(t_i^{\ell_i}) = \sum_{i=1}^{m} \ell_i = \deg f^*(Q).$$

  On the other hand, by a preceding lemma we also have

$$\widetilde{\mathscr{O}} \cong \mathscr{O}_{C',Q}^d \quad d = \deg f.$$

  Thus, we get

$$\widetilde{\mathscr{O}}/(t) \cong (\mathscr{O}_{C',Q}/(t))^d \cong k^d.$$

  These equations imply that $d = \deg f^*(Q)$.

Principal Divisors of Smooth Projective Curves

Theorem (Degree of Principal Divisors)

If C is a smooth projective curve, then every principal divisor on C has degree 0.

- Let $f \in k(C)$ be a nonconstant function on C.
  Then f defines a rational map $f : C \dashrightarrow \mathbb{P}_k^1$.
  By a previous lemma, f is a morphism.
  But, then, another lemma asserts that f is surjective.
  By definition of (f) and f^*, we have
  $$(f) = f^*(0) - f^*(\infty).$$
  A previous proposition implies that
  $$\deg f^*(0) = \deg f^*(\infty) = \deg f.$$
  So
  $$\deg(f) = \deg f^*(0) - \deg f^*(\infty) = 0.$$

Remark: The Case $k = \mathbb{C}$

- In the case $k = \mathbb{C}$ one can also give an analytic proof of the theorem.
- For this one considers the integral

$$\int_\gamma \frac{df}{f}$$

  over a suitable closed path γ.
- By Cauchy's integral theorem this integral counts the difference between the number of zeros and poles of f in the "interior" of γ.
- By following the path in the opposite direction, the integral also counts the difference between the number of zeros and poles in the region "outside" γ.
- This gives the value 0 for the degree of the principal divisor.

The Jacobian Variety of C

- Since every principal divisor has degree 0, the degree function induces a homomorphism
$$\deg : \mathrm{Cl}\,C \to \mathbb{Z}.$$

Definition (The Jacobian Variety)

We define the **Jacobian variety** of C (**of degree** 0) by
$$\mathrm{Jac}^0 C := \mathrm{Cl}^0 C := \{D \in \mathrm{Cl}\,C : \deg D = 0\}.$$

- There is an exact sequence
$$0 \longrightarrow \mathrm{Cl}^0 C \longrightarrow \mathrm{Cl}\,C \xrightarrow{\deg} \mathbb{Z} \longrightarrow 0.$$
- Torelli's theorem says that the polarized abelian variety $\mathrm{Cl}^0 C$ determines the curve.

The Jacobian of Rational Curves

Proposition

The Jacobian $\mathrm{Cl}^0 C$ is trivial if and only if C is rational (i.e., isomorphic to $\mathbb{P}^1_k$).

- We have already seen that $\mathrm{Cl}^0(\mathbb{P}^1_k) = \{0\}$.

  Conversely, suppose $\mathrm{Cl}^0(C) = \{0\}$. Then any two divisors D and D' of the same degree are linearly equivalent.

  Let $P \neq Q$ be two distinct points on C. Since $P \sim Q$, there is a nonzero rational function $f \in k(C)$, with

  $$(f) = P - Q.$$

  By a preceding lemma, the rational map $f : C \dashrightarrow \mathbb{P}^1_k$ is a regular map.

  We have $f^*(0) = P$, $f^*(\infty) = Q$. In particular f has degree 1.

  So it induces an isomorphism of the function fields.

  Thus, by a preceding corollary, is an isomorphism of C with $\mathbb{P}^1_k$.

The Structure of the Jacobian

- We saw $\mathrm{Cl}^0 C$ is the kernel of the morphism $\deg : \mathrm{Cl} C \to \mathbb{Z}$.
- So the Jacobian $\mathrm{Cl}^0 C$ has a group structure.
- The group structure on $\mathrm{Cl}^0 C$ is given simply by adding divisors.
- This is clearly compatible with linear equivalence.
- Thus, if C is not a rational curve, then $\mathrm{Cl}^0 C$ is a g-dimensional projective abelian variety.
- That is, a projective variety with the structure of an abelian group, with addition,
$$(a, b) \mapsto a + b,$$
and inversion
$$a \mapsto a^{-1},$$
being morphisms.
- Over $\mathbb{C}$, an abelian variety is a g-dimensional torus, which is also a projective variety.

The Genus of a Curve

- The dimension g of $\mathrm{Cl}^0 C$ is the **genus** of the curve C.
- Over the ground field $\mathbb{C}$, it is a nontrivial result that this is the same as the topological genus, which is given by the number of holes of C as a Riemann surface.
- We will give a definition of the genus of C in terms of the number of independent regular differentials on C.
- We will then relate it to the degree of the **canonical divisor** of C.
- The genus will also appear in the statement of the Riemann-Roch Theorem.

The Case of a Smooth Plane Cubic

- Consider the case of a smooth plane cubic $C \subseteq \mathbb{P}^2_{\mathbb{C}}$.
- We gave a topological description for the case of cubics in Legendre normal form.
- We also saw that such a curve is homeomorphic to a torus.
- Hence C has genus 1.
- Consider a fixed point $O \in C$ (which for simplicity we will take to be a point of inflection).
- It can then be seen that the map

$$\begin{array}{rcl} C & \to & \mathrm{Cl}^0 C \\ P & \mapsto & P - O \end{array}$$

defines an isomorphism of C with $\mathrm{Cl}^0 C$.

Subsection 3

Bézout's Theorem

Bézout's Theorem for One Smooth Curve

Theorem

Let C and C' be two plane curves of degrees d and d', respectively. Assume that C is smooth, and that C is not a component of C'. Then C and C' intersect in precisely dd' points, that is,

$$C.C' = \sum_P I_P(C, C') = dd'.$$

- Suppose that
$$C = \{f(x_0, x_1, x_2) = 0\} \subseteq \mathbb{P}_k^2,$$
where f is a homogeneous polynomial of degree d.
Suppose
$$C' = \{g(x_0, x_1, x_2) = 0\} \subseteq \mathbb{P}_k^2,$$
where g is a homogeneous polynomial of degree d'.
By assumption, C' does not contain C.

Bézout's Theorem for One Smooth Curve (Cont'd)

- For every point $P \in C$, we can view the equation g of C' as a regular function in an affine neighborhood of P.

  That is, we can view g as an element of $\mathscr{O}_{C,P}$ (well defined up to multiplication by nonzero scalars).

  In this way, C' defines a divisor

  $$D = \sum_{P \in C} v_P(g) P \in \operatorname{Div} C.$$

  From the definition of local intersection multiplicity, it follows that

  $$I_P(C, C') = v_P(g).$$

  Thus,

  $$C.C' = \sum_P I_P(C, C') = \deg D.$$

Bézout's Theorem for One Smooth Curve (Cont'd)

- Now consider the rational function
$$h = \frac{g}{x_0^{d'}}.$$
Here we may assume that $C \neq \{x_0 = 0\}$.
The rational function h defines a principal divisor on C,
$$(h) = D - d' D_0,$$
where D is as above, and D_0 is the divisor on C defined by x_0.
If L is the line $\{x_0 = 0\}$, then
$$\begin{aligned} \deg D_0 &= L.C \\ &= d. \quad \text{(previous proposition)} \end{aligned}$$
Since (h) is a principal divisor, by the theorem,
$$0 = \deg(h) = \deg D - d' \deg D_0.$$
So $\deg D = d' \deg D_0 = dd'$.

Remarks on the General Case

- The same method can also be used to prove the general version of Bézout's theorem.
- In general one must decompose C into irreducible components
$$C_1, \ldots, C_n.$$
- If they are not smooth, one must consider the normalization
$$\nu_i : \widetilde{C}_i \to C_i.$$
- As before, the proof in the general case is based on the fact that principal divisors on curves have degree zero.

Subsection 4

Linear Systems on Curves

Ordering on Divisors

Definition (Divisor Ordering)

There is a partial ordering on divisors on curves, defined as follows. Consider divisors

$$D_1 = \sum n_P P \quad \text{and} \quad D_2 = \sum m_P P.$$

We write

$$D_1 \geq D_2 \quad \text{iff} \quad n_P \geq m_P, \text{ for all } P \in C.$$

Vector Space Associated With a Divisor

Definition

Let D be a divisor. We define

$$L(D) := \{0 \neq f \in k(C) : (f) \geq -D\} \cup \{0\}.$$

- Clearly $L(D)$ is a k-vector space.
- We define

$$\ell(D) := \dim_k L(D).$$

Support of a Divisor and Effective Divisors

Definition

The **support** of a divisor $D = \sum n_P P$ is defined by

$$\operatorname{supp} D := \{P : n_P \neq 0\}.$$

Definition

A divisor D is called **effective** if

$$D \geq 0.$$

Degree of D and Dimension of $L(D)$

Lemma

(1) If $\deg D < 0$, then $L(D) = \{0\}$.

(2) For every effective divisor D, we have $\ell(D) \leq \deg D + 1$.
In particular, $L(D)$ is a finite dimensional vector space.
Equality holds only if C is rational or $D = 0$.

(1) Suppose $\deg D < 0$.
Consider $0 \neq f \in L(D)$.
Then f has more zeros than poles.
This contradicts $\deg(f) = 0$.

Degree of D and Dimension of $L(D)$ (Cont'd)

(2) Suppose $\deg D = 0$.
By hypothesis, D is effective.
So we have $D = 0$.
In this case $L(D)$ is the space of constants.
So $\ell(D) = 1$.
Next, suppose that $d := \deg D \geq 1$.
Let $P_1, \ldots, P_{d+1}$ be distinct points that do not lie in the support of D.
Consider the map

$$\begin{array}{rcl} L(D) & \to & k^{d+1}; \\ f & \mapsto & (f(P_1), \ldots, f(P_{d+1})). \end{array}$$

Its kernel is $L(D - P_1 - \cdots - P_{d+1})$.
So $L(D - P_1 - \cdots - P_{d+1})$ has codimension at most $d+1$ in $L(D)$.
By Part (1), $L(D - P_1 - \cdots - P_{d+1}) = \{0\}$.
So $\dim L(D) \leq d + 1$.

Degree of D and Dimension of $L(D)$ (Cont'd)

- Now suppose that $\ell(D) = d + 1$.

  This is equivalent to the above map being surjective.

  Then the composition with the projection onto the first $d-1$ components is also surjective.

  Moreover, its kernel is $L(D - P_1 - \cdots - P_{d-1})$.

  Thus,

$$\dim L(D - P_1 - \cdots - P_{d-1}) = 2.$$

  So, there exist linearly independent functions

$$f, g \in L(D - P_1 - \cdots - P_{d-1}).$$

Degree of D and Dimension of $L(D)$ (Cont'd)

- Now $\deg(D - P_1 - \cdots - P_{d-1}) = 1$.

  So, for some points $P \neq Q$ on C with $P \sim Q$,

$$\begin{array}{rcl} (f) & = & P_1 + \cdots + P_{d-1} - D + P; \\ (g) & = & P_1 + \cdots + P_{d-1} - D + Q. \end{array}$$

  The proof of the preceding proposition then shows that there is an isomorphism $C \cong \mathbb{P}_k^1$.

  Suppose, conversely, that C is rational.

  Then we can always find a rational function with:

  - Poles (counted with multiplicity) given by the divisor D;
  - Zeros at exactly d of the points $P_1, \ldots, P_{d+1}$.

  This implies that the map $L(D) \to k^{d+1}$ is surjective.

Spaces Associated With Equivalent Divisors

Proposition

If $D_1 \sim D_2$, then $L(D_1) \cong L(D_2)$.

- Let $D_1 - D_2 = (f)$.
  If $g \in L(D_1)$, then
  $$(gf) = (g) + (f) \geq -D_2.$$
  In this way, we obtain an isomorphism
  $$\begin{array}{rcl} L(D_1) & \to & L(D_2); \\ g & \mapsto & gf. \end{array}$$

Complete Linear Systems Associated With Divisors

Definition

A divisor D defines a **complete linear system**

$$|D| := \{D' \geq 0 : D' \sim D\}.$$

- We know that:
  - Equivalent divisors have the same degree (since principal divisors have degree 0);
  - Effective divisors have nonnegative degree.
- So, if $\deg D < 0$, we have $|D| = \emptyset$.

Bijection Between $|D|$ and $\mathbb{P}(L(D))$

Proposition

There is a natural bijection between the complete linear system $|D|$ and the projective space $\mathbb{P}(L(D))$.

- Let $f \in L(D)$ be nonzero.
  Define
  $$D_f := (f) + D.$$
  Then we have:
  - $D_f \geq 0$;
  - $D_f \sim D$;
  - For $\lambda \in k^*$, $(f) = (\lambda f)$.

  Thus, we obtain a map
  $$\begin{array}{rcl} \mathbb{P}(L(D)) & \to & |D|; \\ f & \mapsto & D_f. \end{array}$$

Bijection Between $|D|$ and $\mathbb{P}(L(D))$ (Cont'd)

- We first show surjectivity.

  Suppose that $D' \geq 0$ and $D' \sim D$.

  Let f be a rational function, with $(f) = -D + D'$.

  Since $D' \geq 0$, it follows that $f \in L(D)$.

  Next, we show injectivity.

  Suppose that f and g are rational functions with $(f) = (g)$.

  Then $\frac{f}{g}$ is an everywhere regular function.

  Hence, $\frac{f}{g}$ is constant.

  I.e., $f = \lambda g$, for some $\lambda \in k^*$.
- In the following we will identify $|D|$ and $\mathbb{P}(L(D))$.
- So $|D|$ is considered to have the structure of a projective space.

Linear System and Base Points

Definition (Linear System)

A **linear system** ϑ on C is a projective subspace of a complete linear system $|D|$.

Definition (Base Point)

A point $P \in C$ is called a **base point** of the linear system ϑ if

$$\vartheta = \vartheta \cap |D - P|.$$

A linear system ϑ is called **base point free** if it has no base points.

- We may consider only base point free linear systems.

  Suppose P is a base point of $|D|$. Then $L(D) = L(D - P)$.

  By subtracting all the base points of D, we obtain a base point free linear system $|D'|$ with $L(D) = L(D')$.

Linear System Defined By Hyperplane Sections

- Suppose that $C \subseteq \mathbb{P}^r_k$ is a smooth curve not contained in a hyperplane.
- Then the set of hyperplanes H in $\mathbb{P}^r_k$ defines a set of divisors of the form $C \cap H$ on C, called **hyperplane sections**.
- More precisely, $C \cap H$ is defined by restricting an equation $\{s = 0\}$ for H to the curve C.
- On every open set $C \setminus \{x_i = 0\}$, we can view s as a regular function.
- Taking the zeros of s, counted with multiplicity, defines an effective divisor $D := C \cap H$.

Linear System Defined By Hyperplane Sections (Cont'd)

- Consider two hyperplanes:
  - $H_1 = \{s_1 = 0\}$;
  - $H_2 = \{s_2 = 0\}$.
- Then $\frac{s_1}{s_2}$ is a rational function.
- So the divisors D_1 and D_2 are linearly equivalent.
- The set of hyperplane sections $C \cap H$ on C is a base point free linear system ϑ, which is not necessarily complete.
- The divisors of the linear system ϑ all have the same degree d.
- We call d the **degree** of the embedded curve $C \subseteq \mathbb{P}_k^r$.
- In the case of a plane curve this agrees with the definition of degree previously given.

The Map Defined by a Complete Linear System

Definition (The Map Defined by a Complete Linear System)

Let D be a divisor on a curve C.
Let $|D|$ be a base point free complete linear system.
Then the map φ_D, called **the map defined by the complete linear system** $|D|$, is defined as follows.
Let $\ell = \ell(D) > 0$. Choose a basis $f_0, \ldots, f_{\ell-1}$ of $L(D)$.
We define φ_D by

$$\begin{array}{rccl} \varphi_D : & C & \to & \mathbb{P}_k^{\ell-1}; \\ & P & \mapsto & (f_0(P) : \ldots : f_{\ell-1}(P)). \end{array}$$

- By a previous lemma, the map φ_D is a morphism.

Remarks

- Clearly φ_D depends on the choice of basis.
- However, two different bases give rise to maps differing only by a projective automorphism of $\mathbb{P}_k^{\ell-1}$.
- If $\ell \geq 2$, then the map
$$\varphi_D : C \to \varphi_D(C)$$
has finite fibers.
- Similarly, for any base point free linear system $\vartheta \subseteq |D|$ of projective dimension r, by making a choice of basis of ϑ, we obtain a morphism
$$\varphi_\vartheta : C \to \mathbb{P}_k^r.$$
- If $\vartheta = |D|$, then $\varphi_\vartheta = \varphi_{|D|} = \varphi_D$.

Example

- Let $C = \mathbb{P}^1_k$, with homogeneous coordinates x_0, x_1.
- Consider the divisor $D = 3\infty$ on C, where $\infty = (1:0)$.
- Then $\ell(D) = 4$.
- Moreover, a basis for $L(D)$ given by

$$\frac{x_0^3}{x_1^3}, \quad \frac{x_0^2 x_1}{x_1^3} = \frac{x_0^2}{x_1^2}, \quad \frac{x_0 x_1^2}{x_1^3} = \frac{x_0}{x_1}, \quad \frac{x_1^3}{x_1^3} = 1 \quad \in L(D).$$

- The map φ_D is given by

$$\begin{array}{rccl} \varphi_D : & \mathbb{P}^1_k & \to & \mathbb{P}^3_k; \\ & (x_0 : x_1) & \mapsto & (\frac{x_0^3}{x_1^3} : \frac{x_0^2}{x_1^2} : \frac{x_0}{x_1} : 1) = (x_0^3 : x_0^2 x_1 : x_0 x_1^2 : x_1^3). \end{array}$$

- Thus $|D|$ defines the embedding of $\mathbb{P}^1_k$ as the cubic normal curve in $\mathbb{P}^3_k$.

Hyperplane Sections and Base Point Free Linear Systems

- Let δ be an arbitrary base point free linear system on C.
- We show the hyperplane sections of $\varphi_\delta(C)$ give rise to elements of δ.
- Let δ be an arbitrary base point free linear system of projective dimension r, on a curve C.
- Choose a basis $f_0,\ldots,f_r$ for δ.
- We get a map

$$\begin{array}{rccl} \varphi_\delta : & C & \to & \mathbb{P}^r_k; \\ & P & \mapsto & (f_0(P):\ldots:f_r(P)). \end{array}$$

- Let $x_0,\ldots,x_r$ be the projective coordinates of $\mathbb{P}^r_k$.
- Consider in $\mathbb{P}^r_k$ a hyperplane

$$H = \left\{\sum \lambda_i x_i = 0\right\}.$$

- The hyperplane section $\varphi_\delta(C) \cap H$ may be pulled back to a divisor $\varphi^*_\delta(H)$ on C, given by the points in $\varphi^{-1}(H)$, counted with multiplicity.

Hyperplane Sections and Linear Systems (Cont'd)

- The multiplicities are defined as follows.
- Consider the function
$$(\sum \lambda_i x_i) \circ \varphi_\delta.$$
- It can be viewed as a local function at every point $P \in C$.
- The multiplicity of the point P in $\varphi_\delta^*(H)$ is the vanishing order of this function at P.
- The divisor obtained in this way is nothing but the divisor $D_{f_H} \in \delta$ of the function
$$f_H = \sum \lambda_i f_i.$$
- Thus every hyperplane section $H \cap \varphi_\delta(C)$ gives rise to $D_{f_H} \in \delta$.
- If φ_δ is an embedding, then all elements of δ correspond to hyperplane sections of the embedded curve in this way.

Differential Forms on a Smooth Curve

- Let $U \subseteq C$ be an open set.
- Consider the vector space

$$\phi(U) := \{\varphi : U \to \bigcup m_x/m_x^2 : \varphi(x) \in m_x/m_x^2\}.$$

- Let $f \in \mathscr{O}_C(U)$ be a regular function.
- Define an element $df \in \phi(U)$ by

$$df(x) := f - f(x) \mod m_x^2.$$

Properties of Differential Forms

- As in elementary calculus, the following identities hold.
  (1) $d(f+g) = df + dg$;
  (2) $d(fg) = fdg + gdf$;
  (3) $d(\frac{f}{g}) = \frac{gdf - fdg}{g^2}$, whenever $g \neq 0$.

  Property (1) is obvious.

  Properties (2) and (3) follow by calculating and comparing both sides of the equality modulo m_x^2.
- If $F \in k[x_1, \ldots, x_n]$, then for regular functions $f_1, \ldots, f_n$ on U,
  (4) $dF(f_1, \ldots, f_n) = \sum_{i=1}^n \frac{\partial F}{\partial x_i}(f_1, \ldots, f_n) df_i$.

  In view of Property (1), it is enough to prove this for monomials.

  Using Property (2), this can be shown by induction on the degree of the monomials.
- Property (3) generalizes to rational functions in their domain of definition.

Regular Differential Forms

Definition (Regular Differential Form)

An element $\varphi \in \phi(U)$ is called a **regular differential form** on U if, for every point $P \in U$, there are a neighborhood V and regular functions $f_1, \ldots, f_\ell, g_1, \ldots, g_\ell \in \mathscr{O}_C(V)$, such that

$$\varphi\,|_V = \sum_{i=1}^{\ell} f_i dg_i.$$

The set of all regular differentials on U is an $\mathscr{O}(U)$-module, which we denote by $\Omega^1[U]$.

Cotangent Bundles and Regular Differential Forms

- Recall that m_x/m_x^2 can be identified with the dual of the tangent space of C at x.
- The disjoint union of these spaces is called the **cotangent bundle over** U.
- The elements of $\phi(U)$ are **sections** of the cotangent bundle.
- The definition of regular differential forms tells us which sections should be called regular.

Example

- Let t be the coordinate of $\mathbb{A}_k^1$.
- Then dt is a basis of m_x/m_x^2, for every $x \in \mathbb{A}_k^1$.
- Hence, every element $\varphi \in \phi(\mathbb{A}_k^1)$can be written as

$$\varphi = fdt, \text{ for some function } f \text{ on } \mathbb{A}_k^1.$$

- If $\varphi \in \Omega^1[\mathbb{A}_k^1]$, then the formula for $dF(f_1,\dots,f_n)$ implies that

$$\varphi|_V = gdt, \text{ for some function } g \text{ regular on } V.$$

- Comparing this with $\varphi|_V = f|_V\, dt$ shows that f is regular.
- Hence, we find that

$$\Omega^1[\mathbb{A}_k^1] = k[\mathbb{A}_k^1]dt.$$

Characterization of $\Omega^1[U]$

Proposition

Let C be a smooth curve and let $P \in C$. Then there exists an affine neighborhood U of P, such that, as $\mathscr{O}(U)$-modules,

$$\Omega^1[U] \cong \mathscr{O}(U).$$

- We can assume that $C \subseteq \mathbb{A}_k^n$ has coordinates $x_1, \ldots, x_n$, such that $t_1 := x_1 |_C$ is a local parameter near the point P.
  Choose a basis $F_1, \ldots, F_\ell$ of the ideal $I(C)$ of C in $\mathbb{A}_k^n$.
  Since the F_i are in the ideal $I(C)$, we have $dF(f_1, \ldots, f_n) = 0$.
  Hence, for $1 \leq i \leq \ell$, we have

$$\sum_{j=1}^{n} \left. \frac{\partial F_i}{\partial x_j} \right|_C dt_j = 0, \quad t_j = x_j |_C .$$

Characterization of $\Omega^1[U]$ (Cont'd)

- Since C has dimension 1, the matrix

$$\left(\left(\frac{\partial F_i}{\partial x_j}\right)(P)\right)_{i,j}$$

has rank $n-1$ at P.

We use the preceding equation to obtain local representations

$$dt_i = g_i dt_1, \quad i = 2, \ldots, n,$$

where the g_i are regular in a neighborhood of P.

Characterization of $\Omega^1[U]$ (Cont'd)

- Let
$$U' = \bigcap_{i=2}^{n} \mathrm{dom} g_i.$$
Choose some affine set $U \subseteq U'$.
Now $dx_1, \ldots, dx_n$ span the cotangent space in $\mathbb{A}_k^n$ at every point.
So dt_1 is nonzero in m_x/m_x^2, for all $x \in U$.
Therefore, every element $\varphi \in \Omega^1[U]$ has a representation
$$\varphi = \psi dt_1,$$
for some k-valued function ψ on U.
On the other hand, $dt_i = g_i dt_1$.
So there is a local representation $\varphi = g dt_1$, where g is regular.
This shows that $\psi = g$ is regular on U. I.e., $\varphi \in \mathscr{O}(U)$.
Hence, $\Omega^1[U] \cong \mathscr{O}(U)$ as an $\mathscr{O}(U)$-module.

$\Omega^1[U]$ and Local Parameters

Corollary

Let t be a local parameter at a point $P \in C$. Then there exists an affine neighborhood V of P, such that

$$\Omega^1[V] = \mathscr{O} dt.$$

- Let U be a neighborhood of P.

  Let $t' \in \mathscr{O}(U)$ be as in the preceding proof.

  Then $dt = g dt'$, for some $g \in \mathscr{O}(U)$.

  Now both t and t' are local parameters at the point P.

  So we have $g(P) \neq 0$.

  We can then take as V any affine neighborhood of P where g does not vanish.

Zeroes of Regular Differential Forms

- Let $\omega \in \Omega^1[U]$ be a regular differential form on U.
- We look at the zeros of ω, i.e., the points where $\omega(x) = 0 \in m_x/m_x^2$.
- Let $\omega = gdt$ be a local representative, as in the proposition.
- Then the zeros of ω are just the zeros of g.
- If $\omega \not\equiv 0$, then this defines an effective divisor on U.
- Note that if two regular differential forms $\omega, \omega' \in \Omega^1[U]$ coincide on a nonempty open set in U, then $\omega = \omega'$.

Equivalence of Regular Differential Forms

- Consider pairs (U,ω), where:
  - $U \subseteq C$ is open and nonempty;
  - ω is a regular differential form on U.
- We define an equivalence relation by

$$(U,\omega) \sim (U',\omega') \quad \text{iff} \quad \text{for some nonempty open } V \subseteq U, \ \omega|_V = \omega'|_V .$$

- Previous results assert this is an equivalence relation.

Rational Differential Forms

Definition (Rational Differential Form)

A **rational differential form** on C is an equivalence class of pairs

$$(U,\omega),$$

where:

- U is a nonempty open set in C;
- ω is a regular differential form on U.

We denote the set of rational differential forms on C by $\Omega^1(C)$.

- Let $f \in k(C)$ is a rational function.
- Then df defines a rational differential form on C.
- Moreover, $\Omega^1(C)$ is a $k(C)$-vector space.

Domain of Regularity of a Rational Differential Forms

- Let $\omega \in \Omega^1(C)$ be a rational differential form.
- Define
$$U_\omega$$
to be the union of all open sets U, such that ω has a representative (U, ω').
- Then $\omega \in \Omega^1[U_\omega]$.
- U_ω is called the **domain of regularity** of ω.

The Vector Space $\Omega^1(C)$

Proposition

$\Omega^1(C)$ is a 1-dimensional vector space over $k(C)$.

- Let $U \subseteq C$ be open, as in the preceding proposition, so that, for some local parameter t,
$$\Omega^1[U] = \mathscr{O}(U)dt.$$
Let $\omega \in \Omega^1(C)$.
Then, there exists some open $V \subseteq U$, such that $\omega|_V$ is regular.
We still have $\Omega^1[V] = \mathscr{O}(V)dt$
Hence, $\omega|_V = gdt$, for some $g \in k[V] \subseteq k(C)$.
The map $\omega \mapsto g$ gives the desired isomorphism.

Divisors Associated With Rational Differential Forms

- Let $\omega \in \Omega^1(C)$ be a nonzero rational differential form on C.
- Let $P \in C$.
- Then there is a neighborhood U of P, such that

$$\omega\,|_U = g dt,$$

for some local parameter t at P, and some rational function g.
- If dt' is some other local parameter, then locally

$$dt = u dt',$$

for some unit u in a neighborhood of P.
- Hence if we associate to P the order of g at P, this is well defined.
- In this way we define a divisor $(\omega) \in \mathrm{Div}\, C$.

Canonical Divisor Class

- Let $\omega \in \Omega^1(C)$ be a nonzero rational differential form on C.
- Suppose ω' is any other nonzero rational differential form on C.
- We know that
  $$\omega' = f\omega,$$
  for some $f \in k(C)^*$.
- This implies that $(\omega) \sim (\omega')$.
- In this way we obtain a well defined divisor class
  $$K := (\omega) \in \mathrm{Cl}C.$$

Definition (Canonical Divisor Class)

The divisor K is called the **canonical divisor class**, or the **canonical divisor**, of C.

The Genus

Definition (Genus)

The **genus** of the curve C is defined to be the integer

$$g := \ell(K).$$

- Thus the genus of a smooth curve C is equal to the number of linearly independent regular differential forms on C.
- If the ground field is $\mathbb{C}$, then g is equal to the topological genus of the compact Riemann surface C.
- A smooth curve C has genus 0 if and only if it is isomorphic to $\mathbb{P}^1_k$.
- A smooth plane cubic has genus 1.

Riemann-Roch Theorem

Theorem (Riemann-Roch)

If C is a projective curve of genus g and D is a divisor of degree d on C, then

$$\ell(D) - \ell(K - D) = 1 + d - g.$$

- Over $\mathbb{C}$, the Riemann-Roch Theorem establishes a link between the algebraic and topological properties of the curve.

Riemann's Theorem

Corollary (Riemann's Theorem)

If D is a divisor of degree $d > 2g - 2$, then

$$\ell(D) = d + 1 - g.$$

- By hypothesis $d > 2g - 2$.

  So $\deg(K - D) < 0$.

  Then, by a previous lemma, $L(K - D) = \{0\}$.

  Thus, by the Riemann-Roch Theorem,

$$\ell(D) = 1 + d - g.$$

Degree of Canonical Divisor

Corollary

The canonical divisor has even degree, given by

$$\deg K = 2g - 2.$$

- Set $D = K$.
  By a previous lemma,
  $$\ell(K - D) = \ell(0) = 1.$$
  By definition $\ell(K) = g$.
  So, by the Riemann-Roch Theorem,
  $$\begin{aligned} \ell(K) - \ell(0) &= 1 + d - g \\ g - 1 &= 1 + \deg(K) - g \\ \deg(K) &= 2g - 2. \end{aligned}$$

Subsection 5

Projective Embeddings of Curves

Differential of a Morphism Between Varieties

- Let $f : X \to Y$ be a morphism between varieties with $f(P) = Q$.
- By pullback of functions we have a map

$$f^* : m_{Y,Q}/m_{Y,Q}^2 \to m_{X,P}/m_{X,P}^2.$$

- By duality, we get a vector space homomorphism

$$df(P) : T_{X,P} \to T_{Y,Q},$$

  where $df(P)$ is the *differential* of the morphism f at the point P.
- If X and Y are smooth complex varieties, this corresponds to the usual definition of the differential of a holomorphic map.

Projective Embeddings

Definition (Projective Embedding)

A morphism $f : X \to \mathbb{P}_k^n$ of a projective variety X in $\mathbb{P}_k^n$ is called a **projective embedding** of X if:

- f is injective;
- The differential $df(P)$ is injective at every point P of X.

- This terminology is justified by the following result, stated without proof.

Proposition

Let X is a projective variety and $f : X \to \mathbb{P}_k^n$ a projective embedding. Then $f(X)$ is a subvariety of $\mathbb{P}_k^n$ and f induces an isomorphism $f : X \to f(X)$.

Very Ample Divisors

- Let C be a curve.
- Let D be a divisor on C, such that $|D|$ is a base point free complete linear system.
- Recall the construction of the map $\varphi_D : C \to \mathbb{P}_k^{\ell-1}$ defined by the complete linear system $|D|$, where $\ell = \ell(D) > 0$.
- Our goal is to determine which divisors give rise to embeddings.

Definition (Very Ample Divisor)

A divisor D on a curve C is called **very ample** if:

- $|D|$ is base point free;
- The map $\varphi_D : C \to \mathbb{P}_k^{\ell-1}$, where $\ell = \ell(D)$, is an embedding.

Characterization of Very Ample Divisors

Proposition

For a divisor D on a curve C the following are equivalent:

(1) D is very ample.

(2) For any points $P, Q \in C$ (including $P = Q$),

$$\dim|D - P - Q| = \dim|D| - 2.$$

(2)⇒(1): First note that, for any divisor D, and any points P and Q on C, we have

$$\dim|D - P - Q| \leq \dim|D| - 2.$$

To prove this, one applies the same argument that was used in the proof of a previous lemma to show that, for an effective divisor D, $\ell(D) \leq \deg D + 1$.

Characterization of Very Ample Divisors (Cont'd)

- By the same argument, it also follows that,

$$\dim|D-P-Q| = \dim|D|-2, \text{ for all } P \text{ and } Q,$$
$$\text{implies } \dim|D-P| = \dim|D|-1, \text{ for all } P.$$

Equivalently, $|D|$ cannot have a base point.

It remains to show that, under the hypothesis, φ_D is an embedding.

Let P and Q be any two distinct points on C.

Since $|D-P-Q| \neq |D-P|$, there is a function $f \in L(D)$, such that:

  - P lies in the support of the effective divisor $D_f = D+(f)$;
  - Q does not lie in the support of $D_f = D+(f)$.

We can extend f to a basis $f = f_0, \ldots, f_{\ell-1}$ of $L(D)$.

With this choice of basis:

  - The first coordinate of $\varphi_D(P)$ equals 0;
  - The first coordinate of $\varphi_D(Q)$ is not 0.

In particular, $\varphi_D(P) \neq \varphi_D(Q)$.

Thus, φ_D is injective.

Characterization of Very Ample Divisors (Cont'd)

- By the preceding proposition, it suffices now to show that the differential $df(P)$ is injective at a point P.

  We must, in turn, show that there is a function $f \in L(D)$, such that:
  - $(f) + D - P \geq 0$;
  - $(f) + D - 2P$ is not effective.

  That is, f vanishes at P, but not to second order.

  Suppose, this has been shown.

  Consider a basis $f = f_0, \ldots, f_{\ell-1}$ for $L(D)$.

  Then, locally, $\varphi_D^*(x_0) = f_0 \in m_P$, but $\varphi_D^*(x_0) \notin m_P^2$.

  I.e., the map between the duals of the tangent spaces is surjective.

  Thus, the differential is injective.

  For the existence of the required function, note that, by hypothesis (taking $P = Q$), $|D - P| \neq |D - 2P|$.

  The existence of f is equivalent to this statement.

Characterization of Very Ample Divisors (Converse)

(1)⇒(2): Assume that the linear system $|D|$ is base point free and the map $\varphi_D : D \to \mathbb{P}_k^{\ell-1}$ is an embedding.

Let P, Q be two distinct points of C.

By assumption, $\varphi_D(P) \neq \varphi_D(Q)$.

After a transformation of coordinates, we can assume that

$$\varphi_D(P) = (1 : 0 : \ldots : 0) \quad \text{and} \quad \varphi_D(Q) = (0 : 1 : \ldots : 0).$$

Then $\varphi_D^*(x_0)$ defines an effective divisor in $|D|$, but not in $|D - P|$.

I.e.,

$$\dim|D - P| = \dim|D| - 1.$$

Further, $\varphi_D^*(x_1) \in |D - P| \backslash |D - P - Q|$.

Hence, for $P \neq Q$,

$$\dim|D - P - Q| = \dim|D| - 2.$$

Characterization of Very Ample Divisors (Converse Cont'd)

- Now we deal with the case $P = Q$.

  We may assume that:

  - $\varphi_D(P) = (1:0:\ldots:0)$;
  - The tangent to $\varphi_D(C)$ at P is spanned by the line through $(1:0:\ldots:0)$ and $(0:1:\ldots:0)$.

  The hyperplane $\{x_1 = 0\}$:

  - Contains the point $\varphi_D(P) = (1:0:\ldots:0)$;
  - Meets the tangent to $\varphi_D(C)$ at this point transversally.

  So $\varphi_D^*(x_1) \in |D - P| \backslash |D - 2P|$.

  Thus, $\dim|D - 2P| = \dim|D| - 2$.

- The classical language for φ_D:
  - Being injective is that the linear system $|D|$ "separates points";
  - Having an injective differential is that the linear system $|D|$ "separates tangents".

Very Ample Divisors and Genus

Proposition

Let C be a smooth projective curve of genus g.
Let D be a divisor of degree d on C.

(1) If $d \geq 2g$, then $|D|$ is base point free.

(2) If $d \geq 2g+1$, then $|D|$ is very ample.

(1) We know that $\deg K = 2g-2$ and, by hypothesis, $d \geq 2g$.
So the divisors $K-D$ and $K-(D-P)$ have negative degree.
By Riemann's Theorem, $\ell(D) = 1+d-g$ and
$\ell(D-P) = d-g = \ell(D)-1$.
So $|D|$ is base point free.

(2) Since $d \geq 2g+1$, for points $P, Q \in C$, $\deg(K-(D-P-Q)) < 0$.
By Riemann's Theorem, $\ell(D-P-Q) = 1+d-g-2 = \ell(D)-2$.
So the result follows from the preceding proposition.

Non-Constant Morphisms Between Projective Curves

Proposition

Let $f: C \to C'$ be a nonconstant morphism between two projective curves. If C' is nonsingular, then f is surjective.

- Suppose f is nonconstant and $f(C) \neq C'$.
  Choose a point $P \in C'$ which is not in the image of C.
  If we can show that $C' \backslash \{P\}$ is affine, then, we have a contradiction to the fact that every regular function on C is constant.
  Let g be the genus of C'.
  Consider the divisor $D = (2g+1)P$.
  It satisfies the hypothesis of the preceding proposition.
  Hence, D defines an embedding $\varphi_D: C' \to \mathbb{P}_k^{g+1}$.
  By construction of φ_D, there is a hyperplane H, with $C \cap H = \{P\}$ (H touches the curve $\varphi_D(C')$ at the point $\varphi_D(P)$ with order $2g+1$).
  This shows that $C' \backslash \{P\}$ is affine.

The Canonical Linear System $|K|$ on C

Lemma

Let C be a smooth projective curve of genus $g \geq 2$.
Then $|K|$ is base point free.

- By definition we have $\ell(K) = g$.

  We must show that, for every point $P \in C$, $\ell(K - P) = g - 1$.

  By the Riemann-Roch Theorem, we have

  $$\ell(K - P) - \ell(P) = g - 2.$$

  Since P is effective, we have $\ell(P) \geq 1$.

  On the other hand, since $g \geq 2$, the curve C is not rational.

  By a previous lemma, $\ell(P) = 1$.

  Thus, $\ell(K - P) = g - 1$.

Hyperelliptic Curves

Definition

A smooth projective curve C is called **hyperelliptic** if:

- $g(C) \geq 2$;
- There exists a surjective morphism

$$C \to \mathbb{P}_k^1$$

  of degree 2.

Smooth Projective Curves of Genus ≥ 2

Proposition

If C is a smooth projective curve of genus $g \geq 2$, then either $|K|$ is very ample or C is hyperelliptic.

- By the Riemann-Roch Theorem,

$$\ell(K - P - Q) - \ell(P + Q) = g - 3.$$

  - Suppose, first, that, for all p and Q, $\ell(P + Q) = 1$.
    Then it follows that $\ell(K - P - Q) = g - 2$.
    By a previous proposition, $|K|$ is very ample.
  - Suppose, next, $\ell(P + Q) = 2$, for some points P and Q.
    Since C is not rational, a previous lemma implies that any divisor D on C of degree 1 has $\ell(D) \leq 1$.
    So the linear system $|P + Q|$ is base point free.
    Hence, it defines a degree two map $\varphi_{P+Q} : C \to \mathbb{P}^1_k$.
    So C is hyperelliptic.

The Canonical Embedding

Definition (Canonical Embedding)

If $|K|$ is very ample, then

$$\varphi_K : C \to \mathbb{P}_k^{g-1}$$

is called the **canonical embedding** of C.
Moreover, $\varphi_K(C)$ is called the **canonical model** of C.

- We show (using several results about secant varieties and tangent varieties, stated without proofs) that every projective smooth curve C can be embedded in $\mathbb{P}_k^3$.

Projection from a Point

- We considered the projection from a point $P \in \mathbb{P}_k^n$.
- Choose a hyperplane $\mathbb{P}_k^{n-1}$ not containing P.
- The projection from P onto $\mathbb{P}_k^{n-1}$ is given by the map

$$\pi_P : \mathbb{P}_k^n \setminus \{P\} \to \mathbb{P}_k^{n-1}$$

  which takes a point $Q \neq P$ to the intersection of the line $\overline{PQ}$ with the hyperplane $\mathbb{P}_k^{n-1}$.
- Two different choices of hyperplanes give maps differing only by a projective transformation.

Projection from a Point (Cont'd)

- Take coordinates for $\mathbb{P}_k^n$ so that:
  - $P = (1:0:\ldots:0)$;
  - $\mathbb{P}_k^{n-1} = \{x_0 = 0\}$.
- Then π_P is given by

$$\pi_P(x_0 : x_1 : \ldots : x_n) = (x_1 : \ldots : x_n).$$

- For a curve $C \subseteq \mathbb{P}_k^n$, with $P \notin C$, by restricting π_P to C we obtain a projection $\pi_P : C \to \mathbb{P}_k^{n-1}$ from P.
- In terms of linear systems, this map is defined by the linear system

$$\vartheta = \{H \cap C : P \in H\} \cong \mathbb{P}_k^{n-1}.$$

Secant of a Curve

- We determine when the projection $\pi_P : C \to \mathbb{P}_k^{n-1}$ is an embedding.
- We find a sequence of embeddings.
- In the end we will end up with C embedding in $\mathbb{P}_k^3$.

Definition (Secant)

If Q and R are two distinct points of C, then the line

$$\overline{QR}$$

is called a **secant** of C.

Secant Variety of a Curve

- If the point P lies on $\overline{QR}$ then clearly $\pi_P(Q) = \pi_P(R)$.
- So the projection $\pi_P : C \to \mathbb{P}_k^{n-1}$ is not injective.
- For π_P to be an embedding we must choose P not lying on any secant of C.
- This leads to the concept of a **secant variety**.
- As a set, it is the union of the secants and tangents of C.
- To give a definition from which it is clear that the secant variety is a variety, we must introduce *Grassmannian varieties*.

Grassmannain Varieties

Definition (Grassmannian Variety)

The **Grassmannian variety** of lines in $\mathbb{P}_k^n$ is given by

$$\mathrm{Gr}(1,n) := \{L : L \text{ is a line in } \mathbb{P}_k^n\}.$$

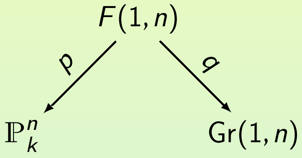
- The Grassmannian $\mathrm{Gr}(1,n) \subseteq \mathbb{P}$ can be identified with the subset of $(\bigwedge^2 k^{n+1})$ of tensors of the form $v \wedge w$.
- In fact, $\mathrm{Gr}(1,n)$ is a (smooth) projective variety defined by quadratic equations.

Flag Variety

- The set

$$F(1,n) := \{(P,L) \in \mathbb{P}_k^n \times \mathrm{Gr}(1,n) : P \in L\}$$

  is called a **flag variety**.
- $F(1,n)$ is also a smooth projective variety.
- By definition, $F(1,n) \subseteq \mathbb{P}_k^n \times \mathrm{Gr}(1,n)$.
- So we have projection maps p and q,



- The fibers of the projection q are isomorphic to $\mathbb{P}_k^1$.

The Mappings t and s

- We define a map from the curve C to the Grassmannian,

$$\begin{array}{rccc} t: & C & \to & \mathrm{Gr}(1,n); \\ & Q & \mapsto & T_Q C. \end{array}$$

- t associates to a point $Q \in C$ the tangent to C at the point Q.
- We define a map s from $C \times C$ to the Grassmannian,

$$\begin{array}{rccc} s: & C \times C & \to & \mathrm{Gr}(1,n); \\ & s(Q,R) & \mapsto & \begin{cases} \overline{QR}, & \text{if } Q \neq R, \\ T_Q C, & \text{if } Q = R. \end{cases} \end{array}$$

- Both t and s are morphisms of projective varieties.

The Tangent Surface and the Secant Variety

Definition (Tangent Surface)

The **tangent surface** of C is defined by

$$\mathrm{Tan}\, C := p(q^{-1}(t(C))).$$

Definition (Secant Variety)

The **secant variety** of a curve C is defined by

$$\mathrm{Sec}\, C := p(q^{-1}(s(C \times C))).$$

- As sets:
  - $\mathrm{Tan}\, C$ is the union of the tangents to the curve C;
  - $\mathrm{Sec}\, C$ is the union of the secants and tangents of C.
- We have $C \subseteq \mathrm{Tan}\, C \subseteq \mathrm{Sec}\, C$.

Dimensions of Tan C and Sec C

- Tan C is defined as the image of the projective variety $q^{-1}(t(C))$ under the morphism p.
- So it can be shown that Tan C is a projective variety in $\mathbb{P}_k^n$.
- C is a curve and q is a dominant morphism with one-dimensional fibers.
- We use the fact that for a dominant morphism of projective varieties, the dimension of the image plus the dimension of the general fiber equals the dimension of the domain.
- We conclude that $q^{-1}(t(C))$ has dimension ≤ 2.
- Thus, $\dim \mathrm{Tan}\, C \leq 2$.
- In fact, we always have $\dim \mathrm{Tan}\, C = 2$, unless C is a line.
- Similarly, Sec C is a projective variety with $\dim \mathrm{Sec}\, C \leq 3$.

Characterization of π_P Being an Embedding

Proposition

The projection $\pi_P : C \to \mathbb{P}_k^{n-1}$ of a curve C from a point $P \notin C$ is an embedding if and only if:

(1) P lies on no secant of C;

(2) P lies on no tangent of C.

- As discussed above, the map π_P is given by the linear system

$$\vartheta = \{H \cap C : P \in H\} \cong \mathbb{P}_k^{n-1}.$$

Since $P \notin C$, this is a base point free linear system.

Characterization of π_P Being an Embedding (Cont'd)

- We point out the following properties about the base point free linear system ϑ.

  It separates points (i.e., φ_ϑ is injective) if and only if, for any two points $Q \neq R$ of C, there is a hyperplane H through P, such that:
  - H contains Q;
  - H does not contain R.

  This is equivalent to saying that P does not lie on the secant $\overline{QR}$.

  The linear system separates tangents at a point $R \in C$ if and only if there is a hyperplane H through P and R which intersects the tangent $T_R C$ transversally, i.e., $T_R C \nsubseteq H$.

  This is the case if and only if P does not lie on the tangent $T_R C$.

Embeddability of Smooth Projective Curves in $\mathbb{P}_k^3$

Corollary

Every smooth projective curve C can be embedded in $\mathbb{P}_k^3$.

- Let $g = g(C)$ be the genus of the curve C.
  Let D be a divisor of degree $d = 2g + 1$.
  By a preceding proposition,
  $$\varphi_D : C \to \mathbb{P}_k^{g+1}$$
  is an embedding.
  - If $g \leq 2$, then there is no more to prove.
  - Suppose $g \geq 3$. Since $\dim \mathrm{Sec} C \leq 3$, there is a point $P \notin \mathrm{Sec} C$.
    By the proposition. the projection from P gives an embedding
    $$C \to \mathbb{P}_k^g.$$
    We can continue to project from points $P \notin \mathrm{Sec} C$ in this way.
    In the end, we obtain an embedding in $\mathbb{P}_k^3$.