Introduction to Algebraic Number Theory

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1. Analytic Methods

Subsection 1

The Riemann Zeta Function

The Riemann Zeta Function

- The **zeta function** was introduced by Euler as a function on real numbers $s > 1$, defined by

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

- Euler computed the values of $\zeta(2k)$ (for $k \geq 1$), and was aware of the behavior of the function as s gets closer to 1.
- Riemann developed the theory of the zeta function and seems to have been the first to consider s as a complex variable.
- Riemann realized that questions about the distribution of primes are inextricably linked with the complex behavior of the zeta function.
- In particular, with the values it takes to the left of the line $\mathrm{Re}(s) = 1$, where the definition above no longer converges.

The Functional Equation

- The zeta function has a meromorphic continuation to the entire complex plane, with a simple pole at $s=1$ (with residue 1).
- This continuation has a symmetry, relating the values of ζ at s and at $1-s$, known as the **functional equation**.
- Consider the Gamma function
$$\Gamma(z)=\int_0^\infty e^{-t}t^{z-1}dt.$$
- Write
$$\xi(s)=\pi^{-s/2}\Gamma(s/2)\zeta(s).$$
- Then
$$\xi(s)=\xi(1-s).$$
- Using classical formulae for the Gamma function, we get
$$\zeta(s)=2^s\pi^{s-1}\sin(\pi s/2)\Gamma(1-s)\zeta(1-s).$$

Behavior of $\zeta(s)$ for Real s

Proposition

$\sum_{n=1}^{\infty} \frac{1}{n^s}$ converges absolutely for any real $s > 1$, and is not convergent for $s \leq 1$.

- We prove the two statements using the comparison test.

  First, consider the case $s \leq 1$.

  Then $\frac{1}{n^s} \geq \frac{1}{n}$.

  We use grouping of the terms,

$$\begin{array}{rcl} \sum_{n=1}^{\infty} \frac{1}{n} & = & 1 + \frac{1}{2} + (\frac{1}{3} + \frac{1}{4}) + (\frac{1}{5} + \cdots + \frac{1}{8}) + (\frac{1}{9} + \cdots + \frac{1}{16}) + \cdots \\ & > & 1 + \frac{1}{2} + (\frac{1}{4} + \frac{1}{4}) + (\frac{1}{8} + \cdots + \frac{1}{8}) + (\frac{1}{16} + \cdots + \frac{1}{16}) + \cdots \\ & = & 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots . \end{array}$$

  The last sum diverges.

  So the sum $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges also.

Behavior of $\zeta(s)$ for Real s (Cont'd)

- Next, we treat the case $s > 1$.

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{1}{n^s} &= 1 + \left(\frac{1}{2^s} + \frac{1}{3^s}\right) + \left(\frac{1}{4^s} + \cdots + \frac{1}{7^s}\right) + \left(\frac{1}{8^s} + \cdots + \frac{1}{15^s}\right) + \cdots \\
&< 1 + \left(\frac{1}{2^s} + \frac{1}{2^s}\right) + \left(\frac{1}{4^s} + \cdots + \frac{1}{4^s}\right) + \left(\frac{1}{8^s} + \cdots + \frac{1}{8^s}\right) + \cdots \\
&= 1 + 2 \cdot \frac{1}{2^s} + 4 \cdot \frac{1}{4^s} + 8 \cdot \frac{1}{8^s} + \cdots \\
&= 1 + \frac{1}{2^{s-1}} + \frac{1}{4^{s-1}} + \frac{1}{8^{s-1}} + \cdots \\
&= 1 + \frac{1}{2^{s-1}} + \frac{1}{(2^{s-1})^2} + \frac{1}{(2^{s-1})^3} + \cdots \\
&= \frac{1}{1 - \frac{1}{2^{s-1}}}. \\
&= \frac{2^{s-1}}{2^{s-1} - 1}.
\end{aligned}$$

Estimating $\zeta(s)$

- One method to estimate the value of $\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$ is to compare it with the integral
$$\int_1^{\infty} \frac{dx}{x^s}.$$
- We have
$$\int_n^{n+1} \frac{dx}{x^s} = \left.\frac{x^{1-s}}{1-s}\right|_n^{n+1} = \frac{n^{1-s} - (n+1)^{1-s}}{s-1}.$$
- Moreover, x^{-s} is a decreasing function.
- So the area under the graph on this interval of length 1 lies in between the values of the function at n and at $n+1$,
$$\frac{1}{(n+1)^s} < \frac{n^{1-s} - (n+1)^{1-s}}{s-1} < \frac{1}{n^s}.$$

Estimating $\zeta(s)$ (Cont'd)

- Summing this over $n = 1, 2, 3, \ldots$ gives

$$\sum_{n=1}^{\infty} \frac{1}{(n+1)^s} < \sum_{n=1}^{\infty} \frac{n^{1-s} - (n+1)^{1-s}}{s-1} < \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

- Equivalently,

$$\zeta(s) - 1 < \frac{1}{s-1} < \zeta(s).$$

- Writing this as two inequalities, and rearranging them gives

$$\frac{1}{s-1} < \zeta(s) < \frac{1}{s-1} + 1.$$

- This gives the rate at which $\zeta(s) \to \infty$ as $s \to 1$.

Euler Product for $\zeta(s)$

Proposition (Euler Product for $\zeta(s)$)

If $\mathrm{Re}(s) > 1$, then

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$

- Observe that $|p^{-s}| = p^{-\mathrm{Re}(s)} < 1$. Thus,

$$\left(1 - \frac{1}{p^s}\right)^{-1} = 1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots.$$

Now multiply all these together:

$$\prod_p \left(1 - \frac{1}{p^s}\right)^{-1} = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \cdots\right) = \zeta(s),$$

as every $\frac{1}{n^s}$ appears exactly once in the product (by uniqueness of prime factorization).

Existence of an Infinity of Primes

- We deduce in two ways from the Euler Product that there are infinitely many prime numbers.
  1. We know that $\zeta(s)=\prod_p\left(1-\frac{1}{p^s}\right)^{-1}\to\infty$ as $s\to 1$.
     Each term tends to the finite limit $\frac{p}{p-1}$.
     So there cannot be only finitely many terms in the product.
  2. You may know that $\zeta(2)=\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$.
     Now π^2 is irrational.
     We have
     $$\prod_p(1-p^{-2})^{-1}=\prod_p\frac{p^2}{p^2-1}=\zeta(2)=\frac{\pi^2}{6}.$$
     So we see that $\prod_p(1-p^{-2})^{-1}$ is irrational.
     So there cannot be only finitely many terms in the product.

Using Logarithms

- By the Euler product
$$\zeta(s) = \prod_p \left(1 - \frac{1}{p^s}\right)^{-1}.$$
- Recall that, if $|z| < 1$,
$$\log(1-z) = -\left(z + \frac{z^2}{2} + \frac{z^3}{3} + \cdots\right).$$
- Putting these together
$$\begin{array}{rcl}\log\zeta(s) & = & \log\prod_p \left(1-\frac{1}{p^s}\right)^{-1} \\ & = & \sum_p -\log(1-p^{-s}) \\ & = & \sum_p \left[p^{-s} + \frac{p^{-2s}}{2} + \frac{p^{-3s}}{3} + \cdots\right] \\ & = & \sum_p p^{-s} + \sum_p \left[\frac{p^{-2s}}{2} + \frac{p^{-3s}}{3} + \frac{p^{-4s}}{4} + \cdots\right].\end{array}$$

Using Logarithms (Cont'd)

- Now we have

$$\begin{array}{rcl}\frac{p^{-2s}}{2}+\frac{p^{-3s}}{3}+\frac{p^{-4s}}{4}+\cdots & < & \frac{p^{-2s}}{2}+\frac{p^{-3s}}{2}+\frac{p^{-4s}}{2}+\cdots \\ & = & \frac{p^{-2s}}{2}(\frac{1}{1-p^{-s}}) \\ & < & p^{-2s}.\end{array}$$

- Moreover, for $s>1$, $p^{-s}<2^{-1}=\frac{1}{2}$.
- So, for $s>1$,

$$\log\zeta(s)\approx\sum_p p^{-s}.$$

- The error is at most

$$\sum_p p^{-2s}<\sum_n n^{-2s}=\zeta(2s)<\zeta(2).$$

Using Logarithms (Cont'd)

- We deduce that, for $s > 1$ but near 1,

$$\sum_p p^{-s} \approx \log \frac{1}{s-1}.$$

- Note that this proves that $\sum_p \frac{1}{p}$ diverges.
- We, thus, get another proof that there are infinitely many primes.

Subsection 2

The Functional Equation of the Riemann Zeta Function

The Function θ

- Set
$$\theta(t) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t^2}, \quad t \in \mathbb{R}.$$
- For $t \neq 0$, the individual terms in the sum converge so fast to 0 that $\theta(t)$ converges for all $t \neq 0$.

Lemma

For $t \neq 0$, we have
$$\theta\left(\frac{1}{t}\right) = t\theta(t).$$

- Recall that
$$\int_{-\infty}^{\infty} e^{-\pi x^2} dx = 1.$$
Fix $t > 0$, and write
$$f(x) = e^{-\pi t^2 x^2}.$$

The Function θ (Cont'd)

- Define
$$F(x) = \sum_{n\in\mathbb{Z}} f(x+n) = \sum_{n\in\mathbb{Z}} e^{-\pi t^2(x+n)^2}.$$
It converges because the terms tend to 0 very quickly.
By definition,
$$F(0) = \theta(t).$$
Also, note that F is periodic, with $F(x) = F(x+1)$.
So it will have a Fourier series
$$F(x) = \sum_{m\in\mathbb{Z}} a_m e^{2\pi imx},$$
where the coefficients a_m are computed as follows.

The Function θ (Cont'd)

- $$\begin{aligned} a_m &= \int_0^1 F(x) e^{-2\pi i m x} dx \\ &= \sum_{n \in \mathbb{Z}} \int_0^1 f(x+n) e^{-2\pi i m x} dx \\ &= \sum_{n \in \mathbb{Z}} \int_0^1 f(x+n) e^{-2\pi i m (x+n)} dx \\ &= \int_{-\infty}^{\infty} f(x) e^{-2\pi i m x} dx \\ &= \int_{-\infty}^{\infty} e^{-\pi t^2 x^2 - 2\pi i m x} dx \\ &= \int_{-\infty}^{\infty} e^{-\pi (tx + i \frac{m}{t})^2} e^{-\pi m^2 / t^2} dx \\ &= e^{-\pi m^2 / t^2} \int_{-\infty}^{\infty} e^{-\pi (tx + i \frac{m}{t})^2} dx \\ &= t^{-1} e^{-\pi m^2 / t^2}, \end{aligned}$$

  by a change of variable $y = tx + i\frac{m}{t}$, and using Cauchy's Theorem to see that the integral along the real axis is the same as the integral along the line $\operatorname{Im}(z) = \frac{m}{t}$.

The Function θ (Cont'd)

- Finally, we get

$$\begin{aligned}\theta(t) &= F(0) \\ &= \sum_{m\in\mathbb{Z}} a_m \\ &= \sum_{m\in\mathbb{Z}} t^{-1}e^{-\pi m^2/t^2} \\ &= t^{-1}\theta\left(\frac{1}{t}\right).\end{aligned}$$

So the result follows.

Relation Between θ and ζ

Proposition

For $\mathrm{Re}(s) > 1$, we have

$$\int_0^\infty (\theta(t) - 1) t^{s-1} dt = \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s),$$

where $\zeta(z)$ is the usual Gamma function, defined by $\Gamma(z) = \int_0^\infty e^{-t} t^{z-1} dt$.

- For $\mathrm{Re}(s) > 1$, and from the definition of $\theta(t)$, the integral is

$$\begin{aligned}
2\int_0^\infty \sum_{n\geq 1} e^{-\pi n^2 t^2} t^{s-1} dt &= 2\sum_{n\geq 1} \int_0^\infty e^{-\pi n^2 t^2} t^{s-1} dt \\
&\overset{u=nt}{=} 2\sum_{n\geq 1} n^{-s} \int_0^\infty e^{-\pi u^2} u^{s-1} du \\
&\overset{v=\pi u^2}{=} 2\zeta(s) \int_0^\infty e^{-v} \left(\frac{v}{\pi}\right)^{s/2-1} \frac{1}{2\pi} dv \\
&= \pi^{-s/2} \Gamma\left(\frac{s}{2}\right) \zeta(s).
\end{aligned}$$

The Functional Equation for ξ

Theorem

Suppose that $\text{Re}(s) > 1$. Write

$$\xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

Then

$$\xi(s) = \xi(1-s).$$

- We break up the integral defining $\xi(s)$ for $\text{Re}(s) > 1$,

$$\xi(s) = \int_1^\infty (\theta(t)-1)t^{s-1}dt + \int_0^1 (\theta(t)-1)t^{s-1}dt.$$

Then, we make the change of variable $u = \frac{1}{t}$ in the second integral.

The Functional Equation for ξ (Cont'd)

- Recalling that $\theta(t) = \frac{1}{t}\theta\left(\frac{1}{t}\right)$, we get

$$\begin{aligned}
\xi(s) &= \int_1^\infty (\theta(t)-1)t^{s-1}dt + \int_0^1 (\theta(t)-1)t^{s-1}dt \\
&= \int_1^\infty (\theta(t)-1)t^{s-1}dt + \int_1^\infty (u\theta(u)-1)u^{-s-1}du \\
&= \int_1^\infty (\theta(t)-1)t^{s-1}dt + \int_1^\infty \theta(u)u^{-s}du - \int_1^\infty u^{-s-1}du \\
&= \int_1^\infty (\theta(t)-1)t^{s-1}dt + \int_1^\infty \theta(u)u^{-s}du - \frac{1}{s} \\
&= \int_1^\infty (\theta(t)-1)t^{s-1}dt + \int_1^\infty (\theta(u)-1)u^{-s}du - \frac{1}{s} - \frac{1}{1-s} \\
&= \int_1^\infty (\theta(t)-1)[t^{s-1}+t^{-s}]dt - \frac{1}{s} - \frac{1}{1-s}.
\end{aligned}$$

This integral converges, for all $s \in \mathbb{C}$, to a holomorphic function.

The final expression being unchanged when $s \leftarrow 1-s$,

$$\xi(s) = \xi(1-s).$$

Also ξ has simple poles at $s=0$ and $s=1$ with residues -1 and $+1$.

Properties of the Gamma Function

- The Gamma function satisfies the identity
$$\Gamma(1-z)\Gamma(z) = \frac{\pi}{\sin \pi z}, \quad z \notin \mathbb{Z}.$$
- Setting $z = \frac{s}{2}$, we get
$$\Gamma\left(1-\frac{s}{2}\right)\Gamma\left(\frac{s}{2}\right) = \frac{\pi}{\sin \frac{\pi s}{2}}.$$
- The Gamma function also satisfies the identity
$$\Gamma(z)\Gamma\left(z+\frac{1}{2}\right) = 2^{1-2z}\sqrt{\pi}\Gamma(2z).$$
- Setting $z = \frac{1-s}{2}$, we get
$$\Gamma\left(\frac{1-s}{2}\right)\Gamma\left(1-\frac{s}{2}\right) = 2^{s}\sqrt{\pi}\Gamma(1-s).$$

The Functional Equation for ζ

- Dividing the two equations in s, we get

$$\frac{\Gamma(\frac{1-s}{2})}{\Gamma(\frac{s}{2})} = \frac{2^s\sqrt{\pi}\Gamma(1-s)\sin\frac{\pi s}{2}}{\pi}.$$

- By the preceding work, $\xi(s)$ is defined for all $s \neq 0, 1$.
- We also have

$$\xi(s) = \xi(1-s).$$

- Moreover,

$$\xi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

- So we can define $\zeta(s)$, for all $s \neq 0, 1$.
- The theorem then gives a relation, known as the **functional equation**, between $\zeta(s)$ and $\zeta(1-s)$.

The Functional Equation for ζ (Cont'd)

- Using the formulas involving the Gamma function, we now have

$$\begin{aligned}\xi(s) &= \xi(1-s)\\ \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s) &= \pi^{-(1-s)/2}\Gamma\left(\frac{1-s}{2}\right)\zeta(1-s)\\ \zeta(s) &= \frac{\pi^{-(1-s)/2}}{\pi^{-s/2}}\frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}\zeta(1-s)\\ \zeta(s) &= \pi^{s-\frac{1}{2}}\frac{2^s\sqrt{\pi}\Gamma(1-s)\sin\frac{\pi s}{2}}{\pi}\zeta(1-s)\\ \zeta(s) &= 2^s\pi^{s-1}\sin\left(\frac{\pi s}{2}\right)\Gamma(1-s)\zeta(1-s).\end{aligned}$$

Subsection 3

Zeta Functions of Number Fields

The Dedekind Zeta Function

Definition

The **Dedekind zeta function** of K is given by

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})^s},$$

where $\mathfrak{a}$ runs through all distinct non-zero integral ideals of the field K (i.e., the ideals of $\mathbb{Z}_K$).

- Note that when $K = \mathbb{Q}$, we get exactly the Riemann zeta function.
- We will see later that the Dedekind zeta function is also convergent for $\text{Re}(s) > 1$.

An Euler Product Formula

- $\zeta_K(s)$ has an Euler product (valid for $\mathrm{Re}(s) > 1$),

$$\zeta_K(s) = \prod_{\mathfrak{p}} \frac{1}{1 - N_{K/\mathbb{Q}}(\mathfrak{p})^{-s}},$$

  where the product is taken over all of the prime ideals $\mathfrak{p}$ of $\mathbb{Z}_K$.
- The proof is identical to that for the Riemann zeta function.
- It is equivalent to unique factorization of ideals.

Properties of $\zeta_K(s)$

- One asks for generalizations of Riemann's results for $\zeta(s)$ to $\zeta_K(s)$.
- Dirichlet's analytic class number formula shows that $\zeta_K(s)$ has a singularity at $s=1$ and computes the limit of $(s-1)\zeta_K(s)$ as $s \to 1$.
- In complex variable language, $\zeta_K(s)$ has a simple pole at $s=1$, and Dirichlet's formula gives the residue.
- The formula for the residue involves many of the arithmetic quantities related to K, such as the class number, the discriminant, the numbers of real and complex embeddings, and so on.
- It is also true that $\zeta_K(s)$ has a meromorphic continuation to the whole complex plane.
- Further, it satisfies a functional equation.

Subsection 4

The Analytic Class Number Formula

Review I

- Let K be a number field with:
  - r_1 real embeddings $\{\rho_1, \dots, \rho_{r_1}\}$;
  - r_2 pairs of complex embeddings $\{\sigma_1, \overline{\sigma}_1, \dots, \sigma_{r_2}, \overline{\sigma}_{r_2}\}$.
- Then there are $r = r_1 + r_2 - 1$ fundamental units $\epsilon_1, \dots, \epsilon_r$, such that, every unit ϵ can be written

$$\epsilon = \zeta \epsilon_1^{v_1} \cdots \epsilon_r^{v_r},$$

  with $\zeta \in \mu(K)$, the roots of unity in K, and $v_i \in \mathbb{Z}$.
- The proof used lattice-theoretic methods (and Minkowski's Theorem).
- Recall, also, that we had a commutative diagram

$$\begin{array}{ccccc}
K^\times & \xrightarrow{i} & K_{\mathbb{R}}^\times & \xrightarrow{\ell} & \mathbb{R}^{r_1+r_2} \\
{\scriptstyle N_{K/\mathbb{Q}}}\downarrow & & {\scriptstyle N}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}} \\
\mathbb{Q}^\times & \longrightarrow & \mathbb{R}^\times & \xrightarrow{\ell} & \mathbb{R}
\end{array}$$

  where $K_{\mathbb{R}} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$.

Review II

- The proof of Dirichlet's Unit Theorem worked by showing that the units $\mathbb{Z}_K^\times \subseteq K^\times$ mapped to a complete lattice in the r-dimensional subspace $H \subseteq \mathbb{R}^{r_1+r_2}$ defined by $H = \{x \in \mathbb{R}^{r_1+r_2} : \text{tr}(x) = 0\}$.
- In particular, the image of $\mathbb{Z}_K^\times$ is an r-dimensional lattice in $\mathbb{R}^{r_1+r_2}$.
- If we write $\lambda = \ell \circ i$, the vectors $\lambda(\epsilon_1), \ldots, \lambda(\epsilon_r)$ are a basis for the lattice, and so span H.
- The analytic class number formula will also involve a term describing how "widely spaced" the units are in H (in a similar way to how the discriminant describes how widely spaced the integers $\mathbb{Z}_K$ are).
- Recall that if $x \in \mathbb{Z}_K$,

$$\lambda(x) = (\log|\rho_1(x)|, \ldots, \log|\rho_{r_1}(x)|, \log|\sigma_1(x)|^2, \ldots, \log|\sigma_{r_2}(x)|^2).$$

- We can measure how widely spaced the units are in H by applying λ to the fundamental units and taking a determinant.

Regulators

Definition

Let $\epsilon_1,\ldots,\epsilon_r$ denote a set of fundamental units, where $r=r_1+r_2-1$.
Consider the map $\lambda : K \to \mathbb{R}^{r_1+r_2}$ and write

$$\lambda(x)=(\lambda_1(x),\ldots,\lambda_{r_1+r_2}(x)),$$

so that

$$\lambda_i(x)=\begin{cases} \log|\rho_i(x)|, & \text{if } 1\le i\le r_1; \\ \log|\sigma_{i-r_1}(x)|^2, & \text{if } i>r_1. \end{cases}$$

Consider the $(r+1)\times r$-matrix whose entries are $\lambda_i(\epsilon_j)$.
Define the **regulator** R_K to be the absolute value of the determinant of any $r\times r$-minor of this matrix.

The Analytic Class Number Formula

Theorem (Analytic Class Number Formula)

Consider again

$$\zeta_K(s) = \sum_{\mathfrak{a}} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})^s},$$

where $\mathfrak{a}$ runs through all distinct non-zero integral ideals of the field K. $\zeta_K(s)$ converges for all $\mathrm{Re}(s) > 1$. It has a simple pole at $s = 1$, and

$$\lim_{s \to 1} (s-1)\zeta_K(s) = \frac{2^{r_1+r_2}\pi^{r_2} R_K}{m|D_K|^{1/2}} h_K,$$

where:

- R_K is the regulator of K;
- h_K is the class number of K;
- $m = |\mu(K)|$, the number of roots of unity in K.

Cones

- Our goal is to translate the preceding result into a calculation of volumes of certain regions in $K_{\mathbb{R}} \cong \mathbb{R}^n$.

Definition

We say that a **cone** in $\mathbb{R}^n$ is a subset $X \subseteq \mathbb{R}^n$, such that

$$x \in X \quad \text{and} \quad \lambda \in \mathbb{R}_{>0}, \quad \text{imply} \quad \lambda x \in X.$$

- The same definition applies to any real vector space, such as $K_{\mathbb{R}}$.

Functions on Cones

Proposition

Let X be a cone in $\mathbb{R}^n$. Let $F : X \to \mathbb{R}_{>0}$ be a function satisfying

$$F(\xi x) = \xi^n F(x), \quad x \in X, \xi \in \mathbb{R}_{>0}.$$

Suppose that $T = \{x \in X : F(x) \le 1\}$ is bounded, with non-zero volume $v = \text{vol}(T)$. Let Γ be a lattice in $\mathbb{R}^n$, with $\Delta = \text{vol}(\Gamma)$. Consider the function

$$Z(s) = \sum_{\Gamma \cap X} \frac{1}{F(x)^s}.$$

It converges for $\text{Re}(s) > 1$ and

$$\lim_{s \to 1} (s-1) Z(s) = \frac{v}{\Delta}.$$

Functions on Cones (Cont'd)

- Note that, for all $r \in \mathbb{R}_{>0}$,

$$\text{vol}\left(\frac{1}{r}\Gamma\right) = \frac{\Delta}{r^n}.$$

Suppose $N(r)$ denotes the number of points in $\frac{1}{r}\Gamma \cap T$. Then

$$v = \text{vol}(T) = \lim_{r\to\infty} N(r)\frac{\Delta}{r^n} = \Delta \lim_{r\to\infty} \frac{N(r)}{r^n}.$$

But $N(r)$ is also the number of points in

$$\{x \in \Gamma \cap X : F(x) \le r^n\},$$

at least for the nice F we consider.

Functions on Cones (Cont'd)

- Order the points of $\Gamma \cap X$ so that
$$0 < F(x_1) \leq F(x_2) \leq \cdots .$$
Let
$$r_k = F(x_k)^{1/n}.$$
Then, for all $\epsilon > 0$,
$$N(r_k - \epsilon) < k \leq N(r_k).$$
It follows that
$$\frac{N(r_k - \epsilon)}{(r_k - \epsilon)^n}\left(\frac{r_k - \epsilon}{r_k}\right)^n < \frac{k}{r_k^n} \leq \frac{N(r_k)}{r_k^n}.$$
Thus, since the two outer terms have the same limit,
$$\lim_{r_k \to \infty} \frac{k}{r_k^n} = \lim_{k \to \infty} \frac{k}{F(x_k)} = \frac{v}{\Delta}.$$

Functions on Cones (Cont'd)

- We use

$$\lim_{r_k\to\infty}\frac{k}{r_k^n}=\lim_{k\to\infty}\frac{k}{F(x_k)}=\frac{v}{\Delta}$$

to approximate the terms in the sum $Z(s)$.

Given $\epsilon>0$, there exists k_0, such that for all $k\geq k_0$, one has

$$\left(\frac{v}{\Delta}-\epsilon\right)\frac{1}{k}<\frac{1}{F(x_k)}<\left(\frac{v}{\Delta}+\epsilon\right)\frac{1}{k}.$$

Summing,

$$\left(\frac{v}{\Delta}-\epsilon\right)^s\sum_{k=k_0}^{\infty}\frac{1}{k^s}<\sum_{k=k_0}^{\infty}\frac{1}{F(x_k)^s}<\left(\frac{v}{\Delta}+\epsilon\right)^s\sum_{k=k_0}^{\infty}\frac{1}{k^s}.$$

We know that the Riemann zeta function converges for $\mathrm{Re}(s)>1$.

So the same holds for $Z(s)$.

Functions on Cones (Conclusion)

- We also know the residue of $\zeta(s)$ at $s=1$.

  We multiply through by $(s-1)$, and let s tend to 1 from above.

  We have
  $$\lim_{s\to 1}(s-1)\zeta(s)=1.$$
  We observe that
  $$\lim_{s\to 1}(s-1)[\text{a finite sum}]=0.$$
  We conclude that
  $$\frac{v}{\Delta}-\epsilon\le\lim_{s\to 1}(s-1)Z(s)\le\frac{v}{\Delta}+\epsilon.$$
  But this holds for all $\epsilon>0$.

  It follows that
  $$\lim_{s\to 1}(s-1)Z(s)=\frac{v}{\Delta}.$$

Expression for $\zeta_K(s)$

- Let C_K be the set of ideal classes in the class group.
- Set
$$f_C(s) = \sum_{\mathfrak{a} \in C} \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{a})^s}.$$
- Write
$$\zeta_K(s) = \sum_{C \in C_K} f_C(s),$$
the sum running over ideal classes in the class group.
- We will compute, for each ideal class C,
$$\lim_{s \to 1} (s-1) f_C(s).$$
- We will observe that the result is independent of C.
- This accounts for the factor h_K in the formula.

Expression for $f_C(s)$

- Choose any integral $\mathfrak{b}$ in the class C^{-1}, the inverse class.
- Then, for all $\mathfrak{a} \in C$, $\mathfrak{a}\mathfrak{b}$ is principal, $\langle \alpha \rangle$, say.
- The association $\mathfrak{a} \mapsto \langle \alpha \rangle$ gives a bijection between integral ideals $\mathfrak{a} \in C$, and principal ideals $\langle \alpha \rangle$ divisible by $\mathfrak{b}$ (i.e., elements $\alpha \in \mathfrak{b}$).
- It follows that

$$f_C(s) = N_{K/\mathbb{Q}}(\mathfrak{b})^s \sum_{\mathfrak{b} | \langle \alpha \rangle} \frac{1}{|N_{K/\mathbb{Q}}(\alpha)|^s}.$$

- Note that $\langle \alpha \rangle = \langle \alpha' \rangle$ if and only if α and α' are associate.
- We may therefore assume that α runs over a complete set $\mathscr{B}$ of non-associate members of $\mathfrak{b}$.

Rewriting $f_C(s)$

- Let
$$\Gamma = i(\mathfrak{b}) = \{x \in K_{\mathbb{R}} : x = i(\alpha), \text{ for some } \alpha \in \mathfrak{b}\}.$$
- Write
$$\Theta = \{x \in K_{\mathbb{R}} : x = i(\alpha), \text{ for some } \alpha \in \mathscr{B}\}.$$
- Thus,
$$f_C(s) = N_{K/\mathbb{Q}}(\mathfrak{b})^s \sum_{x \in \Theta} \frac{1}{|N(x)|^s}.$$
- We will find a cone $X \subseteq K_{\mathbb{R}}$, such that every $\alpha \in \mathscr{B}$ has $i(\alpha)$ associate to precisely one member of X.
- It will follow that
$$\Theta = \Gamma \cap X.$$
- Then, we may apply the preceding proposition with $F(x) = |N(x)|$.

Setting Up the Cone

- We define the cone X.
- Let $\epsilon_1, \ldots, \epsilon_r$ be fundamental units (where $r = r_1 + r_2 - 1$).
- Write
$$\lambda = (1, \ldots, 1, 2, \ldots, 2)$$
be the vector in $\mathbb{R}^{r_1+r_2}$ whose components are:
  - $\lambda_i = 1$, if $i \leq r_1$ (corresponding to the real components in $K_{\mathbb{R}}$);
  - $\lambda_i = 2$, if $i > r_1$ (corresponding to the complex embeddings).
- The vectors $\lambda(\epsilon_1), \ldots, \lambda(\epsilon_r)$ span H, as we saw previously.
- Thus, the set $\{\lambda, \lambda(\epsilon_1), \ldots, \lambda(\epsilon_r)\}$ are a basis for $\mathbb{R}^{r_1+r_2}$.

Setting Up the Cone (Cont'd)

- The set $\{\lambda, \lambda(\epsilon_1), \ldots, \lambda(\epsilon_r)\}$ are a basis for $\mathbb{R}^{r_1+r_2}$.
- So for all $\ell(x) \in \mathbb{R}^{r_1+r_2}$, we can write

$$\ell(x) = \xi\lambda + \xi_1\lambda(\epsilon_1) + \cdots + \xi_r\lambda(\epsilon_r),$$

  for some coefficients $\xi, \xi_i \in \mathbb{R}$.
- Observe that $\mathrm{tr}\lambda(\epsilon_i) = 0$ (as $\lambda(\epsilon_i) \in H$).
- So

$$\mathrm{tr}\ell(x) = \xi \cdot \mathrm{tr}\lambda = \xi n.$$

- But $\mathrm{tr}\ell(x) = \log|N(x)|$.
- So

$$\xi = \frac{1}{n}\log|N(x)|.$$

The Cone X

Definition

The cone $X \subseteq K_{\mathbb{R}}$ will be defined to consist of all x such that:

1. $N(x) \neq 0$;
2. The coefficients ξ_i, $i = 1, \ldots, r$, of $\ell(x)$ satisfy $0 \leq \xi_i < 1$;
3. $0 \leq \arg(x_1) < \frac{2\pi}{m}$, where x_1 is the first component of x.

- We show that X is a cone in $K_{\mathbb{R}}$.

  I.e., that, if $x \in X$ and $\xi > 0$, then $\xi x \in X$.
  - $N(\xi x) = \xi^n N(x) \neq 0$.
  - $\ell(\xi x) = (\log \xi)\lambda + \ell(x)$.
    So the coefficients of $\lambda(\epsilon_i)$ are unchanged.
  - $\arg(\xi x_1) = \arg(x_1)$.

  Thus, if $x \in X$, and $\xi \in \mathbb{R}_{>0}$, then $\xi x \in X$.

Property of the Cone X

Lemma

Let $y \in \mathbb{R}^n$, with $N(y) \neq 0$. Then y is uniquely of the form

$$x \cdot i(\epsilon), \quad x \in X,\ \epsilon \in \mathbb{Z}_K^\times.$$

- One has

$$\ell(y) = \gamma\lambda + \gamma_1\lambda(\epsilon_1) + \cdots + \gamma_r\lambda(\epsilon_r).$$

Write $\gamma_i = k_i + \xi_i$, with $k_i \in \mathbb{Z}$, $\xi_i \in [0,1)$.

Let

$$\eta = \epsilon_1^{k_1} \cdots \epsilon_r^{k_r}.$$

Set

$$z = y \cdot i(\eta^{-1}).$$

Property of the Cone X (Cont'd)

- Suppose $\arg(z_1) = \phi$.
  Write
  $$0 \le \phi - \frac{2\pi k}{m} < \frac{2\pi}{m}, \quad \text{for some } k \in \mathbb{Z}.$$
  Choose $\zeta \in \mu(K)$, such that
  $$\tau_1(\zeta) = e^{\frac{2\pi i}{m}},$$
  where τ_1 gives the first component of the map $K \mapsto K_{\mathbb{R}}$.
  Then
  $$x = y \cdot i(\eta^{-1}) \cdot i(\zeta^{-k}) \in X.$$
  Clearly then $y = x \cdot i(\epsilon)$, for a unit ϵ.
  This decomposition is clearly unique from the construction.

Conclusions

- It follows that in every class of associate members of $\mathbb{Z}_K$, there is a unique one whose image in $\mathbb{R}^n$ lies in X.
- Moreover, we have
$$f_C(s) = N_{K/\mathbb{Q}}(\mathfrak{b})^s \sum_{x \in \Gamma \cap X} \frac{1}{|N(x)|^s}.$$
- We can then evaluate the sum as in the previous proposition.

Volumes

- Recall that:
  - $T = \{x \in X : |N(x)| \leq 1\}$;
  - $v = \text{vol}(T)$;
  - $\Delta = \text{vol}(\Gamma)$.
- We needed to calculate v and Δ.
- For the latter, we already know

$$\Delta = N_{K/\mathbb{Q}}(\mathfrak{b})|D_K|^{1/2}.$$

- It merely remains to calculate v.

Computing vol(T)

Proposition

vol(T) is given by

$$v = \frac{2^{r_1+r_2}\pi^{r_2}R_K}{m}.$$

- If $\epsilon \in \mathbb{Z}_K^\times$, then multiplication by ϵ is volume preserving.
  This is simply because the volume form is multiplied by value of the determinant of the transformation $x \mapsto x \cdot i(\epsilon)$, which is $|N_{K/\mathbb{Q}}(\epsilon)| = 1$.
  Put
  $$\widetilde{T} = \bigcup_{k=0}^{m-1} T \cdot i(\zeta^k).$$
  Then $\widetilde{T}$ corresponds to the cone X defined only by Conditions 1 and 2 of the preceding definition. It follows that
  $$\mathrm{vol}(\widetilde{T}) = m \cdot \mathrm{vol}(T).$$

Computing $\mathrm{vol}(\overline{T})$ (Cont'd)

- Let $\overline{T}$ denote the set

$$\overline{T} = \{x \in \widetilde{T} : x_i > 0, \text{for all } i = 1, \ldots, r_1\}.$$

It follows that

$$\mathrm{vol}(T) = \frac{2^{r_1}}{m} \mathrm{vol}(\overline{T}).$$

Thus it suffices to calculate $\mathrm{vol}(\overline{T})$.

We make several changes of variables, before computing the volume.

Firstly, we consider the isomorphism

$$\begin{array}{rcl} K_{\mathbb{R}} & \to & \mathbb{R}^n; \\ (x_1, \ldots, x_{r_1}, z_1, \ldots, z_{r_2}) & \mapsto & (x_1, \ldots, x_{r_1}, R_1, \phi_1, \ldots, R_{r_2}, \phi_{r_2}), \end{array}$$

where

$$z_k = R_k e^{i\phi_k}.$$

Computing vol($\overline{T}$) (Cont'd)

- Then

$$\ell(x_1,\ldots,x_{r_1},z_1,\ldots,zr_2)=(\log x_1,\ldots,\log x_{r_1},\log R_1^2,\ldots,\log R_{r_2}^2).$$

The Jacobian of this change of variables is $R_1\cdots R_{r_2}$.

Then $\overline{T}$ is given by:

1. $x_1>0,\ldots,x_{r_1}>0$, $R_1>0,\ldots,R_{r_2}>0$ and $x_1\cdots x_{r_1}(R_1\cdots R_{r_2})^2\leq 1$.
   Note that the last quantity is $N(x)$.
2. In the formula giving the j-th component of $\ell(x)$,

$$\ell(x)=\xi\lambda+\xi_1\lambda(\epsilon_1)+\cdots+\xi_r\lambda(\epsilon_r),$$

   one has $0\leq\xi_k<1$.

Computing vol($\overline{T}$) (Cont'd)

- The $\phi_1,\ldots,\phi_{r_2}$ independently take values in $[0,2\pi)$.

  We replace the variables $x_1,\ldots,x_{r_1},R_1,\ldots,R_{r_2}$ by $\xi,\xi_1,\ldots,\xi_r$, got from the formula $\ell(x)=\xi\lambda+\xi_1\lambda(\epsilon_1)+\cdots+\xi_r\lambda(\epsilon_r)$, so that $\xi=N(x)$.

  Now the image of $\overline{T}$ is given simply by $0<\xi\le 1, 0\le\xi_k<1$, for all k.

  We need to compute the Jacobian of this change of variable.

  Considering the j-th components, we get

$$\begin{array}{rcl}\log x_j & = & \frac{1}{n}\log\xi+\sum_{k=1}^r\xi_k\lambda_j(\epsilon_k);\\ \log R_j^2 & = & \frac{2}{n}\log\xi+\sum_{k=1}^r\xi_k\lambda_{r_1+j}(\epsilon_k).\end{array}$$

  We can read off

$$\frac{\partial x_j}{\partial\xi}=\frac{x_j}{n\xi};\quad \frac{\partial x_j}{\partial\xi_k}=x_j\lambda_j(\epsilon_k);\quad \frac{\partial R_j}{\partial\xi}=\frac{R_j}{n\xi};\quad \frac{\partial R_j}{\partial\xi_k}=\frac{R_j}{2}\lambda_{r_1+j}(\epsilon_k).$$

Computing vol($\overline{T}$) (The Jacobean)

- The Jacobian of this change of variables is given by

$$J = \begin{vmatrix} \frac{x_1}{n\xi} & x_1\lambda_1(\epsilon_1) & \cdots & x_1\lambda_1(\epsilon_r) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{x_{r_1}}{n\xi} & x_{r_1}\lambda_{r_1}(\epsilon_1) & \cdots & x_1\lambda_{r_1}(\epsilon_r) \\ \frac{R_1}{n\xi} & \frac{R_1}{2}\lambda_{r_1+1}(\epsilon_1) & \cdots & \frac{R_1}{2}\lambda_{r_1+1}(\epsilon_r) \\ \vdots & \vdots & \ddots & \vdots \\ \frac{R_{r_2}}{n\xi} & \frac{R_{r_2}}{2}\lambda_{r_1+r_2}(\epsilon_1) & \cdots & \frac{R_{r_2}}{2}\lambda_{r_1+r_2}(\epsilon_r) \end{vmatrix}$$

$$= \frac{x_1\cdots x_{r_1}R_1\cdots R_{r_2}}{2^{r_2}n\xi} \begin{vmatrix} 1 & \lambda_1(\epsilon_1) & \cdots & \lambda_1(\epsilon_r) \\ \vdots & \vdots & \ddots & \vdots \\ 1 & \lambda_{r_1}(\epsilon_1) & \cdots & \lambda_{r_1}(\epsilon_r) \\ 2 & \lambda_{r_1+1}(\epsilon_1) & \cdots & \lambda_{r_1+1}(\epsilon_r) \\ \vdots & \vdots & \ddots & \vdots \\ 2 & \lambda_{r_1+r_2}(\epsilon_1) & \cdots & \lambda_{r_1+r_2}(\epsilon_r) \end{vmatrix}.$$

Computing $\text{vol}(\overline{T})$ (The Jacobean Cont'd)

- One adds all rows to the top one to get

$$J = \frac{x_1 \cdots x_{r_1} R_1 \cdots R_{r_2}}{2^{r_2} n\xi} \begin{vmatrix} n & 0 & \cdots & 0 \\ \cdot & \lambda_2(\epsilon_1) & \cdots & \lambda_2(\epsilon_r) \\ \vdots & \vdots & \ddots & \vdots \\ \cdot & \lambda_{r_1+r_2}(\epsilon_1) & \cdots & \lambda_{r_1+r_2}(\epsilon_r) \end{vmatrix}.$$

  Expanding along the top row, we see that this determinant is exactly nR_K, where R_K denotes the regulator.

  Recall that $\xi = x_1 \cdots x_{r_1} (R_1 \cdots R_{r_2})^2$.

  It follows that

$$|J| = \frac{R_K}{2^{r_2} R_1 \cdots R_{r_2}}.$$

Computing vol(T) (Conclusion)

- We can now deduce the result:

$$\begin{aligned} \text{vol}(\overline{T}) &= 2^{r_2}\text{vol}_{\mathbb{R}}(\overline{T}) \\ &= 2^{r_2}\int_{\overline{T}} dx_1\cdots dx_{r_1}dy_{r_1+1}dz_{r_1+1}\cdots dy_{r_1+r_2}dz_{r_1+r_2} \\ &= 2^{r_2}\int_{\overline{T}} R_1\cdots R_{r_2}dx_1\cdots dx_{r_1}dR_1\cdots dR_{r_2}d\phi_1\cdots d\phi_{r_2} \\ &= 2^{r_2}(2\pi)^{r_2}\int R_1\cdots R_{r_2}dx_1\cdots dx_{r_1}dR_1\cdots dR_{r_2} \\ &= 2^{r_2}(2\pi)^{r_2}\int |J|R_1\cdots R_{r_2}d\xi d\xi_1\cdots d\xi_r \\ &= 2^{r_2}\pi^{r_2}R_K. \end{aligned}$$

Thus,

$$\text{vol}(T) = \frac{2^{r_1+r_2}\pi^{r_2}R_K}{m}.$$

The Analytic Class Number Formula

- Now we obtain

$$\lim_{s\to 1}(s-1)f_C(s) = N_{K/\mathbb{Q}}(\mathfrak{b})\frac{v}{\Delta} = N_{K/\mathbb{Q}}(\mathfrak{b})\frac{\frac{2^{r_1+r_2}\pi^{r_2}R_K}{m}}{N_{K/\mathbb{Q}}(\mathfrak{b})|D_K|^{1/2}}.$$

So

$$\lim_{s\to 1}(s-1)f_C(s) = \frac{2^{r_1+r_2}\pi^{r_2}R_K}{m|D_K|^{1/2}}.$$

This is independent of C.

Summing over the ideal classes gives

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1+r_2}\pi^{r_2}R_K}{m|D_K|^{1/2}}h_K.$$

This is the analytic class number formula.

Subsection 5

Explicit Class Number Formulae

Principal Ideals in Quadratic Fields

- We give explicit expressions for class numbers of quadratic fields.
- Let $K = \mathbb{Q}(\sqrt{d})$, with d squarefree.
- The principal ideal $\langle p \rangle$ for a prime p of $\mathbb{Z}$ can factorize in $\mathbb{Z}_K$ in three different ways.
  1. p can split, so that $\langle p \rangle = \mathfrak{p}_1\mathfrak{p}_2$ with $\mathfrak{p}_1 \neq \mathfrak{p}_2$, and
  $$N_{K/\mathbb{Q}}(\mathfrak{p}_1) = N_{K/\mathbb{Q}}(\mathfrak{p}_2) = p;$$
  2. p can be inert, so that $\langle p \rangle$ remains a prime ideal in $\mathbb{Z}_K$, with norm p^2;
  3. p can ramify, so that $\langle p \rangle = \mathfrak{p}^2$, for some prime ideal $\mathfrak{p}$ of norm p.

Character of a Prime in a Quadratic Field

- We define

$$\chi(p) = \begin{cases} 1, & \text{if } p \text{ splits,} \\ -1, & \text{if } p \text{ is inert,} \\ 0, & \text{if } p \text{ ramifies.} \end{cases}$$

- χ is actually a Dirichlet character modulo D_K.
- This means that:
  - $\chi(p)$ depends only on the value of $p \pmod{D_K}$;
  - $\chi(p)$ can be extended to all integers n, in such a way that if m and n are coprime, then

$$\chi(mn) = \chi(m)\chi(n).$$

Factors in the Euler Product

- Consider the factors corresponding to the primes dividing $\langle p \rangle$ in the Euler product
$$\zeta_K(s) = \prod_{\mathfrak{p}} \left(1 - \frac{1}{N_{K/\mathbb{Q}}(\mathfrak{p})^s}\right)^{-1}.$$
- By the remark above, these are
$$\left(1 - \frac{1}{p^s}\right)^{-2}, \quad \left(1 - \frac{1}{p^{2s}}\right)^{-1}, \quad \left(1 - \frac{1}{p^s}\right)^{-1},$$
in the split, inert and ramified cases, respectively.
- In each case, there is a factor $\left(1 - \frac{1}{p^s}\right)^{-1}$.
- This is the Euler factor of the Riemann zeta function at p.
- The other factor is given by
$$\left(1 - \frac{\chi(p)}{p^s}\right)^{-1}.$$

The Dirichlet L-Function

- We define the **Dirichlet L-function**
$$L(s,\chi)=\prod_p\left(1-\frac{\chi(p)}{p^s}\right)^{-1}.$$
- Then
$$\zeta_K(s)=\zeta(s)L(s,\chi).$$
- By the multiplicativity of χ and unique factorization in $\mathbb{Z}$, we get the alternative expression
$$L(s,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n^s}.$$

Computing $L(1,\chi)$

- We can multiply through by $(s-1)$ and let $s \to 1$ in this expression.
- The Riemann zeta function has a simple pole at $s=1$ with residue 1.
- The residue of $\zeta_K(s)$ is given by the analytic class number formula,

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^{r_1+r_2}\pi^{r_2}R_K}{m|D_K|^{1/2}}h_K.$$

- We get

$$\frac{2^{r_1}(2\pi)^{r_2}h_K R_K}{m|D_K|^{1/2}} = L(1,\chi).$$

The Real and Imaginary Cases

- In the case of a real quadratic field, we have

$$r_1 = 2, \quad r_2 = 0, \quad m = 2, \quad R_K = \log \epsilon,$$

  where $\epsilon > 1$ is a fundamental unit.
  We conclude that

$$h_K = \frac{\sqrt{|D_K|}}{2 \log \epsilon} L(1, \chi).$$

- If K is imaginary quadratic, then

$$r_1 = 0, \quad r_2 = 1, \quad R_K = 1.$$

  So we get

$$h_K = \frac{m\sqrt{|D_K|}}{2\pi} L(1, \chi).$$

  Recall that, in this case, $m = 2$ except for the fields $\mathbb{Q}(i)$ and $\mathbb{Q}(\sqrt{-3})$, both of which have class number one.

Computing h_K Using $L(1,\chi)$

- The quantities $L(1,\chi)$ are not so easy to compute exactly.
- However, it is sometimes relatively easy to compute enough terms in the sum
$$L(1,\chi)=\sum_{n=1}^{\infty}\frac{\chi(n)}{n}.$$
- We may, thus, get an idea of the value of h_K (especially in the imaginary quadratic case), recalling that it must be integral.

Example

- Consider $K = \mathbb{Q}(\sqrt{2})$.

  The fundamental unit is $\epsilon = 1 + \sqrt{2}$.

  The discriminant is $D_K = 8$.

  Then

$$h_K = \frac{\sqrt{8}}{2\log(1+\sqrt{2})} L(1,\chi) \approx 1.605 L(1,\chi).$$

  But, by grouping terms suitably,

$$L(1,\chi) = 1 - \frac{1}{3} - \frac{1}{5} + \frac{1}{7} + \cdots < 1.$$

  So h_K is a positive integer less than 1.605.

  We conclude that $h_K = 1$.