Introduction to Algebraic Topology

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1. Simplicial Homology Groups
   - Chains, Cycles, Boundaries and Homology Groups
   - Examples of Homology Groups
   - The Structure of Homology Groups
   - The Euler-Poincaré Theorem
   - Pseudomanifolds and the Homology Groups of S^n

Subsection 1

Chains, Cycles, Boundaries and Homology Groups

Chains and Chain Groups

Definition

Let K be an oriented simplicial complex. If p is a positive integer, a p-**dimensional chain**, or p-**chain**, is a function c_p from the family of oriented p-simplexes of K to the integers such that, for each p-simplex σ^p,

$$c_p(-\sigma^p) = -c_p(+\sigma^p).$$

A 0-**dimensional chain** or 0-chain is a function from the 0-simplexes of K to the integers.

With the operation of pointwise addition induced by the integers, the family of p-chains forms a group, the p-**dimensional chain group** of K. This group is denoted by $C_p(K)$.

Elementary Chains

Definition (Cont'd)

Let K be an oriented simplicial complex. Let p be a positive integer.
An **elementary p-chain** is a p-chain c_p for which there is a p-simplex σ^p such that

$$c_p(\tau^p) = 0,$$

for each p-simplex τ^p distinct from σ^p. Such an elementary p-chain is denoted by

$$g \cdot \sigma^p,$$

where $g = c_p(+\sigma^p)$.
With this notation, an arbitrary p-chain d_p can be expressed as a formal finite sum

$$d_p = \sum g_i \cdot \sigma_i^p$$

of elementary p-chains, where the index i ranges over all p-simplexes of K.

Remarks on the Group $C_p(K)$

(a) Let

$$\begin{aligned} c_p &= \sum f_i \cdot \sigma_i^p; \\ d_p &= \sum g_i \cdot \sigma_i^p \end{aligned}$$

be two p-chains on K.

Then

$$c_p + d_p = \sum (f_i + g_i) \cdot \sigma_i^p.$$

(b) The additive inverse of the chain $c_p = \sum f_i \cdot \sigma_i^p$ in the group $C_p(K)$ is the chain

$$-c_p = \sum -f_i \cdot \sigma_i^p.$$

Remarks on the Group $C_p(K)$ (Cont'd)

(c) The chain group $C_p(K)$ is isomorphic to the direct sum of the group $\mathbb{Z}$ of integers over the family of p-simplexes of K.

In more detail, suppose K has α_p p-simplexes.

Then $C_p(K)$ is isomorphic to the direct sum of α_p copies of $\mathbb{Z}$.

One isomorphism is given by the correspondence

$$\sum_{i=1}^{\alpha_p} g_i \cdot \sigma_i^p \quad \leftrightarrow \quad (g_1, g_2, \ldots, g_{\alpha_p}).$$

- Any commutative group, commutative ring or field could be used as the coefficient set for the p-chains, thus making $C_p(K)$ a commutative group, a module or a vector space.

Boundaries

Definition

If $g \cdot \sigma^p$ is an elementary p-chain with $p \geq 1$, the **boundary** of $g \cdot \sigma^p$, denoted by $\partial(g \cdot \sigma^p)$, is defined by

$$\partial(g \cdot \sigma^p) = \sum [\sigma^p, \sigma_i^{p-1}] g \cdot \sigma_i^{p-1}, \quad \sigma_i^{p-1} \in K.$$

The boundary operator ∂ is extended by linearity to a homomorphism

$$\partial : C_p(K) \to C_{p-1}(K).$$

I.e., if $c_p = \sum g_i \cdot \sigma_i^p$ is an arbitrary p-chain, then

$$\partial(c_p) = \sum \partial(g_i \cdot \sigma_i^p).$$

The **boundary of a** 0**-chain** is defined to be zero.

A Comment on Boundaries

- Strictly speaking, the boundary homomorphism should be denoted by

$$\partial_p : C_p(K) \to C_{p-1}(K).$$

- However, the dimension involved is indicated by the chain group $C_p(K)$.
- So it is customary to omit the subscript.

Composition of Boundary Operators

Theorem

If K is an oriented complex and $p \geq 2$, then the composition $\partial\partial : C_p(K) \to C_{p-2}(K)$ in the diagram

$$C_p(K) \xrightarrow{\partial} C_{p-1}(K) \xrightarrow{\partial} C_{p-2}(K)$$

is the trivial homomorphism.

- We show that $\partial\partial(c_p) = 0$ for each p-chain.
  It suffices to show $\partial\partial(g \cdot \sigma^p) = 0$, for every elementary p-chain $g \cdot \sigma^p$.
  Observe that

$$\begin{array}{rcl} \partial\partial(g \cdot \sigma^p) & = & \partial(\sum_{\sigma_i^{p-1} \in K} [\sigma^p, \sigma_i^{p-1}] g \cdot \sigma_i^{p-1}) \\ & = & \sum_{\sigma_i^{p-1} \in K} \partial([\sigma^p, \sigma_i^{p-1}] g \cdot \sigma_i^{p-1}) \\ & = & \sum_{\sigma_i^{p-1} \in K} \sum_{\sigma_j^{p-2} \in K} [\sigma^p, \sigma_i^{p-1}][\sigma_i^{p-1}, \sigma_j^{p-2}] g \cdot \sigma_j^{p-2}. \end{array}$$

Composition of Boundary Operators (Cont'd)

- We have $\partial\partial(g\cdot\sigma^p)=\sum_{\sigma_i^{p-1}\in K}\sum_{\sigma_j^{p-2}\in K}[\sigma^p,\sigma_i^{p-1}][\sigma_i^{p-1},\sigma_j^{p-2}]g\cdot\sigma_j^{p-2}$.

  Reversing the order of summation and collecting coefficients of each simplex σ_j^{p-2} gives

$$\partial\partial(g\cdot\sigma^p)=\sum_{\sigma_j^{p-2}\in K}\left(\sum_{\sigma_i^{p-1}\in K}[\sigma^p,\sigma_i^{p-1}][\sigma_i^{p-1},\sigma_j^{p-2}]g\cdot\sigma_j^{p-2}\right).$$

  By a preceding theorem, for every σ_j^{p-2},

$$\sum_{\sigma_i^{p-1}\in K}[\sigma^p,\sigma_i^{p-1}][\sigma_i^{p-1},\sigma_j^{p-2}]=0.$$

  Therefore, $\partial\partial(g\cdot\sigma^p)=0$.

Cycles and Cycle Groups

Definition

Let K be an oriented complex. Let p is a positive integer.
A p-**dimensional cycle** on K, or p-**cycle**, is a p-chain z_p such that

$$\partial(z_p) = 0.$$

The family of p-cycles is thus the kernel of the homomorphism

$$\partial : C_p(K) \to C_{p-1}(K)$$

and is a subgroup of $C_p(K)$. This subgroup, denoted by $Z_p(K)$, is called the p-**dimensional cycle group** of K.
Recall that, by definition, the boundary of every 0-chain is 0.
So we define a 0-**cycle** to be synonymous with a 0-chain.
Thus the group $Z_0(K)$ of 0-cycles is the group $C_0(K)$ of 0-chains.

Boundaries and Boundary Groups

Definition

If $p \geq 0$, a p-chain b_p is a p-**dimensional boundary** on K, or p-**boundary**, if there is a $(p+1)$-chain c_{p+1} such that

$$\partial(c_{p+1}) = b_p.$$

The family of p-boundaries is the homomorphic image $\partial(C_{p+1}(K))$ and is a subgroup of $C_p(K)$. This subgroup is called the p-**dimensional boundary group** of K and is denoted by $B_p(K)$.

If n is the dimension of K, then there are no p-chains on K for $p > n$.

In this case we say that $C_p(K)$ is the trivial group $\{0\}$.

In particular, note that there are no $(n+1)$-chains on K, which implies:

- $C_{n+1}(K) = \{0\}$;
- $B_n(K) = \{0\}$.

Cycle Groups and Boundary Groups

Theorem

If K is an oriented complex, then $B_p(K) \subseteq Z_p(K)$, for each integer p such that $0 \leq p \leq n$, where n is the dimension of K.

- Suppose that b_p is a p-boundary of K.

  Thus, there exists a $(p+1)$-chain c_{p+1}, such that

  $$\partial c_{p+1} = b_p.$$

  But then, we get

  $$\partial b_p = \partial \partial c_{p+1} = 0.$$

  Thus, b_p is also a p-cycle of K.

Introducing Homology

- We think of a p-cycle as a linear combination of p-simplexes which makes a complete circuit.
- The p-cycles which enclose "holes" are the interesting cycles.
- They are the ones which are not boundaries of $(p+1)$-chains.
- We use the relation of homology between cycles to restrict our attention to nonbounding cycles and to weed out the bounding ones.

Homologous Cycles

Definition

Two p-cycles w_p and z_p on a complex K are **homologous**, written

$$w_p \sim z_p,$$

provided that there is a $(p+1)$-chain c_{p+1}, such that

$$\partial(c_{p+1}) = w_p - z_p.$$

If a p-cycle t_p is the boundary of a $(p+1)$-chain, we say that t_p is **homologous to zero** and write $t_p \sim 0$.

Homology Classes and Group Structure

- The relation $\sim$ of homology for p-cycles is an equivalence relation.
- So it partitions $Z_p(K)$ into **homology classes**

$$[z_p] = \{w_p \in Z_p(K) : w_p \sim z_p\}.$$

- The homology class $[z_p]$ is actually the coset

$$z_p + B_p(K) = \{z_p + \partial(c_{p+1}) : \partial(c_{p+1}) \in B_p(K)\}.$$

- So the homology classes are the members of the quotient group $Z_p(K)/B_p(K)$.
- We can use the quotient group structure to add homology classes.

The Homology Groups

Definition

Let K is an oriented complex. Let p be a non-negative integer. The p-**dimensional homology group** of K is the quotient group

$$H_p(K) = Z_p(K)/B_p(K).$$

Subsection 2

Examples of Homology Groups

Example I

- Let K be the closure of a 2-simplex $\langle a_0 a_1 a_2 \rangle$ with orientation induced by the ordering $a_0 < a_1 < a_2$.
- K has:
  - 0-simplexes $\langle a_0 \rangle, \langle a_1 \rangle$ and $\langle a_2 \rangle$;
  - Positively oriented 1-simplexes $\langle a_0 a_1 \rangle, \langle a_1 a_2 \rangle$ and $\langle a_0 a_2 \rangle$;
  - A positively oriented 2-simplex $\langle a_0 a_1 a_2 \rangle$.
- A 0-chain on K is a sum of the form

$$c_0 = g_0 \cdot \langle a_0 \rangle + g_1 \cdot \langle a_1 \rangle + g_2 \cdot \langle a_2 \rangle,$$

  where g_0, g_1 and g_2 are integers.
- Hence, $C_0(K) = Z_0(K)$ is isomorphic to the direct sum $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$ of three copies of the group of integers.

Example I (1-Chains)

- A 1-chain on K is a sum of the form

$$c_1 = h_0 \cdot \langle a_0 a_1 \rangle + h_1 \cdot \langle a_1 a_2 \rangle + h_2 \cdot \langle a_0 a_2 \rangle,$$

  where h_0, h_1 and h_2 are integers.
- So $C_1(K)$ is isomorphic to $\mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.

Example I (1-Cycles)

- We have

$$\partial(c_1) = (-h_0 - h_2)\cdot\langle a_0\rangle + (h_0 - h_1)\cdot\langle a_1\rangle + (h_1 + h_2)\cdot\langle a_2\rangle.$$

- Hence c_1 is a 1-cycle if and only if h_0, h_1 and h_2 satisfy the equations

$$-h_0 - h_2 = 0, \quad h_0 - h_1 = 0, \quad h_1 + h_2 = 0.$$

- This system gives $h_0 = h_1 = -h_2$.
- So the 1-cycles are chains of the form

$$h\cdot\langle a_0 a_1\rangle + h\cdot\langle a_1 a_2\rangle - h\cdot\langle a_0 a_2\rangle,$$

  where h is any integer.
- Thus $Z_1(K)$ is isomorphic to the group $\mathbb{Z}$ of integers.

Example I (2-Homology and 1-Homology)

- The only 2-simplex of K is $\langle a_0 a_1 a_2 \rangle$.
- So the only 2-chains are the elementary ones

$$h \cdot \langle a_0 a_1 a_2 \rangle,$$

where h is an integer.
- Thus, $C_2(K) \cong \mathbb{Z}$.
- We have

$$\partial(h \cdot \langle a_0 a_1 a_2 \rangle) = h \cdot \langle a_0 a_1 \rangle + h \cdot \langle a_1 a_2 \rangle - h \cdot \langle a_0 a_2 \rangle.$$

So $\partial(h \cdot \langle a_0 a_1 a_2 \rangle) = 0$ only when $h = 0$. Thus, $Z_2(K) = \{0\}$.
- This gives $H_2(K) = \{0\}$.
- The 1-cycles and the 1-boundaries have precisely the same form,

$$h \cdot \langle a_0 a_1 \rangle + h \cdot \langle a_1 a_2 \rangle - h \cdot \langle a_0 a_2 \rangle.$$

- So we get $Z_1(K) = B_1(K)$.
- This gives $H_1(K) = \{0\}$.

Example I (0-Homology)

- A 0-cycle $g_0 \cdot \langle a_0 \rangle + g_1 \cdot \langle a_1 \rangle + g_2 \cdot \langle a_2 \rangle$ is a 0-boundary if and only if there are integers h_0, h_1 and h_2, such that

$$-h_0 - h_2 = g_0, \quad h_0 - h_1 = g_1, \quad h_1 + h_2 = g_2.$$

- Then we have $g_0 + g_1 = -g_2$.
- So, for 0-boundaries, two coefficients are arbitrary, and the third is determined by the first two.
- This gives $B_0(K) \cong \mathbb{Z} \oplus \mathbb{Z}$.
- But $B_0(K) \cong \mathbb{Z} \oplus \mathbb{Z}$ and $Z_0(K) \cong \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$.
- So we suspect $H_0(K) \cong \mathbb{Z}$.

Example I (0-Homology Cont'd)

- To complete the proof, consider a 0-cycle

$$g_0 \cdot \langle a_0 \rangle + g_1 \cdot \langle a_1 \rangle + g_2 \cdot \langle a_2 \rangle.$$

- Write

$$\begin{aligned} & g_0 \cdot \langle a_0 \rangle + g_1 \cdot \langle a_1 \rangle + g_2 \cdot \langle a_2 \rangle \\ & \qquad = \partial(g_1 \cdot \langle a_0 a_1 \rangle + g_2 \cdot \langle a_0 a_2 \rangle) + (g_0 + g_1 + g_2) \cdot \langle a_0 \rangle. \end{aligned}$$

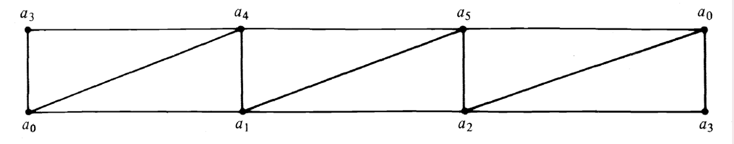
- This shows that any 0-cycle is homologous to a 0-cycle of the form $t \cdot \langle a_0 \rangle$, t an integer.
- Hence, each 0-homology class has a representative $t \cdot \langle a_0 \rangle$.
- So $H_0(K)$ is isomorphic to $\mathbb{Z}$.

Example I (Summary)

- Summarizing, we have:
  - $H_0(K) \cong Z$;
  - $H_1(X) = \{0\}$;
  - $H_2(K) = \{0\}$.
- The trivial groups $H_1(X)$ and $H_2(K)$ indicate the absence of holes in the polyhedron $|K|$.
- As we shall see later, the fact that $H_0(K)$ is isomorphic to $\mathbb{Z}$ indicates that $|K|$ has one component.

Example II (Möbius Strip)

- Let M denote the triangulation of the Möbius strip.



- Assume the orientation induced by

$$a_0 < a_1 < a_2 < a_3 < a_4 < a_5.$$

- There are no 3-simplexes in M.
- So $B_2(M) = \{0\}$.

Example II (2-Homology)

- Consider a 2-cycle

$$\begin{aligned} w &= g_0 \cdot \langle a_1 a_3 a_4 \rangle + g_1 \cdot \langle a_0 a_1 a_4 \rangle + g_2 \cdot \langle a_1 a_4 a_5 \rangle \\ & g_3 \cdot \langle a_1 a_2 a_5 \rangle + g_4 \cdot \langle a_0 a_2 a_5 \rangle + g_5 \cdot \langle a_0 a_2 a_3 \rangle. \end{aligned}$$

- In $\partial(w)$, the coefficient that appears with $\langle a_3 a_4 \rangle$ is g_0.
- To have $\partial(w) = 0$, it must be true that $g_0 = 0$.
- Similar reasoning applied to the other horizontal 1-simplexes shows that each coefficient in w must be 0.
- Thus $Z_2(M) = \{0\}$.
- This shows that $H_2(M) = \{0\}$.

Example II (1-Homology)

- Using a bit of intuition, we suspect that the 1-chains

$$\begin{aligned} z &= 1\cdot\langle a_0a_1\rangle + 1\cdot\langle a_1a_2\rangle + 1\cdot\langle a_2a_3\rangle - 1\cdot\langle a_0a_3\rangle, \\ z' &= 1\cdot\langle a_0a_3\rangle + 1\cdot\langle a_3a_4\rangle + 1\cdot\langle a_4a_5\rangle - 1\cdot\langle a_0a_5\rangle \end{aligned}$$

  are 1-cycles (both make complete circuits beginning at a_0).
- Direct computation verifies that z and z' are cycles.
- However, $z - z'$ traverses the boundary of M.
- So $z - z'$ should be the boundary of some 2-chain.
- A straightforward computation shows that

$$\begin{aligned} z - z' = {} & \partial(1\cdot\langle a_0a_1a_4\rangle + 1\cdot\langle a_1a_2a_5\rangle + 1\cdot\langle a_0a_2a_3\rangle \\ & -1\cdot\langle a_0a_2a_5\rangle - 1\cdot\langle a_1a_4a_5\rangle - 1\cdot\langle a_0a_3a_4\rangle). \end{aligned}$$

- So $z \sim z'$.

Example II (1-Homology Cont'd))

- A similar calculation verifies that any 1-cycle is homologous to a multiple of z.

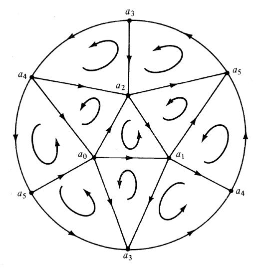
  Hence,
  $$H_1(M) = \{[g \cdot z] : g \text{ is an integer}\}.$$
- So $H_1(M) \cong \mathbb{Z}$.
- This result indicates that the polyhedron $|M|$ has one hole bounded by 1-simplexes.

Example II (0-Homology)

- Observe that any two elementary 0-chains $1 \cdot \langle a_i \rangle$ and $1 \cdot \langle a_j \rangle$ (i, j range from 0 to 5) are homologous.
- E.g.,
$$1 \cdot \langle a_5 \rangle - 1 \cdot \langle a_0 \rangle = \partial(1 \cdot \langle a_0 a_4 \rangle + 1 \cdot \langle a_4 a_5 \rangle).$$
- Hence,
$$H_0(M) = \{[g \cdot \langle a_0 \rangle] : g \text{ is an integer}\}.$$
- So $H_0(M) \cong \mathbb{Z}$.
- This indicates that $|M|$ has only one component.

Example III (The Projective Plane)

- The projective plane is obtained from a finite disk by identifying each pair of diametrically opposite points.
- A triangulation P of the projective plane, with orientations indicated by the arrows, is shown in the figure.



- To compute $Z_2(P)$, observe that each 1-simplex σ^1 of P is a face of exactly two 2-simplexes σ_1^2 and σ_2^2.
  - When σ^1 is $\langle a_3 a_4 \rangle, \langle a_4 a_5 \rangle$ or $\langle a_5 a_3 \rangle$, both incidence numbers $[\sigma_1^2, \sigma^1]$ and $[\sigma_2^2, \sigma^1]$ are $+1$.
  - For all other choices of σ^1, the two incidence numbers are negatives of each other.

Example III (2-Homology)

- Let us call $\langle a_3 a_4 \rangle, \langle a_4 a_5 \rangle$ and $\langle a_5 a_3 \rangle$ 1-simplexes of type I and the others 1-simplexes of type II.
- Suppose that w is a 2-cycle.
- In order for the coefficients of the type II 1-simplexes in $\partial(w)$ to be 0, all the coefficients in w must have a common value, say g.
- But then

$$\partial(w) = 2g \cdot \langle a_3 a_4 \rangle + 2g \cdot \langle a_4 a_5 \rangle + 2g \cdot \langle a_5 a_3 \rangle,$$

  since both incidence numbers for the type I 1-simplexes are +1.
- Hence w is a 2-cycle only when $g = 0$.
- So $Z_2(P) = \{0\}$.
- Thus, $H_2(P) = \{0\}$.

Example III (Cont'd)

- Observe that any 1-cycle is homologous to a multiple of

$$z = 1 \cdot \langle a_3 a_4 \rangle + 1 \cdot \langle a_4 a_5 \rangle + 1 \cdot \langle a_5 a_3 \rangle.$$

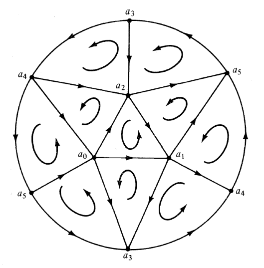
- Furthermore, the previous equation shows that any even multiple of z is a boundary.
- Thus $H_1(P) \cong \mathbb{Z}_2$, the group of integers modulo 2.
- This result indicates the twisting that occurs around the "hole" in the polyhedron $|P|$.

Chains in terms of Negatively Oriented Simplexes

- In the computation of homology groups, it is sometimes convenient to express an elementary chain in terms of a negatively oriented simplex.
- We use $g \cdot (-\sigma^p)$ to denote the elementary p-chain $-g \cdot \sigma^p$.
- In other words, if $\langle a_0 \dots a_p \rangle$ represents a positively or negatively oriented p-simplex, then $g \cdot \langle a_0 \dots a_p \rangle$ denotes the elementary p-chain which assigns value:
  - g to the orientation determined by the class of even permutations of the given ordering;
  - $-g$ to the orientation determined by the class of odd permutations of the given ordering.

Example

- Consider the Projective Plane triangulation.
- $\langle a_5 a_3 \rangle$ denotes a positively oriented 1-simplex.
- The symbols $g \cdot \langle a_5 a_3 \rangle$ and $-g \cdot \langle a_3 a_5 \rangle$ now denote the same elementary 1-chain.



- An elementary 2-chain $h \cdot \langle a_0 a_1 a_2 \rangle$ may be written in any of six ways:

$$h \cdot \langle a_0 a_1 a_2 \rangle = h \cdot \langle a_1 a_2 a_0 \rangle = h \cdot \langle a_2 a_0 a_1 \rangle$$
$$= -h \cdot \langle a_1 a_0 a_2 \rangle = -h \cdot \langle a_0 a_2 a_1 \rangle = -h \cdot \langle a_2 a_1 a_0 \rangle.$$

Subsection 3

The Structure of Homology Groups

Form of Chains, Cycles and Boundaries

- Suppose that K has α_p p-simplexes.
- Then $C_p(K)$ is isomorphic to $\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}$ (α_p summands).
- So $C_p(K)$ is a free abelian group on α_p generators.
- Every subgroup of a free abelian group is a free abelian group.
- So $Z_p(K)$ and $B_p(K)$ are both free abelian groups.

Structure of the Homology Groups

- The quotient group

$$H_p(K) = Z_p(K)/B_p(K)$$

may not be free.

- Its structure is given by the Decomposition Theorem for Finitely Generated Abelian Groups,

$$H_p(K) = G \oplus T_1 \oplus \cdots \oplus T_m,$$

where:
  - G is a free abelian group;
  - Each T_i is a finite cyclic group.
- The direct sum

$$T_1 \oplus \cdots \oplus T_m$$

is called the **torsion subgroup** of $H_p(K)$.
- The torsion subgroup describes the "twisting" in the polyhedron $|K|$.

Homology and Orientations

Theorem

Let K be a geometric complex with two orientations, and let K_1, K_2 denote the resulting oriented geometric complexes. Then, for each dimension p, the homology groups $H_p(K_1)$ and $H_p(K_2)$ are isomorphic.

- For a p-simplex σ^p of K, let ${}^i\sigma^p$ denote the positive orientation of σ^p in the complex K_i, $i = 1, 2$. Then there is a function α defined on the simplexes of K, such that $\alpha(\sigma^p)$ is ± 1 and

$${}^1\sigma^p = \alpha(\sigma^p)\ {}^2\sigma^p.$$

Define a sequence $\varphi_p = \{\varphi_p\}$ of homomorphisms $\varphi_p : C_p(K_1) \to C_p(K_2)$ by

$$\varphi_p\left(\sum g_i \cdot {}^1\sigma_i^p\right) = \sum \alpha(\sigma_1^p) g_i \cdot {}^2\sigma_i^p,$$

where $\sum g_i \cdot {}^1\sigma_i^p$ represents a p-chain on K_1.

Homology and Orientations (Cont'd)

- For an elementary p-chain $g \cdot {}^1\sigma^p$ on K_1 with $p \geq 1$,

$$\begin{array}{rcl}
\varphi_{p-1}(\partial(g \cdot {}^1\sigma^p)) & = & \varphi_{p-1}(\sum_{\sigma^{p-1} \in K} g[{}^1\sigma^p, {}^1\sigma^{p-1}] \cdot {}^1\sigma^{p-1}) \\
& = & \sum_{\sigma^{p-1} \in K} \alpha(\sigma^{p-1}) g[{}^1\sigma^p, {}^1\sigma^{p-1}] \cdot {}^2\sigma^{p-1} \\
& = & \sum_{\sigma^{p-1} \in K} \alpha(\sigma^{p-1}) g \alpha(\sigma^{p-1}) \alpha(\sigma^p) [{}^2\sigma^p, {}^2\sigma^{p-1}] \cdot {}^2\sigma^{p-1} \\
& = & \alpha(\sigma^p) g \sum_{\sigma^{p-1} \in K} [{}^2\sigma^p, {}^2\sigma^{p-1}] \cdot {}^2\sigma^{p-1} \\
& = & \partial(\alpha(\sigma^p) g \cdot {}^2\sigma^p) \\
& = & \partial \varphi_p(g \cdot {}^1\sigma^p).
\end{array}$$

Thus the relation $\varphi_{p-1}\partial = \partial\varphi_p$ holds in the diagram

$$\begin{array}{ccc}
Cp(K_1) & \xrightarrow{\varphi_p} & C_p(K_2) \\
\partial \downarrow & & \downarrow \partial \\
C_{p-1}(K_1) & \xrightarrow[\varphi_{p-1}]{} & C_{p-1}(K_2)
\end{array}$$

Homology and Orientations (Cont'd)

- If $z_p \in Z_p(K_1)$, then

$$\partial\varphi_p(z_p) = \varphi_{p-1}\partial(z_p) = \varphi_{p-1}(0) = 0.$$

So $\varphi_p(z_p) \in Z_p(K_2)$. Hence, $\varphi_p(Z_p(K_1))$ is a subset of $Z_p(K_2)$.
If $\partial(c_{p+1}) \in B_p(K_1)$, then

$$\varphi_p\partial(c_{p+1}) = \partial\varphi_{p+1}(c_{p+1}).$$

So $\varphi_p\partial(c_{p+1})$ is in $B_p(K_2)$. Thus, φ_p maps $B_p(K_1)$ into $B_p(K_2)$.
Hence, φ_p induces a homomorphism φ_p^* from the quotient group $H_p(K_1) = Z_p(K_1)/B_p(K_1)$ to $H_p(K_2) = Z_p(K_2)/B_p(K_2)$ defined by

$$\varphi_p^*([z_p]) = [\varphi_p(z_p)],$$

for each homology class $[z_p]$ in $H_p(K_1)$.

Homology and Orientations (Conclusion)

- Now reverse the roles of K_1 and K_2.

  We, thus, get a sequence $\psi = \{\psi_p\}$ of homomorphisms

  $$\psi_p : C_p(K_2) \to C_p(K_1),$$

  such that φ_p and ψ_p are inverses of each other for each p.

  This implies that ψ_p^* is the inverse of φ_p^*.

  Hence,

  $$\varphi_p^* : H_p(K_1) \to H_p(K_2)$$

  is an isomorphism for each dimension p.

Connectivity and Combinatorial Components

- We next show that there is no torsion in dimension zero.
- Moreover, the rank of the free abelian group $H_0(K)$ is the number of components of the polyhedron $|K|$.

Definition

Let K be a complex. Two simplexes s_1 and s_2 are **connected** if either of the following conditions is satisfied:

(a) $s_1 \cap s_2 \neq \emptyset$;

(b) There is a sequence $\sigma_1, \ldots, \sigma_p$ of 1-simplexes of K, such that $s_1 \cap \sigma_1$ is a vertex of s_1, $s_2 \cap \sigma_p$ is a vertex of s_2 and, for $1 \leq i < p$, $\sigma_i \cap \sigma_{i+1}$ is a common vertex of σ_1 and σ_{i+1}.

This concept of connectedness is an equivalence relation.
Its equivalence classes are called the **combinatorial components** of K.
K is said to be **connected** if it has only one combinatorial component.

$H_0(K)$ and Number of Combinatorial Components

Theorem

Let K be a complex with r combinatorial components. Then $H_0(K)$ is isomorphic to the direct sum of r copies of the group $\mathbb{Z}$ of integers.

- Let K' be a combinatorial component of K.

  Let $\langle a' \rangle$ a 0-simplex in K'.

  By assumption, given any 0-simplex $\langle b \rangle$ in K', there is a sequence of 1-simplexes $\langle ba_0 \rangle, \langle a_0 a_1 \rangle, \langle a_1 a_2 \rangle, \ldots, \langle a_p a' \rangle$ from b to a', such that each two successive 1-simplexes have a common vertex.

  If g is an integer, we define a 1-chain c_1 on the sequence of 1-simplexes by assigning either g or $-g$ to each simplex (depending on orientation) so that $\partial(c_1)$ is $g \cdot \langle b \rangle - g \cdot \langle a' \rangle$ or $g \cdot \langle b \rangle + g \cdot \langle a' \rangle$.

$H_0(K)$ and Number of Combinatorial Components (Cont'd)

- Hence, any elementary 0-chain $g \cdot \langle b \rangle$ is homologous to one of the 0-chains $g \cdot \langle a' \rangle$ or $-g \cdot \langle a' \rangle$.

  It follows that any 0-chain on K' is homologous to an elementary 0-chain $h \cdot \langle a' \rangle$, where h is some integer.

  Now apply the result to each combinatorial component $K_1, \ldots, K_r$ of K.

  It shows that, there is a vertex a^i of K_i, such that any 0-cycle on K_i is homologous to a 0-chain of the form $h_i \cdot \langle a^i \rangle$, where h_i is an integer.

  So, given any 0-cycle c_0 on K, there are integers $h_1, \ldots, h_r$, such that

$$c_0 \sim \sum_{i=1}^{r} h_i \cdot \langle a^i \rangle.$$

$H_0(K)$ and Number of Combinatorial Components (Cont'd)

- Suppose that two such 0-chains

$$\sum h_i \cdot \langle a^i \rangle \quad \text{and} \quad \sum g_i \cdot \langle a^i \rangle$$

represent the same homology class.

Then $\sum (g_i - h_i)\langle a^i \rangle = \partial(c_1)$, for some 1-chain c_1.

But, for $i \neq j$, a^i and a^j belong to different combinatorial components.

So the last equation is impossible unless $g_i = h_i$, for each i.

Hence, each homology class $[c_0]$ in $H_0(K)$ has a unique representative of the form $\sum h_i \cdot \langle a^i \rangle$.

The function

$$\sum h_i \cdot \langle a^i \rangle \to (h_1, \ldots, h_r)$$

is the required isomorphism between $H_0(K)$ and the direct sum of r copies of $\mathbb{Z}$.

Number of Components of Polyhedra

- The components of $|K|$ and the geometric carriers of the combinatorial components of K are identical.
- So we obtain the following

Corollary

If a polyhedron $|K|$ has r components and triangulation K, then $H_0(K)$ is isomorphic to the direct sum of r copies of $\mathbb{Z}$.

Subsection 4

The Euler-Poincaré Theorem

Linear Independence with Respect to Homology

Definition

Let K be an oriented complex. A family $\{z_p^1, \ldots, z_p^r\}$ of p-cycles is **linearly independent with respect to homology**, or **linearly independent mod** $B_p(K)$, means that there do not exist integers $g_1, \ldots, g_r$ not all zero, such that the chain $\sum g_i z_p^i$ is homologous to 0.

The largest integer r for which there exist r p-cycles linearly independent with respect to homology is denoted by $R_p(K)$ and called the p**-th Betti number** of the complex K.

- In the theorem that follows, we assume that the coefficient group has been chosen to be the rational numbers and not the integers.
  - Linear independence with integral coefficients is equivalent to linear independence with rational coefficients.
  - Moreover, this change does not alter the values of the Betti numbers.

The Euler-Poincaré Theorem

Theorem (The Euler-Poincaré Theorem)

Let K be an oriented geometric complex of dimension n. For $p = 0, 1, \ldots, n$, let α_p denote the number of p-simplexes of K. Then

$$\sum_{p=0}^{n} (-1)^p \alpha_p = \sum_{p=0}^{n} (-1)^p R_p(K),$$

where $R_p(K)$ denotes the p-th Betti number of K.

- Since K is the only complex under consideration, the notation will be simplified by omitting reference to it in the group notations.

  C_p, Z_p, and B_p are vector spaces over the field of rational numbers.

  Let $\{d_p^i\}$ be a maximal set of p-chains such that no proper linear combination of the d_p^i is a cycle.

  Let D_p be the linear subspace of C_p spanned by $\{d_p^i\}$.

The Euler-Poincaré Theorem (Cont'd)

- By definition of D_p, $D_p \cap Z_p = \{0\}$.

  So, as a vector space, C_p is the direct sum of Z_p and D_p.

  Now we have

  $$\alpha_p = \dim C_p = \dim D_p + \dim Z_p.$$

  That is,

  $$\dim Z_p = \alpha_p - \dim D_p, \quad 1 \leq p \leq n,$$

  where dim denotes vector space dimension.

  For $p = 0, \ldots, n-1$, let $b_p^i = \partial(d_{p+1}^i)$.

  The set $\{b_p^i\}$ forms a basis for B_p.

  Let

  $$\{z_p^i\}, \quad i = 1, \ldots, R_p,$$

  be a maximal set of p-cycles linearly independent mod B_p.

The Euler-Poincaré Theorem (Cont'd)

- The cycles in $\{z_p^i\}$ span a subspace G_p of Z_p, and

$$Z_p = G_p \oplus B_p, \quad 0 \le p \le n-1.$$

But $R_p = \dim G_p$.
So we get

$$\dim Z_p = \dim G_p + \dim B_p = R_p + \dim B_p.$$

Then

$$R_p = \dim Z_p - \dim B_p = \alpha_p - \dim D_p - \dim B_p, \quad 1 \le p \le n-1.$$

Observe that B_p is spanned by the boundaries of elementary chains

$$\partial(1 \cdot \sigma_i^{p+1}) = \sum \eta_{ij}(p) \cdot \sigma_j^p,$$

where $(\eta_{ij}(p)) = \eta(p)$ is the p-th incidence matrix.
Thus, $\dim B_p = \operatorname{rank} \eta(p)$.

The Euler-Poincaré Theorem (Conclusion)

Since the number of d^i_{p+1} is the same as the number of b^i_p,

$$\dim D_{p+1} = \dim B_p = \text{rank}\eta(p), \quad 0 \le p \le n-1.$$

Then

$$\begin{aligned} R_p &= \alpha_p - \dim D_p - \dim B_p \\ &= \alpha_p - \text{rank}\eta(p-1) - \text{rank}\eta(p), \quad 1 \le p \le n-1. \end{aligned}$$

Note also that:

$$\begin{aligned} R_0 &= \dim Z_0 - \dim B_0 = \alpha_0 - \text{rank}\eta(0); \\ R_n &= \dim Z_n = \alpha_n - \dim D_n = \alpha_n - \text{rank}\eta(n-1). \end{aligned}$$

In the alternating sum $\sum_{p=0}^n (-1)^p R_p$, all the terms $\text{rank}\eta(p)$ cancel. So we obtain $\sum_{p=0}^n (-1)^p R_p = \sum_{p=0}^n (-1)^p \alpha_p$.

The Euler Characteristic

Definition

If K is a complex of dimension n, the number

$$\chi(K) = \sum_{p=0}^{n} (-1)^p R_p$$

is called the **Euler characteristic** of K.

- Chains, cycles, boundaries, the homology relation, and Betti numbers were defined by Poincaré in his paper *Analysis Situs* in 1895.
- The proof of the Euler-Poincaré Theorem given is essentially Poincaré's original one.
- Complexes (in slightly different form) and incidence numbers were defined in *Complément à l'Analysis Situs* in 1899.

Betti Numbers and Topological Invariance

- The Betti numbers were named for Enrico Betti (1823-1892).
- J. W. Alexander (1888-1971) in 1915 showed that the Betti numbers are topological invariants.
- This means that, if the geometric carriers $|K|$ and $|L|$ are homeomorphic, then $R_p(K) = R_p(L)$ in each dimension p.
- Topological invariance of the homology groups was proved by Oswald Veblen in 1922.
- These results ensure that the homology characters $H_p(|K|), R_p(|K|)$ and $\chi(|K|)$ are well-defined, i.e., independent of the triangulation of the polyhedron $|K|$.
- The p-th Betti number $R_p(K)$ of a complex K is the rank of the free part of the p-th homology group $H_p(K)$.
- The p-th Betti number indicates the number of "p-dimensional holes" in the polyhedron $|K|$.

Rectilinear Polyhedra

Definition

A **rectilinear polyhedron** in Euclidean 3-space $\mathbb{R}^3$ is a solid bounded by properly joined convex polygons.

- The bounding polygons are called **faces**.
- The intersections of the faces are called **edges**.
- The intersections of the edges are called **vertices**.

A **simple polyhedron** is a rectilinear polyhedron whose boundary is homeomorphic to the 2-sphere S^2.

A **regular polyhedron** is a rectilinear polyhedron whose faces are regular plane polygons and whose polyhedral angles are congruent.

Betti Numbers and Euler Characteristic of Sphere

- The tetrahedron forms a triangulation of S^2.
- Using brute force, we may compute

$$H_2 = \mathbb{Z}, \quad H_1 = 0, \quad H_0 = \mathbb{Z}.$$

- So the Betti numbers of the 2-sphere S^2 are

$$R_0(S^2) = 1, \quad R_1(S^2) = 0, \quad R_2(S^2) = 1.$$

- Thus, S^2 has Euler characteristic

$$\chi(S^2) = \sum_{p=0}^{2} (-1)^p R_p(S^2) = 1 - 0 + 1 = 2.$$

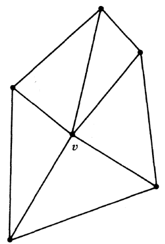
Euler's Theorem

Theorem (Euler's Theorem)

If S is a simple polyhedron with V vertices, E edges and F faces, then $V - E + F = 2$.

- S may have some non-triangular faces.
  This situation can be dealt with as follows.
  Consider a face τ of S having n_0 vertices and n_1 edges.
  Computing *vertices* – *edges* + *faces* gives $n_0 - n_1 + 1$ for τ.

  Choose a new vertex v in the interior of τ.
  Join the new vertex to each of the original vertices by a line segment as illustrated in the figure.
  In the triangulation of τ:
  - One new vertex and n_0 new edges were added;
  - Face τ is replaced by n_0 new faces.



τ Triangulated

Euler's Theorem (Cont'd)

- Then

$$\text{vertices} - \text{edges} + \text{faces} = (n_0 + 1) - (n_1 + n_0) + n_0 = n_0 - n_1 + 1.$$

So the sum $V - E + F$ is not changed in the triangulation process.

Let α_i, $i = 0, 1, 2$, denote the number of i-simplexes in the triangulation of S obtained in this way.

BY the preceding argument,

$$V - E + F = \alpha_0 - \alpha_1 + \alpha_2.$$

The Euler-Poincaré Theorem shows that

$$\alpha_0 - \alpha_1 + \alpha_2 = R_0(S^2) - R_1(S^2) + R_2(S^2) = 2.$$

Hence $F - E + F = 2$ for any simple polyhedron.

The Regular Simple Polyhedra

Theorem

There are only five regular, simple polyhedra.

- Suppose S is such a polyhedron with V vertices, E edges and F faces. Denote by:
  - m the number of edges meeting at each vertex;
  - n the number of edges of each face.

  We have the following:
  - $n \geq 3$;
  - $mV = 2E = nF$;
  - $V - E + F = 2$.

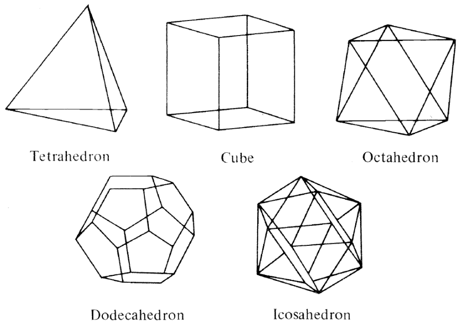
  So we obtain
  $$\frac{nF}{m} - \frac{nF}{2} + F = 2.$$
  Equivalently,
  $$F(2n - mn + 2m) = 4m.$$

The Regular Simple Polyhedra (Cont'd)

- We got $F(2n-mn+2m)=4m$.
  It must be true that $2n-mn+2m>0$.
  Since $n\geq 3$, this gives
  $$2m>n(m-2)\geq 3(m-2)=3m-6 \;\Rightarrow\; m<6.$$
  Thus m can only be 1, 2, 3, 4, or 5.
  The three relations
  $$F(2n-mn+2m)=4m, \quad n\geq 3, \quad m<6,$$
  produce the following possible values for (m,n,F):
  (a) $(3,3,4)$;
  (b) $(3,4,6)$;
  (c) $(4,3,8)$;
  (d) $(3,5,12)$;
  (e) $(5,3,20)$.
  For example, $m=4$ implies $F(8-2n)=16$ implies $F=8, n=3$.

The Regular Simple Polyhedra (Cont'd)

- The five possibilities for (m, n, F) are realized as follows:
  - $(3, 3, 4)$ in the tetrahedron;
  - $(3, 4, 6)$ in the cube;
  - $(4, 3, 8)$ in the octahedron;
  - $(3, 5, 12)$ in the dodecahedron;
  - $(5, 3, 20)$ in the icosahedron.



Subsection 5

Pseudomanifolds and the Homology Groups of S^n

Manifolds

- "Manifolds" can be traced to the work of G. F. B. Riemann (1826-1866) on differentials and multivalued functions.
- A manifold is a generalization of an ordinary surface like a sphere or a torus and its primary characteristic is its "local" Euclidean structure.

Definition

An n-**dimensional manifold**, or n-**manifold**, is a compact, connected Hausdorff space each of whose points has a neighborhood homeomorphic to an open ball in Euclidean n-space $\mathbb{R}^n$.

Pseudomanifolds

Definition

An n-**pseudomanifold** is a complex K with the following properties:

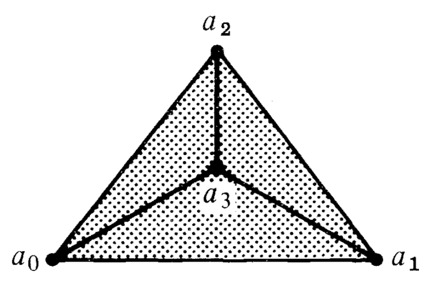
(a) Each simplex of K is a face of some n-simplex of K.

(b) Each $(n-1)$-simplex is a face of exactly two n-simplexes of K.

(c) Given a pair σ_1^n and σ_2^n of n-simplexes of K, there is a sequence of n-simplexes beginning with σ_1^n and ending with σ_2^n, such that any two successive terms of the sequence have a common $(n-1)$-face.

Examples

(a) Consider the complex K consisting of all proper faces of a 3-simplex $\langle a_0 a_1 a_2 a_3 \rangle$.

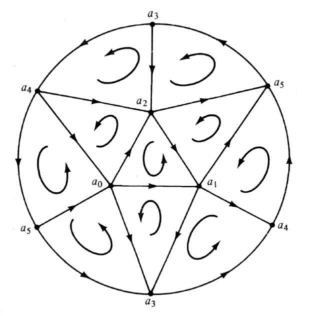


It is a 2-pseudomanifold.

Moreover, it is a triangulation of the 2-sphere S^2.

Examples (Cont'd)

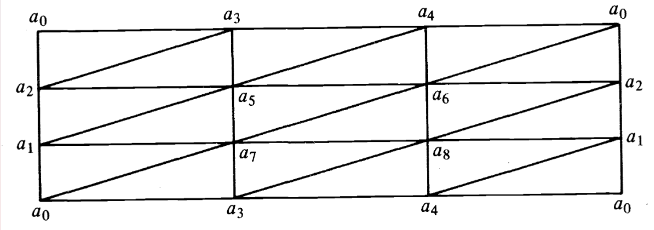
(b) Consider the triangulation of the projective plane.



It is also a 2-pseudomanifold.

Examples (Cont'd)

(c) Consider the triangulation of the torus.

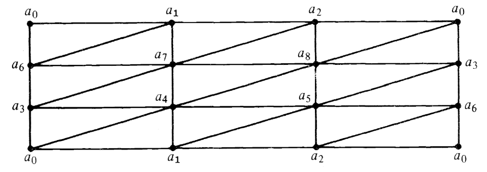


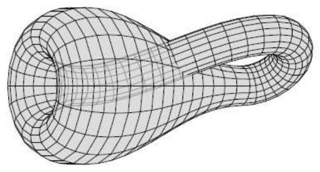
It is a 2-pseudomanifold.

Examples (Cont'd)

(d) The **Klein Bottle** is constructed from a cylinder by identifying opposite ends with the orientations of the circles reversed.

Consider a triangulation of the Klein Bottle shown on the left.





It is a 2-pseudomanifold.

The Klein Bottle cannot be embedded in Euclidean 3-space without self-intersection.

Allowing self-intersection, we get the figure on the right.

Remarks on Terminology

- Each space in the preceding examples is a 2-manifold.
- The n-sphere S^n, $n \geq 1$, is an n-manifold.
- This indicates why the unit sphere in $\mathbb{R}^{n+1}$ is called the "n-sphere" and not the "$(n+1)$-sphere".
- The integer n refers to the local dimension as a manifold and not to the dimension of the containing Euclidean space.
- Note that:
  - Each point of a circle has a neighborhood homeomorphic to an open interval in $\mathbb{R}$;
  - Each point of S^2 has a neighborhood homeomorphic to an open disk in $\mathbb{R}^2$;

    $\vdots$

Manifolds versus Pseudomanifolds

- The relation between manifold (a type of topological space) and pseudomanifold (a type of geometric complex) is as follows.
  - If X is a triangulable n-manifold, then each triangulation K of X is an n-pseudomanifold.
  - The homology groups of the pseudomanifold K reflect the connectivity, the "holes" and "twisting", of the associated manifold X.
- The computation of homology groups of pseudomanifolds is thus a worthwhile project.
- If X is a space each of whose triangulations is a pseudomanifold, it is sometimes said that "X is a pseudomanifold".
  - A space and a triangulation of the space are different.
    So this is an abuse of language.
  - In some situations, such as in the computation of homology groups, the distinction between space and complex is not important.

Number of Simplexes in 2-Pseudomanifolds

Theorem

Let K be a 2-pseudomanifold with α_0 vertices, α_1 1-simplexes, and α_2 2-simplexes. Then:

(a) $3\alpha_2 = 2\alpha_1$;

(b) $\alpha_1 = 3(\alpha_0 - \chi(K))$;

(c) $\alpha_0 \geq \frac{1}{2}(7 + \sqrt{49 - 24\chi(K)})$.

- Each 1-simplex is a face of exactly two 2-simplexes.

  Each 2-simplex has exactly 3 1-simplexes as 1-faces.

  It follows that $3\alpha_2 = 2\alpha_1$. Hence $\alpha_2 = \frac{2}{3}\alpha_1$.

  The Euler-Poincaré Theorem gives $\alpha_0 - \alpha_1 + \alpha_2 = \chi(K)$.

  Then $\alpha_0 - \alpha_1 + \frac{2}{3}\alpha_1 = \chi(K)$. Hence, $\alpha_1 = 3(\alpha_0 - \chi(K))$.

Number of Simplexes in 2-Pseudomanifolds (Part (c))

- To prove (c), note that $\alpha_0 \geq 4$ and that $\alpha_1 \leq \binom{\alpha_0}{2} = \frac{1}{2}\alpha_0(\alpha_0 - 1)$. By elementary algebra,

$$\begin{array}{rcl} 3\alpha_2 & = & 2\alpha_1 \\ 6\alpha_2 & = & 4\alpha_1 \\ 2\alpha_1 & = & 6\alpha_1 - 6\alpha_2 \\ \alpha_0(\alpha_0 - 1) & \geq & 6\alpha_1 - 6\alpha_2 \\ \alpha_0^2 - \alpha_0 - 6\alpha_0 & \geq & 6\alpha_1 - 6\alpha_2 - 6\alpha_0 = -6\chi(K) \\ 4\alpha_0^2 - 28\alpha_0 + 49 & \geq & 49 - 24\chi(K) \\ (2\alpha_0 - 7)^2 & \geq & 49 - 24\chi(K) \\ \alpha_0 & \geq & \frac{1}{2}(7 + \sqrt{49 - 24\chi(K)}). \end{array}$$

Example (The 2-Sphere)

- We use the theorem to determine the 2-pseudomanifold triangulation of a polyhedron having the minimum number of simplexes in each dimension.

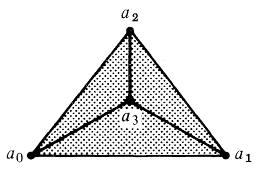
  Example: Consider the 2-sphere S^2. We know $\chi(S^2) = 2$.

  So we get

$$\begin{aligned} \alpha_0 &\geq \tfrac{1}{2}(7 + \sqrt{49 - 24\chi(K)}) = 4; \\ \alpha_1 &= 3(\alpha_0 - \chi(K)) \geq 3(4-2) = 6; \\ \alpha_2 &= \tfrac{2}{3}\alpha_1 \geq \tfrac{2}{3} \cdot 6 = 4. \end{aligned}$$

  Hence any triangulation of S^2 must have at least four vertices, at least six 1-simplexes, and at least four 2-simplexes.

  This minimal triangulation is achieved by the boundary complex of a tetrahedron (proper faces of a 3-simplex).



Example: The Projective Plane

- Consider the projective plane P, a 2-manifold.

  We know that $H_2(P) = \{0\}$, $H_1(P) \cong \mathbb{Z}_2$.

  Since P is connected, by a preceding theorem, $H_0(P) \cong \mathbb{Z}$.

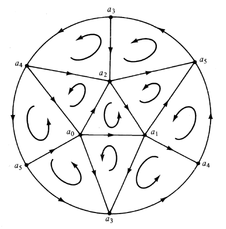
  So we have $R_2(P) = R_1(P) = 0$, $R_0(P) = 1$. Hence, $\chi(P) = 1$.

  This gives

$$\begin{aligned} \alpha_0 &\geq \tfrac{1}{2}(7 + \sqrt{49 - 24\chi(P)}) = 6; \\ \alpha_1 &\geq 3(6-1) = 15; \\ \alpha_2 &\geq \tfrac{2}{3} \cdot 5 = 10. \end{aligned}$$

  Thus, any triangulation of P must have at least six vertices, fifteen 1-simplexes, and ten 2-simplexes.

  So the triangulation shown is minimal.



Orientable and Nonorientable Pseudomanifolds

Definition

Let K be an n-pseudomanifold. For each $(n-1)$-simplex σ^{n-1} of K, let σ_1^n and σ_2^n denote the two n-simplexes of which σ^{n-1} is a face. An orientation for K having the property

$$[\sigma_1^n, \sigma^{n-1}] = -[\sigma_2^n, \sigma^{n-1}],$$

for each $(n-1)$-simplex σ^{n-1} of K is a **coherent orientation**.
An n-pseudomanifold is **orientable** if it can be assigned a coherent orientation. Otherwise, it is **nonorientable**.

- Orientability is a topological property of the underlying polyhedron $|K|$ and is not dependent on the particular triangulation K.
- We shall assume this statement without proof.

Examples of Orientable and Nonorientable Pseudomanifolds

- Orientable pseudomanifolds include:
  - The 2-sphere;
  - The torus.
- Nonorientable pseudomanifolds include:
  - The projective plane;
  - The Klein Bottle.

Orienting the n-Skeleton of an $(n+1)$-Simplex

- Let σ^{n+1} be an $(n+1)$-simplex in $\mathbb{R}^{n+1}$, $n \geq 1$.
- Denote by K the n-skeleton of the closure of σ^{n+1}.
- Then K is an n-pseudomanifold.
- Moreover, it is a triangulation of the n-sphere S^n.
- The following notation is used only in this example.
- For an integer j, with $0 \leq j \leq n+1$, let

$$\sigma_j = \langle a_0 \ldots \widehat{a}_j \ldots a_{n+1} \rangle,$$

  where the symbol $\widehat{a}_j$ indicates that the vertex a_j is deleted.
- The positively oriented simplex $+\sigma_j$ has:
  - The given ordering when j is even;
  - The opposite ordering (an odd permutation of the given ordering) when j is odd.
- The $(n-1)$-simplex $+\sigma_{ij} = +\langle a_0 \ldots \widehat{a}_i \ldots \widehat{a}_j \ldots a_{n+1} \rangle$ is then a face of the two n-simplexes σ_i and σ_j.

Orienting the n-Skeleton of an $(n+1)$-Simplex (Cont'd)

Claim: This orientation for the n-simplexes and $(n-1)$-simplexes gives $[\sigma_i, \sigma_{ij}] = -[\sigma_j, \sigma_{ij}]$ in each case.

Let us look at two examples regarding $\langle a_0 a_1 a_2 a_3 a_4 a_5 \rangle$ in $\mathbb{R}^6$.

We calculate the following:

$$[\sigma_1, \sigma_{14}] = \text{sgn}\langle a_4 a_0 a_2 a_3 a_5 \rangle = -1 = +\langle a_0 a_2 a_3 a_4 a_5 \rangle;$$
$$[\sigma_4, \sigma_{14}] = \text{sgn}\langle a_1 a_0 a_2 a_3 a_5 \rangle = -1 = -\langle a_0 a_1 a_2 a_3 a_5 \rangle;$$
$$[\sigma_2, \sigma_{24}] = \text{sgn}\langle a_4 a_0 a_1 a_3 a_5 \rangle = -1 = -\langle a_0 a_1 a_3 a_4 a_5 \rangle;$$
$$[\sigma_4, \sigma_{24}] = \text{sgn}\langle a_2 a_0 a_1 a_3 a_5 \rangle = +1 = +\langle a_0 a_1 a_2 a_3 a_5 \rangle.$$

We may check that

$$\begin{aligned} [\sigma_1, \sigma_{14}] &= -[\sigma_4, \sigma_{14}]; \\ [\sigma_2, \sigma_{24}] &= -[\sigma_4, \sigma_{24}]. \end{aligned}$$

Orienting the n-Skeleton of an $(n+1)$-Simplex (Cont'd)

- It follows that any n-chain of the form

$$\sum_{\sigma_i \in K} g \cdot \sigma_i, \quad g \text{ an integer},$$

is an n-cycle.

Furthermore, suppose $z = \sum_{\sigma_i \in K} g \cdot \sigma_i$ is an n-cycle.

Then

$$0 = \partial(z) = \sum_{\sigma_{ij} \in K} h_{ij} \cdot \sigma_{ij},$$

where h_{ij} is either $g_i - g_j$ or $g_j - g_i$.

Hence, z is an n-cycle if and only if all g_i are equal.

Thus $Z_n(S^n) \in \mathbb{Z}$.

Since $B_n(S^n) = \{0\}$, we get $H_n(S^n) \cong \mathbb{Z}$.

The Homology Groups of S^n

Theorem

The homology groups of the n-sphere, $n \geq 1$, are

$$H_p(S^n) \cong \begin{cases} \mathbb{Z}, & \text{if } p = 0 \text{ or } p = n \\ \{0\}, & \text{if } 0 < p < n \end{cases}$$

- Since S^n is connected, a preceding theorem implies that $H_0(S^n) \cong \mathbb{Z}$.
  The above example shows that $H_n(S^n) \cong \mathbb{Z}$.
  In handling the case $0 < p < n$, we use the following notation.
  Suppose $+\sigma^p = \langle a_0 \dots a_p \rangle$ and v is a vertex for which the set $\{v, a_0, \dots, a_p\}$ is geometrically independent.
  Then $v\sigma^p$ denotes the positively oriented $(p+1)$-simplex $+\langle va_0 \dots a_p \rangle$.
  For $c = \sum g_i \cdot \sigma_i^p$ a p-chain, vc denotes the $(p+1)$-chain $vc = \sum g_i \cdot v\sigma_i^p$.
  Then
  $$\partial(1 \cdot v\sigma^p) = 1 \cdot \sigma^p - v\partial(1 \cdot \sigma^p).$$

The Homology Groups of S^n (Cont'd)

- Let v be a vertex in the triangulation of S^n of the preceding example.

  Any p-simplex containing v can be expressed in the form $v\sigma^{p-1}$.

  So any p-cycle z can be written

  $$z = \sum g_i \cdot \sigma_i^p + \sum h_j \cdot v\sigma_j^{p-1},$$

  where:
  - Simplexes in the second sum have v as a vertex;
  - Simplexes in the first sum do not have v as a vertex.

  Since z is a p-cycle,

  $$\begin{array}{rcl} 0 & = & \partial(z) \\ & = & \partial(\sum g_i \cdot \sigma_i^p) + \partial(\sum h_j \cdot v\sigma_j^{p-1}) \\ & = & \partial(\sum g_i \cdot \sigma_i^p) + \sum h_j \cdot \sigma_j^{p-1} - v(\partial \sum h_j \cdot \sigma_j^{p-1}). \end{array}$$

The Homology Groups of S^n (Cont'd)

- We found
$$\partial(\sum g_i \cdot \sigma_i^p) + \sum h_j \cdot \sigma_j^{p-1} - v(\partial \sum h_j \cdot \sigma_j^{p-1}) = 0.$$
Thus,
$$\begin{aligned} \partial(\sum h_j \cdot \sigma_j^{p-1}) &= 0; \\ \partial(\sum g_i \cdot \sigma_i^p) &= -\sum h_j \cdot \sigma_j^{p-1}. \end{aligned}$$
This gives
$$\begin{aligned} \partial(\sum g_i \cdot v\sigma_i^p) &= \sum g_i \cdot \sigma_i^p - v\partial(\sum g_i \cdot \sigma_i^p) \\ &= \sum g_i \cdot \sigma_i^p + v\sum h_j \cdot \sigma_j^{p-1} \\ &= z. \end{aligned}$$
Thus, every p-cycle on S^n is a boundary.
It follows that $H_p(S^n) = \{0\}$, for $0 < p < n$.

Orientability and Homology Groups

Theorem

An n-pseudomanifold K is orientable if and only if the n-th homology group $H_n(K)$ is not the trivial group.

- Assume first that K is orientable.

  Assign K a coherent orientation.

  If the $(n-1)$-simplex σ^{n-1} is a face of σ_1^n and σ_2^n, we have

$$[\sigma_1^n, \sigma^{n-1}] = -[\sigma_2^n, \sigma^{n-1}].$$

  So any n-chain of the form $c = \sum_{\sigma^n \in K} g \cdot \sigma^n$ is an n-cycle.

  Thus, $Z_n(K) \neq \{0\}$.

  On the other hand, $B_n(K) = \{0\}$.

  Therefore, $H_n(K) \neq \{0\}$.

Orientability and Homology Groups (Cont'd)

Claim: If $H_n(K) \neq \{0\}$, then K is orientable.

Suppose that $z = \sum_{\sigma_i^n \in K} g_i \cdot \sigma_i^n$ is a nonzero n-cycle.

By the definition of a pseudomanifold:

- Each pair of n-simplexes in K can be joined by a sequence of n-simplexes;
- Each $(n-1)$-simplex is a face of exactly two n-simplexes.

So any two coefficients in z can differ only in sign.

That is to say, $g_i = \pm g_0$ if $\partial(z) = 0$.

By reorienting σ_i^n if $g_i = -g_0$, we obtain an n-cycle

$$\sum_{\sigma_i^n \in K} g_0 \cdot \sigma_i^n = g_0 \left(\sum_{\sigma_i^n \in K} 1 \cdot \sigma_i^n \right).$$

It follows that $\sum 1 \cdot \sigma_i^n$ is an n-cycle.

Orientability and Homology Groups (Cont'd)

- We showed that $\sum 1 \cdot \sigma_i^n$ is an n-cycle.

  This means that each $(n-1)$-simplex must have:
  - Positive incidence number with one of the n-simplexes of which it is a face;
  - Negative incidence number with the other of the n-simplexes of which it is a face.

  We conclude that K is orientable.

Corollary

An n-pseudomanifold L is nonorientable if and only if $H_n(L) = \{0\}$.

Closed Surfaces

- A 2-manifold is called a **closed surface**.
- The topological power of the homology groups is demonstrated by the following classification theorem for closed surfaces.

Theorem

Two closed surfaces are homeomorphic if and only if they have the same Betti numbers in corresponding dimensions.

- The proof of the theorem is omitted.
- The proof requires a lengthy digression into the theory of closed surfaces.
- The theorem preceded Poincaré's formalization of algebraic topology.
- It was a motivating force behind Poincaré's work.