Introduction to Algebraic Topology

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LSSU Math 500

1 The Fundamental Group

- Introduction
- Homotopic Paths and the Fundamental Group
- The Covering Homotopy Property for S^1
- Examples of Fundamental Groups
- The Relation Between $H_1(K)$ and $\pi_1(|K|)$

Subsection 1

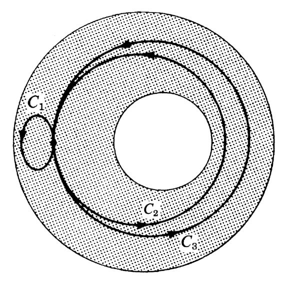
Introduction

Idea of Homotopic Paths

- Recall that two closed paths in a space are homotopic provided that each of them can be "continuously deformed into the other".

  Example:

  - Paths C_2 and C_3 are homotopic to each other.
  - C_1 is homotopic to a constant path.
  - Path C_1 is not homotopic to either C_2 or C_3, since neither C_2 nor C_3 can be pulled across the hole that they enclose.



- The basic idea is a special case of the homotopy relation for continuous functions considered in the proof of the Simplicial Approximation Theorem.

Subsection 2

Homotopic Paths and the Fundamental Group

Paths, Equivalence of Paths and Homotopies

Definition

A **path** in a topological space X is a continuous function α from the closed unit interval $I = [0,1]$ into X. The points $\alpha(0)$ and $\alpha(1)$ are the **initial point** and **terminal point** of α, respectively.

Paths α and β with common initial point $\alpha(0) = \beta(0)$ and common terminal point $\alpha(1) = \beta(1)$ are **equivalent** provided that there is a continuous function $H : I \times I \to X$, such that

$$H(t,0) = \alpha(t), \quad H(t,1) = \beta(t), \quad t \in I,$$
$$H(0,s) = \alpha(0) = \beta(0), \quad H(1,s) = \alpha(1) = \beta(1), \quad s \in I.$$

The function H is called a **homotopy** between α and β. For a given value of s, the restriction of H to $I \times \{s\}$ is called the s-**level** of the homotopy and is denoted $H(\cdot,s)$.

Loops, Equivalence of Loops

Definition

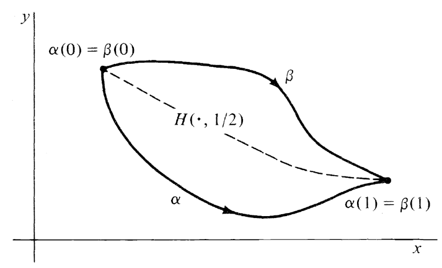
A **loop** in a topological space X is a path α in X with $\alpha(0) = \alpha(1)$.
The common value of the initial point and terminal point is referred to as the **base point** of the loop.
Two loops α and β having common base point x_0 are **equivalent** or **homotopic modulo** x_0 provided that they are equivalent as paths.
In other words, α and β are homotopic modulo x_0 (denoted $\alpha \sim_{x_0} \beta$) provided that there is a homotopy $H : I \times I \to X$, such that

$$H(\cdot, 0) = \alpha, \quad H(\cdot, 1) = \beta, \quad H(0, s) = H(1, s) = x_0, \quad s \in I.$$

Since $H(0, s)$ and $H(1, s)$ always have value x_0 regardless of the choice of s in $[0, 1]$, it is sometimes said that the base point "stays fixed throughout the homotopy".

Example

- The paths α and β are equivalent.



A homotopy H demonstrating the equivalence is defined by

$$H(t,s) = s\beta(t) + (1-s)\alpha(t), \quad (s,t) \in I \times I.$$

The homotopy "pulls α across to β" leaving the end points fixed.

- If the space had a "hole" between the ranges of α and β, then the paths would not be equivalent.

The Continuity Lemma

The Continuity Lemma

Let X be a topological space with closed subsets A and B, such that $A \cup B = X$. Let $f : A \to Y$ and $g : B \to Y$ be continuous maps to a space Y, such that

$$f(x) = g(x), \quad \text{for all } x \text{ in } A \cap B.$$

Then the map $h : X \to Y$ defined by

$$h(x) = \begin{cases} f(x), & \text{if } x \in A \\ g(x), & \text{if } x \in B \end{cases}$$

is continuous.

- If $x \notin A$, then $x \in A^c$, which is an open subset of B.

  Therefore, by the continuity of g, h is continuous.

  The case $x \notin B$ is treated similarly.

The Continuity Lemma

- In the remaining case $x \in A \cap B$.
  Let $\varepsilon > 0$.
  - By continuity of f, there exists $\delta_1 > 0$, such that, for all $y \in X \cap A$,
  $$|x - y| < \delta_1 \quad \text{implies} \quad |f(x) - f(y)| < \varepsilon;$$
  - By continuity of g, there exists $\delta_2 > 0$, such that, for all $y \in X \cap A$,
  $$|x - y| < \delta_2 \quad \text{implies} \quad |g(x) - g(y)| < \varepsilon.$$

  It follows that, for all $y \in X$,
  $$|x - y| < \min\{\delta_1, \delta_2\} \quad \text{implies} \quad |h(x) - h(y)| < \epsilon.$$

  This shows that h is continuous.

Equivalence of Paths and of Loops are Equivalences

Theorem

(a) Equivalence of paths is an equivalence relation on the set of paths in a space X.

(b) Equivalence of loops is an equivalence relation on the set of loops in X with base point x_0.

- We only prove Part (b).

  Consider the set of loops in X having base point x_0.

  Any such loop α is equivalent to itself under the homotopy

$$F(t,s) = \alpha(t), \quad (t,s) \in I \times I.$$

  Thus, the relation $\sim_{x_0}$ is reflexive.

Equivalence of Paths and of Loops (Cont'd)

- Suppose $\alpha \sim_{x_0} \beta$.

  Then there is a homotopy $H : I \times I \to X$ satisfying

  $$H(\cdot, 0) = a, \quad H(\cdot, 1) = \beta, \quad H(0, s) = H(1, s) = x_0, \quad s \in I.$$

  Consider the homotopy

  $$\overline{H}(t, s) = H(t, 1 - s), \quad (s, t) \in I \times I.$$

  It shows that $\beta \sim_{x_0} \alpha$.

  Hence, that equivalence of loops is a symmetric relation.

Equivalence of Paths and of Loops (Cont'd)

- Suppose that for loops α, β, and γ, we have $\alpha \sim_{x_0} \beta$ and $\beta \sim_{x_0} \gamma$. Then there are homotopies H and K, such that

$$\begin{array}{llll} H(\cdot,0) = \alpha, & H(\cdot,1) = \beta, & H(0,s) = H(1,s) = x_0, & s \in I, \\ K(\cdot,0) = \beta, & K(\cdot,1) = \gamma, & K(0,s) = K(1,s) = x_0, & s \in I. \end{array}$$

Consider the homotopy L between α and γ defined by

$$L(ts) = \begin{cases} H(t,2s), & \text{if } 0 \le s \le \frac{1}{2} \\ K(t,2s-1), & \text{if } \frac{1}{2} \le s \le 1 \end{cases}$$

The continuity of L follows from the Continuity Lemma with $A = I \times [0, \frac{1}{2}]$ and $B = I \times [\frac{1}{2}, 1]$.

This shows that $\alpha \sim_{x_0} \gamma$.

So $\sim_{x_0}$ is an equivalence relation.

Path Products

Definition

If α and β are paths in X with $\alpha(1)=\beta(0)$, then the **path product** $\alpha\star\beta$ is the path defined by

$$\alpha\star\beta(t)=\begin{cases}\alpha(2t), & \text{if } 0\leq t\leq\frac{1}{2}\\ \beta(2t-1), & \text{if } \frac{1}{2}\leq t\leq 1\end{cases}$$

- The continuity of $\alpha\star\beta$ is a consequence of the Continuity Lemma.

Visualizing Path Products

- Thinking of t as time, a path α in X can be visualized by the motion of a point beginning at $\alpha(0)$ and tracing a continuous route to $\alpha(1)$.
- A product $\alpha \star \beta$ is then visualized as follows.
    - The moving point begins at $\alpha(0)$ and follows path α at twice the normal rate, arriving at $\alpha(1)$ when $t = \frac{1}{2}$;
    - The point then follows path β at twice the normal rate and arrives at $\beta(1)$ at time $t = 1$.
- The condition $\alpha(1) = \beta(0)$ is required for the product of paths in order to avoid discontinuities.

Congruence Property of Loop Homotopy

- We focus on products of loops α and β having common base point x_0.
- In this case the product $\alpha \star \beta$ is also a loop with base point x_0.

Lemma

Suppose that loops $\alpha, \alpha', \beta, \beta'$ in a space X all have base point x_0 and satisfy the relations $\alpha \sim_{x_0} \alpha'$ and $\beta \sim_{x_0} \beta'$. Then the products $\alpha \star \beta$ and $\alpha' \star \beta'$ are homotopic modulo x_0.

- Since $\alpha \sim_{x_0} \alpha'$ and $\beta \sim_{x_0} \beta'$, there are homotopies H and K, such that

$$H(\cdot,0) = \alpha, \quad H(\cdot,1) = \alpha', \quad H(0,s) = H(1,s) = x_0;$$
$$K(\cdot,0) = \beta, \quad K(\cdot,1) = \beta', \quad K(0,s) = K(1,s) = x_0.$$

Congruence Property of Loop Homotopy (Cont'd)

- Define the homotopy

$$L(t,s)=\begin{cases} H(2t,s), & \text{if } 0\leq t\leq \frac{1}{2}, \\ K(2t-1,s), & \text{if } \frac{1}{2}\leq t\leq 1. \end{cases}$$

Then $L(t,s)$ is continuous.

Moreover, we have

$$\begin{array}{rcl} L(\cdot,0) & = & \left.\begin{cases} \alpha(2t), & \text{if } 0\leq t\leq \frac{1}{2} \\ \beta(2t-1), & \text{if } \frac{1}{2}\leq t\leq 1 \end{cases}\right\}=(\alpha\star\beta)(t); \\ L(\cdot,1) & = & \left.\begin{cases} \alpha'(2t), & \text{if } 0\leq t\leq \frac{1}{2} \\ \beta'(2t-1), & \text{if } \frac{1}{2}\leq t\leq 1 \end{cases}\right\}=(\alpha'\star\beta')(t); \\ L(0,s) & = & H(0,s)=x_0; \\ L(1,s) & = & K(1,s)=x_0. \end{array}$$

So L is the required homotopy from $\alpha\star\beta$ to $\alpha'\star\beta'$.

The Fundamental Group

Definition

Consider the family of loops in X with base point x_0.
Homotopy modulo x_0 is an equivalence relation on this family.
Therefore, it partitions it into disjoint equivalence classes.
We denote by

$$[\alpha]$$

the equivalence class determined by loop α.
The class $[\alpha]$ is called the **homotopy class** of α.
The set of such homotopy classes is denoted by

$$\pi_1(X, x_0).$$

The Fundamental Group (Cont'd)

Definition (Cont'd)

If $[\alpha]$ and $[\beta]$ belong to $\pi_1(X, x_0)$, then the product $[\alpha] \circ [\beta]$ is defined by

$$[\alpha] \circ [\beta] = [\alpha \star \beta].$$

Thus the product of two homotopy classes is the class determined by the path product of their representative elements.

By the preceding lemma, the product $\circ$ is well-defined on $\pi_1(X, x_0)$.

The set $\pi_1(X, x_0)$ with the $\circ$ operation is called the **fundamental group** of X at x_0, the **first homotopy group** of X at x_0, or the **Poincaré group** of X at x_0.

The Group Property of $\pi_1(X, x_0)$

Theorem

The set $\pi_1(X, x_0)$ is a group under the $\circ$ operation.

- To show that $\pi_1(X, x_0)$ is a group, we must show that:
  - There is a loop c for which $[c]$ is an identity element;
  - Each homotopy class $[\alpha]$ has an inverse $[\overline{\alpha}] = [\alpha]^{-1}$;
  - The multiplication $\circ$ is associative.

  We prove each of these as a separate lemma.

Lemma A: Existence of Identity

Lemma A

$\pi_1(X,x_0)$ has an identity element $[c]$, where c is the constant loop whose only value is x_0.

- The constant loop c is defined by $c(t)=x_0,\ t\in I$.

  Let α be a loop in X based at x_0.

  Then

  $$c\star\alpha(t)=\begin{cases} x_0, & \text{if } 0\le t\le\frac{1}{2},\\ \alpha(2t-1), & \text{if } \frac{1}{2}\le t\le 1.\end{cases}$$

  We must show that $[c\star\alpha]=[\alpha]$.

  We need a homotopy $H:I\times I\to X$, such that

  $$H(\cdot,0)=c\star\alpha,\quad H(\cdot,1)=\alpha,\quad H(0,s)=H(1,s)=x_0,\quad s\in I.$$

Lemma A: Existence of Identity (Cont'd)

- We define

$$H(s,t)=\begin{cases} x_0, & \text{if } 0 \le t \le \frac{1-s}{2}, \\ \alpha\left(\frac{2t+s-1}{s+1}\right), & \text{if } \frac{1-s}{2} \le t \le 1. \end{cases}$$

Then we have

$$H(t,0)=\left.\begin{cases} x_0, & \text{if } 0 \le t \le \frac{1}{2} \\ \alpha(2t-1), & \text{if } \frac{1}{2} \le t \le 1 \end{cases}\right\} = c \star \alpha(t);$$
$$H(t,1)=\left.\begin{cases} x_0, & \text{if } 0 \le t \le 0 \\ \alpha(t), & \text{if } 0 \le t \le 1 \end{cases}\right\} = \alpha(t);$$

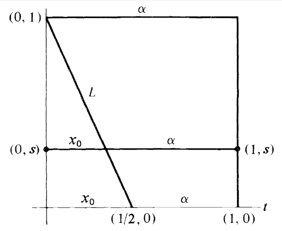
$$H(0,s)=x_0, \quad H(1,s)=\alpha\left(\frac{2+s-1}{s+1}\right)=\alpha(1)=x_0, \quad s \in I.$$

Continuity of H is assured by the Continuity Lemma, since $\frac{2t+s-1}{s+1}$ is a continuous function of (t,s) and the two parts of the definition of H agree when $t=\frac{1-s}{2}$.

The Homotopy $H(s,t)$ of Lemma A

- The homotopy H was obtained from the diagram as follows.

We wish to define a homotopy H on the unit square which agrees with $c \star \alpha$ on the bottom and with α on the top.



We define the s-level $H(\cdot,s)$ to have value x_0 at each point (t,s) from $t=0$ out to the diagonal line L.

Then $H(\cdot,s)$ follows the route of α.

Since L has equation $t=\frac{1-s}{2}$ and the "time" remaining when $t=\frac{1-s}{2}$ is $1-\frac{1-s}{2}=\frac{1+s}{2}$, the desired effect is accomplished by defining

$$H(t,s)=\begin{cases} x_0, & \text{if } 0 \le t \le \frac{1-s}{2}, \\ \alpha((t-\frac{1-s}{2})\frac{2}{1+s}), & \text{if } \frac{1-s}{2} \le t \le 1. \end{cases}$$

This expression reduces to the formula for H given previously.

The Right Identity Property

- We have now proved the following.

  If $[\alpha] \in \pi_1(X, x_0)$, then

  $$[c] \circ [\alpha] = [c \star \alpha] = [\alpha].$$

  So $[c]$ is a left identity for $\pi_1(X, x_0)$.

  To show that $[c]$ is also a right identity, we must show $[\alpha \star c] = [\alpha]$.

  This is accomplished by the homotopy

  $$H'(t,s) = \begin{cases} \alpha(\frac{2t}{s+1}), & \text{if } 0 \le t \le \frac{s+1}{2}, \\ x_0, & \text{if } \frac{s+1}{2} \le t \le 1. \end{cases}$$

  The details are similar.

Existence of Inverses

Lemma B

For each homotopy class $[\alpha]$ in $\pi_1(X, x_0)$, the inverse of $[\alpha]$ with respect to the operation $\circ$ and the identity element $[c]$ is the class $[\overline{\alpha}]$, where

$$\overline{\alpha}(t) = \alpha(1-t), \quad t \in I.$$

- The path $\overline{\alpha}(t) = \alpha(1-t)$ is called the **reverse** of the path α. It begins at $\alpha(1) = x_0$ and traces the route of α backwards. We must prove that

$$[\alpha] \circ [\overline{\alpha}] = [\overline{\alpha}] \circ [\alpha] = [c].$$

Note that $[\alpha] \circ [\overline{\alpha}] = [\alpha * \overline{\alpha}]$, where

$$\alpha * \overline{\alpha}(t) = \begin{cases} \alpha(2t), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \alpha(2-2t), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Existence of Inverses (Cont'd)

- The path $\alpha * \overline{\alpha}$ follows α and then the reverse of α back to x_0.
  We shall define a homotopy K for which the s-level $K(\cdot, s)$:
  - Follows route α out to $\alpha(s)$;
  - Then retraces its steps back to x_0.

  This is accomplished by defining

$$K(t,s) = \begin{cases} \alpha(2ts), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \alpha(2s - 2ts), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Existence of Inverses (Cont'd)

- Observe that:
  - $K(\cdot,0) = \alpha(0) = c$;
  - $K(\cdot,1) = \alpha * \overline{\alpha}$;
  - $K(0,s) = K(1,s) = x_0$, for $s \in I$.

  Moreover, K is continuous.

  Thus,
  $$[\alpha] \circ [\overline{\alpha}] = [\alpha * \overline{\alpha}] = [c].$$

  So $[\overline{\alpha}]$ is a right inverse for $[\alpha]$.

  Since the reverse of the reverse of α is α (i.e., $\overline{\overline{\alpha}} = \alpha$), the same proof shows that
  $$[\overline{\alpha}] \circ [\alpha] = [\overline{\alpha}] \circ \left[\overline{\overline{\alpha}}\right] = [c].$$

  Hence $[\overline{\alpha}] = [\alpha]^{-1}$ is a two-sided inverse for $[\alpha]$.

Associativity of Multiplication

Lemma C

The multiplication $\circ$ is associative.

- Let $[\alpha], [\beta]$ and $[\gamma]$ be members of $\pi_1(X, x_0)$.

  We must prove that $([\alpha] \circ [\beta]) \circ [\gamma] = [\alpha] \circ ([\beta] \circ [\gamma])$.

  Equivalently, we must show

  $$[(\alpha \star \beta) \star \gamma] = [\alpha \star (\beta \star \gamma)].$$

  A little arithmetic shows that

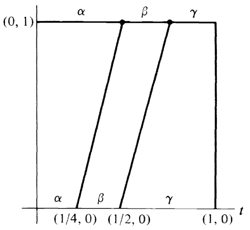
  $$(\alpha * \beta) * \gamma(t) = \begin{cases} \alpha(4t), & \text{if } 0 \leq t \leq \frac{1}{4}, \\ \beta(4t-1), & \text{if } \frac{1}{4} \leq t \leq \frac{1}{2}, \\ \gamma(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

Associativity of Multiplication (Cont'd)

- Similarly,

$$\alpha * (\beta * \gamma)(t) = \begin{cases} \alpha(2t), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ \beta(4t-2), & \text{if } \frac{1}{2} \leq \frac{3}{4}, \\ \gamma(4t-3), & \text{if } \frac{3}{4} \leq t \leq 1. \end{cases}$$

  We use the figure on the right.
  We obtain the homotopy



$$L(t,s) = \begin{cases} \alpha(\frac{4t}{s+1}), & \text{if } 0 \leq t \leq \frac{s+1}{4}, \\ \beta(4t-1-s), & \text{if } \frac{s+1}{4} \leq t \leq \frac{s+2}{4}, \\ \gamma(\frac{4t-2-s}{2-s}), & \text{if } \frac{s+2}{4} \leq t \leq 1. \end{cases}$$

  One can verify that L is a homotopy modulo x_0 between $(\alpha \star \beta) \star \gamma$ and $\alpha \star (\beta \star \gamma)$.

Path Connected Spaces

Definition

A space X is **path connected** if each pair of points in X can be joined by a path. In other words, if x_0 and x_1 are points in X, then there is a path in X with initial point x_0 and terminal point x_1.

Path Connected Spaces and Fundamental Groups

Theorem

If a space X is path connected and x_0, x_1 are points in X, then the fundamental groups $\pi_1(X,x_0)$ and $\pi_1(X,x_1)$ are isomorphic.

- Let $\rho : I \to X$ be a path such that $\rho(0) = x_0$, $\rho(1) = x_1$.
  Let α be a loop based at x_0.
  Then $(\overline{\rho} \star \alpha) \star \rho$ is a loop based at x_1, where $\overline{\rho}$ is the reverse of ρ,

$$\overline{\rho}(t) = \rho(1-t), \quad 0 \le t \le 1.$$

  We define a function $P : \pi_1(X,x_0) \to \pi_1(X,x_1)$ by

$$P([\alpha]) = [(\overline{\rho} \star \alpha) \star \rho], \quad [\alpha] \in \pi_1(X,x_0).$$

  The image of $[\alpha]$ is independent of the choice of path in $[\alpha]$.
  So P is well defined.

Path Connected Spaces (Obsevations)

- We make some observations before continuing with the proof.
- Lemma B with minor modifications shows that $[\rho \star \overline{\rho}]$ and $[\overline{\rho} \star \rho]$ are the identity elements of $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$, respectively.
- Lemma C can be easily modified to show that for any paths α, β, γ for which $(\alpha \star \beta) \star \gamma$ and $\alpha \star (\beta \star \gamma)$ are defined, the indicated triple products are equivalent.
- Thus, in $[(\overline{\rho} \star \alpha) \star \rho]$ we may ignore the inner parentheses and simply write $[\overline{\rho} \star \alpha \star \rho]$, since the equivalence class is the same regardless of the way in which the terms of the product are associated.

Path Connected Spaces (Homomorphism)

- Consider $[\alpha], [\beta]$ in $\pi_1(X, x_0)$.

  Then we have

$$\begin{array}{rcl} P([\alpha] \circ [\beta]) & = & P([\alpha \star \beta]) \\ & = & [\overline{\rho} \star \alpha \star \beta \star \rho] \\ & = & [\overline{\rho} \star \alpha \star \rho \star \overline{\rho} \star \beta \star \rho] \\ & = & [\rho \star \alpha \star \rho] \circ [\overline{\rho} \star \beta \star \rho] \\ & = & P([\alpha]) \circ P([\beta]). \end{array}$$

  Thus, P is a homomorphism.

Path Connected Spaces (Isomorphism)

- The inverse of P is the function $Q:\pi_1(X,x_1)\to\pi_1(X,x_0)$ defined by

$$Q([\sigma])=[\rho\star\sigma\star\overline{\rho}],\quad [\sigma]\in\pi_1(X,x_1).$$

To see this, observe that for $[\alpha]\in\pi_1(X,x_0)$,

$$\begin{array}{rcl}QP([\alpha]) & = & Q([\overline{\rho}\star\alpha\star\rho])\\ & = & [\rho\star\overline{\rho}\star\alpha\star\rho\star\overline{\rho}]\\ & = & [\rho\star\overline{\rho}]\circ[\alpha]\circ[\rho\star\overline{\rho}]\\ & = & [\alpha].\end{array}$$

Thus, QP is the identity map on $\pi_1(X,x_0)$.

By symmetry, PQ must be the identity map on $\pi_1(X,x_1)$.

Thus the indicated fundamental groups are isomorphic.

Remarks on Fundamental Groups and Base Points

- By the preceding theorem, when X is path connected, the same fundamental group results regardless of the choice of the base point.
- So, when X is path connected, we refer to "the fundamental group of X", written $\pi_1(X)$.
- This applies primarily to the process of computing the fundamental group of a given space.
- Note, however, that the theorem does not guarantee that the isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ is unique.
- Quite often different paths lead to different isomorphisms.
- For this reason, there are many applications of the fundamental group in which the specification of a base point is important.
- E.g., when comparing fundamental groups of two spaces X and Y on the basis of a continuous map $f : X \to Y$, it is usually necessary to specify a base point for each space.

Simply Connected Spaces

Definition

A path connected space X is **simply connected** provided that $\pi_1(X)$ is the trivial group.

Theorem

Every contractible space is simply connected.

- Let X be a contractible space.

  There is a point x_0 in X and a homotopy $H : X \times I \to X$, such that

$$H(x,0) = x, \quad H(x,1) = x_0, \quad x \in X.$$

Simply Connected Spaces (A Special Case)

Claim: X is path connected.

Consider a point $x \in X$.

Define the function

$$\alpha_x = H(x,\cdot) : I \to X.$$

α_x is a path from $H(x,0) = x$ to $H(x,1) = x_0$.

Thus, any two points x and y are joined by the path

$$\alpha_x \star \overline{\alpha}_y,$$

where $\overline{\alpha}_y$ is the reverse of α_y.

Simply Connected Spaces (A Special Case)

- Assume for a moment that H has the additional property

$$H(x_0, s) = x_0, \quad s \in I.$$

For $[\alpha] \in \pi_1(X, x_0)$, define a homotopy $h : I \times I \to X$ by

$$h(t, s) = H(\alpha(t), s).$$

Then

$$h(t, 0) = \alpha(t), \quad h(t, 1) = x_0, \quad t \in I,$$
$$h(0, s) = h(1, s) = x_0, \quad s \in I.$$

The extra assumption $H(x_0, s) = x_0$ gives $h(0, s) = h(1, s) = x_0$.

Thus, h demonstrates that α is equivalent to c, the constant loop whose only value is x_0.

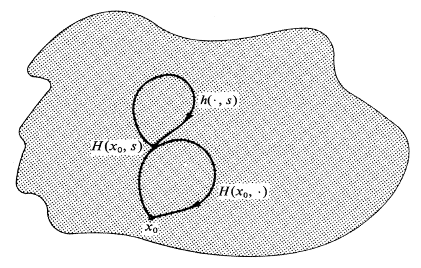
So $[\alpha] = [c]$ and $\pi_1(X, x_0)$ consists only of an identity element.

Simply Connected Spaces (General Case)

- The process needs tweaking if the path $H(x_0, \cdot) : I \to X$ is not constant.

  We must modify each level of the homotopy h to produce at each level a loop based at x_0.

  The procedure is illustrated in the figure.



  The details are omitted.

Subsection 3

The Covering Homotopy Property for S^1

The Function exp

- We determine the fundamental group of the circle.
- We consider the unit circle S^1 as a subset of the complex plane.
- Consider $\mathbb{R}^2$ as the set of all complex numbers $x = x_1 + ix_2$.
- We refer several times to the function $p : \mathbb{R} \to S^1$, defined by

$$p(t) = \exp(2\pi it), \quad t \in \mathbb{R}.$$

- Here exp denotes the exponential function on the complex plane.
- If t is in the set $\mathbb{R}$ of real numbers, then

$$\exp(2\pi it) = \cos(2\pi t) + i\sin(2\pi t).$$

  - p maps each integer n in $\mathbb{R}$ to 1 in S^1.
  - p wraps each interval $[n, n+1]$ exactly once around S^1 in the counterclockwise direction.

Covering Paths and Homotopies

Definition

Let $\sigma : I \to S^1$ be a path. Then a path $\widetilde{\sigma} : I \to \mathbb{R}$, such that

$$p\widetilde{\sigma} = \sigma$$

is called a **covering path** of σ or a **lifting** of σ to the real line $\mathbb{R}$.
Let $F : I \times I \to S^1$ be a homotopy. Then a homotopy $\widetilde{F} : I \times I \to \mathbb{R}$, such that

$$p\widetilde{F} = F$$

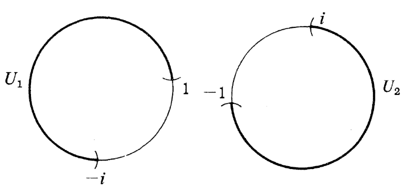
is called a **covering homotopy** or a **lifting** of F.

The Covering Path Property

Theorem (The Covering Path Property)

If $\sigma : I \to S^1$ is a path in S^1 with initial point 1, then there is a unique covering path $\widetilde{\sigma} : I \to \mathbb{R}$ with initial point 0.

- Consider the following open arcs in S^1.



  - U_1 begins at 1 and extends counterclockwise to $-i$;
  - U_2 begins at -1 and extends counterclockwise to i.

The Covering Path Property (Cont'd)

- By definition, U_1 and U_2 are open sets in S^1.

  Moreover, $U_1 \cup U_2 = S^1$.

  We also have

$$p^{-1}(U_1) = \bigcup_{n=-\infty}^{\infty} \left(n, n + \frac{3}{4}\right), \quad p^{-1}(U_2) = \bigcup_{n=-\infty}^{\infty} \left(n - \frac{1}{2}, n + \frac{1}{4}\right).$$

  Note that p maps:
  - Each interval $(n, n + \frac{3}{4})$ homeomorphically onto U_1;
  - Each interval $(n - \frac{1}{2}, n + \frac{1}{4})$ homeomorphically onto U_2.

Covering Path Property (Idea of Proof)

- We summarize the idea of the proof of the Covering Path Property.
- We subdivide the range of the path α into sections so that each section is contained either in U_1 or in U_2.
  - Suppose a particular section is contained in U_1.
    Then we choose one of the intervals $V = (n, n + \frac{3}{4})$.
    We consider the restriction $p|_V$ of p to this interval.
    We compose $(p|_V)^{-1}$ with this section of the path.
    This "lifts" the section to a section of a path in $\mathbb{R}$.
  - The same method applies to sections lying in U_2.
- For continuity, we must ensure that the initial point of a given lifted section is the terminal point of the lifted section that precedes it.

Covering Paths Property (Proof)

- The following is applied inductively.
  Consider the open cover $\{\sigma^{-1}(U_1), \sigma^{-1}(U_2)\}$ of I.
  Let ϵ be a Lebesgue number for the open cover.
  Choose a sequence
  $$0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$$
  of numbers in I, with each successive pair differing by less than ϵ.
  Then the image $\sigma([t_i, t_{i+1}])$ of any subinterval $[t_i, t_{i+1}]$, $0 \leq i \leq n-1$, must be contained in either U_1 or U_2
  Note that $\sigma(t_0) = \sigma(0) = 1 \notin U_1$.
  So $\sigma([t_0, t_1])$ must be contained in U_2.
  Let $V_1 = (-\frac{1}{2}, \frac{1}{4})$.
  Define $\widetilde{\sigma}$ on $[t_0, t_1]$ by
  $$\widetilde{\sigma}(t) = (p|_{V_1})^{-1}\sigma(t).$$

Covering Paths Property (Proof)

- Suppose that σ has been defined on the interval $[t_0, t_k]$.
  Then $\sigma([t_k, t_{k+1}]) \subseteq U$, where U is either U_1 or U_2.
  Let V_{k+1} be the component of $p^{-1}(U)$ to which $\widetilde{\sigma}(t_k)$ belongs.
  Note that V_{k+1} is one of the intervals $(n, n+\frac{3}{4})$ or $(n-\frac{1}{2}, n+\frac{1}{4})$.
  Then $p|_{V_{k+1}}$ is a homeomorphism.
  The desired extension of $\widetilde{\sigma}$ to $[t_k, t_{k+1}]$ is obtained by defining

  $$\widetilde{\sigma}(t) = (p|_{V_{k+1}})^{-1}\sigma(t), \quad t \in [t_k, t_{k+1}].$$

  The continuity of $\widetilde{\sigma}$ is guaranteed by the Continuity Lemma since the lifted sections agree at the endpoints t_k.
  This extends $\widetilde{\sigma}$ to $[t_0, t_n] = I$.

Covering Paths Property (Uniqueness)

- To prove that $\widetilde{\sigma}$ is the only such covering path, suppose that σ' also satisfies the required properties $p\sigma' = \sigma$ and $\sigma'(0) = 0$.

  Then the path $\widetilde{\sigma} - \sigma'$ has initial point 0 and, for all $t \in I$,

  $$p(\widetilde{\sigma}(t) - \sigma'(t)) = \frac{p\widetilde{\sigma}(t)}{p\sigma'(t)} = \frac{\sigma(t)}{\sigma(t)} = 1.$$

  So $\widetilde{\sigma} - \sigma'$ is a covering path of the constant path with value is 1.

  But p maps only integers to 1.

  So $\widetilde{\sigma} - \sigma'$ must have only integral values.

  Since I is connected, $\widetilde{\sigma} - \sigma'$ can have only one integral value.

  This one value must be the initial value, 0.

  Therefore, $\widetilde{\sigma} - \sigma' = 0$. So $\widetilde{\sigma} = \sigma'$.

  So the required lifting $\widetilde{\sigma}$ is unique.

The Generalized Covering Path Property

Corollary (The Generalized Covering Path Property)

Let σ be a path in S^1 and r is a real number such that $p(r) = \sigma(0)$. Then there is a unique covering path $\widetilde{\sigma}$ of σ with initial point r.

- The path $\frac{\sigma}{\sigma(0)}$ is a path in S^1 with initial point $\frac{\sigma(0)}{\sigma(0)} = 1$.

  Therefore, it has a unique covering path η with initial point 0.

  Consider the path $\widetilde{\sigma} : I \to \mathbb{R}$ defined by

  $$\widetilde{\sigma}(t) = r + \eta(t), \quad t \in I.$$

  It is the required covering path of σ with initial point r.

  The uniqueness of $\widetilde{\sigma}$ follows from that of η.

The Covering Homotopy Property

Theorem (The Covering Homotopy Property)

Let $F : I \times I \to S^1$ be a homotopy such that $F(0,0) = 1$. Then there is a unique covering homotopy $\widetilde{F} : I \times I \to \mathbb{R}$, such that $F(0,0) = 0$.

- The proof is similar to that of the Covering Path Property.
  We use the same open sets U_1, U_2 in S^1.
  By a Lebesgue number argument, there must exist numbers
  $$0 = t_0 < t_1 < \cdots < t_n = 1, \quad 0 = s_0 < s_1 < \cdots < s_m = 1,$$
  such that F maps any $[t_i, t_{i+1}] \times [s_k, s_{k+1}]$ into either U_1 or U_2.
  Since $F(0,0) = 1 \notin U_1$, $F([t_0, t_1] \times [s_0, s_1])$ must be contained in U_2.
  Let $V_1 = (-\frac{1}{2}, \frac{1}{4})$.
  Define $\widetilde{F}$ on $[t_0, t_1] \times [s_0, s_1]$ by
  $$\widetilde{F}(t,s) = (p\,|_{V_1})^{-1} F(t,s).$$

The Covering Homotopy Property (Existence)

- Extend the definition of $\widetilde{F}$ over the rectangles $[t_1, t_{i+1}] \times [s_0, s_1]$ in succession as in the proof of the Covering Path Property.

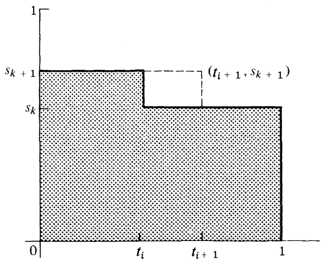
  Take care that, at each step, the definitions agree on common edges of adjacent rectangles.

  This will define $\widetilde{F}$ on $I \times [s_0, s_1]$.

  Proceeding inductively, suppose that $\widetilde{F}$ has been defined on

$$(I \times [s_0, s_k]) \cup ([t_0, t_i] \times [s_k, s_{k+1}]).$$

  We wish to extend the domain to include $[t_i, t_{i+1}] \times [s_k, s_{k+1}]$, as shown.



The Covering Homotopy Property (Existence Cont'd)

- Let
$$A = \{(x,y) \in [t_i, t_{i+1}] \times [s_k, s_{k+1}] : x = t_i \text{ or } y = s_k\}$$
be the boundary of the present domain of F with $[t_i, t_{i+1}] \times [s_k, s_{k+1}]$.
Now, $F([t_i, t_{i+1}] \times [s_k, s_{k+1}])$ is contained in either U_1 or U_2.
Denote this containing set by U.
Let V be the component of $p^{-1}(U)$ which contains $\widetilde{F}(A)$.
Define $\widetilde{F}$ on $[t_i, t_{i+1}] \times [s_k, s_{k+1}]$ by
$$\widetilde{F}(t,s) = (p\,|_V)^{-1} F(t,s).$$
The continuity of $\widetilde{F}$ follows from the Continuity Lemma since the old and new definitions of $\widetilde{F}$ agree on the closed set A.
This induction extends the domain of $\widetilde{F}$ to $[t_0, t_n] \times [s_0, s_m] = I \times I$.

The Covering Homotopy Property (Uniqueness)

- $\widetilde{F}$ is the only covering homotopy of F with $\widetilde{F}(0,0)=0$.

  Suppose that F' is another one.

  Then the homotopy $\widetilde{F}-F'$ has the properties

$$(\widetilde{F}-F')(0,0)=\widetilde{F}(0,0)-F'(0,0)=0$$

  and, for all (t,s) in $I\times I$,

$$p(\widetilde{F}-F')(t,s)=\frac{p\widetilde{F}(t,s)}{pF'(t,s)}=\frac{F(t,s)}{F(t,s)}=1.$$

  Thus, as in the case of covering paths, $\widetilde{F}-F'$ can have only one integral value, namely 0.

  Then $\widetilde{F}=F'$ and the covering homotopy is unique.

Degree of a Loop

Definition

Let α be a loop in S^1 with base point 1.
By the Covering Path Property, there is exactly one covering path

$$\widetilde{\alpha}$$

of α with initial point 0. We have

$$1 = \alpha(1) = p\widetilde{\alpha}(1) = \exp(2\pi i\widetilde{\alpha}(1)).$$

So $\widetilde{\alpha}(1)$ must be an integer.
This integer is called the **degree** of the loop α.

Equivalent Loops

Theorem

Two loops α and β in S^1 with base point 1 are equivalent if and only if they have the same degree.

- Let $\widetilde{\alpha}$ and $\widetilde{\beta}$ denote the covering paths of α and β, respectively, having initial point 0 in $\mathbb{R}$.

  Suppose that α and β have the same degree so that $\widetilde{\alpha}(1) = \widetilde{\beta}(1)$.

  Define a homotopy $H : I \times I \to \mathbb{R}$ by

  $$H(t,s) = (1-s)\widetilde{\alpha}(t) + s\widetilde{\beta}(t), \quad (t,s) \in I \times I.$$

  Then H demonstrates the equivalence of $\widetilde{\alpha}$ and $\widetilde{\beta}$ as paths in $\mathbb{R}$.

  Note that $H(1,s)$ is the common degree of α and β for each s in I.

  The homotopy $pH : I \times I \to S^1$ shows the equivalence of α and β as loops in S^1.

Equivalent Loops (Cont'd)

- Suppose, next, that α and β are equivalent loops in S^1.
  Let $K : I \times I \to S^1$ be a homotopy such that
  $$K(\cdot,0) = \alpha, \quad K(\cdot,1) = \beta, \quad K(0,s) = K(1,s) = 1, \quad s \in I.$$
  By the Covering Homotopy Property, there is a covering homotopy $\widetilde{K} : I \times I \to \mathbb{R}$, such that $\widetilde{K}(0,0) = 0$, $p\widetilde{K} = K$. Then, for all $s \in I$,
  $$p\widetilde{K}(0,s) = K(0,s) = 1.$$
  So $\widetilde{K}(0,s)$ must be an integer for each value of s.
  Since I is connected, $\widetilde{K}(0,\cdot)$ must have only the value $\widetilde{K}(0,0) = 0$.
  A similar argument shows that $\widetilde{K}(1,\cdot)$ is also a constant function.

Equivalent Loops (Cont'd)

- We have
  - $p\widetilde{K}(\cdot,0) = K(\cdot,0) = \alpha$;
  - $p\widetilde{K}(\cdot,1) = K(\cdot,1) = \beta$.

  Hence, $\widetilde{K}(\cdot,0) = \widetilde{\alpha}$ and $\widetilde{K}(\cdot,1) = \widetilde{\beta}$ are the unique covering paths of α and β, respectively, with initial point 0.

  It follows that

$$\text{degree}\alpha = \widetilde{\alpha}(1) = \widetilde{K}(1,0) = \widetilde{K}(1,1) = \widetilde{\beta}(1) = \text{degree}\beta.$$

  So α and β must have the same degree.

The Fundamental Group $\pi_1(S^1)$

Corollary

The fundamental group $\pi_1(S^1)$ is isomorphic to the group $\mathbb{Z}$ of integers under addition.

- Consider $\pi_1(S^1, 1)$.

  Define a function $\deg : \pi_1(S^1, 1) \to \mathbb{Z}$ by $\deg[\alpha] = \text{degree}\alpha$.

  By the preceding theorem, deg is well-defined and one-to-one.

  We show that deg maps $\pi_1(S^1, 1)$ onto $\mathbb{Z}$.

  Let n be an integer.

  The loop γ in S^1 defined by

  $$\gamma(t) = \exp(2\pi int)$$

  is covered by the path $t \mapsto nt$, $t \in I$.

  Therefore, it has degree n. Thus, $\deg[\gamma] = n$.

The Fundamental Group $\pi_1(S^1)$ (Cont'd)

- Suppose now that $[\alpha]$ and $[\beta]$ are in $\pi_1(S^1, 1)$. We must show that
$$\deg([\alpha] \circ [\beta]) = \deg[\alpha] + \deg[\beta].$$
Let $\widetilde{\alpha}$, $\widetilde{\beta}$ be the unique covering paths of α, β which begin at 0. The path $f : I \to \mathbb{R}$ defined by
$$f(t) = \begin{cases} \widetilde{\alpha}(2t), & \text{if } 0 \le t \le \frac{1}{2}, \\ \widetilde{\alpha}(1) + \widetilde{\beta}(2t-1), & \text{if } \frac{1}{2} \le t \le 1, \end{cases}$$
is the covering path of $\alpha \star \beta$ with initial point 0. Thus
$$\text{degree}(\alpha \star \beta) = f(1) = \widetilde{\alpha}(1) + \widetilde{\beta}(1) = \text{degree}\alpha + \text{degree}\beta.$$
Then
$$\begin{array}{rcl} \deg([\alpha] \circ [\beta]) & = & \text{degree}(\alpha \star \beta) \\ & = & \text{degree}\alpha + \text{degree}\beta \\ & = & \deg[\alpha] + \deg[\beta]. \end{array}$$

Subsection 4

Examples of Fundamental Groups

Deformation Retracts

Definition

Let X be a space and A a subspace of X.
The subspace A is a **deformation retract** of X if there exists a homotopy $H: X \times I \to X$, such that

$$H(x,0) = x, \quad H(x,1) \in A, \quad x \in X,$$
$$H(a,t) = a, \quad a \in A, \ t \in I.$$

The homotopy H is called a **deformation retraction**.

Deformation Retracts and Fundamental Groups

Theorem

Let A is a deformation retract of a space X and x_0 is a point of A. Then $\pi_1(X, x_0)$ is isomorphic to $\pi_1(A, x_0)$.

- Let $H : X \times I \to X$ be a deformation retraction of X onto A.

  Consider a loop α in X with base point x_0.

  Then

  $$H(\alpha(\cdot), 1)$$

  is a loop in A with base point x_0.

  We, therefore, define $h : \pi_1(X, x_0) \to \pi_1(A, x_0)$ by

  $$h([\alpha]) = [H(\alpha(\cdot), 1)].$$

Deformation Retracts and Fundamental Groups (Cont'd)

- For $[a],[\beta]$ in $\pi_1(X,x_0)$,

$$\begin{array}{rcl} h([\alpha]\circ[\beta]) & = & h([\alpha\star\beta]) \\ & = & [H(\alpha\star\beta(\cdot),1)] \\ & = & [H(\alpha(\cdot),1)\star H(\beta(\cdot),1)] \\ & = & h([\alpha])\circ h([\beta]). \end{array}$$

  So h is a homomorphism.

  Now $H(\alpha(\cdot),1)$ is equivalent to $H(\alpha(\cdot),0)=a$ as loops in X.

  Therefore, h is one-to-one.

  Suppose, next, that $[\gamma]$ is in $\pi_1(A,x_0)$.

  Then γ determines a homotopy class (still called $[\gamma]$) in $\pi_1(A,x_0)$.

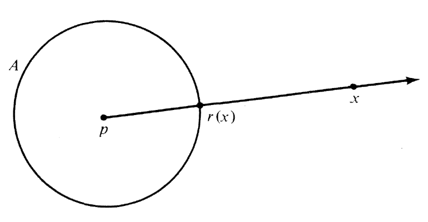
  Since H leaves each point of A fixed, then

$$h([\gamma])=H(\gamma(\cdot),1)=[\gamma].$$

  So h maps $\pi_1(X,x_0)$ onto $\pi_1(A,x_0)$. Thus, h is an isomorphism.

Fundamental Group of the Punctured Plane

- Consider the punctured plane $\mathbb{R}^2 \setminus \{p\}$ consisting of all points in $\mathbb{R}^2$ except a particular point p. Let A be a circle with center p. Let $x \in \mathbb{R}^2 \setminus \{p\}$.



The half line from p through x intersects A at a point $r(x)$.

This function r is clearly a retraction of $\mathbb{R}^2 \setminus \{p\}$ onto A.

Define a homotopy $H : (\mathbb{R}^2 \setminus \{p\}) \times I \to \mathbb{R}^2 \setminus \{p\}$ by

$$H(x,t) = tr(x) + (1-t)x, \quad x \in \mathbb{R}^2 \setminus \{p\}, \ t \in I.$$

We can show that H is a deformation retraction.

So A is a deformation retract of $\mathbb{R}^2 \setminus \{p\}$.

It follows that $\pi_1(\mathbb{R}^2 \setminus \{p\}) \cong \pi_1(A) \cong \mathbb{Z}$.

Examples

Example: Consider an annulus X in the plane.

Both the inner and outer circles of X are deformation retracts.

So $\pi_1(X)$ is the group of integers.

Example: Each of the following spaces is contractible.

(a) A single point;
(b) An interval on the real line;
(c) The real line;
(d) Euclidean n-space $\mathbb{R}^n$;
(e) Any convex set in $\mathbb{R}^n$.

It follows that each of these spaces has fundamental group $\{0\}$.

Fundamental Groups and Product Spaces

Theorem

Let X and Y be spaces with points x_0 in X and y_0 in Y. Then

$$\pi_1(X \times Y, (x_0, y_0)) \cong \pi_1(X, x_0) \oplus \pi_1(Y, y_0).$$

- Let p_1 and p_2 denote the projections of the product space $X \times Y$ on X and Y, respectively:

$$p_1(x, y) = x, \quad p_2(x, y) = y, \quad (x, y) \in X \times Y.$$

Any loop α in $X \times Y$ based at (x_0, y_0) determines loops

$$\alpha_1 = p_1 \alpha, \quad \alpha_2 = p_2 \alpha$$

in X and Y based at x_0 and y_0, respectively.

Fundamental Groups and Product Spaces (Cont'd)

- Conversely, consider a pair of loops

$$\alpha_1 \quad \text{and} \quad \alpha_2$$

in X and Y based at x_0 and y_0, respectively.

Such a pair determines a loop

$$\alpha = (\alpha_1, \alpha_2)$$

in $X \times Y$ based at (x_0, y_0).

The function $h : \pi_1(X \times Y, (x_0, y_0)) \to \pi_1(X, x_0) \oplus \pi_1(Y, y_0)$ defined by

$$h([\alpha]) = ([\alpha_1], [\alpha_2]), \quad [\alpha] \in \pi_1(X \times Y, (x_0, y_0)),$$

is the required isomorphism.

Examples

Example: The torus T is homeomorphic to the product $S^1 \times S^1$. Hence

$$\pi_1(T) \cong \pi_1(S^1) \oplus \pi_1(S^1) \cong \mathbb{Z} \oplus \mathbb{Z}.$$

Example: An n-**dimensional torus** T^n is the product of n unit circles. Hence, $\pi_1(T^n)$ is isomorphic to the direct sum of n copies of the group of integers.

Example: A **closed cylinder** C is the product of a circle S^1 and a closed interval $[a, b]$. Thus,

$$\pi_1(C) \cong \pi_1(S^1) \oplus \pi_1([a, b]) \cong \mathbb{Z} \oplus \{0\} \cong \mathbb{Z}.$$

Connectivity

Theorem

Let X be a space for which there is an open cover $\{V_i\}$ of X such that:

(a) $\bigcap V_i \neq \emptyset$;

(b) Each V_i is simply connected;

(c) For $i \neq j$, $V_i \cap V_j$ is path connected.

Then X is simply connected.

- By hypothesis:
  - Each of the open sets V_i is path connected;
  - Their intersection is not empty.

  It follows easily that X is path connected.

  Let x_0 be a point in $\bigcap V_i$.

  We must show that $\pi_1(X, x_0)$ is the trivial group.

Connectivity (Cont'd)

- Let $[\alpha]$ be a member of $\pi_1(X, x_0)$.

  Then $\alpha : I \to X$ is a continuous map.

  So the set of all inverse images

  $$\{\alpha^{-1}(V_i)\}$$

  is an open cover of the unit interval I.

  Since I is compact, this open cover has a Lebesgue number ϵ.

  Thus, there is a partition

  $$0 = t_0 < t_1 < t_2 < \cdots < t_n = 1$$

  of I, such that if $0 \leq j \leq n-1$, $\alpha([t_j, t_{j+1}])$ is a subset of some V_i.

  Indeed, we need only require that successive terms of the partition differ by less than ϵ.

Connectivity (Cont'd)

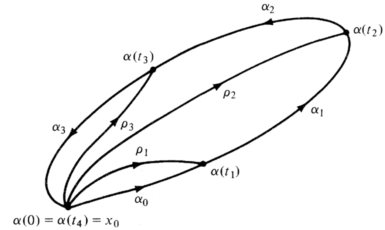
- We alter the notation of the open cover $\{V_i\}$, if necessary, so that

$$\alpha([t_j, t_{j+1}]) \subseteq V_j, \quad 0 \le j \le n-1.$$

Let, for $s \in I$,

$$\alpha_j(s) = \alpha((1-s)t_j + st_{j+1}).$$

We have a sequence $\{\alpha_j\}_{j=0}^{n-1}$ of paths in X, such that:



  - $\alpha_j(I)$ is a subset of the simply connected set V_j;
  - $[\alpha] = [\alpha_0 \star \alpha_1 \star \cdots \star \alpha_{n-1}]$.

But $V_{j-1} \cap V_j$ is path connected. So there is a path ρ_j from x_0 to $\alpha(t_j)$, $1 \le j \le n-1$, lying entirely in $V_{j-1} \cap V_j$ (note that $\alpha(t_j)$ is the terminal point of α_{j-1} and the initial point of α_j).

Connectivity (Conclusion)

- The product $\overline{\rho}_j \star \rho_j$ of ρ_j is equivalent to the constant loop at x_0.

  Thus,

$$\begin{aligned}[\alpha] &= [\alpha_0 \star \overline{\rho}_1 \star \rho_1 \star \alpha_1 \star \overline{\rho}_2 \star \rho_2 \star \alpha_2 \star \cdots \star \overline{\rho}_{n-1} \star \rho_{n-1} \star \alpha_{n-1}] \\ &= [\alpha_0 \star \overline{\rho}_1] \circ [\rho_1 \star \alpha_1 \star \overline{\rho}_2] \circ \cdots \\ &\qquad \circ [\rho_{n-1} \star \alpha_{n-2} \star \overline{\rho}_{n-1}] \circ [\rho_{n-1} \star \alpha_{n-1}].\end{aligned}$$

  The term determined by α_j in this product is the homotopy class of a loop lying in the simply connected set V_j.

  Hence, each term of the product represents the identity class.

  So $[\alpha]$ must be the identity class as well.

  Thus, $\pi_1(X) = \{0\}$, and X is simply connected.

Example

Claim: The space $S^n, n > 1$, has an open cover with two members satisfying the requirements of the theorem.

It follows that

$$\pi_1(S^n) = \{0\}, \quad \text{for } n > 1.$$

Subsection 5

The Relation Between $H_1(K)$ and $\pi_1(|K|)$

On the Relation Between $H_1(K)$ and $\pi_1(|K|)$

- The fundamental group is defined for every topological space.
- We have defined homology groups for polyhedra.
- Suppose $|K|$ is a polyhedron with triangulation K.
- We ask how the groups $H_1(K)$ and $\pi_1(|K|)$ are related.
- For our examples thus far (interval, circle, torus, cylinder, annulus and n-sphere),
$$\pi_1(|K|) \cong H_1(K).$$
- However, this is not true in general.
- The precise answer is given by the following theorem.
- Only an outline of the proof is given.

The Relation Between $H_1(K)$ and $\pi_1(|K|)$

Theorem

Let K be a connected complex. Then $H_1(K)$ is isomorphic to the quotient group $\pi_1(|K|)/F$, where F is the commutator subgroup of $\pi_1(|K|)$. Thus whenever $\pi_1(|K|)$ is abelian, $\pi_1(|K|)$ and $H_1(K)$ are isomorphic.

- Choose a vertex v of K as the base point for the fundamental group.

  Let σ_i be an oriented 1-simplex of K.

  Let α_i denote a linear homeomorphism from $[0,1]$ onto σ_i.

  The α_i are called **elementary edge paths**.

  An **edge loop** is a product of elementary edge paths with v as initial point and terminal point.

  Note that an edge loop $\alpha_1 \star \alpha_2 \star \cdots \star \alpha_n$ corresponds in a natural way to a 1-cycle $1 \cdot \sigma_1 + 1 \cdot \sigma_2 + \cdots + 1 \cdot \sigma_n$.

The Relation Between $H_1(K)$ and $\pi_1(|K|)$ (Cont'd)

- Omitting lengthy details, the following hold.
  (a) If an edge loop is equivalent to the constant loop at v, then the corresponding 1-cycle is a boundary.
  (b) If two edge loops are equivalent, then their corresponding 1-cycles are homologous.
  (c) Each loop in $|K|$ with base point v is equivalent to an edge loop.

  A homomorphism $f : \pi_1(|K|, v) \to H_1(K)$ may be defined as follows.

  Let $[\alpha] \in \pi_1(|K|, v)$.
  Let $\widehat{\alpha} = \alpha_1 \star \alpha_2 \star \cdots \star \alpha_n$ be an edge loop equivalent to α.
  Define the value $f([\alpha])$ to be the homology class determined by the 1-cycle which corresponds to $\widehat{\alpha}$.

  Then f is a homomorphism from $\pi_1(|K|, v)$ onto $H_1(K)$.

  Moreover, its kernel is the commutator subgroup F.

  It follows, by the First Homomorphism Theorem, that the quotient group $\pi_1(|K|, v)/F$ is isomorphic to $H_1(K)$.

Comments

- The fundamental group was defined by Poincaré in *Analysis Situs*.
- The relation between homology and homotopy given in the theorem was known to him.
- Poincaré did not prove the relation, but he stated in Analysis Situs that "fundamental equivalence" of paths in the homotopy sense corresponded precisely to homological equivalence of 1-chains except for commutativity.
- Since the commutator subgroup F of a group G is the smallest subgroup for which G/F is abelian, it is sometimes said that

$$H_1(K) \text{ is "}\pi_1(|K|) \text{ made abelian".}$$