Introduction to Algebraic Topology

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1. Covering Spaces
   - The Definition and Some Examples
   - Basic Properties of Covering Spaces
   - Classification of Covering Spaces
   - Universal Covering Spaces
   - Applications

Subsection 1

The Definition and Some Examples

Local Path Connectivity

- Recall that a space X is **path connected** provided that each pair of points in X can be joined by a path in X.
- A space that satisfies this property locally is called "*locally path connected*".

Definition

A topological space X is **locally path connected** means that X has a basis of path connected open sets.
In other words, if $x \in X$ and O is an open set containing x, then there exists an open set U containing x and contained in O such that U is path connected.

Path Components

Definition

A maximal path connected subset of a space X is called a **path component**.

Thus, A is a path component of X means that A is path connected and is not a proper subset of any path connected subset of X.

The **path components of a subset** B of X are the path components of B in its subspace topology.

- All spaces considered in this set are assumed path connected and locally path connected unless stated otherwise.

Covering Spaces

Definition

Let E and B be spaces and $p : E \to B$ a continuous map.
The pair (E, p) is a **covering space** of B if, for each point x in B, there is a path connected open set $U \subseteq B$, such that $x \in U$ and p maps each path component of $p^{-1}(U)$ homeomorphically onto U.
Such an open set U is called an **admissible neighborhood** or an **elementary neighborhood**.
The space B is the **base space** and p is a **covering projection**.

- In cases where the covering projection is clearly understood, one sometimes refers to E as the covering space.
- We avoid ambiguity by referring to the covering space as (E, p).

Example

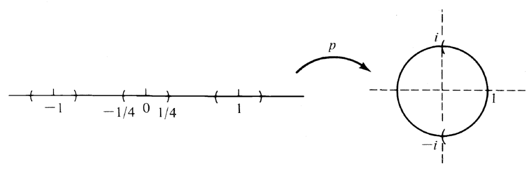
- Consider the map $p:\mathbb{R}\to S^1$ from the real line to the unit circle

$$p(t)=e^{2\pi it}=\cos(2\pi t)+i\sin(2\pi t),\quad t\in\mathbb{R}.$$

Then p is a covering projection.

Any proper open interval or arc on S^1 can serve as an elementary neighborhood.

For the particular point 1 in S^1, let U denote the right hand open interval on S^1 from $-i$ to i.



Then

$$p^{-1}(U)=\bigcup_{n=-\infty}^{\infty}\left(n-\frac{1}{4},n+\frac{1}{4}\right).$$

The path components of $p^{-1}(U)$ are the real intervals $(n-\frac{1}{4},n+\frac{1}{4})$. p maps each of these homeomorphically onto U.

Example

- For any positive integer n, let $q_n : S^1 \to S^1$ be the map defined by

$$q_n(z) = z^n, \quad z \in S^1,$$

where z^n is the n-th power of the complex number z.

Then (S^1, q_n) is a covering space of S^1.

In polar coordinates, the action of q_n is described by

$$q_n \text{ takes any point } (1,\theta) \text{ to } (1, n\theta).$$

Let U be an open interval on S^1 subtended by an angle θ, $0 \leq \theta \leq 2\pi$, and containing a point x. Then $p^{-1}(U)$ consists of n open intervals each determining an angle $\frac{\theta}{n}$ and each containing one n-th root of x.

These n intervals are the path components of $p^{-1}(U)$.

Each is mapped by p homeomorphically onto U.

Thus, any proper open interval in S^1 is an admissible neighborhood.

Example

- Consider a space X.

  Recall that, according to our general assumption, X must be path connected and locally path connected.

  It follows that the identity map

  $$i : X \to X$$

  is a covering projection.

  So (X, i) is a covering space of X.

Example

- Let P denote the projective plane.

  Consider the natural map

  $$p: S^2 \to P$$

  which identifies each pair of antipodal or diametrically opposite points.

  We show the existence of admissible neighborhoods.

  Let w be a point in P which is the image of two antipodal points x and $-x$.

  Let O be a path connected open set in S^2 containing x such that O does not contain any pair of antipodal points.

  E.g., a small disc centered at x will do nicely.

Example (Cont'd)

- Then $p(O)$ is an open set containing w.

  Moreover, $p^{-1}p(O)$ has path components O and the set of points antipodal to points in O.

  Note that p maps each of these path components homeomorphically onto $p(O)$.

  So $p(O)$ is an admissible neighborhood.

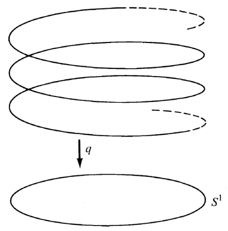
  Thus, (S^2, p) is a covering space of P.

Example

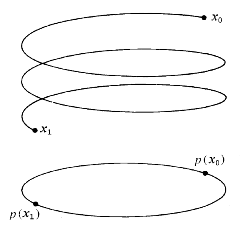
- Consider the map $r : \mathbb{R}^2 \to S^1 \times S^1$ from the plane to the torus defined by
$$r(t_1, t_2) = (e^{2\pi i t_1}, e^{2\pi i t_2}), \quad (t_1, t_2) \in \mathbb{R}^2.$$
Then $(\mathbb{R}^2, r)$ is a covering space of $S^1 \times S^1$.
This example is essentially a generalization of the covering projection $p : \mathbb{R} \to S^1$ of a previous example.
For any point (z_1, z_2) in $S^1 \times S^1$, let U denote a small open rectangle formed by the product of two proper open intervals in S^1 containing z_1 and z_2, respectively.
Then U is an admissible neighborhood whose inverse image consists of a countably infinite family of open rectangles in the plane.

Example

- Let Q denote an infinite spiral.
  Let $q : Q \to S^1$ denote the projection shown.
  Each point on the spiral is projected to the point on the circle directly beneath it.
  We can show that (Q, q) is a covering space of S^1.



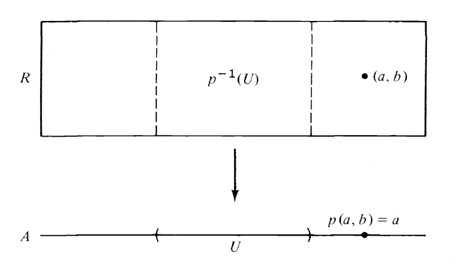
It is important that the spiral be infinite.
A finite spiral projected in the same manner is not a covering space.
The points $p(x_0)$ and $p(x_1)$ lying under the ends of the spiral do not have admissible neighborhoods.



Example

- The following is not an example of a covering space.

  Let R be a rectangle which is mapped by the projection onto the first coordinate to an interval A.



  Let U be an open interval in A.

  Then $p^{-1}(U)$ is a strip in R consisting of all points above U.

  This strip cannot be mapped homeomorphically onto U.

  So this situation does not allow admissible neighborhoods.

Subsection 2

Basic Properties of Covering Spaces

Characterization of Local Path Connectedness

Lemma

A space X is locally path connected if and only if each path component of each open subset of X is open.

- Suppose X is locally path connected.

  Take an open set A and a path component C of A.

  We must show that $A \setminus C$ is closed in A.

  Take a point x in $\overline{A \setminus C}$.

  By local path connectedness we can find a path connected neighborhood N of x.

  If $x \in C$, then $N \subseteq C$. But N intersects $A \setminus C$, a contradiction.

  So C is open in A. Hence, it is open in X.

  The converse follows from the obvious fact that every open set is the union of its components.

Openness of Covering Projections

Theorem

Every covering projection is an open mapping.

- Let $p : E \to B$ be a covering projection.

  We show that for each open set F in E, $p(V)$ is open in B.

  Let $x \in p(V)$.

  Let $\widetilde{x}$ be a point of F, such that $p(\widetilde{x}) = x$.

  Let U be an admissible neighborhood for x.

  Let W be the path component of $p^{-1}(U)$ which contains x.

Openness of Covering Projections (Cont'd)

- By hypothesis, E is locally path connected.

  So, by the preceding lemma, W is open in E.

  Also p maps W homeomorphically onto U.

  So p maps the open set $W \cap V$ to an open subset $p(W \cap V)$ in B.

  Thus, $x \in p(W \cap V)$.

  So $p(W \cap V)$ is an open set contained in $p(V)$.

  Since x was an arbitrary point of $p(V)$, it follows that $p(V)$ is a union of open sets.

  Thus, $p(V)$ is an open set.

  So p is an open mapping.

Continuous Maps to a Covering Space

Theorem

Let (E, p) be a covering space of B. Let X be a space and f and g be continuous maps from X into E for which $pf = pg$. Then the set of points at which f and g agree is an open and closed subset of X. (Here, we do not assume that X is path connected or locally path connected.)

- Denote by A the set of points at which f and g agree,

$$A = \{x \in X : f(x) = g(x)\}.$$

To see that A is open, let x be a member of A.
Let U an admissible neighborhood of $pf(x)$.
Let V be the path component of $p^{-1}(U)$ to which $f(x)$ belongs.
V is an open set in E.
Hence, $f^{-1}(V)$ and $g^{-1}(V)$ are open in X.
But $f(x) \in V$ and $f(x) = g(x)$. So x belongs to $f^{-1}(V) \cap g^{-1}(V)$.

Continuous Maps to a Covering Space (Cont'd)

Claim: $f^{-1}(V) \cap g^{-1}(V)$ is a subset of A.

Note that the claim implies that A is open.

Indeed, by the claim, A contains a neighborhood of each of its points.

Let $t \in f^{-1}(V) \cap g^{-1}(V)$.

Then $f(t)$ and $g(t)$ are in V.

Moreover, they are mapped by p to the common point $pf(t) = pg(t)$.

But p maps V homeomorphically onto U.

Thus, it must be true that $f(t) = g(t)$.

Then $t \in A$.

We conclude that A is an open set.

Continuous Maps to a Covering Space (Claim)

- Suppose that A is not closed.

  Let y be a limit point of A not in A.

  Then $f(y) \neq g(y)$.

  The point $pf(y) = pg(y)$ has an elementary neighborhood W.

  Moreover, $f(y)$ and $g(y)$ must belong to distinct path components V_0 and V_1 of $p^{-1}(W)$.

  But y belongs to the open set $f^{-1}(V_0) \cap g^{-1}(V_0)$.

  So $f^{-1}(V_0) \cap g^{-1}(V_1)$ must contain a point $t \in A$.

  This is a contradiction, since the point $f(t) = g(t)$ would have to belong to the disjoint sets V_0 and V_1.

  Thus, A contains all its limit points.

  Hence, A is a closed set.

Continuous Maps from a Connected to a Covering Space

Corollary

Let (E,p) be a covering space of B. Let f,g be continuous maps from a connected space X into E, such that $pf=pg$. If f and g agree at a point of X, then $f=g$.

- In a connected space X, the only sets that are both open and closed are X and the empty set $\varnothing$. Thus, for the set A of the theorem,

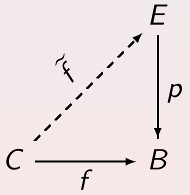
$$A=X \quad \text{or} \quad A=\varnothing.$$

  So f and g must be precisely equal or must disagree at every point.
- Note that the corollary requires only that X be connected.

  It does not require path connectedness or local path connectedness.

Lifting or Covering of a Map

- Suppose that spaces E and B are to be compared using a continuous map $p : E \to B$.
- Assume, further that there is given another map $f : C \to B$ from a space C into B.
- Then a map $\widetilde{f} : C \to E$ for which the diagram on the right is commutative, that is for which $p\widetilde{f} = f$, is called a **lifting** or **covering** of f.



- We are interested in lifting two kinds of maps:
  - Paths;
  - Homotopies between paths.

The Covering Path Property

Definition

Let (E, p) be a covering space of B, and let $\alpha : I \to B$ be a path.
A path $\widetilde{\alpha} : I \to E$, such that

$$p\widetilde{\alpha} = \alpha,$$

is called a **lifting** or **covering path** of α.
If $F : I \times I \to B$ is a homotopy, then a homotopy $\widetilde{F} : I \times I \to E$, such that

$$p\widetilde{F} = F,$$

is called a **lifting** or **covering homotopy** of F.

Covering Paths and Covering Homotopies

Theorem (The Covering Path Property)

Let (E, p) be a covering space of B. Let $\alpha : I \to B$ be a path in B beginning at a point b_0. If e_0 is a point in E with $p(e_0) = b_0$, then there is a unique covering path of α beginning at e_0.

Basic idea of the proof: Subdivide the range of the path α into sections so that each section lies in an admissible neighborhood.

If U is one of these admissible neighborhoods, then p maps each path component of $p^{-1}(U)$ homeomorphically onto U.

We can then choose a path component V of $p^{-1}(U)$ and consider the restriction $p|_V$ of p to V, a homeomorphism from V onto U.

Composing with $(p|_V)^{-1}$ "lifts" one section of α to E.

The Covering Path Property (Cont'd)

- Let $\{U_j\}$ be an open cover of B by admissible neighborhoods.
  Let ϵ be a Lebesgue number for the open cover $\{\alpha^{-1}(U_j)\}$ of I.
  Choose a sequence
  $$0 = t_0 < t_1 < \cdots < t_{n-1}$$
  of numbers in I with each successive pair differing by less than ϵ.
  Then each subinterval $[t_i, t_{i+1}]$, $0 \le i \le n-1$, is mapped by α into an admissible neighborhood U_{i+1}.
  First consider $\alpha([t_0, t_1])$, which is contained in U_1.
  Let V_1 denote the path component of $p^{-1}(U_1)$ to which the desired initial point e_0 belongs.
  Then, for $t \in [t_0, t_1]$, define
  $$\widetilde{\alpha}(t) = (p\,|_{V_1})^{-1}\alpha(t).$$

The Covering Path Property (Conclusion)

- Suppose now that $\widetilde{\alpha}$ has been defined on the interval $[t_0, t_k]$.

  Then $\alpha([t_k, t_{k+1}]) \subseteq U_{k+1}$.

  Let V_{k+1} be the path component of $p^{-1}(U_{k+1})$ to which $\widetilde{\alpha}(t_k)$ belongs. $p|_{V_{k+1}}$ is a homeomorphism.

  So the desired extension of $\widetilde{\alpha}$ to $[t_k, t_{k+1}]$ is obtained by defining

  $$\widetilde{\alpha}(t) = (p|_{V_{k+1}})^{-1}\alpha(t), \quad t \in [t_k, t_{k+1}].$$

  The continuity of $\widetilde{\alpha}$ follows from the Continuity Lemma since the lifted sections match properly at the end points.

  The uniqueness of the covering path $\widetilde{\alpha}$ can be proved from the uniqueness of each lifted section.

  However, it is simpler to apply the preceding corollary.

  Suppose α' is another covering path of α with $\alpha'(0) = e_0$.

  Then $\widetilde{\alpha}$ and α' agree at 0. Hence, they must be identical.

The Covering Homotopy Property

Theorem (The Covering Homotopy Property)

Let (E,p) be a covering space of B. Let $F: I \times I \to B$ be a homotopy such that $F(0,0) = b_0$. If e_0 is a point of E with $p(e_0) = b_0$, then there is a unique covering homotopy $\widetilde{F}: I \times I \to E$, such that $\widetilde{F}(0,0) = e_0$.

- One can piece together the proof from:
  - The special case presented in the preceding chapter;
  - The Covering Path Property for covering spaces.
- The proof follows that of the Covering Path Property by subdividing $I \times I$ into rectangles in the way that I was subdivided into intervals.

The Monodromy Theorem

Theorem (The Monodromy Theorem)

Let (E, p) be a covering space of B. Suppose that $\widetilde{\alpha}$ and $\widetilde{\beta}$ are paths in E with common initial point e_0. Then $\widetilde{\alpha}$ and $\widetilde{\beta}$ are equivalent if and only if $p\widetilde{\alpha}$ and $p\widetilde{\beta}$ are equivalent paths in B. In particular, if $p\widetilde{\alpha}$ and $p\widetilde{\beta}$ are equivalent, then $\widetilde{\alpha}$ and $\widetilde{\beta}$ must have common terminal point.

- If $\widetilde{\alpha}$ and $\widetilde{\beta}$ are equivalent by a homotopy G, then the homotopy pG demonstrates the equivalence of $p\widetilde{\alpha}$ and $p\widetilde{\beta}$.

  Conversely, let b_0 and b_1 denote the common initial point and common terminal point respectively of $p\widetilde{\alpha}$ and $p\widetilde{\beta}$.

  Let $H : I \times I \to B$ be a homotopy witnessing $p\widetilde{\alpha} \sim p\widetilde{\beta}$,

$$H(\cdot, 0) = p\widetilde{\alpha}, \quad H(\cdot, 1) = p\widetilde{\beta},$$
$$H(0, t) = b_0, \quad H(1, t) = b_1, \quad t \in I.$$

The Monodromy Theorem (Cont'd)

- By the Covering Homotopy Property, there is a covering homotopy $\widetilde{H}$ of H, with $\widetilde{H}(0,0) = e_0$.

  Both $\widetilde{\alpha}$ and the initial level $\widetilde{H}(\cdot,0)$ are covering paths of $p\widetilde{\alpha}$, and they have common value e_0 at 0. By the preceding corollary, $\widetilde{H}(\cdot,0) = \widetilde{\alpha}$.

  Similarly, we conclude that $\widetilde{H}(\cdot,1) = \widetilde{\beta}$.

  It remains to be seen that $\widetilde{H}(0,\cdot)$ and $\widetilde{H}(1,\cdot)$ are constant paths.

  $\widetilde{H}(0,\cdot)$ is a lifting of the constant path $H(0,\cdot)$, with $\widetilde{H}(0,0) = e_0$.

  The unique lifting of a constant path is obviously a constant path.

  So $\widetilde{H}(0,\cdot)$ must be the constant path whose only value is e_0.

  Similarly, $\widetilde{H}(1,\cdot)$ must be the constant path whose only value is

$$\widetilde{\alpha}(1) = \widetilde{H}(1,0) = \widetilde{H}(1,1) = \widetilde{\beta}(1).$$

  Thus, $\widetilde{H}$ is a homotopy that demonstrates the equivalence of $\widetilde{\alpha}$ and $\widetilde{\beta}$.

Covering Spaces and Pre-Images of Points

Theorem

Let (E, p) is a covering space of B. Then all the sets $p^{-1}(b)$, $b \in B$, have the same cardinal number.

- Let b_0 and b_1 be points in B.
  We define a one-to-one correspondence between $p^{-1}(b_0)$ and $p^{-1}(b_1)$.
  Let α be a path in B from b_0 to b_1. Let $x \in p^{-1}(b_0)$.
  Let $\widetilde{\alpha}_x$ denote the unique covering path of α beginning at x.
  The terminal point $\widetilde{\alpha}_x(1)$ is a point in $p^{-1}(b_1)$.
  So, for each x in $p^{-1}(b_0)$, set $f(x) = \widetilde{\alpha}_x(1)$, a point in $p^{-1}(b_1)$.
  By considering the reverse path from b_1 to b_0, one can define in the same manner a function $g : p^{-1}(b_1) \to p^{-1}(b_0)$.
  The functions f and g can be shown to be inverses of each other.
  So $p^{-1}(b_0)$ and $p^{-1}(b_1)$ must have the same cardinal number.

Number of Sheets of a Covering

- We showed that for (E,p) a covering space of B, all sets

$$p^{-1}(b), \quad b \in B,$$

have the same cardinal number.

Definition

Let (E,p) be a covering space of B.
The common cardinal number of the sets $p^{-1}(b)$, $b \in B$, is called the **number of sheets** of the covering.
A covering of n sheets is called an n-**fold covering**.

Example

- Consider the covering projection

$$p: S^2 \to P$$

of the projective plane that identifies each pair of antipodal or diametrically opposite points.

Now p identifies pairs of antipodal points.

So the number of sheets of this covering is two.

Thus, (S^2, p) is referred to as the "double covering" of the projective plane.

Example

- Recall the covering projection $p : \mathbb{R} \to S^1$, given by

$$p(t) = e^{2\pi it} = \cos(2\pi t) + \sin(2\pi t), \quad t \in \mathbb{R}.$$

It maps each integer and only the integers to $1 \in S^1$.

Thus the number of sheets of this covering is countably infinite.

Coverings and Homomorphisms of Fundamental Groups

- Let (E, p) be a covering of a space B.
- We consider the fundamental groups of E and B.
- Choose base points e_0 in E and $b_0 = p(e_0)$ in B.
- Suppose α is a loop in E based at e_0.
- The composition $p\alpha$ is a loop in B with base point b_0.
- Thus, p induces a function

$$p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$$

defined by

$$p_*([\alpha]) = [p\alpha], \quad [\alpha] \in \pi_1(E, e_0).$$

- This function p_* is a group homomorphism.
- It is called the **homomorphism induced by** p.

Injectivity of the Homomorphisms of Fundamental Groups

Theorem

Let (E, p) be a covering space of B. The induced homomorphism

$$p_* : \pi_1(E, e_0) \to \pi_1(B, b_0)$$

is one-to-one.

- The proof uses the Monodromy Theorem.

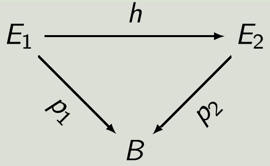
Subsection 3

Classification of Covering Spaces

Homomorphisms of Covering Spaces

Definition

Let (E_1, p_1) and (E_2, p_2) be covering spaces of the same space B. A **homomorphism** from (E_1, p_1) to (E_2, p_2) is a continuous map $h: E_1 \to E_2$ for which $p_2 h = p_1$.



A homomorphism $h: E_1 \to E_2$ of covering spaces which is also a homeomorphism is called an **isomorphism**.
If there is an isomorphism from one covering space to another, the two covering spaces are called **isomorphic**.

- A homomorphism of covering spaces is actually a covering projection.
- That is, if $h: E_1 \to E_2$ is a homomorphism, then (E_1, h) is a covering space of E_2.

Coverings and Conjugacy Classes of Subgroups

Theorem

Let (E, p) be a covering space of B. If $b_0 \in B$, then the groups $p_* \pi_1(E, e)$, as e varies over $p^{-1}(b_0)$, form a conjugacy class of subgroups of $\pi_1(B, b_0)$.

- Recall that subgroups H and K of a group G are conjugate subgroups if and only if

$$H = x^{-1} K x, \quad \text{for some } x \in G.$$

The theorem then makes two assertions.

(a) For any e_0, e_1 in $p^{-1}(b_0)$, the subgroups $p_* \pi_1(E, e_0)$ and $p_* \pi_1(E, e_1)$ are conjugate;

(b) Any subgroup of $\pi_1(B, b_0)$ conjugate to $p_* \pi_1(E, e_0)$ must equal $p_* \pi_1(E, e)$, for some e in $p^{-1}(b_0)$.

Coverings and Conjugacy Classes of Subgroups (a)

Claim: For any e_0, e_1 in $p^{-1}(b_0)$, the subgroups $p_*\pi_1(E, e_0)$ and $p_*\pi_1(E, e_1)$ are conjugate.

Consider two points e_0 and e_1 in $p^{-1}(b_0)$.

Let $\rho : I \to E$ be a path from e_0 to e_1.

By a previous theorem, $P : \pi_1(E, e_0) \to \pi_1(E, e_1)$ defined by

$$P([\alpha]) = [\overline{\rho} \star \alpha \star \rho], \quad [\alpha] \in \pi_1(E, e_0),$$

is an isomorphism.

In particular, $\pi_1(E, e_1) = P\pi_1(E, e_0)$.

So $p_*\pi_1(E, e_1) = p_* P\pi_1(E, e_0)$.

It follows from the definition of P, however, that

$$p_* P\pi_1(E, e_0) = [p\rho]^{-1} \circ \pi_1(E, e_0) \circ [p\rho].$$

Note that $[p\rho]$ is an element of $\pi_1(B, b_0)$.

So $p_*\pi_1(E, e_0)$ and $p_*\pi_1(E, e_1)$ are conjugate subgroups of $\pi_1(B, b_0)$.

Coverings and Conjugacy Classes of Subgroups (b)

Claim: Any subgroup of $\pi_1(B, b_0)$ conjugate to $p_*\pi_1(E, e_0)$ must equal $p_*\pi_1(E, e)$ for some e in $p^{-1}(b_0)$.
Suppose that H is a subgroup conjugate to $p_*\pi_1(E, e_0)$ by some element $[\delta]$ in $\pi_1(B, b_0)$,

$$H = [\delta]^{-1} \circ p_*\pi_1(E, e_0) \circ [\delta].$$

Let $\widetilde{\delta}$ be the unique covering path of δ beginning at e_0.
Then $\widetilde{\delta}$ has a terminal point $e \in p^{-1}(b_0)$.
The argument for Part (a) shows that

$$p_*\pi_1(E, e) = [p\widetilde{\delta}]^{-1} \circ p_*\pi_1(E, e_0) \circ [p\widetilde{\delta}] = [\delta]^{-1} \circ p_*\pi_1(E, e_0) \circ [\delta] = H.$$

Thus, $p_*\pi_1(E, e) = H$.
It follows that the set

$$\{p_*\pi_1(E, e) : e \in p^{-1}(b_0)\}$$

is precisely a conjugacy class of subgroups of $\pi_1(B, b_0)$.

Conjugacy Classes Determined by a Covering Space

Definition

Let (E,p) be a covering space of B and $b_0 \in B$.
The conjugacy class of subgroups

$$\{p_*\pi_1(E,e) : e \in p^{-1}(b_0)\}$$

is called the **conjugacy class determined by the covering space** (E,p).

Characterization of Isomorphic Covering Spaces

Theorem

Let B be a space with base point b_0. Covering spaces (E_1, p_1) and (E_2, p_2) of B are isomorphic if and only if they determine the same conjugacy class of subgroups of $\pi_1(B, b_0)$.

- We present the "if" part of the proof.

  Suppose the conjugacy classes of the covering spaces are identical.

  Then there must be points $e_1 \in p_1^{-1}(b_0)$ and $e_2 \in p_2^{-1}(b_0)$, such that

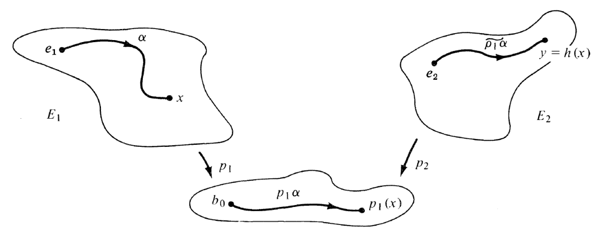
  $$p_{1*}\pi_1(E_1, e_1) = p_{2*}\pi_1(E_2, e_2).$$

  We define the covering space isomorphism $h : E_1 \to E_2$.

Characterization of Isomorphic Covering Spaces (Cont'd)

- We define the covering space isomorphism $h: E_1 \to E_2$.

  Let $x \in E_1$. Let α be a path in E_1 from e_1 to x.



  Then $p_1\alpha$ is a path in B from b_0 to $p_1(x)$.

  This path has a unique covering path $\widetilde{p_1\alpha}$ in E_2 beginning at e_2 and ending at some point y in E_2.

  We then define $h(x) = y$.

Characterization (h is well-defined)

- We show that h is well-defined.

  Let β be another path in E_1 from e_1 to x.

  Note that α and β both begin at e_1 and terminate at x.

  So the product path $\alpha \star \overline{\beta}$ is a loop in E_1 based at e_1.

  Thus,

$$p_{1*}([\alpha \star \overline{\beta}]) = [p_1\alpha \star p_1\overline{\beta}] \in p_{1*}\pi_1(E_1, e_1).$$

  But $p_{1*}\pi_1(E_1, e_1)$ and $p_{2*}\pi_1(E_2, e_2)$ are equal.

  So there is a member $[\gamma] \in \pi_1(E_2, e_2)$, such that

$$[p_1\alpha \star p_1\overline{\beta}] = [p_2\gamma].$$

  Thus, the loops $p_1\alpha \star p_1\overline{\beta}$ and $p_2\gamma$ are equivalent loops in B.

  Using the Covering Homotopy Property to lift a homotopy between $p_1\alpha \star p_1\overline{\beta}$ and $p_2\gamma$ to E_2, we obtain a loop γ' in E_2 based at e_2 for which $p_2\gamma' = p_1\alpha \star p_1\overline{\beta}$.

Characterization (h is well-defined Cont'd)

- Divide γ' into the product of two paths α' and $\overline{\beta}'$ as follows:

$$\alpha'(t) = \gamma'(t/2), \quad \beta'(t) = \gamma'((2-t)/2), \quad t \in I.$$

It is a simple matter to observe that $p_2\alpha' = p_1\alpha$, $p_2\beta' = p_1\beta$.

But α' and β' have initial point e_2.

So they are the unique covering paths of $p_1\alpha$ and $p_1\beta$ with respect to the covering (E_2, p_2). I.e., $\alpha' = \widetilde{p_1\alpha}$ and $\beta' = \widetilde{p_1\beta}$.

Then

$$\widetilde{p_1\alpha}(1) = \alpha'(1) = \gamma'\left(\frac{1}{2}\right), \quad \widetilde{p_1\beta}(1) = \beta'(1) = \gamma'\left(\frac{1}{2}\right).$$

So the same value $h(x) = \gamma'(\frac{1}{2})$ results regardless of the choice of the path from e_1 to x. This concludes the proof that h is well-defined.

Characterization (h is continuous)

- In showing that h is continuous, we use the fact that the admissible neighborhoods form a basis for the topology of B.

  Let O be an open set in E_2 and x a member of $h^{-1}(O)$.

  We show there is an open set V in E_1, with $x \in V$ and $h(V) \subseteq O$.

  By the definition of h, $p_2 h = p_1$.

  Since p_2 is an open mapping, $p_1(x)$ is in the open set $p_2(O) \subseteq B$.

  The admissible neighborhoods form a basis for B.

  So there is an admissible neighborhood U, with $p_1(x) \in U \subseteq p_2(O)$.

  Let W be the path component of $p_2^{-1}(U)$ to which $h(x)$ belongs.

  Then $h(x)$ belongs to the open set $O' = O \cap W$.

  Moreover, the restriction $f = p_2|_{O'} : O' \to p_2(O')$ is a homeomorphism.

  Since $p_2(O')$ is open in B, $p_1^{-1} p_2(O')$ is open in E_1.

  Let V be a path connected open set in E_1 which contains x and is contained $p_1^{-1} p_2(O')$.

Characterization (h is continuous Cont'd)

Claim: $h(V) \subseteq O$.

Let $t \in V$. Let:

- α be a path in E_1 from e_1 to x;
- β a path in V from x to t.

Then

$$h(x) = \widetilde{p_1\alpha}(1), \quad h(t) = \widetilde{p_1\alpha \star p_1\beta}(1).$$

Now $f = p_2|_{O'}$ is a homeomorphism.

So the covering path of $p_1\alpha \star p_1\beta$ is $\widetilde{p_1\alpha} \star f^{-1}p_1\beta$.

Thus,

$$h(t) = f^{-1}p_1\beta(1) = f^{-1}p_1(t).$$

But f is a homeomorphism between O' and $p_2(O')$.

So, since $p_1(t) \in p_2(O')$, $f^{-1}p_1(t) \in O'$.

Since $O' \subseteq O$, it follows that $h(t) \in O$. Hence, $h(V) \subseteq O$.

Characterization (h has an inverse)

- The proof thus far has shown that there is a covering space homomorphism h from E_1 to E_2.
  By looking at constant paths, it is easy to see that $h(e_1) = e_2$.
  We must also show the existence of a continuous inverse for h.
  The proof thus far has essentially done that.
  Reversing the roles of E_1 and E_2, there must exist a continuous map $g : E_2 \to E_1$, such that

$$p_1 g = p_2, \quad g(e_2) = e_1.$$

  Consider the composite map gh from E_1 to E_1,

$$p_1 gh = p_2 h = p_1 i_1,$$

  where i_1 is the identity map on E_1. Also, gh and i_1 agree at e_1.
  By a previous corollary, gh is the identity map on E_1.
  By symmetry, hg must be the identity map on E_2.
  Thus, h is an isomorphism between (E_1, p_1) and (E_2, p_2).

Notation

- It is often necessary to make the statement

  "f is a function from space X to space Y which maps a particular point x_0 in X to the point y_0 in Y".

- We shall shorten this by referring to f as a function from the "pair" (X, x_0) to the pair (Y, y_0).
- In this case we write
$$f : (X, x_0) \to (Y, y_0).$$

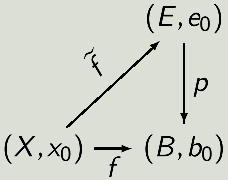
A Lifting Theorem

- Minor modifications in the proof of the preceding theorem establish

Theorem

Let E, B, and X be spaces with base points e_0, b_0, and x_0 respectively, and suppose that (E, p) is a covering space of B with $p(e_0) = b_0$.

Suppose $f : (X, x_0) \to (B, b_0)$ is a continuous map for which $f_* \pi_1(X, x_0) \subseteq p_* \pi_1(E, e_0)$. Then there is a continuous map $\widetilde{f} : (X, x_0) \to (E, e_0)$, for which $p\widetilde{f} = f$.



- The theorem remains valid if the requirement on X is reduced to connectedness in place of path connected and locally path connected.

Example

- In all our examples of covering spaces, the fundamental group of each base space is abelian.

  It follows that each conjugacy class has only one member.

  Example: Consider the covering $(\mathbb{R}, p)$ over S^1.

  The fundamental group of $\mathbb{R}$ is trivial.

  So $p_{1*}\pi_1(\mathbb{R}) = \{0\}$.

  Thus, the conjugacy class consists of only the trivial subgroup of $\pi_1(S^1)$.

Example

- The map $q_n : S^1 \to S^1$ defined by

$$q_n(z) = z^n, \quad z \in S^1,$$

wraps S^1 around itself n times.

Thus, if $[\alpha] \in \pi_1(S^1)$, the loop $q_n\alpha$ has degree $\deg(q_n\alpha) = n\deg\alpha$.

Represent $\pi_1(S^1)$ as the group of integers.

Then $q_{n*}\pi_1(S^1, 1)$ is the subgroup of $\mathbb{Z}$, consisting of all multiples of the integer n.

Examples

Example: Suppose $i: X \to X$ is the identity map.

Then

$$i_* \pi_1(X) = \pi_1(X).$$

So the conjugacy class contains only the fundamental group of X.

Example: Consider the double covering

$$(S^2, p)$$

over the projective plane P.

The 2-sphere is simply connected.

So the conjugacy class contains only the trivial subgroup.

Examples

Example: The plane is simply connected.

So the conjugacy class of $(\mathbb{R}^2, r)$ over the torus also contains only the trivial subgroup.

Example: The infinite spiral Q is contractible.

Thus, it has a trivial fundamental group.

Then (Q, q) determines the conjugacy class of $i_1(S^1)$ consisting of only the trivial subgroup.

This is the conjugacy class determined in the first example.

So, by the last theorem, (Q, q) and $(\mathbb{R}, p)$ are isomorphic covering spaces of S^1.

Classifying Covering Spaces of S^1

- The only subgroups of $\pi_1(S^1) = \mathbb{Z}$ are the groups W_n of all multiples of the non-negative integer n.
- Since $\mathbb{Z}$ is abelian, each singleton set $\{W_n\}$ is a conjugacy class.
  - The subgroup $W_0 = \{0\}$ corresponds to the covering space $(\mathbb{R}, p)$ of the first example.
  - W_n corresponds to the covering (S^1, q_n), $n = 1, 2, \ldots$.
- By the classification of covering spaces, any covering space of S^1 must be isomorphic either to $(\mathbb{R}, p)$ or to one of the coverings (S^1, q_n).

Subsection 4

Universal Covering Spaces

Universal Covering Spaces

- Let B be a topological space.
- Then there is always a covering space corresponding to the conjugacy class of the entire fundamental group.
- This is the pair (B, i), where i is the identity map on B.
- This covering space is of little interest for obvious reasons.
- At the other extreme, the covering space corresponding to the conjugacy class of the trivial subgroup $\{0\}$ of $\pi_1(B)$ is the most interesting.
- This covering space, if it exists for a particular base space, is called the "universal covering space".

Definition

Let B be a space. A covering space (U, q) of B for which U is simply connected is called the **universal covering space** of B.

Universality of Universal Covering Spaces

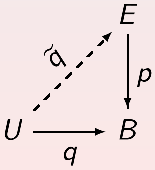
Theorem

(a) Any two universal covering spaces of a base space B are isomorphic.

(b) If (U,q) is the universal covering space of B and (E,p) is a covering space of B, then there is a continuous map $r:U\to E$, such that (U,r) is a covering space of E.

(a) This follows immediately from a preceding theorem, since any universal covering space determines the conjugacy class of the trivial subgroup.

(b) Consider the diagram. Choose base points u_0, e_0 and b_0 in U, E and B, respectively, for which $q(u_0) = p(e_0) = b_0$. Since $\pi_1(U)$ is trivial, then $q_*\pi_1(U,u_0) \subseteq p_*\pi_1(E,e_0)$. The preceding theorem guarantees the existence of a continuous map $\widetilde{q}:(U,q_0)\to(E,e_0)$ for which $p\widetilde{q}=q$. This means that $r=\widetilde{q}$ is a covering space homomorphism. Thus, it is a covering projection for U over E.



Group of Automorphisms of a Covering Space

Definition

Let (E, p) be a covering space of B. An isomorphism from (E, p) to itself is called an **automorphism**. Under the operation of composition, the set of automorphisms of (E, p) forms a group. This group is called the **group of automorphisms** of (E, p) and is denoted by $A(E, p)$.

- Concerning automorphisms of covering spaces, the following hold.
  (a) If f and g are automorphisms of (E, p), such that
  $$f(x) = g(x), \quad \text{for some } x,$$
  then $f = g$.
  (b) The only member of $A(E, p)$ that has a fixed point is the identity map.

Group of Automorphisms and Fundamental Group

Theorem

Let (U,q) be the universal covering space of B. Then $A(U,q)$ is isomorphic to $\pi_1(B)$. Moreover, the order of $\pi_1(B)$ is the number of sheets of the universal covering space.

- Choose a base point b_0 in B and a u_0 in U, such that $q(u_0) = b_0$. We define a function $T : A(U,q) \to \pi_1(B)$.

  For $f \in A(U,q)$, $f(u_0)$ is a point in U.
  Let γ be a path in U from u_0 to $f(u_0)$.
  Since $qf = q$, $f(u_0) \in q^{-1}(b_0)$.
  Hence, $q\gamma$ is a loop in B, with base point b_0.
  We thus define T by $T(f) = [q\gamma]$, $f \in A(U,q)$.

  Since U is simply connected, the choice of path γ from u_0 to $f(u_0)$ does not affect the homotopy class $[q\gamma]$. Thus, T is well-defined.

Automorphisms and Fundamental Group (Cont'd)

Claim: T is a homomorphism.

Let $f_1, f_2 \in A(U, q)$.

Let γ_1, γ_2 be paths in U from u_0 to $f_1(u_0)$ and $f_2(u_0)$, respectively.

Then

$$T(f_1) = [q\gamma_1], \quad T(f_2) = [q\gamma_2].$$

The product path $\gamma_1 \star f_1\gamma_2$ is a path from u_0 to $f_1 f_2(u_0)$.

Thus

$$\begin{array}{rcl} T(f_1 f_2) & = & [q(\gamma_1 \star f_1\gamma_2)] \\ & = & [q\gamma_1 \star qf_1\gamma_2] \\ & = & [q\gamma_1 \star q\gamma_2] \\ & = & [q\gamma_1] \circ [q\gamma_2] \\ & = & T(f_1) \circ T(f_2). \end{array}$$

So T is a homomorphism.

Automorphisms and Fundamental Group (Cont'd)

Claim: T is one-to-one.

Suppose that $T(f_1) = T(f_2)$.

Thus, the loops $q\gamma_1$ and $q\gamma_2$ determined by f_1 and f_2 are equivalent.

By the Monodromy Theorem,

$$f_1(u_0) = f_2(u_0).$$

But distinct automorphisms must disagree at every point.

So $f_1 = f_2$.

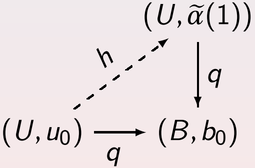
Automorphisms and Fundamental Group (Cont'd)

Claim: T maps $A(U,q)$ onto $\pi_1(B,b_0)$.

Let $[\alpha] \in \pi_1(B,b_0)$.

Let $\widetilde{\alpha}$ denote the unique covering path of α beginning at u_0.

Since U is simply connected, we can apply a preceding theorem to the right diagram to obtain a continuous lifting h of q, such that $h(u_0) = \widetilde{\alpha}(1)$. By commutativity of the diagram, $qh = q$. Therefore, h is a homomorphism.



Reversing the roles of $\widetilde{\alpha}(1)$ and u_0 determines a homomorphism k on (U,q), such that $k(\widetilde{\alpha}(1)) = u_0$. Then hk and kh are homomorphisms which agree with the identity at some point. So hk and kh are the identity map on U. Thus, $k = h^{-1}$. So h is an automorphism.

It follows that $T(h) = [q\widetilde{\alpha}] = [\alpha]$.

Automorphisms and Fundamental Group (Conclusion)

- The proof that the order of $\pi_1(B)$ is the number of sheets of the universal covering space is based on what has already been done.

  The fact that T is one-to-one establishes a one-to-one correspondence between $q^{-1}(b_0)$ and a subset of $\pi_1(B, b_0)$.

  In proving that T is onto, we showed that every homotopy class $[\alpha]$ in $\pi_1(B, b_0)$ corresponds to a point $\widetilde{\alpha}(1)$ in $q^{-1}(b_0)$.

  Thus, the cardinal number of $q^{-1}(b_0)$ equals the order of $\pi_1(B)$.

  This cardinal number is exactly the number of sheets of (U, q).

Examples

Example: The real line is simply connected.

Consider the covering space $(\mathbb{R}, p)$ over S^1.

$$\begin{array}{c} \mathbb{R} \\ \big\downarrow p \\ S^1 \end{array}$$

It is the universal covering space of the unit circle.

Example: The plane is simply connected.

Consider the covering space $(\mathbb{R}^2, r)$ over the torus $\mathbb{T}$.

$$\begin{array}{c} \mathbb{R}^2 \\ \big\downarrow r \\ \mathbb{T} \end{array}$$

It is the universal covering space of the torus.

Projective Plane

- Consider the double covering (S^2, p) of the projective plane P.

  Since $\pi_1(S^2) = \{0\}$, (S^2, p) is the universal covering space of P.

  The theorem allows us to determine $\pi_1(P)$ by determining $A(S^2, p)$.

  p identifies pairs of antipodal points.

  Hence, (S^2, p) has two automorphisms.
  - The identity map;
  - The antipodal map.

  Thus, $A(S^2, p)$ is the cyclic group of order two.

  It follows that $\pi_1(P)$ is the same group.

  Thus, $\pi_1(P)$ is essentially the group of integers modulo 2.

Projective n-Space

Definition

Let P^n denote the quotient space of the n-sphere S^n obtained by identifying each pair of antipodal points x and $-x$.
Then P^n is called **projective n-space**.

- The quotient map $p : S^n \to P^n$ is a covering projection.
- By the same reasoning applied to the projective plane, the fundamental group of each projective space P^n, $n \geq 2$, is isomorphic to the group of integers modulo 2.
- A moment's reflection will show that P^1 is homeomorphic to S^1.
  Hence, $\pi_1(P^1)$ is not the group of integers mod 2.

On the Existence of Covering Spaces

- The classification of covering spaces shows that two covering spaces of a space B are isomorphic if and only if they determine the same conjugacy class of subgroups of $\pi_1(B)$.
- This leaves open the question of the existence of covering spaces.

  Given a conjugacy class in $\pi_1(B)$, is there a covering space that determines this class?

  In particular, does every space have a universal covering space?
- It turns out the answer is negative for both questions.
- Moreover, there are known necessary and sufficient conditions for the existence of a universal covering space.

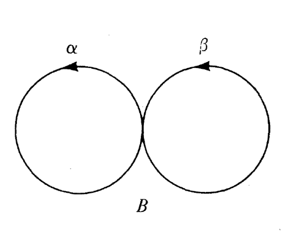
Subsection 5

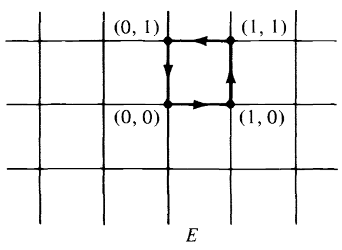
Applications

A Non-Abelian Fundamental Group

- Let the base space B consist of two tangent circles

$$B = \{(z, w) \in S^1 \times S^1 : z = 1 \text{ or } w = 1\}.$$





Let also

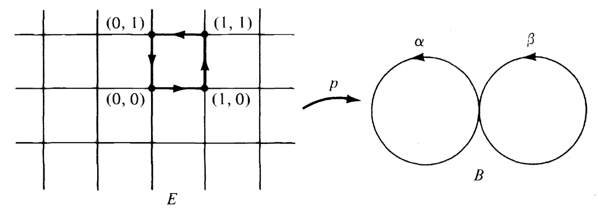
$$E = \{(x, y) \in \mathbb{R}^2 : x \text{ or } y \text{ is an integer}\}.$$

A Non-Abelian Fundamental Group (Cont'd)

- The map $p : E \to B$ defined by

$$p(x,y) = (e^{2\pi ix}, e^{2\pi iy}), \quad (x,y) \in \mathbb{R}^2,$$

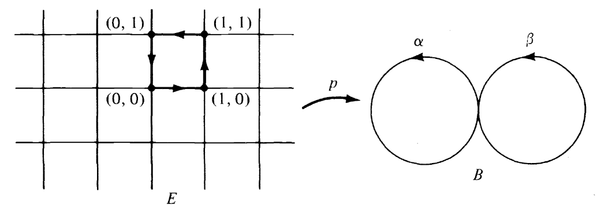
is a covering projection.



The map p maps:

- Each horizontal segment of a square once around the left hand circle;
- Each vertical segment of a square once around the right hand circle.

A Non-Abelian Fundamental Group (Cont'd)



- Let γ denote the loop in E based at $(0,0)$ indicated by the arrows.
- Let $[\alpha]$ and $[\beta]$ denote generators of the fundamental groups of the left and right circles of B, respectively.
- Then $[\gamma]$ is not the identity of $\pi_1(E)$.
- Since p_* is one-to-one, $p_*([\gamma]) = [\alpha] \circ [\beta] \circ [\alpha]^{-1} \circ [\beta]^{-1}$ is not the identity in $\pi_1(B)$.
- But, if $\pi_1(B)$ were abelian, the commutator $[\alpha] \circ [\beta] \circ [\alpha]^{-1} \circ [\beta]^{-1}$ would be the identity element of $\pi_1(B)$.
- Thus, $\pi_1(B)$ is not abelian.

The Borsuk-Ulam Theorem

Theorem (The Borsuk-Ulam Theorem)

There is no continuous map $f: S^n \to S^{n-1}$ for which

$$f(-x) = -f(x), \quad \text{for all } x \in S^n,\ n \geq 1.$$

- By the theorem, there is no continuous map from S^n to a sphere of lower dimension which maps antipodal points to antipodal points.
- Such a map would be said to "preserve antipodal points" and would be called "antipode preserving".
- Note that S^0 is a discrete space of two points and, therefore, not connected.

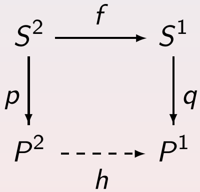
  So the result is clear for the case $n = 1$.

The Borsuk-Ulam Theorem (The Case $n = 2$)

- We prove the case $n = 2$ by contradiction.

  Let $f : S^2 \to S^1$ be continuous, with $f(-x) = -f(x)$, for all $x \in S^2$.

  Consider the diagram, where (S^2, p) and (S^1, q) denote the double coverings of the projective spaces P^2 and P^1. p^{-1} is not single valued. Since f preserves antipodal points, $h = qfp^{-1} : P^2 \to P^1$ is well-defined and continuous. Moreover, the diagram is commutative.

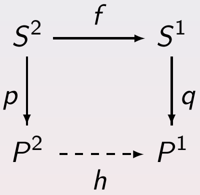
  Now $\pi_1(P^2)$ is cyclic of order 2.

  In addition, $\pi_1(P^1) \cong \pi_1(S^1)$ is infinite and cyclic.

  Therefore, the induced homomorphism $h_* : \pi_1(P^2) \to \pi_1(P^1)$ must be trivial.

The Borsuk-Ulam Theorem (The Case $n = 2$ Cont'd)

- Let y_0 be a point of S^2. Let $b_0 = qf(y_0)$ be the base point of P^1. Suppose α is a path in S^2 from y_0 to $-y_0$. Then $qf\alpha$ is a loop in P^1. This loop is not equivalent to the constant loop c at b_0 for the following reason:



  Suppose $qf\alpha \sim_{b_0} c$. By the Monodromy Theorem, $f\alpha$ is equivalent to the constant loop based at $f(y_0)$. Now f preserves antipodal points. So $f\alpha(1) = f(-y_0) = -f(y_0)$. So $f\alpha$ is not a loop. Hence it cannot possibly be equivalent to a loop. Thus, $[qf\alpha] \neq [c]$.

  Now calculate

$$h_*([p\alpha]) = [hp\alpha] = [qfp^{-1}p\alpha] = [qf\alpha].$$

  So $h_*([p\alpha])$ is not the identity of $\pi_1(B, b_0)$.
  It follows that h_* is not the trivial homomorphism.
  This contradiction shows that no such map as f exists.

A Consequence of the Borsuk-Ulam Theorem

Corollary

Let $g : S^2 \to \mathbb{R}^2$ be a continuous map, such that

$$g(-x) = -g(x), \quad \text{for all } x \text{ in } S^2.$$

Then $g(x) = 0$, for some x in S^2.

- Suppose on the contrary that $g(x)$ is never 0.

  Consider the map $f : S^2 \to S^1$ defined by

  $$f(x) = \frac{g(x)}{\|g(x)\|}, \quad x \in S^2.$$

  This map contradicts the Borsuk-Ulam Theorem for $n = 2$.

Another Consequence of the Borsuk-Ulam Theorem

Corollary

Let $h : S^2 \to \mathbb{R}^2$ be a continuous map. Then there is at least one pair x, $-x$ of antipodal points for which

$$h(x) = h(-x).$$

- Assume to the contrary that for no x in S^2

  $$h(x) = h(-x).$$

  Consider the function $g : S^2 \to \mathbb{R}^2$ defined by

  $$g(x) = h(x) - h(-x), \quad x \in S^2.$$

  This function contradicts the preceding corollary.

Physical Interpretation of the Last Corollary

- The last corollary has an interesting physical interpretation.

  Imagine the surface of the earth to be a 2-dimensional sphere.

  Suppose that the functions $a(x)$ and $t(x)$ which measure the atmospheric pressure and temperature at x are continuous.

  Then the map $h : S^2 \to \mathbb{R}^2$ defined by

$$h(x) = (a(x), t(x)), \quad x \in S^2,$$

  is continuous.

  The corollary guarantees that there is at least one pair of antipodal points on the surface of the earth having identical atmospheric pressures and identical temperatures!