Introduction to Algebraic Topology

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LSSU Math 500

Subsection 1

Introduction

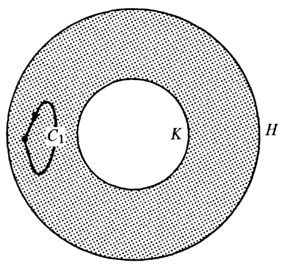
Introduction

- **Point-Set Topology**
  - Georg Cantor 1880: Theory of Sets;
  - Maurice Frechet 1906: Theory of Metric Spaces;
  - Felix Hausdorff 1912: "Basics of Set Theory".
- **Algebraic Topology**
  - Henri Poincaré 1895-1901:
    - Analysis Situs;
    - Complément à l'Analysis Situs;
    - Deuxième Complément;
    - Cinquième Complément.

Subsection 2

Examples

Equivalence of Paths I



- Suppose we evaluate curve integrals

$$\int_C pdx + qdy,$$

  where $p = p(x,y)$ and $q = q(x,y)$ are continuous functions of two variables whose partial derivatives are continuous and satisfy the relation $\frac{\partial p}{\partial y} = \frac{\partial q}{\partial x}$.
- Since curve C_1 can be continuously deformed to a point in the annulus, we have $\int_{C_1} pdx + qdy = 0$.
- Thus C_1 is considered to be negligible as far as curve integrals are concerned.
- We say that C_1 is "equivalent" to a constant path.

Equivalence of Paths II

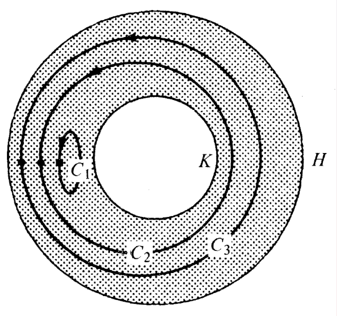
- Consider now the paths C_2 and C_3.
- Recall Green's Theorem

$$\int_{\partial D} (pdx + qdy) = \iint_D \left(\frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} \right) dxdy.$$

- It ensures that the two integrals

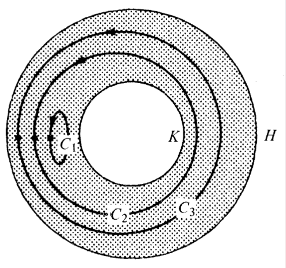
$$\int_{C_2} pdx + qdy,$$
$$\int_{C_3} pdx + qdy$$

are equal.
- So we can consider C_2 and C_3 to be "equivalent".



Basic Idea of Homotopy

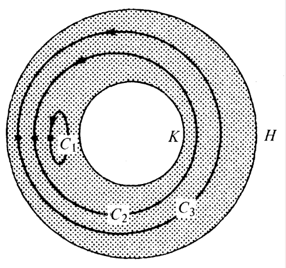
- One way is to consider C_2 and C_3 equivalent because each can be transformed continuously into the other within the annulus.
- This is the basic idea of homotopy theory
- Accordingly, we would say that C_2 and C_3 are homotopic paths.



- Curve C_1 is homotopic to a trivial (or constant) path since it can be shrunk to a point.
- C_2 and C_1 are not homotopic paths since C_2 cannot be pulled across the "hole" that it encloses.
- For the same reason, C_1 is not homotopic to C_3.

Basic Idea of Homology

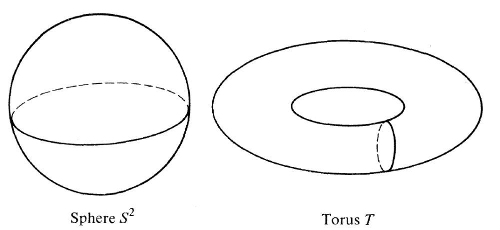
- Another approach is to say that C_2 and C_3 are equivalent because they form the boundary of a region enclosed in the annulus.
- This is the basis of homology theory.
- Accordingly, C_2 and C_3 would be called homologous paths.



- Curve C_1 is homologous to zero since it is the entire boundary of a region enclosed in the annulus.
- C_1 is not homologous to either C_2 or C_3.
- The ideas of homology and homotopy were introduced by Poincaré in his original paper *Analysis Situs* in 1895.

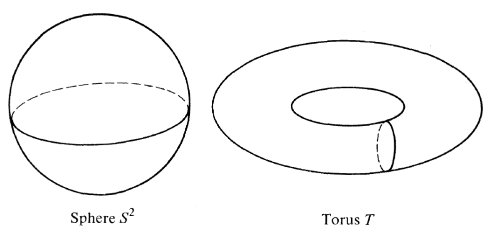
The Sphere and the Torus

- Consider the problem of explaining the difference between a sphere S^2 and a torus T.



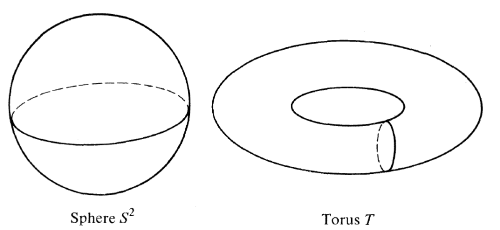
- The difference is that the sphere has one hole, and the torus has two.
- Moreover, the hole in the sphere is somehow different from those in the torus.
- The problem is to explain this difference in a mathematically rigorous way which can be applied to more complicated examples.

Homotopy versus Homology



- Homotopy:
  - Any simple closed curve on the sphere can be continuously deformed to a point on the spherical surface.
  - Meridian and parallel circles on the torus do not have this property.

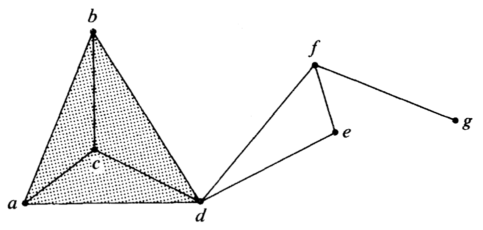
Homotopy versus Homology (Cont'd)



- Homotopy:
  - Every simple closed curve on the sphere is the boundary of the portion of the spherical surface that it encloses and also the boundary of the complementary region.
  - A meridian or parallel circle on the torus is not the boundary of two regions of the torus since such a circle does not separate the torus.
  - Thus any simple closed curve on the sphere is homologous to zero, but meridian and parallel circles on the torus are not homologous to zero.

A Polyhedron

- Consider the configuration shown below:



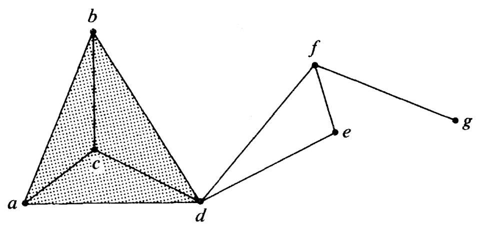
- It consists of:
  - Triangles $\langle abc\rangle, \langle bcd\rangle, \langle abd\rangle$, and $\langle acd\rangle$;
  - Edges $\langle ab\rangle, \langle ac\rangle, \langle ad\rangle, \langle bc\rangle, \langle bd\rangle, \langle cd\rangle, \langle df\rangle, \langle de\rangle, \langle ef\rangle$ and $\langle fg\rangle$;
  - Vertices $\langle a\rangle, \langle b\rangle, \langle c\rangle, \langle d\rangle, \langle e\rangle, \langle f\rangle$ and $\langle g\rangle$.
- The interior of the tetrahedron and the interior of triangle $\langle def\rangle$ are not included.
- This type of space is called a "polyhedron" (the term will be defined formally in the next section).

Chains

- A 0-**chain** is a formal linear combination of vertices, with coefficients modulo 2.
- A 1-**chain** is a formal linear combination of edges with coefficients modulo 2.
- A 2-**chain** is a formal linear combination of triangles with coefficients modulo 2.
- To simplify the notation, we omit those terms with coefficient 0 and consider only those terms in a chain with coefficient 1.

Example

- Consider again the configuration



- An example of a 2-chain is

$$1 \cdot \langle abc \rangle + 1 \cdot \langle abd \rangle + 0 \cdot \langle acd \rangle + 0 \cdot \langle bcd \rangle.$$

- According to the convention, it can be written

$$\langle abc \rangle + \langle abd \rangle.$$

The Boundary Operator

- The **boundary operator** ∂ is defined as follows for chains of length one and extended linearly:

$$\begin{array}{rcl}\partial\langle abc\rangle & = & \langle ab\rangle+\langle ac\rangle+\langle bc\rangle,\\ \partial\langle ab\rangle & = & \langle a\rangle+\langle b\rangle.\end{array}$$

- A p-chain c_p ($p=1$ or 2) is a **boundary** means that there is a $(p+1)$-chain c_{p+1} with

$$\partial c_{p+1}=c_p.$$

- We think of this as indicating that the union of the members of c_p forms the point-set boundary of the union of the members of c_{p+1}.

Example

- Recall that we are operating modulo 2.
- This means that, in a sum, terms occurring twice cancel out.
- So we have the following calculation:

$$\begin{aligned}\langle ab\rangle + \langle bc\rangle + \langle cd\rangle + \langle da\rangle &= \langle ab\rangle + \langle bc\rangle + \langle ca\rangle + \langle ac\rangle + \langle cd\rangle + \langle da\rangle \\ &= \partial(\langle abc\rangle + \langle acd\rangle).\end{aligned}$$

Example

- For any 2-chain c_2, one easily observes that $\partial\partial c_2 = 0$.

  It suffices to consider a single triangle.

  We have

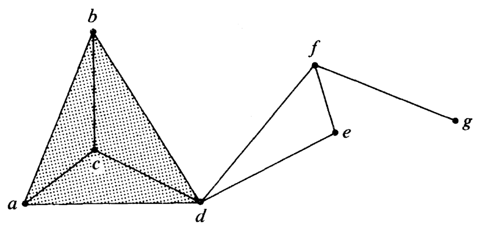
$$\begin{aligned}
\partial\partial\langle abc\rangle &= \partial(\langle ab\rangle + \langle ac\rangle + \langle bc\rangle) \\
&= \partial\langle ab\rangle + \partial\langle ac\rangle + \partial\langle bc\rangle \\
&= \langle a\rangle + \langle b\rangle + \langle a\rangle + \langle c\rangle + \langle b\rangle + \langle c\rangle \\
&= 0.
\end{aligned}$$

Cycles

- A p-**cycle** ($p = 1$ or 2) is a p-chain c_p with
$$\partial c_p = 0.$$
- Since $\partial\partial$ is the trivial operator, every boundary is a cycle.
- Intuitively speaking, a cycle is a chain whose terms fulfill on the following:
  - They close a "hole";
  - They form the boundary of a chain of the next higher dimension.
- We investigate the "holes" in the polyhedron by determining the cycles which are not boundaries.

Example

- Consider again the previous configuration.



- Except for the 2-chain having all coefficients zero, the only 2-cycle is

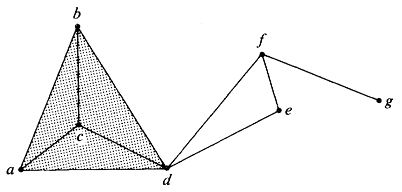
$$\langle abc\rangle + \langle bcd\rangle + \langle acd\rangle + \langle abd\rangle.$$

- It is nonbounding since the interior of the tetrahedron is not included.

Example (Cont'd)

- There is a nonbounding 1-cycle

$$z = \langle df \rangle + \langle fe \rangle + \langle de \rangle.$$



- Any other 1-cycle is of one of the two types:
  - A boundary;
  - The sum of z and a boundary.
- Thus any 1-cycle is homologous to zero or homologous to the fundamental 1-cycle z.
- This indicates the presence of two holes in the polyhedron:
  - One enclosed by the nonbounding 2-cycle;
  - One enclosed by the nonbounding 1-cycle z.

Subsection 3

Geometric Complexes and Polyhedra

Geometric Independence

- For each positive integer n, we shall consider n-dimensional Euclidean space
$$\mathbb{R}^n = \{x = (x_1, x_2, \dots, x_n) : \text{each } x_i \text{ is a real number}\}$$
as a vector space over the field $\mathbb{R}$ of real numbers.
- We will use some basic ideas from the theory of vector spaces.

Definition

A set $A = \{a_0, a_1, \dots, a_k\}$ of $k+1$ points in $\mathbb{R}^n$ is **geometrically independent** means that no hyperplane of dimension $k-1$ contains all the points.

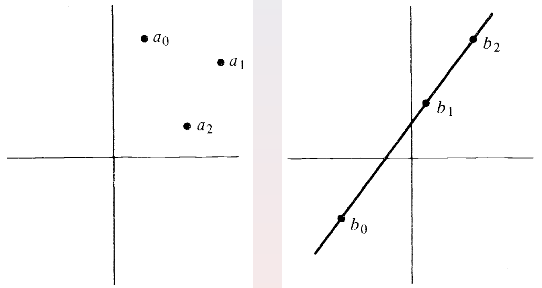
Explanation

- Consider a set
$$\{a_0, a_1, \ldots, a_k\}$$
of points in $\mathbb{R}^n$.
- That this set is geometrically independent means that:
    - All the points are distinct;
    - No three of them lie on a line;
    - No four of them lie in a plane;

      ⋮
    - No $p+1$ of them lie in a hyperplane of dimension $p-1$ or less.

      ⋮

Example



- The set $\{a_0, a_1, a_2\}$ is geometrically independent.

  The only hyperplane in $\mathbb{R}^2$ containing all the points is the entire plane.
- The set $\{b_0, b_1, b_2\}$ is not geometrically independent.

  All three points lie on a line, a hyperplane of dimension 1.

Simplexes

Definition

Let $\{a_0, \ldots, a_k\}$ be a set of geometrically independent points in $\mathbb{R}^n$.
The k-**dimensional geometric simplex** or k-**simplex**, σ^k, spanned by $\{a_0, \ldots, a_k\}$ is the set of all points x in $\mathbb{R}^n$ for which there exist nonnegative real numbers $\lambda_0, \ldots, \lambda_k$, such that

$$x = \sum_{i=0}^{k} \lambda_i a_i, \quad \sum_{i=0}^{k} \lambda_i = 1.$$

The numbers $\lambda_0, \ldots, \lambda_k$ are the **barycentric coordinates** of the point x.
The points $a_0, \ldots, a_k$ are the **vertices** of σ^k.
The set of all points x in σ^k with all barycentric coordinates positive is called the **open geometric** k-**simplex** spanned by $\{a_0, \ldots, a_k\}$.

Simplexes of Small Order

- Simplexes of small orders give us the following:
  - A 0-simplex is simply a singleton set.
  - A 1-simplex is a closed line segment.
  - A 2-simplex is a triangle (interior and boundary).
  - A 3-simplex is a tetrahedron (interior and boundary).
- For open simplexes, we have:
  - An open 0-simplex is a singleton set.
  - An open 1-simplex is a line segment with end points removed.
  - An open 2-simplex is the interior of a triangle.
  - An open 3-simplex is the interior of a tetrahedron.

Faces

Definition

A simplex σ^k is a **face** of a simplex σ^n, $k \leq n$, means that each vertex of σ^k is a vertex of σ^n. The faces of σ^n other than σ^n itself are called **proper faces**.

- If σ^n is the simplex with vertices $a_0, \ldots, a_n$, we write

$$\sigma^n = \langle a_0 \ldots a_n \rangle.$$

Example: The faces of the 2-simplex $\langle a_0 a_1 a_2 \rangle$ are:

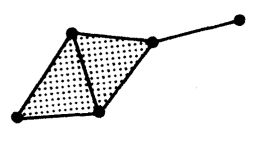
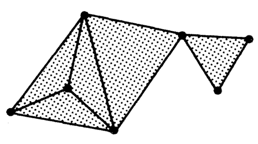
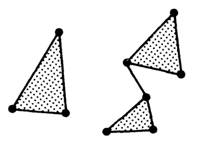
- The 2-simplex itself;
- The 1-simplexes $\langle a_0 a_1 \rangle, \langle a_1 a_2 \rangle$ and $\langle a_0 a_2 \rangle$;
- The 0-simplexes $\langle a_0 \rangle, \langle a_1 \rangle$ and $\langle a_2 \rangle$.

Properly Joined Simplexes

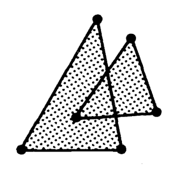
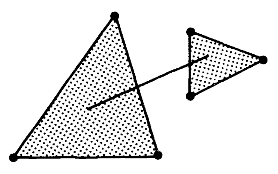
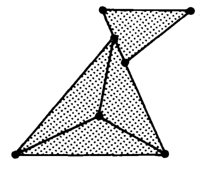
Definition

Two simplexes σ^m and σ^n are **properly joined** provided that they do not intersect or the intersection $\sigma^m \cap \sigma^n$ is a face of both σ^m and σ^n.

Example: The following examples show proper joining:

Example: The following examples show improper joining:

Geometric Complexes

Definition

A **geometric complex** (or **simplicial complex** or **complex**) is a finite family K of geometric simplexes, such that:

- Membes of K are properly joined;
- Each face of a member of K is also a member of K.

The **dimension** of K is the largest positive integer r such that K has an r-simplex.

The union of the members of K with the Euclidean subspace topology is denoted by $|K|$ and is called the **geometric carrier** of K or the **polyhedron associated with** K.

- We shall be concerned, for the purposes of homology, with geometric complexes and polyhedra composed of a finite number of simplexes.
- Greater generality, at the expense of greater complexity, can be obtained by allowing an infinite number of simplexes.

Triangulations, Closures and Skeletons

Definition (Triangulable Spaces and Triangulations)

Let X be a topological space. If there is a geometric complex K whose geometric carrier $|K|$ is homeomorphic to X, then X is said to be a **triangulable space**, and the complex K is called a **triangulation** of X.

Definition (Closure of a Simplex)

The **closure** of a k-simplex σ^k, denoted $\mathrm{Cl}(\sigma^k)$, is the complex consisting of σ^k and all its faces.

Definition (Skeleton of a Complex)

If K is a complex and r a positive integer, the r-**skeleton** of K is the complex consisting of all simplexes of K of dimension less than or equal to r.

Example: 3-Simplex

- Consider a 3-simplex $\sigma^3 = \langle a_0 a_1 a_2 a_3 \rangle$.
- The 2-skeleton of the closure of σ^3 is the complex K whose simplexes are the proper faces of σ^3.
- The geometric carrier of K is the boundary of a tetrahedron.
- It is therefore homeomorphic to the 2-sphere

$$S^2 = \left\{ (x_1, x_2, x_3) \in \mathbb{R}^3 : \sum_{i=1}^{3} x_i^3 = 1 \right\}.$$

- It follows that S^2 is triangulable with K as one triangulation.

Example: n-Sphere

- The n-sphere

$$S^n = \left\{ (x_1, x_2, \dots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum_{i=1}^{n+1} x_i^2 = 1 \right\}$$

  is a triangulable space for $n \geq 0$.
- The n-skeleton of the closure of an $(n+1)$-simplex σ^{n+1} is one triangulation of S^n.

Example: n-Sphere (More Details)

- Let $\sigma^{n+1} = \langle a_0 a_1 \ldots a_n \rangle$ be the $(n+1)$-simplex in $\mathbb{R}^{n+1}$, with:
  - a_0 the origin;
  - a_i, $i > 0$, the point with i-th coordinate 1 and all other coordinates 0.
- If K the n-skeleton of the closure of σ^{n+1}, $|K|$ is its point set boundary.
- Apply in turn the following homeomorphisms:
  - $f : \sigma^{n+1} \to [0,1]^{n+1}$, defined by

$$f(x_0, \ldots, x_n) = \frac{\sum_{i=0}^{n} x_i}{\max_{0 \le i \le n} \{x_i\}} (x_0, \ldots, x_n);$$

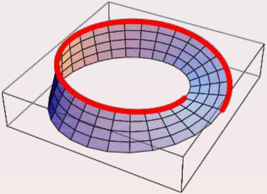
  - $g : [0,1]^{n+1} \to [-1,1]^{n+1}$, defined by

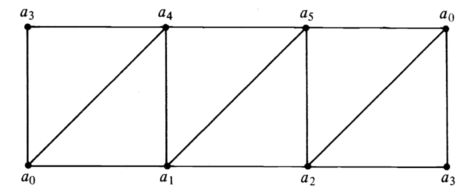
$$g(x_0, \ldots, x_n) = (2x_0 - 1, \ldots, 2x_n - 1);$$

  - $h : [-1,1]^{n+1} \to D^{n+1}$ is normalization, where D^n is the n-disk.
- The composition hgf restricts to a homeomorphism from $|K| \to S^n$.

Example: Möbius Strip

- The **Möbius strip** is obtained by identifying two opposite ends of a rectangle after twisting it through 180 degrees.

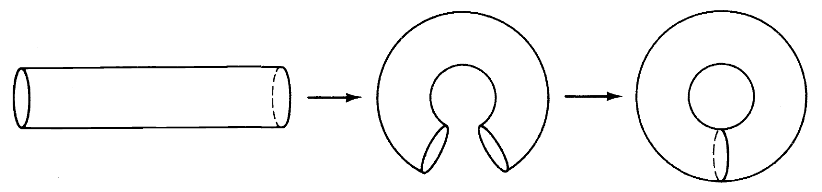




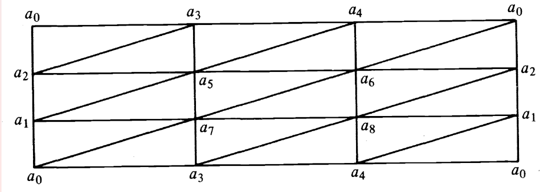
- The right figure shows a triangulation of the Möbius strip. where:
  - The two vertices labeled a_0 are identified, the two vertices labeled a_3 are identified, corresponding points of the two segments $\langle a_0 a_3 \rangle$ are identified;
  - The resulting quotient space, the geometric carrier of the triangulation, is considered as a subspace of $\mathbb{R}^3$.

The Torus

- A **torus** is obtained from a cylinder by identifying corresponding points of the circular ends with no twisting.



- The following diagram, with proper identifications, gives a triangulation of the torus:



Subsection 4

Orientation of Geometric Complexes

Orientation

Definition

An **oriented n-simplex**, $n \geq 1$, is obtained from an n-simplex

$$\sigma^n = \langle a_0 \dots a_n \rangle$$

by choosing an ordering for its vertices.

- The equivalence class of even permutations of the chosen ordering determines the **positively oriented simplex** $+\sigma^n$.
- The equivalence class of odd permutations determines the **negatively oriented simplex** $-\sigma^n$.

An **oriented geometric complex** is obtained from a geometric complex by assigning an orientation to each of its simplexes.

Remarks on Orientation

- If vertices $a_0, \ldots, a_p$ of a complex K are the vertices of a p-simplex σ^p, then the symbol:
  - $+\langle a_0 \ldots a_p \rangle$ denotes the class of even permutations of the indicated order $a_0, \ldots, a_p$;
  - $-\langle a_0 \ldots a_p \rangle$ denotes the class of odd permutations of the indicated order $a_0, \ldots, a_p$.
- If we wanted the class of even permutations of this order to determine the positively oriented simplex, then we would write

$$+\sigma^p = \langle a_0 \ldots a_p \rangle \quad \text{or} \quad +\sigma^p = +\langle a_0 \ldots a_p \rangle.$$

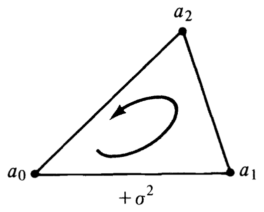
- Since ordering vertices requires more than one vertex, we need not worry about orienting 0-simplexes.
- It is convenient, however, to consider a 0-simplex $\langle a_0 \rangle$ as positively oriented.

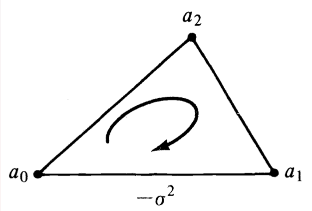
Example: 1-Simplexes

- In the 1-simplex $\sigma^1 = \langle a_0 a_1 \rangle$ let us agree that the ordering is given by $a_0 < a_1$.
- Then
$$+\sigma^1 = \langle a_0 a_1 \rangle, \quad -\sigma^1 = \langle a_1 a_0 \rangle.$$
- If we imagine that the segment $\langle a_i a_j \rangle$ is directed from a_i toward a_j, then $\langle a_0 a_1 \rangle$ and $\langle a_1 a_0 \rangle$ have opposite directions.

Example: 2-Simplexes

- In the 2-simplex $\sigma^2 = \langle a_0 a_1 a_2 \rangle$, assign the order $a_0 < a_1 < a_2$.
- Then:
  - $\langle a_0 a_1 a_2 \rangle, \langle a_1 a_2 a_0 \rangle$ and $\langle a_2 a_0 a_1 \rangle$ all denote $+\sigma^2$;
  - $\langle a_0 a_2 a_1 \rangle, \langle a_2 a_1 a_0 \rangle$ and $\langle a_1 a_0 a_2 \rangle$ all denote $-\sigma^2$.





- In this case

$$+\sigma^2 = +\langle a_0 a_1 a_2 \rangle, \quad -\sigma^2 = -\langle a_0 a_1 a_2 \rangle = +\langle a_0 a_2 a_1 \rangle.$$

- $+\langle a_0 a_2 a_1 \rangle$ denotes the class of even permutations of a_0, a_2, a_1.
- $-\langle a_0 a_1 a_2 \rangle$ denotes the class of odd permutations of a_0, a_1, a_2.

Incidence Number

Definition

Let K be an oriented geometric complex with simplexes σ^{p+1} and σ^p whose dimensions differ by 1. We associate with each such pair (σ^{p+1},σ^p) an **incidence number** $[\sigma^{p+1},\sigma^p]$ defined as follows:

- If σ^p is not a face of σ^{p+1}, then
$$[\sigma^{p+1},\sigma^p]=0.$$
- Suppose σ^p is a face of σ^{p+1}.
  Label the vertices $a_0,\ldots,a_p$ of σ^p so that $+\sigma^p=+\langle a_0\ldots a_p\rangle$.
  Let v denote the vertex of σ^{p+1} which is not in σ^p. Then
$$+\sigma^{p+1}=\pm\langle va_0\ldots a_p\rangle.$$
  - If $+\sigma^{p+1}=+\langle va_0\ldots a_p\rangle$, then $[\sigma^{p+1},\sigma^p]=1$.
  - If $+\sigma^{p+1}=-\langle va_0\ldots a_p\rangle$, then $[\sigma^{p+1},\sigma^p]=-1$.

Example

- Suppose $+\sigma^1 = \langle a_0 a_1 \rangle$.
  Then we have:
  - $[\sigma^1, \langle a_0 \rangle] = -1$;
  - $[\sigma^1, \langle a_1 \rangle] = 1$.
- Consider
  $$+\sigma^2 = +\langle a_0 a_1 a_2 \rangle, \quad +\sigma^1 = \langle a_0 a_1 \rangle \quad +\tau^1 = \langle a_0 a_2 \rangle.$$
  Then:
  - $[\sigma^2, \sigma^1] = 1$;
  - $[\sigma^2, \tau^1] = -1$.

The Incidence Number Formula

Theorem

Let K be an oriented complex, σ^p an oriented p-simplex of K and σ^{p-2} a $(p-2)$-face of σ^p. Then

$$\sum_{\sigma^{p-1}\in K}[\sigma^p,\sigma^{p-1}][\sigma^{p-1}\sigma^{p-2}]=0.$$

- Label the vertices $v_0,\ldots,v_{p-2}$ of σ^{p-2} so that

$$+\sigma^{p-2}=\langle v_0\ldots v_{p-2}\rangle.$$

Then σ^p has two additional vertices a and b.
Without loss of generality, assume

$$+\sigma^p=\langle abv_0\ldots v_{p-2}\rangle.$$

The Incidence Number Formula

- Nonzero terms occur in the sum for only two values of σ^{p-1}.

  These are

  $$\sigma_1^{p-1} = \langle a v_0 \ldots v_{p-2} \rangle, \quad \sigma_2^{p-1} = \langle b v_0 \ldots v_{p-2} \rangle.$$

  We must now treat a total of four cases determined by the orientations of σ_1^{p-1} and σ_2^{p-1}.

  We treat two of them in detail.

  The other two may be treated similarly.

The Incidence Number Formula (Cont'd)

- **Case I**: Suppose that
$$+\sigma_1^{p-1} = +\langle a v_0 \ldots v_{p-2}\rangle, \quad +\sigma_2^{p-1} = +\langle b v_0 \ldots v_{p-2}\rangle.$$
Then
$$\begin{array}{ll} [\sigma^p, \sigma_1^{p-1}] = -1, & [\sigma_1^{p-1}, \sigma^{p-2}] = +1, \\ [\sigma^p, \sigma_2^{p-1}] = +1, & [\sigma_2^{p-1}, \sigma^{p-2}] = +1. \end{array}$$
So the sum in the statement in 0.
- **Case II**: Suppose that
$$+\sigma_1^{p-1} + \langle a v_0 \ldots v_{p-2}\rangle, \quad +\sigma_2^{p-1} = -\langle b v_0 \ldots v_{p-2}\rangle.$$
Then
$$\begin{array}{ll} [\sigma^p, \sigma_1^{p-1}] = -1, & [\sigma_1^{p-1}, \sigma^{p-2}] = +1, \\ [\sigma^p, \sigma_2^{p-1}] = -1, & [\sigma_2^{p-1}, \sigma^{p-2}] = -1. \end{array}$$
So the sum in the statement in 0 also.

The Incidence Matrix of an Oriented Complex

Definition

In the oriented complex K, let $\{\sigma_i^p\}_{i=1}^{\alpha_p}$ and $\{\sigma_i^{p+1}\}_{i=1}^{\alpha_{p+1}}$ denote the p-simplexes and $(p+1)$-simplexes of K, where α_p and α_{p+1} denote the numbers of simplexes of dimensions p and $p+1$, respectively. The matrix

$$\eta(p) = (\eta_{ij}(p)),$$

where

$$\eta_{ij}(p) = [\sigma_i^{p+1}, \sigma_j^p],$$

is called the p-**th incidence matrix** of K.

Comments

- Incidence matrices were used to describe the arrangement of simplexes in a complex during the early days of algebraic or "combinatorial" topology.
- Today group theory has given a much more efficient method of describing the same property.
- The group theoretic formulation seems to have been suggested by Emmy Noether (1882-1935) in around 1925.
- These groups follow quite naturally from Poincaré's original description of homology theory.