Introduction to Algebraic Topology

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 500

Subsection 1

Simplicial Approximation

Chain Mappings

Definition

Let K and L be complexes and $\{\varphi_p\}_0^\infty$ a sequence of homomorphisms $\varphi_p : C_p(K) \to C_p(L)$ such that

$$\partial\varphi_p = \varphi_{p-1}\partial, \quad p \geq 1.$$

Then $\{\varphi_p\}_0^\infty$ is called a **chain mapping** from K into L.

- The sequence $\{\varphi_p\}_0^\infty$ is written as an infinite sequence simply to avoid mention of the dimensions of K and L.
- When p exceeds $\dim K$ and $\dim L$, then $C_p(K)$ and $C_p(L)$ are zero groups and φ_p must be the trivial homomorphism which takes 0 to 0.

Chain Mappings and Homology Mappings

Theorem

A chain mapping $\{\varphi_p\}_0^\infty$ from a complex K into a complex L induces homomorphisms

$$\varphi^* : H_p(K) \to H_p(L)$$

in each dimension p.

- Suppose $b_p = \partial(c_{p+1})$ in $B_p(K)$.

  Then

  $$\varphi_p(b_p) = \varphi_p \partial(c_{p+1}) = \partial \varphi_{p+1}(c_{p+1}).$$

  So $\varphi_p(b_p)$ is the boundary of the $(p+1)$-chain $\varphi_{p+1}(c_{p+1})$.

  Thus, φ_p maps $B_p(K)$ into $B_p(L)$.

Chain Mappings and Homology Mappings (Cont'd)

Claim: φ_p maps $Z_p(K)$ into $Z_p(L)$.

This holds for $p = 0$, since $Z_0(K) = C_0(K)$ and $Z_0(L) = C_0(L)$.

For $p \geq 1$, suppose that $z_p \in Z_p(K)$.

Note that

$$\partial\varphi_p(z_p) = \varphi_{p-1}\partial(z_p) = \varphi_{p-1}(0) = 0.$$

So $\varphi_p(z_p)$ is a p-cycle on L.

Now $H_p(K) = Z_p(K)/B_p(K)$ and $H_p(L) = Z_p(L)/B_p(L)$.

So $\varphi_p^* : H_p(K) \to H_p(L)$ can be defined in the standard way,

$$\varphi_p^*(z_p + B_p(K)) = \varphi_p(z_p) + B_p(L).$$

Equivalently,

$$\varphi_p^*([z_p]) = [\varphi_p(z_p)].$$

Simplicial Mappings

Definition

A **simplicial mapping** from a complex K into a complex L is a function φ from the vertices of K into those of L such that if $\sigma^p = \langle v_0 \ldots v_p \rangle$ is a simplex of K, then the vertices $\varphi(v_i)$, $0 \le i \le p$ (not necessarily distinct) are the vertices of a simplex of L.

If the vertices $\varphi(v_i)$ are all distinct, then the p-simplex

$$\langle \varphi(v_0) \ldots \varphi(v_p) \rangle = \varphi(\sigma^p)$$

is called the **image** of σ^p.

If $\varphi(v_i) = \varphi(v_j)$, for some $i \ne j$, then φ is said to **collapse** σ^p.

Induced Chain Mappings

Definition

Let φ be a simplicial mapping from K into L and p a non-negative integer. If $g \cdot \sigma^p$ is an elementary p-chain on K, define

$$\varphi_p(g \cdot \sigma^p) = \begin{cases} 0, & \text{if } \varphi \text{ collapses } \sigma^p, \\ g \cdot \varphi(\sigma^p), & \text{if } \varphi \text{ does not collapse } \sigma^p. \end{cases}$$

The function φ_p is extended by linearity to a homomorphism $\varphi_p : C_p(K) \to C_p(L)$. I.e., if $\sum g_i \cdot \sigma_i^p$ is a p-chain on K, then

$$\varphi_p(\sum g_i \cdot \sigma_i^p) = \sum \varphi_p(g_i \cdot \sigma_i^p).$$

The sequence $\{\varphi_p\}_0^\infty$ is called the **chain mapping induced by** φ.

Induced Chain Mapping Theorem

Theorem

If $\varphi : K \to L$ is a simplicial mapping, then the sequence $\{\varphi_p\}_0^\infty$ of homomorphisms in the preceding definition is actually a chain mapping.

- Each φ_P is a homomorphism.

  So to show that $\partial\varphi_p = \varphi_{p-1}\partial$, it is sufficient to show that

  $$\partial\varphi_p(g \cdot \sigma^p) = \varphi_{p-1}\partial(g \cdot \sigma^p),$$

  for each elementary p-chain $g \cdot \sigma^p$, $p \geq 1$.

  Let $g \cdot \sigma^p$ be an elementary p-chain on K, where $+\sigma^p = +\langle v_0 \ldots v_p\rangle$.
  - Suppose φ does not collapse σ^p. Then $\varphi(\sigma^p) = \langle\varphi(v_0)\ldots\varphi(v_p)\rangle$.
    Let σ_i^p be the $(p-1)$-face of σ^p obtained by deleting the i-th vertex.
    We define $\varphi(\sigma^p)_i$ similarly.

Induced Chain Mapping Theorem (Cont'd)

Now we have

$$\begin{array}{rcl}\partial\varphi_p(g\cdot\sigma^p) & = & \partial(g\cdot\varphi(\sigma^p)) \\ & = & \sum_{i=0}^p(-1)^i g\cdot\varphi(\sigma^p)_i \\ & = & \sum_{i=0}^p(-1)^i g\cdot\varphi(\sigma_i^p) \\ & = & \varphi_{p-1}\left(\sum_{i=0}^p(-1)^i g\cdot\sigma_i^p\right) \\ & = & \varphi_{p-1}\partial(g\cdot\sigma^p).\end{array}$$

- Suppose that φ collapses σ^p.
  Without loss of generality, assume that $\varphi(v_0)=\varphi(v_1)$.
  Then $\varphi_p(g\cdot\sigma^p)=0$. So $\partial\varphi_p(g\cdot\sigma^p)=0$.
  Moreover,

$$\varphi_{p-1}\partial(g\cdot\sigma^p)=\varphi_{p-1}\left(\sum_{i=0}^p(-1)^i g\cdot\sigma_i^p\right)=\sum_{i=0}^p(-1)^i\varphi_{p-1}(g\cdot\sigma_i^p).$$

Induced Chain Mapping Theorem (Cont'd)

For $i \geq 2$, σ_i^p contains v_0 and v_1.
Since $\varphi(v_0) = \varphi(v_1)$, φ collapses σ_i^p, $i \geq 2$.
So we have

$$\varphi_{p-1}\partial(g \cdot \sigma^p) = \sum_{i=0}^{p}(-1)^i \varphi_{p-1}(g \cdot \sigma_i^p) = \varphi_{p-1}(g \cdot \sigma_0^p) - \varphi_{p-1}(g \cdot \sigma_1^p).$$

But $\sigma_0^p = \langle v_1 v_2 \ldots v_p \rangle$, $\sigma_1^p = \langle v_0 v_2 \ldots v_p \rangle$ and $\varphi(v_0) = \varphi(v_1)$.
So $\varphi_{p-1}(g \cdot \sigma_0^p) = \varphi_{p-1}(g \cdot \sigma_1^p)$.
Hence, $\varphi_{p-1}\partial(g \cdot \sigma^p) = 0$.
Thus, both $\varphi_{p-1}\partial(g \cdot \sigma^p)$ and $\partial\varphi_p(g \cdot \sigma^p)$ are 0.

Therefore, in both cases, $\partial\varphi_p = \varphi_{p-1}\partial$.

So $\{\varphi_p\}_0^\infty$ is a chain mapping.

Simplicial Mappings Between Polyhedra

Definition

Let $|K|$ and $|L|$ be polyhedra with triangulations K and L, respectively. Let φ be a simplicial mapping from the vertices of K into the vertices of L. Then φ is extended to a function $\varphi : |K| \to |L|$ as follows:

> If $x \in |K|$, there is a simplex $\sigma^r = \langle a_0 \dots a_r \rangle$ in K, such that $x \in \sigma^r$. Then
>
> $$x = \sum_{i=0}^{r} \lambda_i a_i,$$
>
> where the λ_i are the barycentric coordinates of x. Define
>
> $$\varphi(x) = \sum_{i=0}^{r} \lambda_i \varphi(a_i).$$

This extended function $\varphi : |K| \to |L|$ is called a **simplicial mapping** from $|K|$ into $|L|$.

Continuity of Simplicial Mappings

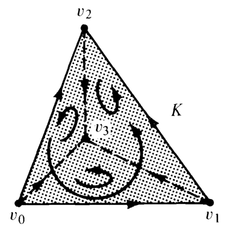
Theorem

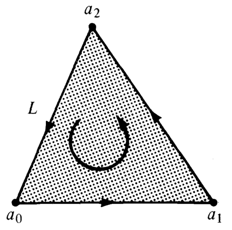
Every simplicial mapping $\varphi : |K| \to |L|$ is continuous.

- Let $\varepsilon > 0$ be arbitrary.
  Suppose $x = \sum_{i=0}^{r} \lambda_i a_i$.
  Take $y = \sum_{i=0}^{r} \mu_i a_i$, such that, for all i,
  $$|\lambda_i - \mu_i| < \frac{1}{rM}\varepsilon,$$
  where $M = \max\{|\varphi(a_i)|\}$.
  Then we have:
  $$|f(x) - f(y)| \leq M \sum_i |\lambda_i - \mu_i| < Mr\frac{1}{rM}\varepsilon = \varepsilon.$$
  Hence, φ is continuous.

Example

- Let K denote the 2-skeleton of a 3-simplex.

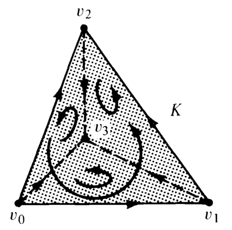


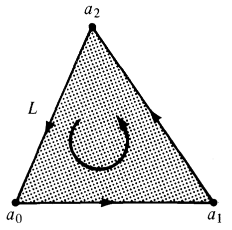


- Let L be the closure of a 2-simplex.
- The orientations are indicated by the arrows.
- Let φ be the simplicial map from K to L defined for vertices by

$$\varphi(v_0) = \varphi(v_3) = a_0, \quad \varphi(v_1) = a_1, \quad \varphi(v_2) = a_2.$$

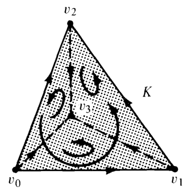
Example (Cont'd)

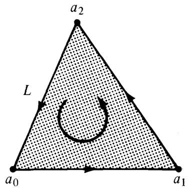




- The extension process for simplicial maps determines a simplicial mapping $\varphi : |K| \to |L|$ which:
  (a) Maps $\langle v_0 v_1 \rangle$, $\langle v_1 v_2 \rangle$ and $\langle v_2 v_0 \rangle$ linearly onto $\langle a_0 a_1 \rangle$, $\langle a_1 a_2 \rangle$ and $\langle a_2 a_0 \rangle$, respectively;
  (b) Maps $\langle v_1 v_3 \rangle$ and $\langle v_2 v_3 \rangle$ linearly onto $\langle a_1 a_0 \rangle$ and $\langle a_2 a_0 \rangle$, respectively;
  (c) Collapses $\langle v_0 v_3 \rangle$ to the vertex a_0;
  (d) Collapses $\langle v_0 v_3 v_2 \rangle$ and $\langle v_0 v_1 v_3 \rangle$;
  (e) Maps each of $\langle v_0 v_1 v_2 \rangle$ and $\langle v_3 v_1 v_2 \rangle$ linearly onto $\langle a_0 a_1 a_2 \rangle$.

Example (Cont'd)





- For the induced homomorphisms $\{\varphi_p\}$ on the chain groups we have:
  (0) $\varphi_0 : C_0(K) \to C_0(L)$ is defined by

$$\varphi_0(g \cdot \langle v_0 \rangle + g_1 \cdot \langle v_1 \rangle + g_2 \cdot \langle v_2 \rangle + g_3 \cdot \langle v_3 \rangle) = (g_0 + g_3) \cdot \langle a_0 \rangle + g_1 \cdot \langle a_1 \rangle + g_2 \cdot \langle a_2 \rangle.$$

  (1) $\varphi_1 : C_1(K) \to C_1(L)$ is defined by

$$\begin{aligned}\varphi_1(h_1 \cdot \langle v_0 v_1 \rangle + h_2 \cdot \langle v_1 v_2 \rangle + h_3 \cdot \langle v_0 v_2 \rangle + h_4 \cdot \langle v_1 v_3 \rangle + h_5 \cdot \langle v_0 v_3 \rangle + h_6 \cdot \langle v_2 v_3 \rangle) \\ = (h_1 - h_4) \cdot \langle a_0 a_1 \rangle + h_2 \cdot \langle a_1 a_2 \rangle + (h_6 - h_3) \cdot \langle a_2 a_0 \rangle.\end{aligned}$$

  (2) $\varphi_2 : C_2(K) \to C_2(L)$ is defined by

$$\begin{aligned}\varphi_2(k_1 \cdot \langle v_0 v_1 v_2 \rangle + k_2 \cdot \langle v_1 v_2 v_3 \rangle + k_3 \cdot \langle v_0 v_1 v_3 \rangle + k_4 \cdot \langle v_0 v_3 v_2 \rangle) \\ = (k_1 + k_2) \cdot \langle a_0 a_1 a_2 \rangle.\end{aligned}$$

Open Simplexes, Stars and Open Stars

Definition

If σ is a geometric simplex, the **open simplex** $o(\sigma)$ associated with σ consists of those points in σ all of whose barycentric coordinates are positive.

If v is a vertex of a complex K, then the **star** of v, $\text{st}(v)$, is the family of all simplexes σ in K of which v is a vertex.

Thus, $\text{st}(v)$ is a subset of K.

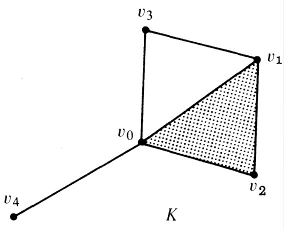
The **open star** of v, $\text{ost}(v)$, is the union of all the open simplexes $o(\sigma)$ for which v is a vertex of σ.

$\text{ost}(v)$ is a subset of the polyhedron $|K|$.

Examples

- If a is a vertex, $o(\langle a \rangle) = \{a\}$.
- For a 1-simplex $\sigma^1 = \langle a_0 a_1 \rangle$, $o(\sigma^1)$ is the open segment from a_0 to a_1 (not including either a_0 or a_1).
- For a 2-simplex σ^2, $o(\sigma^2)$ is the interior of the triangle spanned by the three vertices.
- In the figure $\mathrm{st}(v_0)$ consists of the simplexes $\langle v_0 \rangle, \langle v_0 v_1 \rangle, \langle v_0 v_2 \rangle, \langle v_0 v_3 \rangle, \langle v_0 v_4 \rangle$ and $\langle v_0 v_1 v_2 \rangle$.

  The open star of v_0, $\mathrm{ost}(v_0)$, is the set theoretic union of $\{v_0\}$, the open segments from v_0 to v_1, v_0 to v_2, v_0 to v_3, v_0 to v_4, and the interior of $\langle v_0 v_1 v_2 \rangle$.



- Note that $\mathrm{ost}(v_0)$ is not the interior of $\mathrm{st}(v_0)$ in any sense.
- The star of a vertex is a set of simplexes of K.
- The open star of a vertex is the union of certain point sets in the polyhedron $|K|$.

Star-Related Triangulations

Definition

Let $|K|$ and $|L|$ be polyhedra with triangulations K and L, respectively. Let

$$f : |K| \to |L|$$

be a continuous map.
Then K is **star related to** L **relative to** f means that for each vertex p of K, there is a vertex q of L, such that

$$f(\text{ost}(p)) \subseteq \text{ost}(q).$$

Homotopies

Definition

Let X and Y be topological spaces.
Let f,g continuous functions from X into Y.
Then f is **homotopic to** g means that there is a continuous function

$$H : X \times [0,1] \to Y$$

from the product space $X \times [0,1]$ into Y, such that, for all $x \in X$,

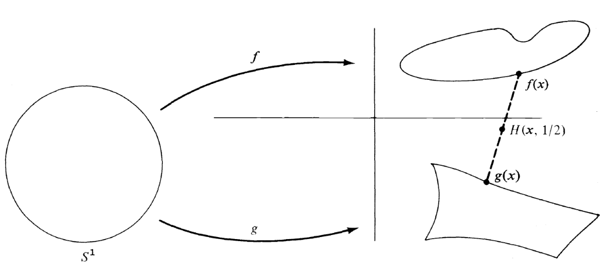
$$H(x,0) = f(x), \quad H(x,1) = g(x).$$

The function H is called a **homotopy** between f and g.

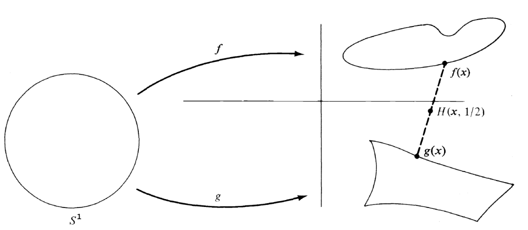
- To simplify notation involving homotopies, we use I to denote the closed unit interval $[0,1]$.

Example

- Consider the functions f and g from the unit circle S^1 into the plane given pictorially in the figure.



Example (Cont'd)



- Using the usual vector addition and scalar multiplication, a homotopy H between f and g is defined by

$$H(x,t)=(1-t)f(x)+tg(x), \quad x\in S^1, t\in I.$$

- H shows how to continuously "deform" $f(x)$ into $g(x)$.
- Observe that if the horizontal axis were removed from the range space, then the indicated functions would not be homotopic.

Simplicial Approximation

Definition

Let K and L be complexes.
Let

$$f : |K| \to |L|$$

be a continuous function.
A simplicial mapping

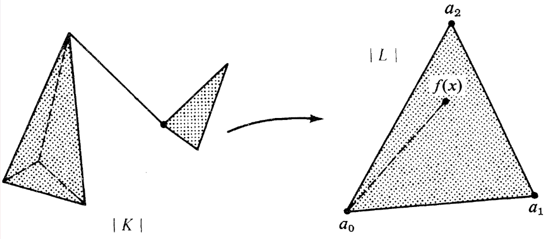
$$g : |K| \to |L|$$

which is homotopic to f is called a **simplicial approximation** of f.

Example

- Let L be the closure of a p-simplex $\sigma^p = \langle a_0 \ldots a_p \rangle$.

  Let K be an arbitrary complex.

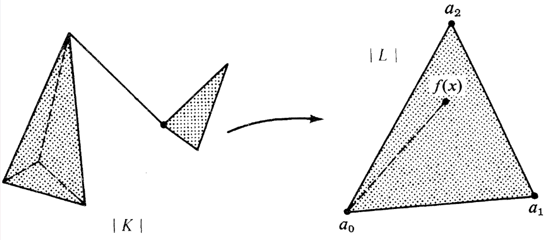


  Then any continuous map $f : |K| \to |L|$ has as a simplicial approximation the constant map

$$g : |K| \to |L|$$

  which collapses all of K to the vertex a_0.

Example (Cont'd)



- Define a homotopy $H : |K| \times I \to |L|$ by

$$H(x,t) = (1-t)f(x) + ta_0, \quad x \in |K|, \ t \in I.$$

Then H is continuous.

Moreover,

$$H(x,0) = f(x), \quad H(x,1) = a_0 = g(x), \quad x \in |K|.$$

Example

- Let both K and L be the 1-skeleton of the closure of a 2-simplex σ^2.
- Then the polyhedra $|K|$ and $|L|$ are both homeomorphic to the unit circle S^1.
- So we may consider any function from $|K|$ to $|L|$ as a function from S^1 to itself.
- For our function f let us choose a rotation through a given angle α.
- We use polar coordinates to refer to points on S^1.

Example (Cont'd)

- Then
$$f : S^1 \to S^1$$
is defined by
$$f(1,\theta) = (1,\theta+\alpha), \quad (1,\theta) \in S^1,\ 0 \le \theta \le 2\pi.$$
- A homotopy H between f and the identity map is defined by
$$H((1,\theta),t) = (1,\theta+t\alpha), \quad (1,\theta) \in S^1,\ t \in I.$$
- Note that H agrees with:
  - The identity map when $t = 0$;
  - f when $t = 1$.
- At any "time" t between 0 and 1 the "t-level of the homotopy" $H(\cdot,t)$, performs a rotation of the circle through the angle $t\alpha$.

Simplicial Map Homotopic to a Continuous Map

- We develop a process of replacing a continuous map

$$f : |K| \to |L|$$

by a homotopic simplicial map g.

- We hall first consider the case in which K is star related to L relative to f.
- We will need a lemma.

Vertices of a Simplex inside a Complex

Lemma

Vertices $v_0, \ldots, v_m$ in a complex K are vertices of a simplex of K if and only if

$$\bigcap_{i=0}^{m} \mathrm{ost}(v_i)$$

is not empty.

- Suppose, first, that $v_0, \ldots, v_m$ are vertices of a simplex of K.

  Then

  $$\sum_{i=0}^{m} \frac{1}{m+1} v_i \in \bigcap_{i=0}^{m} \mathrm{ost}(v_i).$$

  Suppose, conversely, that $\bigcap_{i=0}^{m} \mathrm{ost}(v_i) \neq \emptyset$, say $x \in \bigcap_{i=0}^{m} \mathrm{ost}(v_i)$.

  Consider the simplex of K whose interior contains x.

  Then, by the definition of ost, $v_0, \ldots, v_m$ are vertices of that simplex.

Simplicial Approximations of Star Related Triangulations

Theorem

Let K and L be polyhedra with triangulations K and L, respectively. Let $f:|K|\to|L|$ be a continuous function, such that K is star related to L relative to f. Then f has a simplicial approximation $g:|K|\to|L|$.

- By assumption, K is star related to L relative to f.
  So, for each vertex p of K, there exists a vertex $g(p)$ of L, such that
  $$f(\text{ost}(p))\subseteq\text{ost}(g(p)).$$
  We show that this vertex map g is simplicial.
  Suppose that $v_0,\ldots,v_n$ are vertices of a simplex in K.
  By the preceding lemma, this is equivalent to
  $$\bigcap_{i=0}^n\text{ost}(v_i)\neq\emptyset.$$

Simplicial Approximations (Cont'd)

- Hence,

$$\emptyset \neq f\left(\bigcap_{i=0}^{n} \mathrm{ost}(v_i)\right) \subseteq \bigcap_{i=0}^{n} f(\mathrm{ost}(v_i)) \subseteq \bigcap_{i=0}^{n} \mathrm{ost}(g(v_i)).$$

  So $\bigcap_{i=0}^{n} \mathrm{ost}(g(v_i))$ is not empty.

  By the lemma, $g(v_0),\ldots,g(v_n)$ are vertices of a simplex in L.

  Thus, g is a simplicial vertex map.

  So it has an extension to a simplicial map $g:|K| \to |L|$.

  Let $x \in |K|$.

  Let σ be the simplex of K of smallest dimension which contains x.

  Let a be any vertex of σ.

  Observe that the following hold:
  - $f(x) \in f(\mathrm{ost}(a))$;
  - $f(\mathrm{ost}(a)) \subseteq \mathrm{ost}(g(a))$.

Simplicial Approximations (Cont'd)

- The barycentric coordinate of $g(x)$ with respect to $g(a)$ is greater than or equal to the barycentric coordinate of x with respect to a.

  It follows that $g(x) \in \text{ost}(g(a))$.

  Let $a_0, \ldots, a_k$ denote the vertices of σ.

  By the preceding paragraph, $f(x), g(x) \in \bigcap_{i=0}^{k} \text{ost}(g(a_i))$.

  Thus, $g(a_0), \ldots, g(a_k)$ are vertices of a simplex τ in L containing both $f(x)$ and $g(x)$.

  But each simplex is a convex set.

  So the line segment joining $f(x)$ and $g(x)$ must lie entirely in $|L|$.

  Therefore, the map $H : |K| \times I \to |L|$ defined by

  $$H(x,t) = (1-t)f(x) + tg(x), \quad x \in K, \ t \in I,$$

  is then a homotopy between f and g.

  So g is a simplicial approximation of f.

Idea of Barycentric Subdivision

- The theorem shows that if K is star related to L relative to f, then there is a simplicial map homotopic to f.
- Assume, now, that K is not star related to L relative to f.
- Then K has some vertices $b_0, \ldots, b_n$, such that $f(\text{ost}(b_i))$ is not contained in the open star of any vertex in L.
- We retriangulate K systematically to produce simplexes of smaller and smaller diameters.
- In this way, we reduce the size of $\text{ost}(b_i)$ and the size $f(\text{ost}(b_i))$ to the point that the new complex obtained from K is star related to L relative to f.
- This process of dividing a complex into smaller simplexes is called "barycentric subdivision".

Barycenter and Barycentric Subdivisions

Definition

Let $\sigma^r = \langle a_0 \ldots a_r \rangle$ be a simplex in $\mathbb{R}^n$. The point $\dot{\sigma}^r$ in σ^r all of whose barycentric coordinates with respect to $a_0, \ldots, a_r$ are equal is called the **barycenter** of σ^r.

Note that if σ^0 is a 0-simplex, then $\dot{\sigma}^0$ is the vertex which determines σ^0.

The collection $\{\dot{\sigma}^k : \sigma^k$ is a face of $\sigma^r\}$ of all barycenters of faces of σ^r are the vertices of a complex called the **first barycentric subdivision** of $\mathrm{Cl}(\sigma^r)$.

A subset $\dot{\sigma}_0, \ldots, \dot{\sigma}_p$ of the vertices $\dot{\sigma}^k$ are the vertices of a simplex in the first barycentric subdivision provided that σ_j is a face of σ_{j+1} for $j = 0, \ldots, p-1$.

If K is a geometric complex, the preceding process is applied to each simplex of K to produce the **first barycentric subdivision** $K^{(1)}$ of K.

For $n > 1$, the ***n*-th barycentric subdivision** $K^{(n)}$ of K is the first barycentric subdivision of $K^{(n-1)}$.

Orientation of Barycentric Subdivisions

Definition (Cont'd)

The first barycentric subdivision of K is assigned an orientation consistent with that of K as follows:

Let $\langle \dot{\sigma}^0 \dot{\sigma}^1 \dots \dot{\sigma}^p \rangle$ be a p-simplex of $K^{(1)}$ which occurs in the barycentric subdivision of a p-simplex σ^p of K. Then the vertices of $\sigma^p = \langle v_0 \dots v_p \rangle$ may be ordered so that $\dot{\sigma}_i$ is the barycenter of $\sigma^p = \langle v_0 \dots v_i \rangle$, for $i = 0, \dots, p$. We consider $\langle \dot{\sigma}^0 \dots \dot{\sigma}^p \rangle$ to be:

- Positively oriented if $\langle v_0 \dots v_p \rangle$ is positively oriented;
- Negatively oriented if $\langle v_0 \dots v_p \rangle$ is negatively oriented.

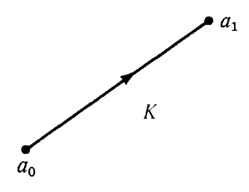
There are other simplexes of $K^{(1)}$ whose orientations are not defined by this process, and they may be oriented arbitrarily.

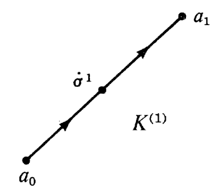
An orientation for $K^{(1)}$ defined in this way is said to be **concordant** with the orientation of K.

The same process applies inductively to higher barycentric subdivisions.

Example

- We shall assume that all barycentric subdivisions are concordantly oriented.
- Consider the complex $K = \mathrm{Cl}(\sigma^1)$ consisting of a 1-simplex $\sigma^1 = \langle a_0 a_1 \rangle$ and two 0-simplexes $\sigma_0^0 = \langle a_0 \rangle$ and $\sigma_1^0 = \langle a_1 \rangle$.



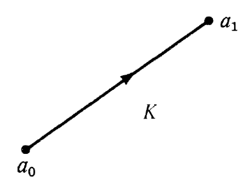


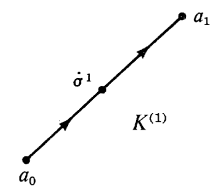
Then $\dot{\sigma}_0^0 = a_0$, $\dot{\sigma}_1^0 = a_1$ and $\dot{\sigma}^1$ is the midpoint of σ^1.

Hence the first barycentric subdivision of K has vertices a_0, a_1 and $\dot{\sigma}^1$.

Since the only faces of σ^1 are $\langle a_0 \rangle$ and $\langle a_1 \rangle$, the only 1-simplexes of $K^{(1)}$ are $\langle a_0 \dot{\sigma}^1 \rangle$ and $\langle a_1 \dot{\sigma}^1 \rangle$.

Example (Cont'd)

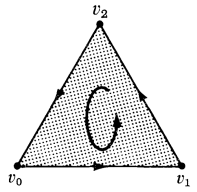


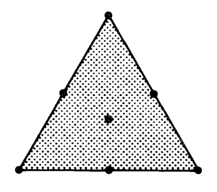


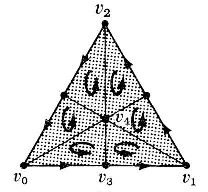
- Consider σ^1 to be oriented by $a_0 < a_1$. Then $\langle a_0 a_1 \rangle$ represents the positive orientation.
  - Then $\langle a_0 \dot{\sigma}^1 \rangle$ occurs in the subdivision of the positively oriented simplex $\langle a_0 a_1 \rangle$. Hence $\langle a_0 \dot{\sigma}^1 \rangle$ is a positively oriented simplex in $K^{(1)}$.
  - $\langle a_1 \dot{\sigma}^1 \rangle$ is produced in the subdivision of the negatively oriented simplex $\langle a_1 a_0 \rangle$. So $\langle a_1 \dot{\sigma}^1 \rangle$ has negative orientation.

Example

- Consider the complex $\mathrm{Cl}(\sigma^2)$ in the first figure.







The barycenters of all simplexes are indicated in the second figure.

The first barycentric subdivision is shown in the last figure.

The orientation for $\langle v_0 v_3 v_4 \rangle$ is determined as follows.

  - Vertex v_3 is the barycenter of $\langle v_0 v_1 \rangle$ and v_4 the barycenter of $\langle v_0 v_1 v_2 \rangle$. Thus, following the definition of concordant orientation, $\langle v_0 v_3 v_4 \rangle$ is assigned positive orientation since it is produced in the subdivision of the positively oriented simplex $\langle v_0 v_1 v_2 \rangle$.

- Note that some simplexes of the barycentric subdivision are not assigned orientations by this process.

The Mesh of a Complex

Definition

If K is a complex, the **mesh** of K is the maximum of the diameters of the simplexes of K.

- It should be obvious that the mesh of the first barycentric subdivision $K^{(1)}$ of a complex K is less than the mesh of K.
- So it is reasonable to expect that the limiting value of mesh $K^{(s)}$ as s increases indefinitely is zero.

The Norm in $\mathbb{R}^n$

Definition

If $x = (x_1, \ldots, x_n)$ is a point in $\mathbb{R}^n$, the **norm** of x is the number

$$\|x\| = \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}.$$

For x, y in $\mathbb{R}^n$, the distance $d(x, y)$ from x to y is

$$\|x - y\|.$$

On Norms

Lemma

If x and y are points in a simplex σ, then there is a vertex v of σ such that

$$\|x-y\| \leq \|x-v\|.$$

- Fix the point x in σ.

  Consider the function

  $$f(y)=\|x-y\|,$$

  defined for points y in σ.

  It is a continuous function defined on a compact set.

  Thus, it attains its maximum at one of the vertices v of the simplex.

  This shows that, for all y, $\|x-y\| \leq \|x-v\|$.

On Diameters

Lemma

The diameter of a simplex of positive dimension is the length of its longest 1-face. Hence the mesh of a complex K of positive dimension is the length of its longest 1-simplex.

- Note that any complex of dimension zero must have mesh zero.
- Suppose σ is a simplex of positive dimension.

  Since σ is compact, its diameter d, i.e., the supremum of the distance between two of its points is actually attained.

  Therefore, there exist points x, y in σ, such that $d = \|x - y\|$.

  By the preceding lemma y must be a vertex.

  Again by the preceding lemma, x must also be a vertex.

  Therefore, d is the length of the longest 1-face of σ.

Limit of the Mesh of $K^{(s)}$

Theorem

For any complex K,

$$\lim_{s\to\infty} \text{mesh} K^{(s)} = 0.$$

- Consider the first barycentric subdivision $K^{(1)}$ of K.

  Let $\langle \dot{\sigma}\dot{\tau}\rangle$ be one of its 1-simplexes.

  Then, by definition, σ is a face of τ.

  The definition of barycenter for the simplex τ insures that

$$\dot{\tau} = \frac{1}{p+1}\sum_{i=0}^{p} v_i,$$

  where $v_0,\ldots,v_p$ are the vertices of τ.

  By a preceding lemma, there must be a vertex v of τ, such that $\|\dot{\tau}-\dot{\sigma}\| \le \|\dot{\tau}-v\|$.

Limit of the Mesh of $K^{(s)}$ (Cont'd)

- Then we have

$$\begin{aligned}\|\dot{\tau}-\dot{\sigma}\| &\leq \left\|\frac{1}{p+1}\sum_{i=0}^{p} v_i - v\right\| \\ &= \left\|\frac{1}{p+1}\sum_{i=0}^{p}(v_i - v)\right\| \\ &\leq \frac{1}{p+1}\sum_{i=0}^{p}\|v_i - v\| \\ &\leq \frac{p}{p+1}\text{mesh}K.\end{aligned}$$

Let n denote the dimension of K.

Then we have $p \leq n$.

So

$$\|\dot{\tau}-\dot{\sigma}\| \leq \frac{n}{n+1}\text{mesh}K.$$

Limit of the Mesh of $K^{(s)}$ (Cont'd)

- We got
$$\|\dot{\tau}-\dot{\sigma}\| \leq \frac{n}{n+1}\operatorname{mesh}K.$$
But the mesh of $K^{(1)}$ is the maximum value of $\|\dot{\tau}-\dot{\sigma}\|$, for all 1-simplexes $\langle\dot{\sigma}\dot{\tau}\rangle$ in $K^{(1)}$.
Therefore,
$$\operatorname{mesh}K^{(1)} \leq \frac{n}{n+1}\operatorname{mesh}K.$$
By the inductive definition of $K^{(s)}$,
$$\operatorname{mesh}K^{(s)} \leq \left(\frac{n}{n+1}\right)^s \operatorname{mesh}K.$$
But
$$\lim_{s\to\infty}\left(\frac{n}{n+1}\right)^s = 0.$$
This yields the result.

The Simplicial Approximation Theorem

Theorem (The Simplicial Approximation Theorem)

Let $|K|$ and $|L|$ be polyhedra with triangulations K and L, respectively. Let $f:|K|\to|L|$ be a continuous function. There is a barycentric subdivision $K^{(k)}$ of K and a continuous function $g:|K|\to|L|$, such that:

(a) g is a simplicial map from $K^{(k)}$ into L;

(b) g is homotopic to f.

- We apply the theorem addressing star related polyhedra.
  - First, an integer k for which $K^{(k)}$ is star related to L relative to f is determined.
  - Then, by the theorem, we obtain the simplicial approximation g.

This is done using a Lebesgue number argument.

Since $|L|$ is a compact metric space, the open cover

$$\{\mathrm{ost}(v): v \text{ is a vertex of } L\}$$

has a Lebesgue number $\eta > 0$.

The Simplicial Approximation Theorem (Cont'd)

- Since f is uniformly continuous (its domain is a compact metric space), there is a positive number δ, such that

$$\|x-y\|<\delta \quad \text{in} \quad |K|,$$

implies

$$\|f(x)-f(y)\|<\eta \quad \text{in} \quad |L|.$$

Thus, if the barycentric subdivision $K^{(k)}$ has mesh less than $\frac{\delta}{2}$, then $K^{(k)}$ is star related to L relative to f.

The function $g:|K|\to|L|$ determined by the star related theorem has the required properties.

Subsection 2

Induced Homomorphisms on the Homology Groups

Continuous Maps and Homology Group Homomorphisms

Definition

Let $|K|$ and $|L|$ be polyhedra with triangulations K and L, respectively.
Let and $f:|K|\rightarrow|L|$ a continuous map.
By the Simplicial Approximation Theorem, there is a barycentric subdivision $K^{(k)}$ of K and a simplicial mapping $g:|K|\rightarrow|L|$ which is homotopic to f.
By previous theorems, g induces homomorphisms $g_p^*:H_p(K)\rightarrow H_p(L)$ in each dimension p. This sequence of homomorphisms $\{g_p^*\}$ is called the **sequence of homomorphisms induced by** f.

Induced Homology Group Homomorphisms

- It can be shown that $\{g_p^* : H_p(K) \to H_p(L)\}$ is unique.
- In particular, does not depend on:
  - The admissible choices for the degree k of the barycentric subdivision;
  - On the admissible choices for the simplicial map g.
- The sequence is, thus, usually written $\{f_p^*\}$ instead of $\{g_p^*\}$, since it is completely determined by f.
- The proof of uniqueness requires some additional concepts.

Composability of Induced Homomorphisms

Lemma

If $f : |K| \to |L|$ and $h : |L| \to |M|$ are continuous maps on the indicated polyhedra, then, in each dimension p, $(hf)_p^* : H_p(K) \to H_p(M)$ is the composition $h_p^* f_p^* : H_p(K) \to H_p(M)$.

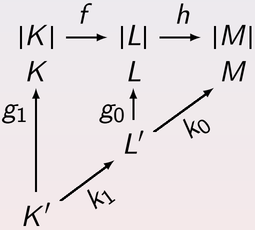
- The identity map $i : |K| \to |K|$ induces the identity homomorphism $i_p^* : H_p(K) \to H_p(K)$.

  Now choose:
  - $k_0 : L' \to M$ a simplicial approximation to h;
  - $g_0 : L' \to L$ a simplicial approximation to $i_{|L|}$.

  Next choose:
  - $k_1 : K' \to L'$ a simplicial approximation to f;
  - $g_1 : K' \to K$ a simplicial approximation to $i_{|K|}$.

Composability of Induced Homomorphisms (Cont'd)

- By choosing simplicial approximations, we have created the diagram



The map $k_0 k_1$ is a simplicial approximation to hf.
By definition,

$$(hf)^* = (k_0 k_1)^* (g_1^*)^{-1}.$$

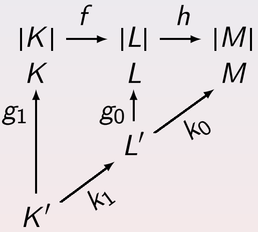
The map $g_0 k_1$ is a simplicial approximation to f.
So, by definition,

$$f^* = (g_0 k_1)^* (g_1^*)^{-1}.$$

Finally, $h^* = (k_0)^* (g_0^*)^{-1}$.

Composability of Induced Homomorphisms (Cont'd)

- We work with the diagram



Since the result holds for simplicial maps, we get

$$\begin{aligned} (hf)^* &= (k_0 k_1)^* (g_1^*)^{-1} \\ &= k_0^* k_1^* (g_1^*)^{-1} \\ &= k_0^* (g_0^*)^{-1} g_0^* k_1^* (g_1^*)^{-1} \\ &= (k_0)^* (g_0^*)^{-1} (g_0 k_1)^* (g_1^*)^{-1} \\ &= f^* h^*. \end{aligned}$$

Invariance of Dimension

Theorem (Invariance of Dimension)

If $m \neq n$, then:

(a) S^m and S^n are not homeomorphic;

(b) $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic.

(a) Suppose, to the contrary, that there is a homeomorphism $h : S^m \to S^n$ from S^m onto S^n with inverse $h^{-1} : S^n \to S^m$.

Now we have:

- $h^{-1}h$ is the identity maps on S^m;
- hh^{-1} is the identity maps on S^n.

But the identity map i on a polyhedron $|K|$ induces the identity isomorphism $i_p^* : H_p(K) \to H_p(K)$ in each dimension p.

Invariance of Dimension (Cont'd)

- Thus, the following are identity isomorphisms in each dimension,

$$(hh^{-1})_p^* = h_p^* h_p^{-1*} : H_p(S^n) \to H_p(S^n),$$
$$(h^{-1}h)_p^* = h_p^{-1*} h_p^* : H_p(S^m) \to H_p(S^m).$$

So h_p^* is an isomorphism between $H_p(S^m)$ and $H_p(S^n)$.

Comparison of homology groups shows this is impossible if $m \neq n$.

Hence, for $m \neq n$, S^m and S^n are not homeomorphic.

(b) Recall from point-set topology that S^n is the one point compactification of $\mathbb{R}^n$.

If $\mathbb{R}^m$ and $\mathbb{R}^n$ are homeomorphic, it must be true that their one point compactifications S^m and S^n are homeomorphic too.

This contradicts Part (a) if $m \neq n$.

Fundamental Class and Degrees

Definition

Let $f : S^n \to S^n$, $n \geq 1$, be a continuous function from the n-sphere into itself. Let K be a triangulation of S^n. Since K is an orientable n-pseudomanifold, a previous theorem shows that it is possible to orient K so that the n-chain

$$z_n = \sum_{\sigma^n \in K} 1 \cdot \sigma^n$$

is an n-cycle whose homology class $[z_n]$ is a generator of the infinite cyclic group $H_n(K)$. This homology class is called a **fundamental class**.
If $f_n^* : H_n(K) \to H_n(K)$ is the homomorphism in dimension n induced by f, then there is an integer ρ, such that

$$f_n^*([z_n]) = \rho [z_n].$$

The integer ρ is called the **degree** of the map f and is denoted $\deg(f)$.

Equivalent (Original) Definition

- The original definition, which is equivalent to the preceding one is:

Alternate Definition

Suppose that $f: S^n \to S^n$ is a continuous map. Suppose S^n is triangulated by a complex K. Choose a barycentric subdivision $K^{(k)}$ of K for which there is a simplicial mapping $\varphi: |K^{(k)}| \to |K|$ homotopic to f.
Let τ be any positively oriented n-simplex in K.

- Let p be the number of positively oriented n-simplexes σ in $K^{(k)}$, such that $\varphi(1 \cdot \sigma) = 1 \cdot \tau$;
- Let q be the number of positively oriented n-simplexes μ in $K^{(k)}$, such that $\varphi(1 \cdot \mu) = -1 \cdot \tau$.

Then the integer $p - q$ is independent of the choice of τ (the same integer $p - q$ results for each n-simplex of K).
It is called the **degree** of the map f.

Comments

- In the last definition, it can be shown that the degree of f is independent of the admissible choices for K, $K^{(k)}$ and φ.
- Intuitively, the definition states that the degree of a map $f : S^n \to S^n$ is the number of times that f "wraps the domain around the range".

Properties of Degrees

Theorem

(a) If $f : S^n \to S^n$ and $g : S^n \to S^n$ are continuous maps, then

$$\deg(gf) = \deg(g)\deg(f).$$

(b) The identity map $i : S^n \to S^n$ has degree $+1$.

(c) A homeomorphism $h : S^n \to S^n$ has degree ± 1.

(a) Choose a triangulation K of S^n with fundamental class $[z_n]$. Consider the induced homomorphisms

$$f^* : H_n(K) \to H_n(K), \quad g^* : H_n(K) \to H_n(K).$$

Then

$$\begin{array}{rcl} (gf)_n^*([z_n]) & = & \mathrm{dg}(gf) \cdot [z_n]; \\ g_n^* f_n^*([z_n]) & = & g_n^*(\deg(f) \cdot [z_n]) = \deg(g) \cdot \deg(f) \cdot [z_n]. \end{array}$$

Properties of Degrees (Cont'd)

By the preceding lemma,

$$(gf)_n^* = g_n^* f_n^*.$$

So $\deg(gf) = \deg(g)\deg(f)$.

(b) In the second definition of degree, it is obvious that for the identity map i, $p = 1$ and $q = 0$. So $\deg(f) = 1 - 0 = 1$.

(c) Letting h^{-1} denote the inverse of h, we have

$$1 = \deg(i) = \deg(hh^{-1}) = \deg(h)\deg(h^{-1}).$$

But $\deg(h)$ must be an integer.

So $\deg(h) = \pm 1$.

It also follows that h and h^{-1} have the same degree.

Brouwer's Degree Theorem

Theorem (Brouwer's Degree Theorem)

If two continuous maps $f, g: S^n \to S^n$ are homotopic, then they have the same degree.

- Let K be a triangulation of S^n.

  Let $h: S^n \times I \to S^n$ be a homotopy such that

  $$h(x,0) = f(x), \quad h(x,1) = g(x), \quad x \in S^n.$$

  Let h_t denote the restriction of h to $S^n \times \{t\}$.

  Thus, $h_0 = f$ and $h_1 = g$.

  Let ϵ be a Lebesgue number for the open cover

  $$\{\text{ost}(w_i) : w_i \text{ is a vertex of } K\}.$$

Brouwer's Degree Theorem (Cont'd)

- Now h is uniformly continuous.

  So there is a positive number δ, such that if A and B are subsets of S^n and I, respectively, with diameters $\text{diam}(A) < \delta$ and $\text{diam}(B) < \delta$, then $\text{diam}(h(A \times B)) < \epsilon$.

  Let $K^{(k)}$ be a barycentric subdivision of K of mesh less than $\frac{\delta}{2}$.

  If v is a vertex of $K^{(k)}$, then $\text{diam}(\text{ost}(v)) < \delta$.

  Let

  $$0 = t_0 < t_1 < \cdots < t_q = 1$$

  be a partition of I for which successive points differ by less than δ.

  Then, for v_i a vertex of $K^{(k)}$ and t_{j-1}, t_j successive members of the partition, the set

  $$h(\text{ost}(v_i) \times [t_{j-1}, t_j]),$$

  has diameter less than ϵ.

  It is therefore contained in $\text{ost}(w_{ij})$ for some vertex w_{ij} of K.

Brouwer's Degree Theorem (Cont'd)

- Now let $t_{j-1} \le t \le t_j$.

  Then the value of the simplicial map φ_t approximating h_t, given by the Simplicial Approximation Theorem, may be defined by letting

  $$\varphi_t(v_i) = w_{ij}.$$

  Therefore, all the maps h_t, for $t_{j-1} \le t \le t_j$ have the same degree.

  But any two successive intervals $[t_{j-1}, t_j]$ and $[t_j, t_{j+1}]$ have t_j in common.

  So the degree of h_t is constant for $0 \le t \le 1$.

  In particular, $h_0 = f$ and $h_1 = g$ have the same degree.

The Hopf Classification Theorem

- The preceding method of proof can be extended to show that homotopic maps from one polyhedron to another induce identical sequences of homomorphisms on the homology groups.

Theorem (The Hopf Classification Theorem)

Two continuous maps f and g from S^n to S^n are homotopic if and only if they have the same degree.

- Hopf considered also maps from polyhedra into spheres.
- He extended the definition of degree to such maps.
- He then proved a more general classification theorem.

  Suppose X is a polyhedron of dimension not exceeding n.
  Two maps f and g from X into S^n are homotopic if and only if they have the same degree.

Subsection 3

The Brouwer Fixed Point Theorem and Related Results

Fixed-Points

Definition

Let $f : X \to X$ be a continuous function from a space X into itself.
A point x_0 in X is a **fixed point of** f if

$$f(x_0) = x_0.$$

Null-Homotopic Maps and Contractible Spaces

Definition

A continuous function $g : X \to Y$ from a space X into a space Y which is homotopic to a constant map is said to be **null-homotopic** or **inessential**.

Definition

A space X is **contractible** means that the identity function $f : X \to X$ is null-homotopic.

In other words, X is contractible if there is a point x_0 in X and a homotopy $H : X \times I \to X$, such that

$$H(x,0) = x, \quad H(x,1) = x_0, \quad x \in X.$$

The homotopy H is called a **contraction** of the space X.

Example: Contractibility of the Disk

Claim: The unit disk

$$D = \{x = (x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 \le 1\}$$

is contractible.

We let $x_0 = (0,0)$ be the origin and define a contraction by

$$H((x_1, x_2), t) = ((1-t)x_1, (1-t)x_2), \quad (x_1, x_2) \in D, t \in I.$$

Imagine the disk as a sheet of rubber.

The contraction essentially "squeezes" the disk to a single point.

Non-Contractibility of the Sphere

Theorem

The n-sphere S^n is not contractible for any $n \geq 0$.

- The identity map on S^n has degree 1, for $n \geq 1$.

  Any constant map has degree 0.

  By a previous theorem, homotopic maps have the same degree.

  So the identity is not null-homotopic.

  Thus, S^n is not contractible, for $n \geq 1$.

  For the case $n = 0$, we observe that

  $$S^0 = \{x \in \mathbb{R} : x^2 = 1\} = \{-1, 1\}.$$

  This is a discrete space.

  Therefore, it is not contractible.

The Brouwer No Retraction Theorem

Theorem (The Brouwer No Retraction Theorem)

There does not exist a continuous function from the $(n+1)$-ball

$$B^{n+1} = \{x = (x_1, x_2, \ldots, x_{n+1}) \in \mathbb{R}^{n+1} : \sum x_i^2 \leq 1\}$$

onto S^n which leaves each point of S^n fixed, $n \geq 0$.

- Let $f : B^{n+1} \to S^n$ be continuous, such that $f(x) = x$, for each x in S^n. Define a homotopy $H : S^n \times I \to S^n$ by

$$H(x,t) = f((1-t)x), \quad x \in S^n, t \in I.$$

  Here $(1-t)x$ denotes the usual scalar product in $\mathbb{R}^n$.

  Then H is a contraction on S^n contradicting the preceding theorem.

  Thus, no f exists with the postulated propeties.

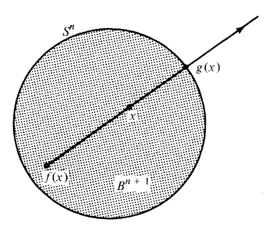
The Brouwer Fixed Point Theorem

Theorem (The Brouwer Fixed Point Theorem)

If $f: B^{n+1} \to B^{n+1}$ is a continuous map from the $(n+1)$-ball into itself and $n \geq 0$, then f has at least one fixed point.

- Suppose on the contrary that f has no fixed point.

  For each $x \in B^{n+1}$, $f(x)$ and x are distinct. For any x, consider the half-line from $f(x)$ through x. Let $g(x)$ denote the intersection of this ray with S^n. Then:

  - $g: B^{n+1} \to S^n$ is continuous;
  - $g(x) = x$, for all $x \in S^n$.

  This contradicts the preceding theorem.

  So the assumption that f has no fixed point must be false.

Reflections of S^n and Antipodal Map

Definition

For each integer i, with $1 \le i \le n+1$, the map $r_i : S^n \to S^n$ defined by

$$r_i(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_{i-1}, -x_i, x_{i+1}, \ldots, x_{n+1}), \quad (x_1, \ldots, x_{n+1}) \in S^n,$$

(with obvious modifications when $i = 1$ or $n+1$) is called the **reflection of S^n with respect to the x_i axis**.

Definition

The map $r : S^n \to S^n$ defined by $r(x) = -x$, $x \in S^n$, is called the **antipodal map** on S^n.

- The antipodal map r is the composition $r_1 r_2 \cdots r_{n+1}$ of the reflections of S^n in the respective axes.

Degrees of Reflections and of the Antipodal Map

Lemma

(a) Each reflection r_i on S^n has degree -1.

(b) The antipodal map on S^n has degree $(-1)^{n+1}$.

(a) Consider a triangulation K of S^n consisting of the n-faces of an $(n+1)$-simplex $a_0 a_1 \ldots a_{n+1}$.

Fix an n-face τ, with positive orientation.

Each reflection interchanges two vertices of the simplex.

Thus, the following hold:

- No n-face of the simplex maps to $1 \cdot \tau$;
- One n-face of the simplex map to $-1 \cdot \tau$.

By the alternative definition of degree, each reflection ha degree -1.

(b) This follows from $r = r_1 r_2 \cdots r_{n+1}$ and a property of degrees.

Continuous Unit Tangent Vector Fields

- Two vectors $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$ in $\mathbb{R}^n$ are **perpendicular** if and only if their dot product

$$x \cdot y = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n = 0.$$

Definition

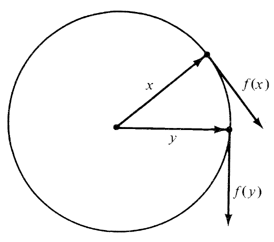
A **continuous unit tangent vector field**, or simply **vector field**, on S^n is a continuous function

$$f : S^n \to S^n.$$

such that x and $f(x)$ are perpendicular for each x in S^n.

Comments

- A vector field f on S^n is a continuous function which associates with vector x of unit length in $\mathbb{R}^{n+1}$ a unit vector $f(x)$ in $\mathbb{R}^{n+1}$, such that x and $f(x)$ are perpendicular.



- If we imagine that $f(x)$ is transposed so that it begins at point x on S^n, then $f(x)$ must be tangent to the sphere S^n.

A Vector Field on S^1

- The following scheme describes a vector field on S^1.

  For each x in S^1, let $f(x)$ denote a vector of unit length beginning at point x and pointing in the clockwise direction tangent to S^1.
- Having all vectors $f(x)$ point in the counterclockwise direction also produces a vector field on S^1.
- The requirement of continuity for f rules out the possibility of having $f(x)$ in the clockwise direction for some values of x and in the counterclockwise direction for others.

The Brouwer-Poincaré Theorem

Theorem (The Brouwer-Poincaré Theorem)

There is a vector field on S^n, $n \geq 1$, if and only if n is odd.

- If n is odd, a vector field f on S^n can be defined by

$$f(x_1, \ldots, x_{n+1}) = (x_2, -x_1, x_4, -x_3, \ldots, x_{n+1}, -x_n), \quad (x_1, \ldots, x_{n+1}) \in S^n.$$

It is clear that f is a continuous function from S^n into S^n.
Next, observe that, for each x in S^n,

$$x \cdot f(x) = (x_1 x_2 - x_1 x_2) + \cdots + (x_n x_{n+1} - x_n x_{n+1}) = 0.$$

So F is indeed a vector field.
Consider, next, an even integer n.
Suppose, towards a contradiction, that $g : S^n \to S^n$ is a vector field.
Define a homotopy $h : S^n \times I \to S^n$ by

$$h(x,t) = x \cos \pi t + g(x) \sin \pi t, \quad x \in S^n, \ t \in I.$$

The Brouwer-Poincaré Theorem (Cont'd)

We have

$$\begin{aligned}\|h(x,t)\|^2 &= h(x,t)\cdot h(x,t)\\ &= \|x\|^2\cos^2\pi t + 2x\cdot g(x)\cos\pi t\sin\pi t + \|g(x)\|^2\sin^2\pi t\\ &= 1^2\cos^2\pi t + 2\cdot 0\cdot\cos\pi t\sin\pi t + 1^2\sin^2\pi t = 1.\end{aligned}$$

So h is a homotopy on S^n.

But

$$h(x,0)=x,\quad h(x,1)=-x,\quad x\in S^n.$$

So h is a homotopy between the identity and the antipodal on S^n.

- The identity map has degree 1;
- The antipodal map has degree $(-1)^{n+1}=-1$, since n is even.

However, by Brouwer's Theorem, homotopic maps have equal degrees.

Thus, S^n has a vector field if and only if n is odd.

Visualization of the Brouwer-Poincaré Theorem

- For $n = 2$, the result can be visualized as follows.
- Imagine a 2-sphere with a unit vector emanating from each point.
- Think of each vector as a hair.
- Finding a vector field for S^2 is equivalent to describing a method for "combing the hairs" so that:
  - Each one is tangent to the sphere;
  - Their directions vary continuously.

  In other words, there must be no parts or whorls in the hairs.
- According to the Brouwer-Poincaré Theorem, such a hairstyle is impossible for spheres of even dimension.
- Because of this analogy, the theorem is sometimes called the "Tennis Ball Theorem".