Introduction to Algebraic Topology

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Subsection 1

Introduction

2-Dimensional Loops: 1st Definition

- We consider in an intuitive way the possible methods of defining the second homotopy group $\pi_2(X,x_0)$ of a space X at a point x_0 in X.
- Recall that $\pi_1(X,x_0)$ is the set of homotopy classes of loops in X based at x_0.
- We would like to define a "2-dimensional loop".
- A "1-dimensional loop" is a continuous map $\alpha : I \to X$ for which the boundary points 0 and 1 have image x_0.
- We might then define a 2-dimensional loop to be a continuous map $\beta : I \times I \to X$ from the unit square into X which maps the boundary of the square to x_0.

2-Dimensional Loops: 2nd Definition

- From a slightly different point of view, we can consider a loop α in X as a continuous map from S^1 to X which takes 1 to x_0.
- This follows from the observation that the quotient space of the unit interval I obtained by identifying 0 and 1 to a single point is simply S^1.
- Thus, another possible definition of 2-dimensional loop is a continuous map from the 2-sphere S^2 into X.
- Both of the preceding definitions of 2-dimensional loop generalize to higher dimensions by considering higher dimensional cubes and spheres.

2-Dimensional Loops: 3rd Definition

- There is a third possibility.
- Perhaps a 2-dimensional loop should be a "loop of loops".
- That is to say, perhaps a 2-dimensional loop should be a function β, having domain I, such that:
  - Each value $\beta(t)$ is a loop in X;
  - $\beta(0) = \beta(1)$.
- To carry out this idea, we must define a topology on the set $\Omega(X, x_0)$ of loops in X with base point x_0.
- Once this topology is determined, one can define $\pi_2(X, x_0)$ to be the fundamental group of $\Omega(X, x_0)$.
- Remarkably, all three approaches lead to the same group $\pi_2(X, x_0)$.

Subsection 2

Equivalent Definitions of $\pi_n(X, x_0)$

The n-Cube

- If n is a positive integer, the symbol I^n denotes the **unit n-cube**

$$I^n = \{t = (t_1, t_2, \ldots, t_n) \in \mathbb{R}^n : 0 \leq t_i \leq 1, \text{ for each } i\}.$$

- ∂I^n, called the **boundary** of I^n, denotes its point set boundary

$$\partial I^n = \{t = (t_1, t_2, \ldots, t_n) \in I^n : \text{some } t_i \text{ is } 0 \text{ or } 1\}.$$

- Note that the boundary symbol ∂ must not be confused with the boundary operator of homology theory.

n-Homotopy Classes (Definition A)

Definition A

Let X be a space and x_0 a point of X.

For a given positive integer n, consider the set $F_n(X, x_0)$ of all continuous maps α from the unit n-cube I^n into X for which $\alpha(\partial I^n) = x_0$.

Define an equivalence relation $\sim_{x_0}$ on $F_n(X, x_0)$ as follows:

> For α and β in $F_n(X, x_0)$, α is **equivalent modulo** x_0 to β, written $\alpha \sim_{x_0} \beta$, if there is a homotopy $H : I^n \times I \to X$, such that
>
> $$\begin{array}{rcl} H(t_1, \ldots, t_n, 0) & = & \alpha(t_1, \ldots, t_n), \\ H(t_1, \ldots, t_n, 1) & = & \beta(t_1, \ldots, t_n), \quad (t_1, \ldots, t_n) \in I^n, \\ H(t_1, \ldots, t_n, s) & = & x_0, \quad (t_1, \ldots, t_n) \in \partial I^n, s \in I. \end{array}$$

n-Homotopy Classes (Definition A Cont'd)

Definition A (Cont'd)

In shorter form the requirements on the homotopy H are

$$H(\cdot,0)=\alpha, \quad H(\cdot,1)=\beta,$$
$$H(\partial I^n \times I)=x_0.$$

Under this equivalence relation on $F_n(X,x_0)$ the equivalence class determined by α is denoted $[\alpha]$ and called the **homotopy class of** α **modulo** x_0 or simply the **homotopy class of** α.

The n-th Homotopy Group

Definition

Define an operation $\star$ on $F_n(X,x_0)$ as follows:

For α,β in $F_n(X,x_0)$,

$$\alpha\star\beta=\begin{cases}\alpha(2t_1,t_2,\dots,t_n), & \text{if } 0\le t_1\le\frac{1}{2}\\ \beta(2t_1-1,t_2,\dots,t_n), & \text{if } \frac{1}{2}\le t_1\le 1\end{cases}$$

Note that the $\star$ operation is completely determined by the first coordinate of the variable point $(t_1,\dots,t_n)$ and that the continuity of $\alpha\star\beta$ follows from the Continuity Lemma.

The n-th Homotopy Group (Cont'd)

Definition (Cont'd)

The $\star$ operation induces an operation $\circ$ on the set of homotopy classes of $F_n(X,x_0)$:

$$[\alpha]\circ[\beta]=[a\star\beta].$$

With this operation, the set of equivalence classes of $F_n(X,x_0)$ is a group. This group is called the **n-th homotopy group of X at x_0** and is denoted by $\pi_n(X,x_0)$.

Details Needing Verification

- As in the case of the fundamental group, the definition requires that some details be verified:
  (1) The relation $\sim_{x_0}$ is an equivalence relation on $F_n(X,x_0)$.
  (2) The operation $\star$ determines the operation $\circ$ completely.
  In other words, if $\alpha\sim_{x_0}\alpha'$ and $\beta\sim_{x_0}\beta'$, then $\alpha\star\beta\sim_{x_0}\alpha'\star\beta'$.
  (3) With the $\circ$ operation, $\pi_n(X,x_0)$ is actually a group.
    - Its identity is the class $[c]$ determined by the constant map $c(I^n)=x_0$.
    - The inverse $[\alpha]^{-1}$ of $[\alpha]$ is the class $[\overline{\alpha}]$, where $\overline{\alpha}$, called the **reverse** of α, is defined by

$$\overline{\alpha}(t_1,t_2,\ldots,t_n)=\alpha(1-t_1,t_2,\ldots,t_n),\quad (t_1,t_2,\ldots,t_n)\in I^n.$$

- The definition of $\pi_n(X,x_0)$ is completely analogous to that of $\pi_1(X,x_0)$ except for the extra coordinates.
- So the proofs of these facts are similar to those for $\pi_1(X,x_0)$.

n-Homotopy Classes (Definition B)

- The quotient space of I^n obtained by identifying ∂I^n to a point is homeomorphic to the n-sphere S^n.
- Assume that the point of identification is the point $1 = (1,0,\ldots,0)$ of S^n having first coordinate unity and all other coordinates zero.
- Then $\pi_n(X,x_0)$ can be defined in terms of maps from

$$(S^n,1) \to (X,x_0)$$

as detailed in the next slide.

n-Homotopy Classes (Definition B Cont'd)

Definition

For a given positive integer n, consider the set $G_n(X,x_0)$ of all continuous maps α from S^n to X, such that $\alpha(1) = x_0$. Define an equivalence relation on $G_n(X,x_0)$ in the following way:

For α, β in $G_n(X,x_0)$, α is **equivalent modulo x_0 to** β, written $\alpha \sim_{x_0} \beta$, if there is a homotopy $H : S^n \times I \to X$, such that

$$H(\cdot,0) = \alpha, \quad H(\cdot,1) = \beta,$$
$$H(1,s) = x_0, \quad s \in I.$$

The equivalence class $[\alpha]$ determined by α is called the **homotopy class of** α. The set of homotopy classes is denoted by $\pi_n(X,x_0)$.

Equivalence of Definitions A and B

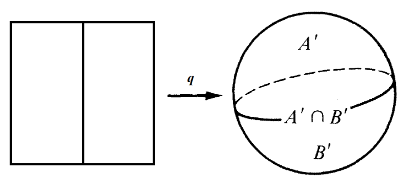
- Recall the discussion preceding the definition.
- There is a natural one-to-one correspondence between $F_n(X, x_0)$ and $G_n(X, x_0)$ under which a map α in $G_n(X, x_0)$ corresponds to the map

$$\alpha' = \alpha q,$$

  where q is the map from I^n to S^n which identifies ∂I^n to the point 1.
- Two members α and β in $G_n(X, x_0)$ are equivalent modulo x_0 if and only if their counterparts α' and β' are equivalent in $F_n(X, x_0)$.
- Thus, Definitions A and B give equivalent definitions of the set $\pi_n(X, x_0)$.
- The elements $[\alpha]$ are usually more easily visualized in terms of Definition B.

The Operation $\circ$ in Definition B

- The $\circ$ operation for Definition B is defined in terms of the identification of I^n to S^n.
- Let $\alpha,\beta \in G_n(X,x_0)$.

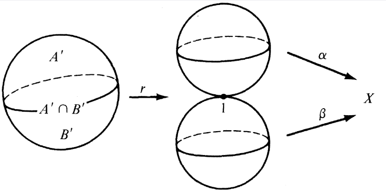


- The identification map q takes the sets

$$\begin{aligned} A &= \{(t_1,\ldots,t_n) \in I^n : t_1 \le \tfrac{1}{2}\}, \\ B &= \{(t_1,\ldots,t_n) \in I^n : t_1 \ge \tfrac{1}{2}\} \end{aligned}$$

  to hemispheres A' and B', respectively, of S^n.
- Their intersection $A' \cap B' = q(A \cap B)$ if homeomorphic to S^{n-1}.

The Operation $\circ$ in Definition B (Cont'd)



- Imagine that $A' \cap B'$ is identified to the base point 1 by an identification map r.
- The resulting space consists of two n-spheres tangent at their common base point.
- The product $\alpha \star \beta$ is now defined by

$$\alpha \star \beta(x) = \begin{cases} \alpha r(x), & \text{if } x \in A' \\ \beta r(x), & \text{if } x \in B' \end{cases}$$

- The group operation $\circ$ is defined by $[\alpha] \circ [\beta] = [\alpha \star \beta]$.

The Compact-Open Topology

Definition

Let F be a collection of continuous functions from a space Y into a space Z. Suppose:

- K is a compact subset of Y;
- U is an open subset of Z.

Define

$$W(K,U) = \{\alpha \in F : \alpha(K) \subseteq U\}.$$

The family of all such sets $W(K,U)$, as K ranges over the compact sets in Y and U ranges over the open sets in Z, is a subbase for a topology for F. This topology is called the **compact-open topology** for F.

The Compact-Open Topology for $\Omega(X,x_0)$

- Since we shall apply the compact-open topology only to the set of loops in a space X, we repeat the definition for this case.

Definition

Let X be a space and x_0 a point of X. Consider the set $\Omega(X,x_0)$ of all loops in X with base point x_0. If K is a compact subset of I and U is open in X, let

$$W(K,U) = \{\alpha \in \Omega(X,x_0) : \alpha(K) \subseteq U\}.$$

The family of all such sets $W(K,U)$, where K is compact in I and U is open in X, is a subbase for a topology for $\Omega(X,x_0)$. This topology is the **compact-open topology** for $\Omega(X,x_0)$. Note that basic open sets in this topology have the form $\bigcap_{i=1}^{r} W(K_i,U_i)$, where $K_1,\ldots,K_r$ are compact sets in I and $U_1,\ldots,U_r$ are open in X. A loop a belongs to this basic open set if and only if $\alpha(K_1) \subseteq U_i$, for each $i = 1,2,\ldots,r$.

Characterization of the Compact-Open Topology

Theorem

If X is a metric space, the compact-open topology for $\Omega(X, x_0)$ is the same as its topology of uniform convergence.

- Let d denote the metric on X.
  Recall that the topology of uniform convergence on $\Omega(X, x_0)$ is determined by the metric ρ defined as follows:

  If α and β are in $\Omega(X, x_0)$, then $\rho(\alpha, \beta)$ is the supremum (or least upper bound) of the distances from $\alpha(t)$ to $\beta(t)$ for t in I:

  $$\rho(\alpha, \beta) = \sup\{d(\alpha(t), \beta(t)) : t \in I\}.$$

  Then the topology of uniform convergence has as a basis the set of all spherical neighborhoods

  $$S(\alpha, r) = \{\beta \in \Omega(X, x_0) : \rho(\alpha, \beta) < r\},$$

  where $\alpha \in \Omega(X, x_0)$ and r is a positive number.

Characterization of the Compact-Open Topology ($T \subseteq T'$)

- Let T denote the compact-open topology on $\Omega(X, x_0)$.
  Let T' denote the topology of uniform convergence on $\Omega(X, x_0)$.
  We show, first, that $T \subseteq T'$.
  Let $W(K, U)$ be a subbasic open set in T, where K is compact in I and U is open in X. Let $\alpha \in W(K, U)$.
  The compact set $\alpha(K)$ is contained in U.
  So, there is a positive number ϵ, such that any point of X at a distance less than ϵ from $\alpha(K)$ is also in U.
  Consider the basic open set $S(\alpha, \epsilon)$ in T'.
  If $\beta \in S(\alpha, \epsilon)$, then for each t in K, $d(\alpha(t), \beta(t)) < \epsilon$.
  Thus, the distance of $\beta(t)$ from a point of $\alpha(K)$ is less than ϵ.
  Hence, $\beta(t)$ must be in U. Thus, $\beta(K) \subseteq U$. So $\beta \in W(K, U)$.
  We get $\alpha \in S(\alpha, \epsilon) \subseteq W(K, U)$. So $W(K, U)$ must be open in T'.
  Then $T \subseteq T'$, since T' contains a subbase for T.

Characterization of the Compact-Open Topology ($T' \subseteq T$)

- We show, next, that $T' \subseteq T$.

  Let $S(\gamma, r)$ with center γ and radius $r > 0$ be a basic open set in T'.

  To prove that $S(\gamma, r)$ is in T, it is sufficient to find a member of T which contains γ and is contained in $S(\gamma, r)$.

  Let $\{U_j\}$ be a cover of X by open sets having diameters less than r.

  Let η be a Lebesgue number for the open cover $\{\gamma^{-1}(U_j)\}$ of I.

  Let

  $$0 = t_0 < t_1 < \cdots < t_n = 1$$

  be a subdivision of I with successive points differing by less than η.

  Then, for $i = 1, 2, \ldots, n$, γ maps each of the compact sets $K_i = [t_{i-1}, t_i]$ into one of the open sets of the cover $\{U_j\}$.

Characterization of the Compact-Open Topology (Cont'd))

- Choose such an open set, say U_i, for each i.

  Then
  $$\gamma(K_i) \subseteq U_i, \quad i = 1, 2, \ldots, n.$$
  So $\gamma \in \bigcap_{i=1}^n W(K_i, U_i)$. The set $\bigcap_{i=1}^n W(K_i, U_i)$ is open in T.

  So it suffices to show that it is contained in $S(\gamma, r)$.

  Let $\beta \in \bigcap_{i=1}^n W(K_i, U_i)$. Then $\rho(\gamma, \beta)$ cannot exceed the maximum of the diameters of $U_1, \ldots, U_n$. Thus, $\rho(\gamma, \beta) < r$. So $\beta \in S(\gamma, r)$.

  It follows that $S(\gamma, r)$ is open in T.

  Now T contains T', since it contains a basis for T'.

  Since we showed that $T \subseteq T'$ and $T' \subseteq T$, we get $T = T'$.

n-Homotopy Classes (Definition C)

Definition

Let X be a space with $x_0 \in X$. Consider the set $\Omega(X,x_0)$ of loops in X based at x_0 with the compact-open topology.
If $n \geq 2$, the ***n*-th homotopy group of X at x_0** is the $(n-1)$-st homotopy group of $\Omega(X,x_0)$ at c, where c is the constant loop at x_0.
Thus,

$$\begin{aligned} \pi_2(X,x_0) &= \pi_1(\Omega(X,x_0),c), \\ &\vdots \\ \pi_n(X,x_0) &= \pi_{n-1}(\Omega(X,x_0),c). \end{aligned}$$

Comments on the Definitions

- To understand homotopy theory, one must know all three definitions and be able to apply the one that fits best in a given situation.
- Definitions A, B, and C of the higher homotopy groups are all equivalent.
- The operation for Definition B has been designed expressly to show that Definitions A and B describe isomorphic groups.
- We will discuss a comparison of Definitions A with C for $n = 2$.
- The extension to higher values of n involves little more than writing additional coordinates.

Comparison of A with C for $n = 2$

- Suppose α is a member of $F_2(X, x_0)$.

  That is:

  - α is a continuous map from the unit square I^2 to X;
  - α takes ∂I^2 to x_0.

  Then α determines a member $\widehat{\alpha}$ of $\Omega(\Omega(X, x_0), c)$ defined by

$$\widehat{\alpha}(t_1)(t_2) = \alpha(t_1, t_2), \quad t_1, t_2 \in I.$$

  By continuity of α, each $\widehat{\alpha}(t_1)$ is continuous from I into X.

  Now $(t_1, 0)$ and $(t_1, 1)$ are in ∂I^2.

  So $\widehat{\alpha}(t_1)(0) = \widehat{\alpha}(t_1)(1) = x_0$.

  Thus, $\widehat{\alpha}(t_1) \in \Omega(X, x_0)$.

  Clearly, $\widehat{\alpha}(0) = \widehat{\alpha}(1)$ is the constant loop c whose only value is x_0.

Comparison of A with C for $n=2$ (Cont'd)

Claim: $\widehat{\alpha}$ is continuous as a function from I into $\Omega(X,x_0)$.

Let $W(K,U)$ be a subbasic open set in $\Omega(X,x_0)$.

As usual, K is compact in I and U is open in X.

Let $t_1 \in \widehat{\alpha}^{-1}(W(K,U))$.

Then $\widehat{\alpha}(t_1)(K) = \alpha(\{t_1\} \times K) \subseteq U$.

But K is compact.

So there is open O in I, such that $t_1 \in O$ and $\alpha(O \times K) \subseteq U$.

Thus, $t_1 \in O \subseteq \widehat{\alpha}^{-1}(W(K,U))$.

So $\widehat{\alpha}^{-1}(W(K,U))$ is an open set and α is continuous.

Thus, each member of $F_2(X,x_0)$ determines in a natural way a member of $\Omega(\Omega(X,x_0),c)$.

Comparison of A with C for $n = 2$ (Cont'd)

- We now reverse the process.

  Start with a member $\widehat{\alpha}$ of $\Omega(\Omega(X, x_0), c)$.

  Then $\widehat{\alpha}$ determines a function $\alpha : I^2 \to X$ defined by

  $$\alpha(t_1, t_2) = \widehat{\alpha}(t_1)(t_2), \quad (t_1, t_2) \in I^2.$$

  We may show that $\alpha \in F_2(X, x_0)$.

  In this way, we have established a one-to-one correspondence between $F_2(X, x_0)$ and $\Omega(\Omega(X, x_0), c)$.

Comparison of A with C for $n=2$ (Conclusion)

- Suppose that $H: I^2 \times I \to X$ is a homotopy demonstrating the equivalence of α and β as prescribed in Definition A.

  Consider the homotopy $\widehat{H}: I \times I \to \Omega(X,x_0)$, defined by

  $$\widehat{H}(t_1,s)(t_2) = H(t_1,t_2,s), \quad t_1,t_2,s \in I.$$

  This demonstrates the equivalence of the loops $\widehat{\alpha}$ and $\widehat{\beta}$.

  The reverse argument shows that $\widehat{\alpha}$ equivalent to $\widehat{\beta}$ implies that α is equivalent to β.

  Thus, there is a one-to-one correspondence between homotopy classes $[\alpha]$ of Definition A and homotopy classes $[\widehat{\alpha}]$ of Definition C.

  Finally, the $\star$ operation in Definition A is completely determined in the first coordinate.

  Thus, for any $\alpha,\beta \in F_2(X,x_0)$, $[\alpha \star \beta]$ corresponds to $[\widehat{\alpha} \star \widehat{\beta}]$.

  So the two definitions of $\pi_2(X,x_0)$ lead to isomorphic groups.

Subsection 3

Basic Properties and Examples

Path Connectedness and n-Homotopy Groups

- The following three results can be proved by methods very similar to those used to prove their analogues for the fundamental group.

Theorem

If the space X is path connected and x_0 and x_1 are points of X, then $\pi_n(X, x_0)$ is isomorphic to $\pi_n(X, x_1)$, for each $n \geq 1$.

- As in the case of the fundamental group, when X is path connected, we refer to the "n-th homotopy group of X" and write $\pi_n(X)$, .

Contractiblity, Products and n-Homotopy Groups

Theorem

If X is contractible by a homotopy that leaves x_0 fixed, then

$$\pi_n(X, x_0) = \{0\}, \quad \text{for each } n \geq 1.$$

Theorem

Let X and Y be spaces with points x_0 in X and y_0 in Y. Then, for all $n \geq 1$,

$$\pi_n(X \times Y, (x_0, y_0)) \cong \pi_n(X, x_0) \oplus \pi_n(Y, y_0).$$

Contractible Spaces

- The following spaces are contractible, so each has n-th homotopy group $\{0\}$, for each value of n:
  (a) The real line;
  (b) Euclidean space of any dimension;
  (c) An interval;
  (d) A convex figure in Euclidean space.
- We saw that the fundamental group is usually difficult to determine.
- This is doubly true of the higher homotopy groups.

  Example: The homotopy groups $\pi_k(S^n)$ of the n-sphere have never been completely determined. (The hard part is the case $k > n$.)

  The groups $\pi_k(S^n)$, for $k \leq n$, are computed in the following examples.

The Groups $\pi_k(S^n)$ for $k < n$

Claim: For $k < n$, the k-th homotopy group $\pi_k(S^n)$ is the trivial group.

Let $[\alpha]$ be a member of $\pi_k(S^n)$.

Consider α as a continuous map from $(S^k, 1)$ to $(S^n, 1)$.

Represent S^k and S^n as the boundary complexes of simplexes of dimensions $k+1$ and $n+1$, respectively.

By the Simplicial Approximation Theorem, α has a simplicial approximation $\alpha' : S^k \to S^n$, such that $[\alpha] = [\alpha']$.

Simplicial maps cannot map a simplex onto one of higher dimension.

Thus, α' is not onto.

The Groups $\pi_k(S^n)$ for $k < n$ (Cont'd)

- Let p be a point in S^n which is not in the range of α'.

  Then $S^n \backslash \{p\}$ is contractible, since it is homeomorphic to $\mathbb{R}^n$.

  So the range of α' is contained in a contractible space.

  It follows that α' is null-homotopic.

  Thus,

$$[\alpha] = [\alpha'] = [c].$$

  So $\pi_k(S^n)$ is the trivial group whose only member is the class $[c]$ determined by the constant map.

The Groups $\pi_n(S^n)$

Claim: For $n \geq 1$, the n-th homotopy group $\pi_n(S^n)$ is isomorphic to the group $\mathbb{Z}$ of integers.

The case $n = 1$ was considered previously.

Consider $\pi_n(S^n)$, $n \geq 2$, as the set of homotopy classes of maps $\alpha : (S^n, 1) \to (S^n, 1)$ as in Definition B.

Define $\rho : \pi_n(S^n) \to \mathbb{Z}$ by

$$\rho([\alpha]) = \text{degree of } \alpha, \quad [\alpha] \in \pi_n(S^n).$$

Brouwer's Degree Theorem insures that ρ is well-defined.

By the Hopf Classification Theorem, it is one-to-one.

The Groups $\pi_n(S^n)$ (Cont'd)

- The identity map $i:(S^n,1)\to(S^n,1)$ has degree 1.

  The description of the $\star$ operation in Definition B shows that the map

  $$i^k = i\star i\star\cdots\star i \quad (k \text{ terms})$$

  has degree k.

  Thus, $[i]$ is a generator of $\pi_n(S^n)$.

  Moreover, we have, for any positive integer k:
  - $\rho([i]^k)=k$;
  - $\rho([i]^{-k})=-k$.

  It follows easily that ρ is an isomorphism.

Fundamental Groups of Topological Groups

Theorem

Let G be a topological group with identity element e. Then $\pi_1(G,e)$ is abelian.

- The operation on G induces an operation $\cdot$ on the set $\Omega(G,e)$ of loops in G based at e defined by

$$\alpha \cdot \beta(t) = \alpha(t)\beta(t), \quad \alpha, \beta \in \Omega(G,e),\ t \in I,$$

where the juxtaposition of $\alpha(t)$ and $\beta(t)$ indicates their product in G.

This operation induces an operation $\square$ on $\pi_1(G,e)$:

$$[\alpha]\square[\beta] = [\alpha \cdot \beta], \quad [\alpha],[\beta] \in \pi_1(G,).$$

Let c denote the constant loop at e.

Let $[\alpha]$ and $[\beta]$ be members of $\pi_1(G,e)$.

Fundamental Groups of Topological Groups (Cont'd)

- Observe the following.

$$(\alpha \star c)\cdot(c \star \beta)(t) = \begin{cases} \alpha(2t)e = \alpha(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ e\beta(2t-1) = \beta(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

$$(c \star \alpha)\cdot(c \star \beta)(t) = \begin{cases} e\beta(2t) = \beta(2t), & \text{if } 0 \leq t \leq \frac{1}{2} \\ \alpha(2t-1)e = \alpha(2t-1), & \text{if } \frac{1}{2} \leq t \leq 1 \end{cases}$$

This gives

$$\begin{aligned} (\alpha \star c)\cdot(c \star \beta) &= \alpha \star \beta, \\ (c \star \alpha)\cdot(\beta \star c) &= \beta \star \alpha. \end{aligned}$$

Fundamental Groups of Topological Groups (Cont'd)

- Then

$$\begin{aligned} [\alpha]\circ[\beta] &= [\alpha\star\beta] \\ &= [(\alpha\star c)\cdot(c\star\beta)] \\ &= [\alpha\star c]\square[c\star\beta] \\ &= [c\star\alpha]\square[\beta\star c] \\ &= [(c\star\alpha)\cdot(\beta\star c)] \\ &= [\beta\star\alpha] \\ &= [\beta]\circ[\alpha]. \end{aligned}$$

  So $\pi_1(G,e)$ is abelian.
- Note that the operations $\circ$ and $\square$ are precisely equal:

$$[\alpha]\circ[\beta]=[\alpha\star\beta]=[(\alpha\star c)\cdot(c\star\beta)]=[\alpha\star c]\square[c\star\beta]=[\alpha]\square[\beta].$$

Introducing Hopf Spaces

- Not all of the group properties were used in the proof of the preceding theorem.
- The existence of a multiplication with identity element e is sufficient.
- In fact, even that assumption can be weakened.
- This motivates the definition of Hopf spaces.

Hopf Spaces

Definition

An **H-space** or **Hopf space** is a topological space Y with a continuous multiplication (indicated by juxtaposition) and a point y_0 in Y for which:

- The map defined by multiplying on the left by y_0,
- The map defined by multiplying on the right by y_0

are both homotopic to the identity map on Y by homotopies that leave y_0 fixed.

In other words, there exist homotopies L and R from $Y \times I$ into Y, such that, for all y in Y and t in I:

$$L(y,0) = y_0 y, \quad L(y,1) = y, \quad L(y_0,t) = y_0;$$
$$R(y,0) = y y_0, \quad R(y,1) = y, \quad R(y_0,t) = y_0.$$

The point y_0 is called the **homotopy unit** of Y.

Example of a Hopf Space

- Let X be a space and x_0 a point of X.

  The loop space $\Omega(X,x_0)$ with the compact-open topology is an H-space.
  - The multiplication is the $\star$ operation;
  - The homotopy unit is the constant map c.

  The required homotopies L and R are defined for α in $\Omega(X,x_0)$ and s in I by

$$L(\alpha,s)(t) = \begin{cases} x_0, & \text{if } 0 \le t \le \frac{1-s}{2} \\ \alpha\left(\frac{2t+s-1}{s+1}\right), & \text{if } \frac{1-s}{2} \le t \le 1 \end{cases}$$

$$R(\alpha,s)(t) = \begin{cases} \alpha\left(\frac{2t}{s+1}\right), & \text{if } 0 \le t \le \frac{s+1}{2} \\ x_0, & \text{if } \frac{s+1}{2} \le t \le 1 \end{cases}$$

  We can show that the multiplication $\star$ and the homotopies L and R are continuous with respect to the compact-open topology.

Fundamental Groups of H-Spaces

Theorem

Let Y be an H-space with homotopy unit y_0. Then $\pi_1(Y, y_0)$ is abelian.

- The operation on Y induces an operation $\cdot$ on $\Omega(Y, y_0)$:

  $$\alpha \cdot \beta(t) = \alpha(t)\beta(t), \quad \alpha, \beta \in \Omega(Y, y_0),\ t \in I.$$

  This operation induces an operation $\square$ on $\pi_1(Y, y_0)$:

  $$[\alpha]\square[\beta] = [\alpha \cdot \beta], \quad [\alpha], [\beta] \in \pi_1(Y, y_0).$$

  Letting c denote the constant loop at y_0,

  $$\begin{aligned}(\alpha \star c)\cdot(c \star \beta)(t) &= \begin{cases} \alpha(2t)y_0, & \text{if } 0 \le t \le \frac{1}{2} \\ y_0\beta(2t-1), & \text{if } \frac{1}{2} \le t \le 1 \end{cases} \\ (c \star \alpha)\cdot(\beta \star c)(t) &= \begin{cases} y_0\beta(2t), & \text{if } 0 \le t \le \frac{1}{2} \\ \alpha(2t-1)y_0, & \text{if } \frac{1}{2} \le t \le 1 \end{cases}\end{aligned}$$

Fundamental Groups of H-Spaces (Cont'd)

- Now multiplication on the left by y_0 and multiplication on the right by y_0 are both homotopic to the identity map on Y.

  So we obtain

$$[(\alpha \star c) \cdot (c \star \beta)] = [\alpha \star \beta], \quad [(c \star \alpha) \cdot (\beta \star c)] = [\beta \star \alpha].$$

  Thus,

$$\begin{aligned} [\alpha] \circ [\beta] &= [\alpha \star \beta] \\ &= [(\alpha \star c) \cdot (c \star \beta)] \\ &= [(c \star \alpha) \cdot (\beta \star c)] \\ &= [\beta \star \alpha] \\ &= [\beta] \circ [\alpha]. \end{aligned}$$

- It follows, as before, that the operations $\circ$ and $\square$ are equal.

Commutativity of Higher Homotopic Groups

Theorem

The higher homotopy groups $\pi_n(X, x_0)$, $n \geq 2$, of any space X are abelian.

- $\Omega(X, x_0)$ is an H-space with the constant loop c as homotopy unit.

  Hence, the second homotopy group

  $$\pi_2(X, x_0) = \pi_1(\Omega(X, x_0), c)$$

  is abelian.

  Proceeding inductively, suppose that, for every space Y, the $(n-1)$-st homotopy group $\pi_{n-1}(Y, y_0)$ is abelian.

  Then

  $$\pi_n(X, x_0) = \pi_{n-1}(\Omega(X, x_0), c)$$

  must be abelian.

The Induced Homomorphism f_*

Definition

Let $f:(X,x_0)\to(Y,y_0)$ be a continuous map on the indicated pairs. If $[\alpha]\in\pi_n(X,x_0)$, $n\geq 1$, then the composition $f\alpha : I^n\to Y$ is a continuous map which takes ∂I^n to y_0. So $f\alpha$ represents an element $[f\alpha]$ in $\pi_n(Y,y_0)$. Thus, f induces a function $f_*:\pi_n(X,x_0)\to\pi_n(Y,y_0)$, defined by

$$f_*([\alpha])=[f\alpha],\quad [\alpha]\in\pi_n(X,x_0).$$

The function f_* is the **homomorphism induced by** f in dimension n.

- To be very precise we should refer to f_*^n, indicating the dimension n, but the dimension becomes known from the subscripts on the homotopy groups involved.
- We can show that f_* is a well-defined homomorphism.

Compositionality of Induced Homomorphisms

Theorem

(a) If $f:(X,x_0)\to(Y,y_0)$ and $g:(Y,y_0)\to(Z,z_0)$ are continuous maps on the indicated pairs, then the induced homomorphism $(gf)_*$ is the composite map
$$g_*f_*:\pi_n(X,x_0)\to\pi_n(Z,z_0)$$
in each dimension n.

(b) If $h:(X,x_0)\to(Y,y_0)$ is a homeomorphism, then the homomorphism h_* induced by h is an isomorphism for each value of n.

(a) If $[\alpha]\in\pi_n(X,x_0)$, then
$$(gf)_*([\alpha])=[gf\alpha]=g_*([f\alpha])=g_*f_*([\alpha]).$$

So $(gf)_*=g_*f_*$.

Compositionality of Induced Homomorphisms (Cont'd)

(b) Suppose that $h^{-1} : (Y, y_0) \to (X, x_0)$ is the inverse of h.
Then for $[\alpha]$ in $\pi_n(X, x_0)$,

$$(h^{-1})_* h_*([\alpha]) = [h^{-1} h\alpha] = [\alpha].$$

So $(h^{-1})_* h_*$ is the identity map on $\pi_n(X, x_0)$.
By symmetry, $h_*(h^{-1})_*$ is the identity map on $\pi_n(Y, y_0)$.
So h_* is an isomorphism.

Covering Spaces and Induced Homomorphisms

- We saw that a covering projection $p: E \to B$ induces a monomorphism (i.e., a one-to-one homomorphism)

$$p_*: \pi_1(E) \to \pi_1(B).$$

Theorem

Let (E, p) be a covering space of B. Let e_0 in E and b_0 in B be points such that $p(e_0) = b_0$. Then the induced homomorphism

$$p_*: \pi_n(E, e_0) \to \pi_n(B, b_0)$$

is an isomorphism for $n \geq 2$.

Covering Spaces and Induced Homomorphisms (Cont'd)

Claim: p_* is onto.

Consider an element $[\alpha]$ in $\pi_n(B, b_0)$.

Think of α as a continuous map from $(S^n, \overline{1})$ to (B, b_0), where $\overline{1}$ is used as the base point of S^n to avoid confusion with the number 1.

Since $n \geq 2$, the fundamental group $\pi_1(S^n, \overline{1})$ is trivial.

Hence, for α_* the homomorphism induced by α on $\pi_1(S^n, \overline{1})$,

$$\alpha_* \pi_1(S^n, \overline{1}) = \{0\} \subseteq p_* \pi_1(E, e_0).$$

By a preceding theorem, α has a continuous lifting $\widetilde{\alpha} : (S^n, \overline{1}) \to (E, e_0)$, such that

$$p\widetilde{\alpha} = \alpha.$$

Then $\widetilde{\alpha}$ determines a member $[\widetilde{\alpha}]$ in $\pi_n(E, e_0)$ for which

$$p_*([\widetilde{\alpha}]) = [p\widetilde{\alpha}] = [\alpha].$$

So p_* maps $\pi_n(E, e_0)$ onto $\pi_n(B, b_0)$.

Covering Spaces and Induced Homomorphisms (Cont'd)

Claim: p_* is one-to-one.

Suppose that $[\beta]$ belongs to its kernel.

I.e., assume $p_*([\beta]) = [p\beta] = [c]$, with c constant, $c(S^n) = b_0$.

As maps from $(S^n, \overline{1})$ to (B, b_0), $p\beta$ and c are equivalent.

So there is a homotopy $H : S^n \times I \to B$ satisfying

$$H(t,0) = p\beta(t), \quad H(t,1) = b_0, \quad t \in S^n,$$
$$H(\overline{1}, s) = b_0, \quad s \in I.$$

The fundamental group $\pi_1(S^n \times I, (\overline{1}, 0))$ is trivial, since $n \geq 2$.

So there exists a lifting $\widetilde{H} : S^n \times I \to E$, with $p\widetilde{H} = H$, $\widetilde{H}(\overline{1}, 0) = e_0$.

Subclaim (To be proven in the next slide): The lifted homotopy $\widetilde{H}$ is a homotopy between β and the constant map $d(S^n) = e_0$.

Then $[\beta] = [d]$ is the identity element of $\pi_n(E, e_0)$.

Thus, the kernel of p_* contains only the identity element of $\pi_n(E, e_0)$.

This shows that p_* is one-to-one.

Covering Spaces and Induced Homomorphisms (Cont'd)

Subclaim: The lifted homotopy $\widetilde{H}$ is a homotopy between β and the constant map $d(S^n) = e_0$.

Observe first that $p\widetilde{H}(\cdot,0) = p\beta$, $\widetilde{H}(\overline{1},0) = \beta(\overline{1})$.

By a previous corollary, $\widetilde{H}(\cdot,0) = \beta$, since S^n is connected.

The same argument shows that $\widetilde{H}(\cdot,1) = d$.

It remains to be seen that $\widetilde{H}(\overline{1},s) = e_0$, for each s in I.

The path $\widetilde{H}(\overline{1},\cdot) : I \to E$ has initial point e_0 and covers $c = H(\overline{1},\cdot)$.

But the unique covering path of c which begins at e_0 is the constant path at e_0, $\widetilde{H}(\overline{1},s) = e_0$, $s \in I$.

Thus, $\widetilde{H} : S^n \times I \to E$ is a homotopy such that

$$\widetilde{H}(\cdot,0) = \beta, \quad \widetilde{H}(\cdot,1) = d,$$
$$\widetilde{H}(\overline{1},s) = e_0, \quad s \in I.$$

Example

- Consider the universal covering space $(\mathbb{R}, p)$ of the unit circle S^1.

  By the preceding theorem,

  $$p_* : \pi_n(\mathbb{R}) \to \pi_n(S^1)$$

  is an isomorphism for $n \geq 2$.

  But all the homotopy groups of the contractible space $\mathbb{R}$ are trivial.

  So

  $$\pi_n(S^1) = \{0\}, \quad n \geq 2.$$

Example

- Consider the double covering (S^n, p) over projective n-space P^n. By the preceding theorem,

$$\pi_k(P^n) \cong \pi_k(S^n), \quad k \geq 2,\ n \geq 2.$$

By a previous example,

$$\pi_n(S^n) \cong \mathbb{Z}, \quad n \geq 2.$$

Therefore, we have

$$\pi_n(P^n) \cong \mathbb{Z}, \quad n \geq 2.$$

Subsection 4

Homotopy Equivalence

Homotopy Equivalent Spaces

- Homotopy equivalence is a relation between topological spaces.
- It is a weaker relation than homeomorphism.
- On the other hand, it is strong enough to ensure that equivalent spaces have isomorphic homotopy groups in corresponding dimensions.

Definition

Let X and Y be topological spaces. Then X and Y are **homotopy equivalent** or have the **same homotopy type** provided that there exist continuous maps $f : X \to Y$ and $g : Y \to X$ for which the composite maps gf and fg are homotopic to the identity maps on X and Y, respectively. The map f is called a **homotopy equivalence**, and g is a **homotopy inverse** for f.

- Homeomorphic spaces are homotopy equivalent.

Homotopy Equivalence is an Equivalence Relation

Theorem

The relation "X is homotopy equivalent to Y" is an equivalence relation for topological spaces.

- The relation is reflexive since the identity map on any space X is a homotopy equivalence.

  The symmetric property is implicit in the definition: Both f and g are homotopy equivalences and each is a homotopy inverse for the other.

  For transitivity transitive, let $f : X \to Y$ and $h : Y \to Z$ be homotopy equivalences with homotopy inverses $g : Y \to X$ and $k : Z \to Y$, respectively. We must show that X and Z are homotopy equivalent.

  The most likely candidate for a homotopy equivalence between X and Z is hf, with gk as the leading contender for homotopy inverse.

Homotopy Equivalence is an Equivalence Relation (Cont'd)

- Let $L: Y \times I \to Y$ be a homotopy such that

$$\begin{array}{rcl} L(\cdot,0) & = & kh; \\ L(\cdot,1) & = & i_Y. \end{array}$$

Consider the map $M: X \times I \to X$, defined by

$$M(x,t) = gL(f(x),t), \quad (x,t) \in X \times I.$$

M is a homotopy, such that

$$\begin{array}{rcl} M(\cdot,0) & = & gL(f(\cdot),0) = (gk)(hf); \\ M(\cdot,1) & = & gL(f(\cdot),1) = gf. \end{array}$$

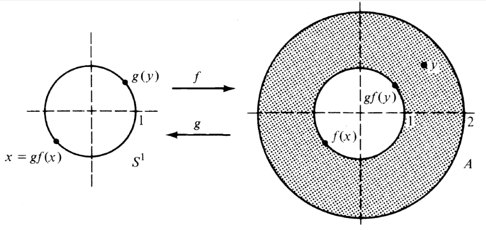
So $(gk)(hf)$ is homotopic to gf. Hence, it is homotopic to i_X.
Similarly, $(hf)(gk)$ is homotopic to the identity on Z.
So X and Z are homotopy equivalent.

Example

Claim: A circle and an annulus are homotopy equivalent.

Consider the unit circle S^1 and the annulus

$$A = \{y \in \mathbb{R}^2 : 1 \leq |y| \leq 2\}.$$



A homotopy equivalence $f : S^1 \to A$ and homotopy inverse $g : A \to S^1$ are defined by $f(x) = x$, $x \in S^1$ and $g(y) = \frac{y}{|y|}$, $y \in A$.

Then gf is the identity map on S^1. Moreover $fg(y) = \frac{y}{|y|}$, $y \in A$.

The required homotopy between fg and the identity on A is given by

$$H(y,t) = ty + (1-t)\frac{y}{|y|}.$$

Homotopy Type of a Contractible Space

Theorem

A space X is contractible if and only if it has the homotopy type of a one point space.

- Suppose X is contractible with homotopy $H : X \times I \to X$ and point x_0 in X, such that

$$H(x,0) = x, \quad H(x,1) = x_0, \quad x \in X.$$

Then X is homotopy equivalent to the singleton space $\{x_0\}$.

The homotopy equivalence $f : X \to \{x_0\}$ and homotopy inverse $g : \{x_0\} \to X$ are defined by

$$\begin{array}{rcl} f(x) & = & x_0, \quad x \in X; \\ g(x_0) & = & x_0. \end{array}$$

Homotopy Type of a Contractible Space (Cont'd)

- Suppose, conversely, that $f : X \to \{a\}$ is a homotopy equivalence between X and the one point space $\{a\}$, with homotopy inverse $g : \{a\} \to X$.

  Then there is a homotopy K between gf and the identity map on X.

  Moreover, by definition,

$$\begin{array}{rcl} K(x,0) & = & x; \\ K(x,1) & = & gf(x) = g(a), \quad x \in X. \end{array}$$

  Thus, the homotopy K is a contraction.

  So the space X is contractible.

Homotopy Equivalence and Deformation Retracts

Theorem

If X is a space and D a deformation retract of X, then D and X are homotopy equivalent.

- There is a homotopy $H: X \times I \to X$, such that

$$H(x,0) = x, \quad H(x,1) \in D, \quad x \in X,$$
$$H(a,t) = a, \quad a \in D, \ t \in I.$$

Let $f: D \to X$ denote the inclusion map $f(a) = a$.

Define $g: X \to D$ by

$$g(x) = H(x,1), \quad x \in X.$$

Then gf is the identity map on D.

H is a homotopy between fg and the identity on X.

Thus, f is a homotopy equivalence, with homotopy inverse g.

Homotopy Equivalence Between Pointed Spaces

Definition

Let X and Y be spaces with points x_0 in X and y_0 in Y. Then the pairs (X,x_0) and (Y,y_0) are **homotopy equivalent** or of the **same homotopy type** means that there exist continuous maps

$$f:(X,x_0)\to(Y,y_0) \quad \text{and} \quad g:(Y,y_0)\to(X,x_0)$$

for which the composite maps gf and fg are homotopic to the identity maps on X and Y, respectively, by homotopies that leave the base points fixed. In other words, it is required that there exist homotopies $H:X\times I\to X$ and $K:Y\times I\to Y$, such that

$$\begin{array}{llll} H(x,0)=gf(x), & H(x,1)=x, & H(x_0,t)=x_0, & x\in X,\ t\in I, \\ K(y,0)=fg(y), & K(y,1)=y, & K(y_0,t)=y_0, & y\in Y,\ t\in I. \end{array}$$

The map f is called a **homotopy equivalence** with **homotopy inverse** g.

Pointed Homotopy Equivalence is an Equivalence

Theorem

Homotopy equivalence between pairs is an equivalence relation.

- The proof is similar to the that of the theorem on homotopy equivalence of topological spaces.

Homotopy Equivalence and Induced Homomorphisms

Theorem

If the map $f:(X,x_0)\to(Y,y_0)$ is a homotopy equivalence between the indicated pairs, then the induced homomorphism

$$f_*:\pi_n(X,x_0)\to\pi_n(Y,y_0)$$

is an isomorphism for each positive integer n.

- Let $g:(Y,y_0)\to(X,x_0)$ be a homotopy inverse for f.
  Let H a homotopy between gf and i_X, which leaves x_0 fixed.
  Let $[\alpha]\in\pi_n(X,x_0)$.
  Consider α as a function from I^n to X, such that

$$\alpha(\partial I^n)=x_0.$$

Homotopy Equivalence and Homomorphisms (Cont'd)

- Define a homotopy $K : I^n \times I \to X$ by

$$K(t,s) = H(\alpha(t),s), \quad t \in I^n,\ s \in I.$$

Then

$$K(\cdot,0) = gf\alpha, \quad K(\cdot,1) = \alpha,$$
$$K(\partial I^n \times I) = H(\{x_0\} \times I) = x_0.$$

So $[gf\alpha] = [\alpha]$.

This means that $g_* f_* [\alpha] = [\alpha]$.

So g_* is a left inverse for f_*.

But f is a homotopy inverse for g.

So, by symmetry, g_* is also a right inverse for f_*.

So f_* is an isomorphism.

Homotopy Equivalence and Homomorphisms (Cont'd)

- The preceding theorem can be strengthened to show that a homotopy equivalence $f: X \to Y$ with $f(x_0) = y_0$ induces an isomorphism between

$$\pi_n(X, x_0) \cong \pi_n(Y, y_0), \quad \text{for each } n.$$

- The proof is more complicated because the homotopies may not leave the base points fixed.

Subsection 5

Homotopy Groups of Spheres

Introduction

- The homotopy groups $\pi_k(S^n)$ are not completely known.
- Previous examples have shown that

$$\pi_k(S^n) = \{0\},\ k < n, \quad \pi_k(S^1) = \{0\},\ k > 1, \quad \pi_n(S^n) \cong \mathbb{Z}.$$

- It may seem natural to conjecture that $\pi_k(S^n)$ is trivial for $k > n$, since the corresponding result holds for the homology groups.
- This would simply mean that every continuous map $f : S^k \to S^n$, where $k > n$ is homotopic to a constant map.
- However, this is not true.
- Hopf showed, in 1931, that $\pi_3(S^2)$ is not trivial.
- We examine some examples.

The Hopf map $p: S^3 \to S^2$

- Let $\mathbb{C}$ denote the field of complex numbers.
- Consider S^3, the unit sphere in Euclidean 4-space, as a set of ordered pairs of complex numbers, each pair having length 1:

$$S^3 = \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} : |z_1|^2 + |z_2|^2 = 1\}.$$

- Define an equivalence relation $\equiv$ on S^3 by

$$(z_1, z_2) \equiv (z_1', z_2')$$

  if and only if there is a complex number λ of length 1 such that $(z_1, z_2) = (\lambda z_1', \lambda z_2')$.
- For (z_1, z_2) in S^3, let $\langle z_1, z_2 \rangle$ denote the equivalence class of (z_1, z_2).
- Let

$$T = \{\langle z_1, z_2 \rangle : (z_1, z_2) \in S^3\}$$

  be the set of equivalence classes.

The Hopf map $p: S^3 \to S^2$ (Cont'd)

- Let $p: S^3 \to T$ be the projection map
$$p(z_1, z_2) = \langle z_1, z_2 \rangle, \quad (z_1, z_2) \in S^3.$$
- Assign T the quotient topology determined by p.
- I.e., a set O is open in T provided that $p^{-1}(O)$ is open in S^3.
- the inverse image $p^{-1}(\langle z_1, z_2 \rangle)$, called the **fiber over** $\langle z_1, z_2 \rangle$.
- For $\langle z_1, z_2 \rangle$ in T, the fiber over $\langle z_1, z_2 \rangle$, is a circle in S^3.
- We pursue the following strategy.
  - Show that T is homeomorphic to S^2;
  - Use the homeomorphism to replace T by S^2;
  - Obtain the Hopf map $p: S^3 \to S^2$.
- Strictly speaking, the Hopf map is the map $hp: S^3 \to S^2$, where $h: T \to S^2$ is the homeomorphism whose existence we must show.

The Hopf map $p: S^3 \to S^2$ (Cont'd)

- Consider the unit disc D in $\mathbb{C}$,

$$D = \{z \in \mathbb{C} : |z| \leq 1\}.$$

- The 2-sphere is the quotient space of D obtained by identifying the boundary of D to a point.
- We show T satisfies the same description.
- Consider the map $f : D \to T$, defined by

$$f(z) = \langle \sqrt{1 - |z|^2}, z \rangle, \quad z \in D.$$

- Then f is a closed, continuous map.
- For $\langle z_1, z_2 \rangle$ in T,

$$\begin{array}{rcl} f^{-1}(\langle z_1, z_2 \rangle) & = & \{z \in D : \langle z_1, z_2 \rangle = \langle \sqrt{1 - |z|^2}, z \rangle\} \\ & = & \{z \in D : \sqrt{1 - |z|^2} = \lambda z_1, z = \lambda z_2, \text{ for some } \lambda \in S^1\}. \end{array}$$

The Hopf map $p: S^3 \to S^2$ (Cont'd)

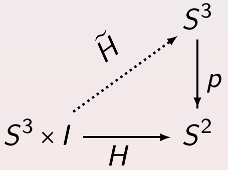
- We got
$$f^{-1}(\langle z_1, z_2\rangle) = \left\{ z \in D : \sqrt{1-|z|^2} = \lambda z_1, z = \lambda z_2, \text{ for some } \lambda \in S^1 \right\}.$$
- We distinguish two cases:
  - Suppose $z_1 \neq 0$.
    The equations above imply $\lambda z_1 = |z_1|$. So $\lambda = \frac{|z_1|}{z_1}$.
    Thus, $f^{-1}(\langle z_1, z_2\rangle)$ is a single point.
  - Suppose $z_1 = 0$. Then
$$\begin{aligned} f^{-1}(\langle z_1, z_2\rangle) &= f^{-1}(\langle 0, z_2\rangle) \\ &= \{z \in D : \sqrt{1-|z|^2} = 0, z = \lambda, \text{ for some } \lambda \in S^1\} \\ &= S^1. \end{aligned}$$
    So $f^{-1}(\langle 0, z_2\rangle)$ is the boundary of D.

The Hopf map $p: S^3 \to S^2$ (Cont'd)

- Using f as quotient map, T is the quotient space of D obtained by identifying the boundary S^1 to a point.
- Then T is homeomorphic to S^2.
- So we replace T by S^2 and have the Hopf map $p: S^3 \to S^2$.
- Showing that p is not homotopic to a constant map requires more background than we have developed.
- So we only sketch the basic idea.

The Hopf map $p: S^3 \to S^2$ (Conclusion)

- Let $H: S^3 \times I \to S^2$ be a homotopy between p and a constant map.
- The Hopf map is not a covering projection.
- But it is close enough to permit a covering homotopy $\widetilde{H}: S^3 \times I \to S^2$,



- The map $\widetilde{H}$ is a homotopy between the identity map on S^3 and a constant map.
- But this implies that S^3 is contractible, an obvious contradiction.
- Thus p is not homotopic to a constant map.
- So $\pi_3(S^2) \neq \{0\}$.

The Hopf maps $S^7 \to S^4$ and $S^{15} \to S^8$

- We switch to the division ring $\mathbb{Q}$ of quaternions.
- We represent S^7, the unit sphere in Euclidean 8-space, as ordered pairs of members of $\mathbb{Q}$,

$$S^7 = \{(z_1, z_2) \in \mathbb{Q} : \|z_1\|^2 + \|z_2\|^2 = 1\}.$$

- The quotient space T in this case is the quotient space of the unit disc

$$D = \{z \in \mathbb{Q} : \|z\| \leq 1\}$$

  obtained by identifying the boundary of D to a single point.
- Now D has real dimension four.
- So this quotient space is homeomorphic to S^4.

The Hopf maps $S^7 \to S^4$ and $S^{15} \to S^8$ (Cont'd)

- The Hopf map
$$p : S^7 \to S^4$$
with fiber S^3 is then defined as in the preceding example.
- This map shows that $\pi_7(S^4) \neq \{0\}$.
- In E^{16}, one can perform a similar construction by representing the unit sphere S^{15} as ordered pairs of Cayley numbers.
- This produces the Hopf map $p : S^{15} \to S^8$ with fiber S^7.
- It shows that $\pi_{15}(S^8) \neq \{0\}$.

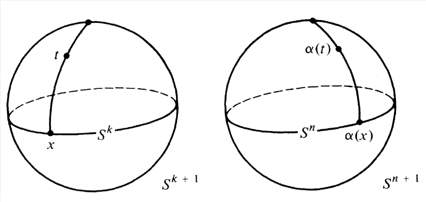
Suspension Homomorphism

- There is for each pair k,n of positive integers a natural homomorphism
$$E:\pi_k(S^n)\to\pi_{k+1}(S^{n+1})$$
called the **suspension homomorphism**.
- Consider $\pi_k(S^n)$ as homotopy classes of maps from $(S^k,1)$ to $(S^n,1)$, where we denote the base point of each sphere by 1.
- Consider S^n as the subspace of S^{n+1} consisting of all points of S^{n+1} having last coordinate 0.
- In this identification, S^n is usually called the "equator" of S^{n+1}.
- Continuing this geographical metaphor:
  - The point $(0,\ldots,0,1)$ is called the "north pole";
  - The point $(0,\ldots,0,-1)$ of S^{n+1} is called the "south pole".

The Suspension

- Suppose that $[\alpha] \in \pi_k(S^n)$.
- Then α is a continuous map from S^k to S^n.
- Extend α to a continuous map $\widehat{\alpha}: S^{k+1} \to S^{n+1}$ as follows.
    - $\widehat{\alpha}|_{S^k}$ is just α, and maps the equator of S^{k+1} to the equator of S^{n+1}. We require that $\widehat{\alpha}$ map the north pole of S^{k+1} to the north pole of S^{n+1} and the south pole of S^{k+1} to the south pole of S^{n+1}.

      The function is then extended radially as in the figure.



- The arc from the north pole to a point x in S^k is mapped linearly onto the arc from the north pole of S^{n+1} to $\alpha(x)$.

  This defines a on the "northern hemisphere".
- The "southern hemisphere" is treated the same way.

The extended map $\widehat{\alpha}$ is called the **suspension** of α.

The Freudenthal Suspension Theorem

- The **suspension homomorphism** E is defined by

$$E([\alpha]) = [\widehat{\alpha}], \quad [\alpha] \in \pi_k(S^n).$$

Claim: E is a homomorphism.

Theorem (The Freudenthal Suspension Theorem)

The suspension homomorphism

$$E : \pi_k(S^n) \to \pi_{k+1}(S^{n+1})$$

is an isomorphism for $k < 2n-1$ and is onto for $k \leq 2n-1$.

A Consequence of the Suspension Theorem

Corollary

The homotopy groups $\pi_k(S^n)$ are trivial for $k < n$.

- For any positive integer $r < k$, we have $k + r + 1 < 2n$.
  Hence, $k - r < 2(n - r) - 1$.
  Then
  $$\pi_k(S^n) \cong \pi_{k-1}(S^{n-1}) \cong \cdots \cong \pi_1(S^{n-k+1}).$$
  Now note that, for $k < n$, we have $n - k + 1 > 1$.
  Therefore, $\pi_1(S^{n-k+1})$ is trivial.
  So its isomorphic image $\pi_k(S^n)$ is also trivial.

Another Consequence of the Suspension Theorem

Corollary

The homotopy groups $\pi_n(S^n)$, $n \geq 1$, are all isomorphic to the group $\mathbb{Z}$ of integers.

- We rely on our previous arguments to show that

$$\pi_1(S^1) \cong \pi_2(S^2) \cong \mathbb{Z}.$$

If $n \geq 2$, then $n < 2n - 1$.

The Freudenthal Suspension Theorem shows that

$$\pi_2(S^2) \cong \pi_3(S^3) \cong \pi_4(S^4) \cong \cdots \cong \pi_n(S^n).$$

Subsection 6

The Relation Between $H_n(K)$ and $\pi_n(|K|)$

The Hurewicz Isomorphism Theorem

- The last theorem of this chapter shows a relationship between the homology groups and the homotopy groups of polyhedra.

Theorem (The Hurewicz Isomorphism Theorem)

Let K be a connected complex and $n \geq 2$ a positive integer. If the first $n-1$ homotopy groups of $|K|$ are trivial, then $H_n(K)$ and $\pi_n(|K|)$ are isomorphic.

Example: Consider the n-sphere S^n, for $n \geq 2$.

We know that $\pi_k(S^n) = \{0\}$, for $k < n$.

So, by the Hurewicz Isomorphism Theorem,

$$\pi_n(S^n) \cong H_n(S^n) \cong \mathbb{Z}.$$

Comment on Classification

- The homotopy groups do not provide for general topological spaces the type of classification already given for 2-manifolds and for covering spaces.
- There are examples of spaces X and Y which have isomorphic homotopy groups in each dimension but which are not homotopy equivalent (and therefore not homeomorphic).
- The induced homomorphism $f_* : \pi_n(X) \to \pi_n(Y)$ has been successful in classifying the homotopy type of spaces known as "CW-complexes".
- These spaces can be used to approximate arbitrary topological spaces.

Non-Homeomorphic Spaces

- Although the homotopy groups have not been completely successful in showing when spaces are homeomorphic, they are extremely useful in showing when spaces are not homeomorphic.
- To show that X and Y are not homeomorphic, compute the homotopy groups $\pi_n(X)$ and $\pi_n(Y)$.
- If $\pi_n(X)$ is not isomorphic to $\pi_n(Y)$, for some n, then X and Y are not homeomorphic.
- The same method can be used with the homology groups.

Example

- Recall that the Poincaré Conjecture asserts that every simply connected 3-manifold is homeomorphic to S^3.
- Our work on homotopy groups shows that the corresponding conjecture in dimension four is false.
- The 4-manifold $S^2 \times S^2$ is simply connected.
- However, we have

$$\pi_2(S^2 \times S^2) \cong \mathbb{Z} \oplus \mathbb{Z} \quad \text{and} \quad \pi_2(S^4) = \{0\}.$$

- So $S^2 \times S^2$ is not homeomorphic to S^4.