Introduction to Algebraic Topology

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LSSU Math 500

Subsection 1

Chain Derivation

Extension of a p-Simplex

- Let $\sigma^p = \langle v_0 \dots v_p \rangle$ be a p-simplex.
- Let v be a vertex, such that $\{v, v_0, \dots, v_p\}$ is geometrically independent.
- The symbol $v\sigma^p$ denotes the $(p+1)$-simplex

$$v\sigma^p = \langle v v_0 \dots v_p \rangle.$$

- Let $c = \sum g_i \cdot \sigma_i^p$ be a p-chain.
- Then vc denotes the $(p+1)$-chain

$$vc = \sum g_i \cdot v\sigma_i^p.$$

Boundary of the Extension

Lemma

Let c be a p-chain on a complex K and v a vertex for which the $(p+1)$-chain vc is defined. Then

$$\partial(vc) = c - v\partial c.$$

- We perform the calculation for a p-simplex $\sigma^p = \langle v_0 \ldots v_p \rangle$ and a geometrically independent vertex v.

$$\begin{array}{rcl}
\partial(v\sigma^p) & = & \partial(\langle vv_0 \ldots v_p \rangle) \\
& = & \langle v_0 \ldots, v_p \rangle - \langle vv_1 \ldots v_p \rangle + \cdots + (-1)^{p+1} \langle vv_0 \ldots v_{p-1} \rangle \\
& = & \sigma^p - v(\langle v_1 \ldots v_p \rangle - \cdots + (-1)^p \langle v_0 \ldots v_{p-1} \rangle) \\
& = & \sigma^p - v\partial\sigma^p.
\end{array}$$

First Chain Derivations

Definition

Let K be a complex. A chain mapping

$$\varphi = \{\varphi_p : C_p(K) \to C_p(K^{(1)})\}$$

is defined inductively as follows:

- Each 0-simplex σ^0 of K is a 0-simplex of the barycentric subdivision $K^{(1)}$. So we may consider $C_0(K)$ as a subgroup of $C_0(K^{(1)})$. Define

$$\varphi_0 : C_0(K) \to C_0(K^{(1)})$$

to be the inclusion map:

$$\varphi_0(c) = c, \quad c \in C_0(K).$$

First Chain Derivations (Cont'd)

Definition (Cont'd)

- For an elementary p-chain $1{\cdot}\sigma^p$ on K, define

$$\varphi_p(1\cdot\sigma^p)=\dot{\sigma}^p\varphi_{p-1}\partial(1\cdot\sigma^p),$$

where $\dot{\sigma}^p$ denotes the barycenter of σ^p.
Extend φ_p by linearity to a homomorphism

$$\varphi_p : C_p(K)\to C_p(K^{(1)});$$
$$\varphi_p(\sum g_i\cdot\sigma_i^p)=\sum\varphi_p(g_i\cdot\sigma_i^p),\quad \sum g_i\cdot\sigma_i^p\in C_p(K).$$

The sequence $\varphi=\{\varphi_p\}$ of homomorphisms defined in this way is called the **first chain derivation** on K.

n-th Chain Derivations

Definition

For $n>1$, the n**-th chain derivation** on k is the composition

$$C_p(K) \xrightarrow{\varphi_p^{(n-1)}} C_p(K^{(n-1)}) \xrightarrow{\varphi_p} C_p(K^{(n)})$$

of $\varphi^{(n-1)}$, the $(n-1)$st chain derivation on K, with the first chain derivation of the $(n-1)$st barycentric subdivision $K^{(n-1)}$.

Thus, the nth chain derivation on K is a chain mapping

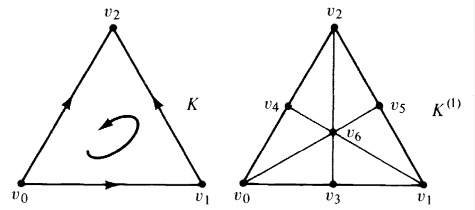
$$\varphi^{(n)} = \{\varphi_p^{(n)} : C_p(K) \to C_p(K^{(n)})\}.$$

Example

- Consider the complex $K = \mathrm{Cl}(\sigma^2)$, the closure of a 2-simplex

$$\sigma^2 = +\langle v_0 v_1 v_2 \rangle.$$

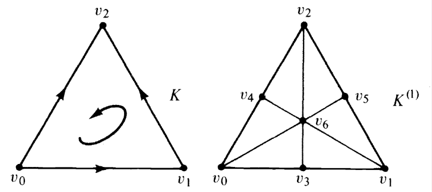
It is shown below with its barycentric subdivision $K^{(1)}$.



We examine the first chain derivation of this complex.

Example (Cont'd)

- Let v_3, v_4, v_5 and v_6 be the barycenters of $\langle v_0 v_1 \rangle$, $\langle v_0 v_2 \rangle$, $\langle v_1 v_2 \rangle$, and $\langle v_0 v_1 v_2 \rangle$, respectively.

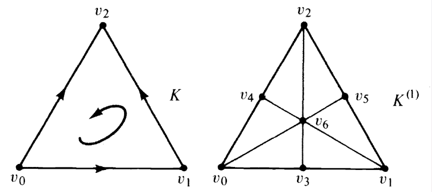


Then $\varphi_0 : C_0(K) \to C_0(K^{(1)})$ is the inclusion map.

For φ_1, we have

$$\begin{array}{rcl}\varphi_1(1 \cdot \langle v_0 v_1 \rangle) & = & v_3 \varphi_0 \partial (1 \cdot \langle v_0 v_1 \rangle) \\ & = & v_3 (1 \cdot \langle v_1 \rangle - 1 \cdot \langle v_0 \rangle) \\ & = & 1 \cdot \langle v_3 v_1 \rangle - 1 \cdot \langle v_3 v_0 \rangle;\end{array}$$

Example (Cont'd)



$$\begin{aligned}
\varphi_1(1\cdot\langle v_0v_2\rangle) &= v_4\varphi_0\partial(1\cdot\langle v_0v_2\rangle) = v_4(1\cdot\langle v_2\rangle - 1\cdot\langle v_0\rangle) \\
&= 1\cdot\langle v_4v_2\rangle - 1\cdot\langle v_4v_0\rangle; \\
\varphi_1(1\cdot\langle v_1v_2\rangle) &= v_5\varphi_0\partial(1\cdot\langle v_1v_2\rangle) = v_5(1\cdot\langle v_2\rangle - 1\cdot\langle v_1\rangle) \\
&= 1\cdot\langle v_5v_2\rangle - 1\cdot\langle v_5v_1\rangle; \\
\varphi_2(1\cdot\langle v_0v_1v_2\rangle) &= v_6\varphi_1\partial(1\cdot\langle v_0v_1v_2\rangle) \\
&= v_6\varphi_1(1\cdot\langle v_1v_2\rangle - 1\cdot\langle v_0v_2\rangle + 1\cdot\langle v_0v_1\rangle) \\
&= 1\cdot\langle v_6v_5v_2\rangle - 1\cdot\langle v_6v_5v_1\rangle - 1\cdot\langle v_6v_4v_2\rangle \\
&\quad + 1\cdot\langle v_6v_4v_0\rangle + 1\cdot\langle v_6v_3v_1\rangle - 1\cdot\langle v_6v_3v_0\rangle.
\end{aligned}$$

Chain Derivations are Chain Mappings

Theorem

Each chain derivation is a chain mapping.

- The composition of chain mappings is a chain mapping.
  So we show that the first chain derivation is a chain mapping.
  Let $\varphi = \{\varphi_p : C_p(K) \to C_p(K^{(1)})\}$ be a chain derivation.
  It must be shown that the following is commutative for $p \geq 1$.

$$\begin{array}{ccc} C_p(K) & \xrightarrow{\varphi_p} & C_p(K^{(1)}) \\ \partial \downarrow & & \downarrow \partial \\ C_{p-1}(K) & \xrightarrow[\varphi_{p-1}]{} & C_{p-1}(K^{(1)}) \end{array}$$

  Thus, we must show that, for each elementary p-chain $1 \cdot \sigma^p$,

$$\partial \varphi_p(1 \cdot \sigma^p) = \varphi_{p-1} \partial (1 \cdot \sigma^p).$$

Chain Derivations are Chain Mappings (Cont'd)

- For $p=1$, we have

$$\begin{array}{ccc} C_1(K) & \xrightarrow{\varphi_1} & C_1(K^{(1)}) \\ \partial \downarrow & & \downarrow \partial \\ C_0(K) & \xrightarrow[\varphi_0]{} & C_0(K^{(1)}) \end{array}$$

$$\begin{array}{rcl} \partial\varphi_1(1\cdot\sigma^1) & \stackrel{\varphi_1}{=} & \partial(\dot{\sigma}^1\varphi_0\partial(1\cdot\sigma^1)) \\ & \stackrel{\text{Lemma}}{=} & \varphi_0\partial(1\cdot\sigma^1)-\dot{\sigma}^1\partial\varphi_0\partial(1\cdot\sigma^1) \\ & \stackrel{\varphi_0}{=} & \varphi_0\partial(1\cdot\sigma^1)-\dot{\sigma}^1\partial\partial(1\cdot\sigma^1) \\ & \stackrel{\partial\partial=0}{=} & \varphi_0\partial(1\cdot\sigma^1). \end{array}$$

Chain Derivations are Chain Mappings (Cont'd)

- Thus $\partial\varphi_1 = \varphi_0\partial$. So the desired conclusion holds for $p=1$.
  Proceeding inductively, let $1\cdot\sigma^p$ be an elementary p-chain on K. Then

$$\begin{array}{ccc} C_p(K) & \xrightarrow{\varphi_p} & C_p(K^{(1)}) \\ \partial\downarrow & & \downarrow\partial \\ C_{p-1}(K) & \xrightarrow[\varphi_{p-1}]{} & C_{p-1}(K^{(1)}) \end{array}$$

$$\begin{array}{rcl} \partial\varphi_p(1\cdot\sigma^p) & \overset{\varphi_p}{=} & \partial(\dot{\sigma}^p\varphi_{p-1}\partial(1\cdot\sigma^p)) \\ & \overset{\text{Lemma}}{=} & \varphi_{p-1}\partial(1\cdot\sigma^p) - \dot{\sigma}^p\partial\varphi_{p-1}\partial(1\cdot\sigma^p) \\ & \overset{\varphi_{p-1}}{=} & \varphi_{p-1}\partial(1\cdot\sigma^p) - \dot{\sigma}^p\varphi_{p-2}\partial\partial(1\cdot\sigma^p) \\ & \overset{\partial\partial=0}{=} & \varphi_{p-1}\partial(1\cdot\sigma^p). \end{array}$$

Thus, $\partial\varphi_p = \varphi_{p-1}\partial$, for elementary p-chains, and, hence, for all p-chains.

Left Invertibility of Chain Derivations

Theorem

Let K be a complex with first chain derivation $\varphi = \{\varphi_p\}$. There is a chain mapping

$$\psi = \{\psi_p : C_p(K^{(1)}) \to C_p(K)\},$$

such that $\psi_p \varphi_p$ is the identity map on $C_p(K)$, for each $p \geq 0$.

- Such a chain mapping ψ is called a **left inverse** for φ.

  Let f be any simplicial map from $K^{(1)}$ to K satisfying:

  If $\dot{\sigma}$ is a vertex of $K^{(1)}$, then $f(\dot{\sigma})$ is a vertex of the simplex σ of which $\dot{\sigma}$ is the barycenter.

  Let $\psi = \{\psi_p\}$ be the chain mapping induced by f.

  Suppose that τ_p is a p-simplex of $K^{(1)}$.

  Then $\psi_p(1 \cdot \tau^p) = \eta \cdot \sigma^p$, where:

  - η is 0, 1 or -1;
  - σ^p is the p-simplex of K which produces τ^p in its barycentric subdivision.

Left Invertibility of Chain Derivations (Cont'd)

- Clearly $\psi_0\varphi_0$ is the identity map on $C_0(K)$.
  Suppose that $\psi_{p-1}\varphi_{p-1} : C_{p-1}(K) \to C_{p-1}(K)$ is the identity.
  Consider $\psi_p\varphi_p : C_p(K) \to C_p(K)$.
  Suppose $1 \cdot \sigma^p$ is an elementary p-chain on K.
  Then
  $$\psi_p\varphi_p(1 \cdot \sigma^p) = \psi_p(\dot{\sigma}\varphi_{p-1}\partial(1 \cdot \sigma^p)) = m \cdot \sigma^p,$$
  for some integer m. But
  $$\begin{aligned} \partial(m \cdot \sigma^p) &= \partial\psi_p\varphi_p(1 \cdot \sigma^p) = \psi_{p-1}\partial\varphi_p(1 \cdot \sigma^p) \\ &= \psi_{p-1}\varphi_{p-1}\partial(1 \cdot \sigma^p) = \partial(1 \cdot \sigma^p). \end{aligned}$$
  So we get $m\partial(1 \cdot \sigma^p) = \partial(m \cdot \sigma^p) = \partial(1 \cdot \sigma^p)$.
  Hence, $m = 1$. Thus, $\psi_p\varphi_p(1 \cdot \sigma^p) = 1 \cdot \sigma^p$.
  So $\psi_p\varphi_p$ is the identity on $C_p(K)$.

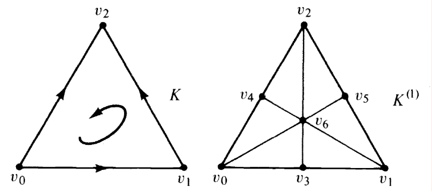
Example

- Consider the chain derivation $\varphi = \{\varphi_p\}_0^2$ of the preceding example.

  We may define the simplicial map f from $K^{(1)}$ to K, the closure of the 2-simplex $\langle v_0 v_1 v_2 \rangle$, in any manner consistent with having $f(v_i)$ a vertex of the simplex of which v_i is the barycenter.

  Thus we must have

$$f(v_0) = v_0, \quad f(v_1) = v_1, \quad f(v_2) = v_2.$$



  One possible definition for f on the remaining vertices is $f(v_3) = f(v_4) = v_0,\ f(v_5) = v_1,\ f(v_6) = v_2.$

Example (Cont'd)

- Let $f(v_3) = f(v_4) = v_0$, $f(v_5) = v_1$, $f(v_6) = v_2$.

  Let $\psi = \{\psi_p\}$ be the chain mapping induced by f as in the proof of the preceding theorem:

$$\psi_0(1\cdot\langle v_0\rangle) = \psi_0(1\cdot\langle v_3\rangle) = \psi_0(1\cdot\langle v_4\rangle) = 1\cdot\langle v_0\rangle;$$
$$\psi_0(1\cdot\langle v_1\rangle) = \psi_0(1\cdot\langle v_5\rangle) = 1\cdot\langle v_1\rangle;$$
$$\psi_0(1\cdot\langle v_2\rangle) = \psi_0(1\cdot\langle v_6\rangle) = 1\cdot\langle v_2\rangle;$$
$$\psi_1(1\cdot\langle v_0 v_4\rangle) = 0; \quad \psi_1(1\cdot\langle v_0 v_6\rangle) = 1\cdot\langle v_6 v_2\rangle; \quad \text{etc.}$$
$$\psi_2(1\cdot\langle v_3 v_1 v_6\rangle) = 1\cdot\langle v_0 v_1 v_2\rangle; \quad \psi_2(1\cdot\langle v_0 v_4 v_6\rangle) = 0; \quad \text{etc.}$$

Example (Cont'd)

- Consider, for example,

$$\begin{aligned}\psi_1\varphi_1(1\cdot\langle v_0v_1\rangle) &= \psi_1(1\cdot\langle v_3v_1\rangle - 1\cdot\langle v_3v_0\rangle)\\ &= 1\cdot\langle v_0v_1\rangle - 0\\ &= 1\cdot\langle v_0v_1\rangle.\end{aligned}$$

We compute $\psi_2\varphi_2(1\cdot\langle v_0v_1v_2\rangle)$, where $\varphi_2(1\cdot\langle v_0v_1v_2\rangle)$ is expressed as in the preceding example,

$$\begin{aligned}\varphi_2(1\cdot\langle v_0v_1v_2\rangle) &= 1\cdot\langle v_6v_5v_2\rangle - 1\cdot\langle v_6v_5v_1\rangle - 1\cdot\langle v_6v_4v_2\rangle\\ &\quad + 1\cdot\langle v_6v_4v_0\rangle + 1\cdot\langle v_6v_3v_1\rangle - 1\cdot\langle v_6v_3v_0\rangle.\end{aligned}$$

But f collapses all 2-simplexes except $\langle v_6v_3v_1\rangle$.
So we get

$$\psi_2\varphi_2(1\cdot\langle v_0v_1v_2\rangle) = \psi_2(1\cdot\langle v_6v_3v_1\rangle) = 1\cdot\langle v_2v_0v_1\rangle = 1\cdot\langle v_0v_1v_2\rangle.$$

Chain Homotopic Mappings and Chain Homotopies

Definition

A pair $\varphi = \{\varphi_p\}_0^\infty$ and $\mu = \{\mu_p\}_0^\infty$ of chain mappings from a complex K to a complex L

$$C_p(K) \underset{\mu_p}{\overset{\varphi_p}{\rightrightarrows}} C_p(L)$$

are **chain homotopic** means that there is a sequence $\mathscr{D} = \{D_p\}_{-1}^\infty$ of homomorphisms $D_p : C_p(K) \to C_{p+1}(L)$, such that

$$C_p(K) \xrightarrow{D_p} C_{p+1}(L) \xrightarrow{\partial} C_p(L) \qquad C_p(K) \xrightarrow{\partial} C_{p-1}(L) \xrightarrow{D_{p-1}} C_p(L)$$

$$\partial D_p + D_{p-1}\partial = \varphi_p - \mu_p, \quad D_{-1} = 0.$$

The sequence $\mathscr{D}$ is called a **deformation operator** or a **chain homotopy**.

Homomorphisms Induced by Chain Homotopies

Theorem

If φ and μ are chain homotopic chain mappings from complex K to complex L, then the induced homomorphisms φ_p^* and μ_p^* from $H_p(K)$ to $H_p(L)$ are equal, $p \geq 0$.

- Suppose φ and μ are chain homotopic.
  By definition, there is a deformation operator $\mathscr{D} = \{D_p\}_{-1}^{\infty}$, such that

  $$\partial D_p + D_{p-1}\partial = \varphi_p - \mu_p, \quad D_{-1} = 0.$$

  For $[z_p] \in H_p(K)$,

  $$\begin{aligned} \varphi_p^*([z_p]) - \mu_p^*([z_p]) &= [\varphi_p(z_p) - \mu_p(z_p)] \\ &= [\partial D_p(z_p) + D_{p-1}(\partial z_p)] = 0, \end{aligned}$$

  since $\partial z_p = 0$, for any cycle, and $\partial D_p(z_p)$ is a boundary.
  Thus, $\varphi_p^* = \mu_p^*$, for each value of p.

Chain Equivalent Complexes

Definition

Complexes K and L are **chain equivalent** means that there are chain mappings

$$\varphi = \{\varphi_p\} : K \to L;$$
$$\psi = \{\psi_p\} : L \to K,$$

such that the composite chain mappings

$$\psi\varphi = \{\psi_p\varphi_p\} \quad \text{and} \quad \varphi\psi = \{\varphi_p\psi_p\}$$

are chain homotopic to the identity chain mappings on K and L, respectively.

- Chain homotopy is an equivalence relation for chain mappings.
- Chain equivalence is an equivalence relation for complexes.

Chain Equivalent Complexes and Homology Groups

Theorem

Chain equivalent complexes K and L have isomorphic homology groups in corresponding dimensions.

- Suppose K and L are chain equivalent complexes.

  Let φ and ψ be the witnessing chain mappings.

  By the preceding theorem,

  $$\psi_p^* \varphi_p^* : H_p(K) \to H_p(K),$$
  $$\varphi_p^* \psi_p^* : H_p(L) \to H_p(L)$$

  are the identity maps.

  So φ_p^* is an isomorphism for each value of p.

Homology Groups of Complex and Barycentric Subdivision

- We aim to prove that the homology groups of a complex K are isomorphic to those of its barycentric subdivision $K^{(1)}$.
- By the preceding theorem, it is sufficient to show that K and $K^{(1)}$ are chain equivalent.
- For this we need chain mappings φ from K to $K^{(1)}$ and ψ from $K^{(1)}$ to K for which $\psi\varphi$ and $\varphi\psi$ are chain homotopic to the appropriate identity chain maps.
- We have φ, the first chain derivation of K.
- We have ψ, the left inverse provided by a previous theorem.
- We know that $\psi\varphi$ is the identity chain map on K.
- So it remains to show that $\varphi\psi$ is chain homotopic to the identity chain map on $K^{(1)}$.

Chain Equivalence of Complex and Barycentric Subdivision

Theorem

A complex K and its first barycentric subdivision are chain equivalent.

- We show that $\varphi\psi$ is chain homotopic to the identity on $K^{(1)}$.

  This requires a deformation operator

  $$\mathscr{D} = \left\{ D_p : C_p(K^{(1)}) \to C_{p+1}(K^{(1)}) \right\},$$

  such that $D_{-1} = 0$ and, for each elementary p-chain $1 \cdot \tau^p$ on $K^{(1)}$,

  $$1 \cdot \tau^p - \varphi_p \psi_p (1 \cdot \tau^p) = \partial D_p (1 \cdot \tau^p) + D_{p-1} \partial (1 \cdot \tau^p).$$

Chain Equivalence (Cont'd)

- We must have $D_{-1} = 0$.
  To define D_0, let w be a vertex of $K^{(1)}$.
  Suppose
  $$\psi_0(1 \cdot \langle w \rangle) = 1 \cdot \langle v \rangle,$$
  v a vertex of a simplex σ of K of which w is the barycenter.
  Then
  $$\varphi_0 \psi_0(1 \cdot \langle w \rangle) = \varphi_0(1 \cdot \langle v \rangle) = 1 \cdot \langle v \rangle.$$
  Thus,
  $$1 \cdot \langle w \rangle - \varphi_0 \psi_0(1 \cdot \langle w \rangle) = 1 \cdot \langle w \rangle - 1 \cdot \langle v \rangle = \partial(1 \cdot \langle vw \rangle).$$
  So we define
  $$D_0(1 \cdot \langle w \rangle) = 1 \cdot \langle vw \rangle.$$
  Being defined for every elementary 0-chain, it can be extended by linearity to a homomorphism $D_0 : C_0(K^{(1)}) \to C_1(K^{(1)})$.

Chain Equivalence (Cont'd)

- Suppose, now, that $D_0, \ldots, D_{p-1}$ have all been defined.
  Let $1 \cdot \tau^p$ be an elementary p-chain on $K^{(1)}$.
  Then, for every $(p-1)$-chain c,
  $$c - \varphi_{p-1}\psi_{p-1}(c) = \partial D_{p-1}(c) + D_{p-2}\partial(c).$$
  So
  $$\partial D_{p-1}(c) = c - \varphi_{p-1}\psi_{p-1}(c) - D_{p-2}\partial(c).$$
  Consider
  $$z := 1 \cdot \tau^p - \varphi_p\psi_p(1 \cdot \tau^p) - D_{p-1}\partial(1 \cdot \tau^p).$$

Chain Equivalence (Cont'd)

- We set $z := 1\cdot\tau^p - \varphi_p\psi_p(1\cdot\tau^p) - D_{p-1}\partial(1\cdot\tau^p)$.

  We compute

$$\begin{aligned}\partial z &= \partial(1\cdot\tau^p) - \partial\varphi_p\psi_p(1\cdot\tau^p) - \partial D_{p-1}\partial(1\cdot\tau^p)\\ &= \partial(1\cdot\tau^p) - \varphi_{p-1}\psi_{p-1}\partial(1\cdot\tau^p)\\ &\quad -(\partial(1\cdot\tau^p) - \varphi_{p-1}\psi_{p-1}\partial(1\cdot\tau^p) - D_{p-2}\partial\partial(1\cdot\tau^p)) = 0.\end{aligned}$$

  This means that z is a cycle on $K^{(1)}$.

  An argument analogous to that used previously shows that z is the boundary of a $(p+1)$-chain c_{p+1} on $K^{(1)}$.

  We then define

$$D_p(1\cdot\tau^p) = c_{p+1}.$$

  Finally, we extend by linearity.

  This completes the definition of the deformation operator $\mathscr{D}$.

  It also shows that K and $K^{(1)}$ are chain equivalent.

Homology Groups of $H_p(K)$ and $H_p(K^{(n)})$

Theorem

The homology groups $H_p(K)$ and $H_p(K^{(n)})$ are isomorphic for all integers $p \geq 0$, $n \geq 1$, and each complex K.

- The inductive definition of $K^{(n)}$ and the preceding theorem show that K and $K^{(n)}$ are chain equivalent for $n \geq 1$.

  By a previous theorem,

$$H_p(K) \cong H_p(K^{(n)}), \quad p \geq 0.$$

Uniqueness of the Induced Homomorphisms

- Let $|K|$ and $|L|$ be polyhedra with triangulations K and L, respectively. and $f:|K|\to|L|$ a continuous map.

  Claim: The induced homomorphisms $f_p^*: H_P(K)\to H_p(L)$ are uniquely determined by f.

  By the Simplicial Approximation Theorem, there are:
  - A barycentric subdivision $K^{(k)}$ of K;
  - A simplicial mapping g from $K^{(k)}$ to L,

  such that, as functions from $|K|$ to $|L|$, f and g are homotopic.

  The theorem allows some freedom in the choices of g and the degree k of the barycentric subdivision.

  k must be large enough so that $K^{(k)}$ is star related to L relative to f.

  The simplicial map g is given, for a vertex u of $K^{(k)}$, by setting $g(u)$ to be any vertex of L, satisfying $f(\text{ost}(w))\subseteq\text{ost}(g(u))$.

Uniqueness of the Induced Homomorphisms (Cont'd)

Claim: The sequence of homomorphisms is independent of the admissible choices for g.

We show that any admissible change in the value of g at one vertex does not alter the induced homomorphisms $g_p^*: H_p(K^{(k)}) \to H_p(L)$.

Then note that any simplicial map satisfying the requirements of the theorem defining g_p^* can be obtained from any other one by a finite sequence of such changes at single vertices.

Let g, h be two simplicial maps from $K^{(k)}$ into L which:

- Have identical values at each vertex of $K^{(k)}$ except for one vertex v;
- For this vertex, $\text{ost}(g(v))$ and $\text{ost}(h(v))$ both contain $f(\text{ost}(v))$.

We show that the chain mappings $\{g_p : C_p(K^{(k)}) \to C_p(L)\}$ and $\{h_p : C_p(K^{(k)}) \to C_p(L)\}$ are chain homotopic.

Then, by the preceding theorem, the induced homomorphisms g_p^* and h_p^* from $H_p(K^{(k)})$ to $H_p(L)$ are identical for each value of p.

Uniqueness of the Induced Homomorphisms ($p = 0$)

- For our deformation operator $\mathscr{D} = \{D_p : C_p(K^{(m)}) \to C_{p+1}(L)\}_{-1}^{\infty}$, we must have $D_{-1} = 0$.
  For any vertex u of $K^{(k)}$, define
  $$\begin{array}{rcl} D_0(1 \cdot \langle u \rangle) & = & 0, \quad u \neq v; \\ D_0(1 \cdot \langle v \rangle) & = & 1 \cdot \langle h(v) g(v) \rangle. \end{array}$$
  Extend D_0 by linearity to a homomorphism from $C_0(K^{(k)})$ to $C_1(L)$.
  Note that
  $$\begin{array}{rcl} \partial D_0(1 \cdot \langle v \rangle) + D_{-1} \partial (1 \cdot \langle v \rangle) & = & \partial (1 \cdot \langle h(v) g(v) \rangle) \\ & = & 1 \cdot \langle g(v) \rangle - 1 \cdot \langle h(v) \rangle \\ & = & g_0(1 \cdot \langle v \rangle) - h_0(1 \cdot \langle v \rangle). \end{array}$$
  If u is a vertex of $K^{(k)}$ different from v, then
  $$g_0(1 \cdot \langle u \rangle) = h_0(1 \cdot \langle u \rangle), \quad D_0(1 \cdot \langle u \rangle) = 0.$$
  So the desired relation $\partial D_p + D_{p-1} \partial = g_p - h_p$ holds for $p = 0$.

Uniqueness of the Induced Homomorphisms ($p \geq 1$)

- Let $1 \cdot \sigma^p$ be an elementary p-chain in $C_p(K^{(k)})$.
  - Suppose v is not a vertex of σ^p.
    Then we define $D_p(1 \cdot \sigma^p) = 0$ in $C_{p+1}(L)$.
  - Suppose v is a vertex of σ^p.
    Then $\sigma^p = v\sigma^{p-1}$, for some $(p-1)$-simplex σ^{p-1}.
    Let τ be the $(p-1)$-simplex in L on which g and h map σ^{p-1}.
    Define
    $$D_p(1 \cdot \sigma^p) = 1 \cdot h(v)g(v)\tau.$$

As usual, D_p is extended linearly to a homomorphism from $C_p(K^{(k)})$ to $C_{p+1}(L)$.

Uniqueness of the Induced Homomorphisms ($p \geq 1$ Cont'd)

- Then for the case in which v is a vertex of σ^p,

$$\begin{aligned}
&\partial D_p(1 \cdot \sigma^p) + D_{p-1}\partial(1 \cdot \sigma^p) \\
&= \partial(1 \cdot h(v)g(v)\tau) + D_{p-1}\partial(1 \cdot v\sigma^{p-1}) \\
&= 1 \cdot g(v)\tau - h(v)\partial(1 \cdot g(v)\tau) + D_{p-1}(1 \cdot \sigma^{p-1} - v\partial(1 \cdot \sigma^{p-1})) \\
&= 1 \cdot g(v)\tau - h(v)[1 \cdot \tau - g(v)\partial(1 \cdot \tau)] - D_{p-1}(v\partial(1 \cdot \sigma^{p-1})) \\
&= 1 \cdot g(v)\tau - 1 \cdot h(v)\tau + h(v)g(v)\partial(1 \cdot \tau) - h(v)g(v)\partial(1 \cdot \tau) \\
&= g_p(1 \cdot v\sigma^{p-1}) - h_p(1 \cdot v\sigma^{p-1}) \\
&= g_p(1 \cdot \sigma^p) - h_p(1 \cdot \sigma^p).
\end{aligned}$$

Thus, $\partial D_p + D_{p-1}\partial = g_p - h_p,\ p \geq 0$.

The chain mappings induced by g and h must be chain homotopic.

By the preceding theorem, $g_p^* = h_p^*$.

So f_p^* is independent of the allowable choices of the simplicial map g.

Independence from the Degree of Subdivision

- The homomorphism $f_p^*: H_p(K) \to H_p(L)$ is actually the composition
$$H_p(K) \xrightarrow{\mu_p^*} H_p(K^{(k)}) \xrightarrow{g_p^*} H_p(L),$$
where μ_p^* is the isomorphism induced by chain derivation.
- Consider a barycentric subdivision $K^{(r)}$ of higher degree.
- Let $\psi_p^*: H_p(K) \to H_p(K^{(r)})$ be the isomorphism induced by chain derivation.
- Let $j_p^*: H_p(K^{(r)}) \to H_p(L)$ be the homomorphism induced by an admissible simplicial map.
- It can be shown that
$$g_p^* \mu_p^* = j_p^* \psi_p^*.$$
- Hence f_p^* is also independent of the allowable choices for the degree of the barycentric subdivision $K^{(k)}$.

Subsection 2

The Lefschetz Fixed Point Theorem

Fixed Simplexes and Weights

- In this section we assume that rational numbers rather than integers are used as the coefficient group for chains.
- Thus, the p-th chain group $C_p(K)$ of a complex K is considered a vector space over the field of rational numbers.

Definition

Let K be a complex with $\{\sigma_i^p\}$ its set of p-simplexes. Let $\varphi = \{\varphi_p\}$ be a chain mapping on K. For a p-simplex σ_i^p of K,

$$\varphi_p(1 \cdot \sigma_i^p) = \sum_{\sigma_j^p \in K} a_{ij}^p \sigma_j^p,$$

for some rational numbers a_{ij}^p one for each p-simplex σ_j^p of K.

Then σ_i^p is a **fixed simplex** of φ provided that a_{ii}^p, the coefficient of σ_i^p in the expansion of $\varphi_p(1 \cdot \sigma_i^p)$, is not zero.

The number $(-1)^p a_{ii}^p$ is called the **weight** of the fixed simplex σ_i^p.

The Lefschetz Number

Definition (Cont'd)

Let $A_p = (a_{ij}^p)$ be the matrix whose entry in row i and column j is a_{ij}^p.
The trace of a square matrix is the sum of its diagonal elements.
We have

$$\text{trace} A_p = \sum a_{ii}^p.$$

Moreover, the number

$$\lambda(\varphi) = \sum_p (-1)^p \text{trace}(A_p)$$

is the sum of the weights of all the fixed simplexes of φ.
The number $\lambda(\varphi)$ is called the **Lefschetz number** of φ.
(Note that, if $\lambda(\varphi) \neq 0$, then φ must have at least one fixed simplex in some dimension p.)

Remarks

- The matrix $A_p = (a_{ij}^p)$ is the matrix of φ_p as a linear transformation from the vector space $C_p(K)$ into itself relative to the basis of elementary p-chains $\{1 \cdot \sigma_i^p\}$.
- The trace of the matrix of a linear transformation is not affected by a change of basis.
- So the Lefschetz number $\lambda(\varphi)$ is independent of the choice of basis for $C_p(K)$.

Lefschetz Number and Euler Characteristic

- Let K be a complex.

  Let $\varphi_p : C_p(K) \to C_p(K)$ be the identity map on $C_p(K)$, $p \geq 0$.

  Then

  $$a_{ii}^p = 1, \quad a_{ij}^p = 0, \text{ for } i \neq j.$$

  Thus, each simplex is a fixed simplex.

  Let:

  - α_p be the number of simplexes of dimension p;
  - $\chi(K)$ the Euler characteristic of K.

  Then we have:

  $$\lambda(\varphi) = \sum(-1)^p \text{trace} A_p = \sum(-1)^p \alpha_p = \chi(K).$$

  Thus, the Lefschetz number is a generalization of the Euler characteristic.

Lefschetz Number and Induced Homomorphisms

Theorem

Let $\varphi = \{\varphi_p\}$ be a chain mapping on a complex K. The Lefschetz number $\lambda(\varphi)$ is completely determined by the induced homomorphisms $\varphi_p^* : H_p(K) \to H_p(K)$ on the homology groups.

- The proof is similar to the proof of the Euler-Poincaré Theorem.

  The same notation is adopted here.
  - $\{z_p^i\} \cup \{b_p^i\}$ is a basis for the cycle vector space Z_p;
  - $\{b_p^i\}$ is a basis for the boundary space B_p;
  - $\{d_p^i\}$ is a basis for D_p;
  - $b_p^i = \partial d_{p+1}^i$;
  - n is the dimension of K.

  Note that $\{b_p^i\} \cup \{z_p^i\} \cup \{d_p^i\}$ is a basis for C_p.

Lefschetz Number and Induced Homomorphisms (Cont'd)

- The linear transformation φ_p takes B_p into B_p.
  So, for any b_p^i,
  $$\varphi_p(b_p^i) = \sum_j a_{ij}^p b_p^j, \quad 0 \leq p \leq n-1,$$
  for some rational coefficients a_{ij}^p.
  For any z_p^i, $0 \leq p \leq n$, $\varphi_p(z_p^i)$ must be a cycle.
  So there are coefficients $a_{ij}'^p$ and e_{ij}^p, such that
  $$\varphi_p(z_p^i) = \sum_j a_{ij}'^p b_p^j + \sum_j e_{ij}^p z_p^j.$$
  For any d_p^i, $1 \leq p \leq n$, there are coefficients $a_{ij}''^p$, $e_{ij}'^p$ and g_{ij}^p, such that
  $$\varphi_p(d_p^i) = \sum_j a_{ij}''^p b_p^j + \sum_j e_{ij}'^p z_p^j + \sum_j g_{ij}^p d_p^j.$$

Lefschetz Number and Induced Homomorphisms (Cont'd)

- Then
$$\lambda(\varphi) = \sum_{i=0}^{n} (-1)^p (\text{trace} A_p + \text{trace} E_p + \text{trace} G_p),$$
where $A_p = (a_{ij}^p)$, $E_p = (e_{ij}^p)$, $G_p = (g_{ij}^p)$, and $A_n = G_0$ is the zero matrix. Now
$$\partial\varphi_{p+1}(d_{p+1}^i) = \varphi_p \partial(d_{p+1}^i) = \varphi_p(b_p^i) = \sum a_{ij}^p b_p^j.$$
Also,
$$\begin{array}{rcl}\partial\varphi_{p+1}(d_{p+1}^i) & = & \partial(\sum a_{ij}''^{p+1} b_{p+1}^j + \sum e_{ij}'^{p+1} z_{p+1}^j + \sum g_{ij}^{p+1} d_{p+1}^j) \\ & = & \sum g_{ij}^{p+1} \partial(d_{p+1}^j) \\ & = & \sum g_{ij}^{p+1} b_p^j.\end{array}$$
Then $a_{ij}^p = g_{ij}^{p+1}$, $A_p = G_{p+1}$, $0 \le p \le n-1$.

Lefschetz Number and Induced Homomorphisms (Cont'd)

- The sum $\lambda(\varphi) = \sum_{i=0}^{n} (-1)^p (\text{trace} A_p + \text{trace} E_p + \text{trace} G_p)$ telescopes to give

$$\lambda(\varphi) = \sum_{i=0}^{n} (-1)^p \text{trace} E_p.$$

This means that the Lefschetz number $\lambda(\varphi)$ is completely determined by the action of the maps φ_p on the generating cycles z_p^i of $H_p(K)$.

But the homology classes $[z_p^i]$ generate $H_p(K)$,

$$\varphi_p^*[z_p^i]) = \sum_j e_{ij}^p [z_p^j].$$

So the coefficients e_{ij}^p are determined by the $\varphi_p^* : H_p(K) \to H_p(K)$.

Thus, the induced homomorphisms completely determine the e_{ij}^p.

And the e_{ij}^p, in turn, completely determine $\lambda(\varphi)$.

The Lefschetz Number of a Continuous Mapping

- We have defined the Lefschetz number for chain mappings.
- The definition is now extended to continuous mappings.

Definition

Let K be a complex and $f : |K| \to |K|$ a continuous function.
Let $K^{(s)}$ be a barycentric subdivision of K.
Let g a simplicial map from $K^{(s)}$ to K which is a simplicial approximation of f. Then g induces a chain mapping

$$\left\{g_p : C_p(K^{(s)}) \to C_p(K)\right\}.$$

Let $\mu = \{\mu_p : C_p(K) \to C_p(K^{(s)})\}$ be the s-th chain derivation on K.
The **Lefschetz number** $\lambda(f)$ of f is the Lefschetz number of the composite chain mapping $\{g_p\mu_p : C_p(K) \to C_p(K)\}$.

Remarks

- It appears that the Lefschetz number $\lambda(f)$ of a continuous $f : |K| \to |K|$ is influenced by the possible choices for g and s.
- However, recall that $\lambda(f)$ is completely determined by the induced homomorphisms
$$f_p^* = g_p^* \mu_p^* : H_p(K) \to H_p(K).$$
- Moreover, f_p^* is independent of the allowable choices for g and s.
- So $\lambda(f)$ is independent of the choices for g and s.

The Lefschetz Fixed Point Theorem

Theorem (The Lefschetz Fixed Point Theorem)

Let K be a complex and $f : |K| \to |K|$ a continuous map. If the Lefschetz number $\lambda(f)$ is not 0, then f has a fixed point.

- Suppose to the contrary that F has no fixed point.

  Since $|K|$ is compact, there is a number $\epsilon > 0$, such that if $x \in |K|$, then the distance $\|f(x) - x\| \geq \epsilon$.

  By replacing K with a suitable barycentric subdivision if necessary, we may assume that $\text{mesh} K < \frac{\epsilon}{3}$.

  By the proof of the Simplicial Approximation Theorem, there are:
  - A positive integer s;
  - A simplicial map g from $K^{(s)}$ to K, homotopic to f,

  such that, for x in $|K|$, $f(x)$ and $g(x)$ lie in a common simplex of K.

  Then $\|f(x) - g(x)\| < \frac{\epsilon}{3}$, for all $x \in |K|$.

The Lefschetz Fixed Point Theorem (Cont'd)

- Suppose that some simplex σ of K contains a point x, such that $g(x)$ is also in σ. Then

$$\|f(x)-x\| \leq \|f(x)-g(x)\| + \|g(x)-x\| < \frac{2\epsilon}{3}.$$

This contradicts the fact that $\|f(x)-x\| > \epsilon$.

Thus, σ and $g(\sigma)$ are disjoint for all σ in K.

Let $\mu = \{\mu_p : C_p(K) \to C_p(K^{(s)})\}$ be the s-th chain derivation.

Let $\{g_p : C_p(K^{(s)}) \to C_p(K)\}$ be the chain mapping induced by g.

The Lefschetz Fixed Point Theorem (Cont'd)

- Suppose σ^p is a p-simplex of K.

  Then $\mu_p(1 \cdot \sigma^p)$ is a chain on $K^{(s)}$ all of whose simplexes with nonzero coefficient are contained in σ^p.

  Now σ^p and $g(\sigma^p)$ are disjoint.

  So $g_p \mu_p(1 \cdot \sigma^p)$ is a p-chain on K none of whose simplexes with nonzero coefficient intersects σ.

  Thus, $g_p \mu_p$ has no fixed simplex.

  So the Lefschetz number of the chain mapping $\{g_p \mu_p\}$ is zero.

  But this is the Lefschetz number of f.

  This contradicts the hypothesis $\lambda(f) \neq 0$.

The Brouwer Fixed Point Theorem

Corollary (The Brouwer Fixed Point Theorem)

If σ^n is an n-simplex, n a positive integer, and $f:\sigma^n \to \sigma^n$ a continuous map, then f has a fixed point.

- Let $K = \mathrm{Cl}(\sigma^n)$. Then $H_0(K) \cong \mathbb{Z}$ and $H_p(K) = \{0\}$, for $p > 0$.
  Let v be a vertex of σ^n so that the homology class $[1 \cdot \langle v \rangle]$ may be considered a generator of $H_0(K)$. Then

  $$f_0^*([1 \cdot \langle v \rangle]) = [1 \cdot \langle v \rangle].$$

  Moreover, the coefficient matrix E_0 (previous theorem) has trace 1.
  Each matrix E_p, for $p > 0$, has only zero entries.
  Hence,

  $$\lambda(f) = \sum (-1)^p \mathrm{trace} E_p = 1.$$

  Thus, $\lambda(f) \neq 0$. So f must have a fixed point.

Continuous Maps from S^n to S^n

Corollary

Every continuous map from S^n to S^n, $n > 1$, whose degree is not 1 or -1 has a fixed point.

- We know $H_0(S^n) \cong H_n(S^n) \cong \mathbb{Z}$ and $H_p(S^n) = \{0\}$, otherwise.
  Let $[1 \cdot \langle v \rangle]$ and $[z_n]$ be generators of $H_0(S^n)$ and $H_n(S^n)$, respectively.
  For d the degree of f,
  $$\begin{array}{rcl} f_0^*([1 \cdot \langle v \rangle]) & = & [1 \cdot \langle v \rangle]; \\ f_n^*([z_n]) & = & d[z_n]. \end{array}$$
  Then $\lambda(f) = 1 + (-1)^n d$.
  So, if $d \neq \pm 1$, $\lambda(f) \neq 0$.

The Degree of the Antipodal Map

Corollary

If $f : S^n \to S^n$ is the antipodal map, then the degree of f is $(-1)^{n+1}$.

- The map f has no fixed point.

  Hence, $\lambda(f) = 0$.

  It follows that, for d is the degree of f,

$$0 = 1 + (-1)^n d.$$

  This gives

$$d = (-1)^{n+1}.$$

Subsection 3

Relative Homology Groups

Relative Homology Groups and Homology Sequence

- Suppose that K is a complex and L is a complex contained in K.
- It often happens that one knows the homology groups of either K of L and needs to know the homology groups of the other.
- The groups $H_p(K)$ and $H_p(L)$ can be compared using the "relative homology groups" $H_p(K/L)$ to which this section is devoted.
- The intuitive idea is to "remove" all chains on L by considering quotient groups.
- The groups $H_p(K)$, $H_p(L)$, and $H_p(K/L)$ form a sequence of groups and homomorphisms called the "homology sequence".
- Using this sequence, one can often compute any one of the groups $H_p(K)$, $H_p(L)$, or $H_p(K/L)$ provided that enough information is known about the others.

Subcomplexes

Definition

A **subcomplex** of a complex K is a complex L with the property that each simplex of L is a simplex of K.

- Note that not every subset of a complex is a subcomplex.
- It is required that the subset be a complex in its own right.
- The p-skeleton of a complex is one type of subcomplex.
- The empty set $\emptyset$ is a subcomplex of each complex K.
- The relative homology groups $H_p(K/L)$ will reduce to $H_p(K)$ when $L = \emptyset$.

Relative Chain Groups

Definition

Let K be a complex with subcomplex L. By assigning value 0 to each simplex of the complement $K \backslash L$, each chain on L can be considered a chain on K. In this way, we can consider $C_p(L)$ as a subgroup of $C_p(K)$, $p \geq 0$. The **relative p-dimensional chain group of K modulo L**, or **relative p-chain group** (with integer coefficients), is the quotient group

$$C_p(K/L) = C_p(K)/C_p(L).$$

Thus, each member of $C_p(K/L)$ is a coset $c_p + C_p(L)$, where $c_p \in C_p(K)$.

The Relative Boundary Operators

Definition

For $p \geq 1$, the **relative boundary operator** $\partial : C_p(K/L) \to C_{p-1}(K/L)$ is defined by

$$\partial(c_p + C_p(L)) = \partial c_p + C_{p-1}(L), \quad (c_p + C_p(L)) \in C_p(K/L),$$

where ∂c_p denotes the usual boundary of the p-chain c_p.
It can be shown that the relative boundary operator is a homomorphism.

Groups of Relative Cycles and Relative Boundaries

Definition (Cont'd)

The group of **relative p-dimensional cycles on K modulo L**, denoted by $Z_p(K/L)$, is the kernel of the relative boundary operator

$$\partial : C_p(K/L) \to C_{p-1}(K/L), \quad p \geq 1.$$

We define $Z_0(K/L)$ to be the chain group $C_0(K/L)$.

For $p \geq 0$, the **group of relative p-dimensional boundaries on K modulo L**, denoted by $B_p(K/L)$, is the image $\partial(C_{p+1}(K/L))$ of $C_{p+1}(K/L)$ under the relative boundary homomorphism.

Relative Simplicial Homology Groups

Definition

The **relative p-dimensional simplicial homology group of K modulo L** is the quotient group

$$H_p(K/L) = \frac{Z_p(K/L)}{B_p(K/L)}, \quad p \geq 0.$$

- The definition of $H_p(K/L)$ makes sense, since $B_p(K/L) \subseteq Z_p(K/L)$.
- The members of $H_p(K/L)$ are denoted

$$[z_p + C_p(L)],$$

  where $z_p + C_p(L)$ is a relative p-cycle.
  - ∂z_p must be a $(p-1)$-chain on L, but z_p may not be an actual cycle;
  - However, if z_p is a cycle, then $z_p + C_p(L)$ is certainly a relative cycle.

Example

- Let K be the 1-skeleton of a 2-simplex $\langle v_0 v_1 v_2 \rangle$.
  Let L be the subcomplex determined by the vertex v_0.
  We determine $H_0(K/L)$ and $H_1(K/L)$.
  Consider, first, the case $p=0$.
  We have
  $$\begin{aligned} C_0(K)=Z_0(K) &\cong \mathbb{Z}\oplus\mathbb{Z}\oplus\mathbb{Z}; \\ C_0(L)=Z_0(L) &\cong \mathbb{Z}. \end{aligned}$$
  Moreover, by definition,
  $$C_0(K/L)=Z_0(K/L)\cong\mathbb{Z}\oplus\mathbb{Z}.$$
  The members of $Z_0(K/L)$ are chains of the form
  $$z=g_1\cdot\langle v_1\rangle+g_2\cdot\langle v_2\rangle+C_0(L), \quad g_1,g_2\in\mathbb{Z},$$
  where $C_0(L)=\{g\cdot\langle v_0\rangle : g \text{ is an integer}\}$.

Example (Cont'd)

- Now we have

$$\partial(g_1 \cdot \langle v_0 v_1 \rangle + g_2 \cdot \langle v_0 v_2 \rangle) = g_1 \cdot \langle v_1 \rangle + g_2 \cdot \langle v_2 \rangle + (-g_1 - g_2) \cdot \langle v_0 \rangle.$$

So

$$\partial(g_1 \cdot \langle v_0 v_1 \rangle + g_2 \cdot \langle v_0 v_2 \rangle + C_1(L)) = g_1 \cdot \langle v_1 \rangle + g_2 \cdot \langle v_2 \rangle + C_0(L).$$

Thus, every relative 0-cycle is a relative 0-boundary.

This means that $Z_0(K/L) = B_0(K/L)$.

Therefore,

$$H_0(K/L) = \{0\}.$$

Example (Cont'd)

- Next, consider the case $p = 1$.

  Consider a relative 1-chain

  $$w = h_1 \cdot \langle v_0 v_1 \rangle + h_2 \cdot \langle v_1 v_2 \rangle + h_3 \cdot \langle v_0 v_2 \rangle + C_1(L).$$

  Since $C_1(L) = \{0\}$, 1-chains and relative 1-chains can be identified.

  We have

  $$\partial w = (h_1 - h_2) \cdot \langle v_1 \rangle + (h_2 + h_3) \cdot \langle v_2 \rangle + C_0(L).$$

  Thus, w is a relative 1-cycle if and only if $h_1 = h_2 = -h_3$.

  It follows that $Z_1(K/L) \cong \mathbb{Z}$.

  Since K has no 2-simplexes, then $B_1(K/L) = \{0\}$ and $H_1(K/L) \cong \mathbb{Z}$.

  Since there are no simplexes of dimension 2 or higher,

  $$H_p(K/L) = \{0\}, \quad p \geq 2.$$

Example

- Let K denote the closure of a 2-simplex $\sigma^2 = \langle v_0 v_1 v_2 \rangle$.

  Let L be its 1-skeleton.

  K and L have precisely the same 0-simplexes and 1-simplexes.

  It follows that

$$C_0(K) = C_0(L), \quad C_0(K/L) = \{0\}, \quad H_0(K/L) = \{0\};$$
$$C_1(K) = C_1(L), \quad C_1(K/L) = \{0\}, \quad H_1(K/L) = \{0\}.$$

  L has no simplexes of dimension two or higher.

  So it might seem that $H_p(K)$ and $H_p(K/L)$ are isomorphic for $p \geq 2$.

  This is true for $p \geq 3$ but not for $p = 2$.

Example (Cont'd)

- It is true that L has no simplexes of dimension two.

  However, the boundary of a 2-chain is a 1-chain.

  So L does affect $Z_2(K/L)$.

  Suppose the 1-chain has nonzero coefficients only for simplexes of L.

  Then the 2-chain is a relative cycle.

  Consider the elementary relative 2-chain $u = g \cdot \langle v_0 v_1 v_2 \rangle + C_2(L)$, $g \in \mathbb{Z}$.

  It has relative boundary

  $$\partial u = g \cdot \langle v_1 v_2 \rangle - g \cdot \langle v_0 v_2 \rangle + g \cdot \langle v_0 v_1 \rangle + C_1(L) = 0,$$

  because all 1-simplexes of K are in L.

  Thus, the subcomplex L produces relative 2-cycles. So $Z_2(K/L) \cong \mathbb{Z}$.

  But $B_2(K/L) = \{0\}$. So $H_2(K/L) \cong \mathbb{Z}$.

  Note that $H_2(K) = \{0\}$. So $H_2(K/L)$ is not isomorphic to $H_2(K)$.

The Homology Sequence (i^*)

- Our next objective is to show that there is a special sequence

$$\cdots \xrightarrow{\partial^*} H_p(L) \xrightarrow{i^*} H_p(K) \xrightarrow{j^*} H_p(K/L) \xrightarrow{\partial^*} H_{p-1}(L) \xrightarrow{i^*} \cdots \xrightarrow{i^*} H_0(K) \xrightarrow{j^*} H_0(K/L),$$

where i^*, j^*, and ∂^* are homomorphisms.

Definition

Let K be a complex with subcomplex L. The inclusion map i from L into K is simplicial and induces a homomorphism

$$i^* : H_p(L) \to H_p(K), \quad p \geq 0.$$

The effect of this homomorphism is easily described:

If $[z_p] \in H_p(L)$ is represented by the p-cycle z_p on L, then z_p can be considered a p-cycle on K. Then z_p determines a homology class $i^*([z_p]) = [z_p]$ in $H_p(K)$.

The Homology Sequence (j^*)

Definition

Let $j : C_p(K) \to C_p(K/L)$ be the homomorphism defined by

$$j(c_p) = c_p + C_p(L), \quad c_p \in C_p(K).$$

Then j induces a homomorphism

$$j^* : H_p(K) \to H_p(K/L), \quad p \geq 0.$$

If $[z_p] \in H_p(K)$, then $z_p + C_p(L)$ is a relative p-cycle.

It determines a member $[z_p + C_p(L)]$ of $H_p(K/L)$.

The homomorphism j^* takes $[z_p]$ to $[z_p + C_p(L)]$.

The Homology Sequence (∂^*)

Definition

We finally define

$$\partial^* : H_p(K/L) \to H_{p-1}(L).$$

- Suppose $[z_p + C_p(L)] \in H_p(K/L)$, $p \geq 1$.
- Then $z_p + C_p(L)$ is a relative p-cycle.
- This means that ∂z_p is in $C_{p-1}(L)$.
- Since $\partial\partial z_p = 0$, ∂z_p is a $(p-1)$-cycle on L.
- So it determines a member $[\partial z_p]$ of $H_{p-1}(L)$.
- We define

$$\partial^*([z_p + C_p(L)]) = [\partial z_p], \quad [z_p + C_p(L)] \in H_p(K/L).$$

The Homology Sequence

Definition

The **homology sequence** of the pair (K, L) is the sequence of groups and homomorphisms

$$\cdots \xrightarrow{\partial^*} H_p(L) \xrightarrow{i^*} H_p(K) \xrightarrow{j^*} H_p(K/L) \xrightarrow{\partial^*} H_{p-1}(L) \xrightarrow{i^*} \cdots \xrightarrow{i^*} H_0(K) \xrightarrow{j^*} H_0(K/L).$$

- We may verify that

$$\begin{array}{rrcl} i^*: & H_p(L) & \to & H_p(K), \\ j^*: & H_p(K) & \to & H_p(K/L), \\ \partial^*: & H_p(K/L) & \to & H_{p-1}(L) \end{array}$$

are well-defined homomorphisms.

Exact Sequences

Definition

A sequence

$$\cdots \xrightarrow{h_{p+1}} G_p \xrightarrow{h_p} G_{p-1} \xrightarrow{h_{p-1}} \cdots \xrightarrow{h_2} G_1 \xrightarrow{h_1} G_0$$

of groups $G_0, G_1, \ldots$ and homomorphisms $h_1, h_2, \ldots$ is **exact** provided that:

- The kernel of h_{p-1} equals the image $h_p(G_p)$, for $p \geq 2$;
- h_1 maps G_1 onto G_0.

Note that requiring that h_1 be onto is equivalent to requiring that G_0 be followed by the trivial group.

Short Exact Sequences

Theorem

Suppose that an exact sequence has a section of four groups

$$\{0\} \xrightarrow{f} A \xrightarrow{g} B \xrightarrow{h} \{0\},$$

where $\{0\}$ denotes the trivial group.
Then g is an isomorphism from A onto B.

- The image $f(\{0\}) = \{0\}$ contains only the identity element of A.

  Exactness then guarantees that g has kernel $\{0\}$.

  So g is one-to-one.

  The kernel of h is all of B.

  This must be the image $g(A)$.

  Thus, g is an isomorphism as claimed.

Exact Sequences with Three Nontrivial Members

Theorem

Suppose that an exact sequence has a section of five groups

$$\{0\} \to A \xrightarrow{f} B \xrightarrow{g} C \to \{0\}.$$

Suppose, in addition, that:

- There is a homomorphism $h : C \to B$, such that gh is the identity on C;
- B is abelian.

Then $B \cong A \oplus C$.

- We define $T : A \oplus C \to B$ by

$$T(a,c) = f(a) \cdot h(c), \quad (a,c) \in A \oplus C.$$

It may be verified that T is the required isomorphism.

Exact Sequences with Three Nontrivial Members (Cont'd)

- It is easily seen that:
  - $A \cong \ker(g)$ via f;
  - $C \cong \operatorname{im}(h)$ via h.

  So it suffices to show that $B \cong \ker(g) \oplus \operatorname{im}(h)$.

  Given $b \in B$, we have

$$b = (b((hg)(b))^{-1}) \cdot (hg)(b),$$

  where $b((hg)(b))^{-1} \in \ker(g)$ and $(hg)(b) \in \operatorname{im}(h)$.
  Finally, if $b \in \ker(g) \cap \operatorname{im}(h)$, then:
  - $g(b) = 1$;
  - $h(c) = b$, for some $c \in C$.

  But then $c = h(h(c)) = g(b) = 0$.

Exactness of the Homology Sequence

Theorem

If K is a complex with subcomplex L, then the homology sequence of (K,L) is exact.

- In the homology sequence

$$\cdots \xrightarrow{\partial^*} H_p(L) \xrightarrow{i^*} H_p(K) \xrightarrow{j^*} H_p(K/L) \xrightarrow{\partial^*} H_{p-1}(L) \xrightarrow{i^*} \cdots \xrightarrow{i^*} H_0(K) \xrightarrow{j^*} H_0(K/L).$$

  we must show that:
  - The last homomorphism j^* maps $H_0(K)$ onto $H_0(K/L)$;
  - The kernel of each homomorphism is the image of the one that precedes it.

  We first show that j^* is onto.

  Let $[z_0 + C_0(L)] \in H_0(K/L)$. Then z_0 is a 0-chain on K.

  Moreover, $j^*[z_0] = [z_0 + C_0(L)]$. So j^* is onto.

Exactness of the Homology Sequence (1)

- The remainder of the proof breaks naturally into six parts:

(1) $\text{im} i^* \subseteq \text{ker} j^*$;
(2) $\text{ker} j^* \subseteq \text{im} i^*$;
(3) $\text{im} j^* \subseteq \text{ker} \partial^*$;
(4) $\text{ker} \partial^* \subseteq \text{im} j^*$;
(5) $\text{im} \partial^* \subseteq \text{ker} i^*$;
(6) $\text{ker} i^* \subseteq \text{im} \partial^*$.

(1) Let $i^*([z_p])$ be in the image of i^*, where z_p is a p-cycle on L. Then

$$j^* i^*([z_p]) = [z_p + C_p(L)] \stackrel{z_p \in C_p(L)}{=} [0 + C_p(L)] = 0.$$

Thus, $\text{im} i^* \subseteq \text{ker} j^*$.

Exactness of the Homology Sequence (2)

(2) Let $[w_p] \in H_p(K)$ be an element of the kernel of j^*.
That is, $j^*([w_p]) = 0$ in $H_p(K/L)$.
We must find an element $[z_p]$ in $H_p(L)$, such that

$$i^*([z_p]) = [w_p].$$

Now $j^*(w_p) = [w_p + C_p(L)] = 0$.
So $w_p + C_p(L)$ is the relative boundary of a relative $(p+1)$-chain $c_{p+1} + C_{p+1}$, i.e., $\partial c_{p+1} + C_p(L) = w_p + C_p(L)$.
It follows that $w_p - \partial c_{p+1}$ is in $C_p(L)$.
Both w_p and ∂c_{p+1} are cycles on K. So $w_p - \partial c_{p+1}$ is also a cycle.
So it determines a member $[w_p - \partial c_{p+1}]$ of $H_p(L)$.
Note that

$$i^*([w_p - \partial c_{p+1}]) = [w_p - \partial c_{p+1}] = [w_p],$$

since w_p and $w_p - \partial c_{p+1}$ are homologous cycles on K.
Thus, $\ker j^* \subseteq \operatorname{im} i^*$.

Exactness of the Homology Sequence ((3)&(4))

(3) Let $j^*([z_p]) = [z_p + C_p(L)]$ be a member of the image of j^*, where z_p is a p-cycle on K. Then

$$\partial^* j^*([z_p]) = \partial^*([z_p + C_p(L)]) = [\partial z_p] \stackrel{\partial z_p = 0}{=} 0.$$

Thus, $\mathrm{im} j^* \subseteq \ker \partial^*$.

(4) Let $[x_p + C_p(L)]$ be in $\ker \partial^*$, where $x_p + C_p(L)$ is a relative p-cycle.
Then $\partial^*([x_p + C_p(L)]) = [\partial x_p] = 0$ in $H_{p-1}(L)$.
This means that $\partial x_p = \partial y_p$, for some p-chain y_p on L.
Then $x_p - y_p$ is a p-cycle on k.
So it determines a member $[x_p - y_p]$ of $H_p(K)$. Note that

$$j^*([x_p - y_p]) = [x_p - y_p + C_p(L)] = [x_p + C_p(L)],$$

since $y_p \in C_p(L)$. Thus, $[x_p + C_p(L)]$ is in the image of j^*.

- Parts (5) and (6) can be proven similarly.

Example

- Let K denote the closure of an n-simplex.
  Let L be its $(n-1)$-skeleton, $n \geq 2$.
  We use the homology sequence to compute $H_p(K/L)$.
  Since $n \geq 2$, K and L have the same 0- and the same 1-chains, and

$$H_0(K/L) = H_1(K/L) = \{0\}.$$

  For $p > 1$, consider the homology sequence

$$\cdots \to H_p(K) \to H_p(K/L) \to H_{p-1}(L) \to H_{p-1}(K) \to \cdots.$$

  We know that $H_{p-1}(K) = H_p(K) = \{0\}$.
  By the Short Exact Sequence Theorem $H_p(K/L) \cong H_{p-1}(L)$, $p > 1$.
  Also $|L|$ is homeomorphic to S^{n-1}. Therefore,

$$H_n(K/L) \cong H_{n-1}(S^{n-1}) \cong \mathbb{Z}.$$

  Moreover, $H_p(K/L) = \{0\}$, if $p \neq n$.

Example

- Let X be the union of two n-spheres tangent at a point.

  Then X has as triangulation K the n-skeleton of the closure of two $(n+1)$-simplexes joined at a common vertex.

  Let L denote the n-skeleton of one of the two $(n+1)$-simplexes.

  Consider the section

$$H_{n+1}(K/L) \xrightarrow{\partial^*} H_n(L) \xrightarrow{i^*} H_n(K) \xrightarrow{j^*} H_n(K/L) \xrightarrow{\partial^*} H_{n-1}(L)$$

  of the homology sequence of (K, L).

  It satisfies the hypotheses of the 3-Member Exact Sequence Theorem.

  So

$$H_n(K) \cong H_n(K/L) \oplus H_n(L).$$

  We can show that $H_n(K/L) \cong H_n(L) \cong \mathbb{Z}$.

  Therefore, $H_n(X) = H_n(K) \cong \mathbb{Z} \oplus \mathbb{Z}$.

Subsection 4

Singular Homology Theory

Singular versus Simplicial Homology

- One method of extending homology groups to spaces other than polyhedra is *singular homology theory*.
- Instead of insisting that the space X be built from properly joined simplexes, one considers continuous maps from standard simplexes into X, called "singular simplexes".
- There are natural definitions of chains, cycles and boundaries paralleling those of simplicial homology.
- The singular and simplicial theories produce isomorphic homology groups when applied to polyhedra.
- However, the singular approach applies to all topological spaces.

  Notation: Points of $\mathbb{R}^{n+1}$ will be written $(x_0, x_1, \ldots, x_n)$ with zeroth coordinate x_0, first coordinate x_1, etc., i.e., with coordinates being numbered 0 through n.

Unit Simplexes

Definition

For $n \geq 0$, the **unit n-simplex** in $\mathbb{R}^{n+1}$ is the set

$$\Delta_n = \left\{(x_0, x_1, \ldots, x_n) \in \mathbb{R}^{n+1} : \sum x_i = 1,\ x_i \geq 0,\ 0 \leq i \leq n\right\}.$$

The point v_i with i-th coordinate 1 and all other coordinates 0 is called the **i-th vertex** of Δ_n. The subset

$$\Delta_n(i) = \{(x_0, x_1, \ldots, x_n) \in \Delta_n : x_i = 0\}$$

is called the **i-th face** of Δ_n or the **face opposite the i-th vertex**. The map $d_i : \Delta_{n-1} \to \Delta_n$ defined by

$$d_i(x_0, \ldots, x_{n-1}) = (x_0, \ldots, x_{i-1}, 0, x_i, \ldots, x_{n-1})$$

is the **i-th inclusion map**.

Comments on the Definition

- Note that Δ_n is simply the simplex in $\mathbb{R}^{n+1}$ whose vertices are the points $v_0 = (1,0,\ldots,0)$, $v_1 = (0,1,0,\ldots,0)$, $\ldots$, $v_n = (0,\ldots,0,1)$.
- The i-th inclusion map d_i maps Δ_{n-1} onto the i-th face of Δ_n.
- For the inclusion maps in the diagram

$$\Delta_{n-2} \xrightarrow{d_j} \Delta_{n-1} \xrightarrow{d_i} \Delta_n$$
$$\Delta_{n-2} \xrightarrow{d_{i-1}} \Delta_{n-1} \xrightarrow{d_j} \Delta_n,\ j < i,$$

we have $d_i d_j = d_j d_{i-1}$.

E.g., we have

$$\begin{aligned} d_5(d_2(x_0,x_1,x_2,x_3,x_4,x_5,x_6,x_7)) &= d_5(x_0,x_1,0,x_2,x_3,x_4,x_5,x_6,x_7) \\ &= (x_0,x_1,0,x_2,x_3,0,x_4,x_5,x_6,x_7) \\ &= d_2(x_0,x_1,x_2,x_3,0,x_4,x_5,x_6,x_7) \\ &= d_2(d_4(x_0,x_1,x_2,x_3,x_4,x_5,x_6,x_7)). \end{aligned}$$

Singular Simplexes

Definition

Let X be a space and n a non-negative integer.
A **singular** n**-simplex** in X is a continuous function $s^n : \Delta_n \to X$.
The set of all singular n-simplexes in X is denoted $S_n(X)$.
For $n > 0$ and $0 \le i \le n$, the composite map

$$s_i^n = s^n d_i : \Delta_{n-1} \to X$$

is a singular $(n-1)$-simplex called the i**-th face** of s^n.
The function from $S_n(X)$ to $S_{n-1}(X)$ which takes a singular n-simplex to its i-th face is called the i**-th face operator** on $S_n(X)$.

Singular Simplexes (Cont'd)

Definition (Cont'd)

The **singular complex** of X is the set

$$S(X) = \bigcup_{n=0}^{\infty} S_n(X)$$

together with its family of face operators.
It is usually denoted by $S(X)$.

A Property of Singular Simplexes

Theorem

Let s^n be a singular n-simplex in a space X, $n > 1$. Then

$$s^n_{i,j} = s^n_{j,i-1}, \quad 0 \leq j < i \leq n.$$

- In the notation of the preceding definitions,

$$\begin{aligned} s^n_{i,j} &= s^n_i d_j \\ &= s^n d_i d_j \\ &= s^n d_j d_{i-1} \\ &= s^n_j d_{i-1} \\ &= s^n_{j,i-1}. \end{aligned}$$

Singular Chains

Definition

For p a nonnegative integer, a ***p*-dimensional singular chain**, or **singular *p*-chain**, is a function

$$c_p : S_p(X) \to \mathbb{Z}$$

from the set of singular p-simplexes of X into the integers, such that

$$c_p(s^p) = 0,$$

for all but finitely many singular p-simplexes.

Under the pointwise operation of addition induced by the integers, the set $C_p(X)$ of all singular p-chains on X forms a group.

This group is the ***p*-dimensional singular chain group** of X.

Comments

- As in the simplicial theory, a singular p-chain can be expressed as a formal linear combination

$$c_p = \sum_{i=0}^{r} g_i \cdot s(i)^p,$$

where:
  - g_i represents the value of c_p at the singular p-simplex $s(i)^p$;
  - c_p has value zero for all p-simplexes not appearing in the sum.
- Since simplicial complexes have only finitely many simplexes, the "finitely nonzero" property of p-chains holds automatically in the simplicial theory.
- As in the simplicial theory, algebraic systems other than the integers can be used as the set of coefficients.

Singular Boundary Homomorphisms

Definition

The **singular boundary homomorphism**

$$\partial : C_p(X) \to C_{p-1}(X)$$

is defined for an elementary singular p-chain $g \cdot s^p$, $p \geq 1$, by

$$\partial(g \cdot s^p) = \sum_{i=0}^{p} (-1)^i g \cdot s_i^p.$$

This function is extended by linearity to a homomorphism ∂ from $C_p(X)$ into $C_{p-1}(X)$.

The **boundary of each singular 0-chain** is defined to be 0.

Triviality of $\partial\partial$

Theorem

If X is a space and $p \geq 2$, then the composition $\partial\partial : C_p(X) \to C_{p-2}(X)$ in the diagram

$$C_p(X) \xrightarrow{\partial} C_{p-1}(X) \xrightarrow{\partial} C_{p-2}(X)$$

is the trivial homomorphism.

- Each p-chain is a linear combination of elementary p-chains.

  So it is sufficient to prove that

$$\partial\partial(g \cdot s) = 0,$$

  for each elementary p-chain $g \cdot s$.

Triviality of $\partial\partial$ (Cont'd)

- Note that

$$
\begin{aligned}
\partial\partial(g\cdot s) &= \partial(\textstyle\sum_{i=0}^{p}(-1)^i g\cdot s_i) \\
&= \textstyle\sum_{i=0}^{p}(-1)^i\sum_{j=0}^{p-1}(-1)^j g\cdot s_{i,j} \\
&= \textstyle\sum_{i=0}^{p}\sum_{j=0}^{p-1}(-1)^{i+j} g\cdot s_{i,j} \\
&= \textstyle\sum_{0\le j<i\le p}(-1)^{i+j} g\cdot s_{i,j} + \sum_{0\le i\le j\le p-1}(-1)^{i+j} g\cdot s_{i,j} \\
&= \textstyle\sum_{0\le j<i\le p}(-1)^{i+j} g\cdot s_{j,i-1} + \sum_{0\le i\le j\le p-1}(-1)^{i+j} g\cdot s_{i,j}.
\end{aligned}
$$

In the left sum on the preceding line, replace $i-1$ by j and j by i.

Then the two sums will cancel completely.

Thus, $\partial\partial = 0$.

Singular Cycles

Definition

Let X be a space and p a positive integer.

A p-**dimensional singular cycle** on X, or **singular** p-**cycle**, is a singular p-chain z_p such that

$$\partial(z_p) = 0.$$

The set of singular p-cycles is, thus, the kernel of the homomorphism

$$\partial : C_p(X) \to C_{p-1}(X)$$

and is a subgroup of $C_p(X)$. This subgroup is denoted $Z_p(X)$ and called the p-**dimensional singular cycle group** of X.

Since the boundary of each singular 0-chain is 0, we define **singular 0-cycle** to be synonymous with singular 0-chain.

Then the group $Z_0(X)$ of singular 0-cycles is the group $C_0(X)$.

Singular Boundaries

Definition

If $p \geq 0$, a singular p-chain b_p is a p-**dimensional singular boundary**, or **singular** p-**boundary**, if there is a singular $(p+1)$-chain c_{p+1} such that

$$\partial(c_{p+1}) = b_p.$$

The set $B_p(X)$ of singular p-boundaries is the image $\partial(C_{p+1}(X))$ and is a subgroup of $C_p(X)$. This subgroup is called the p-**dimensional singular boundary group of** X.

Now $\partial\partial : C_p(X) \to C_{p-2}(X)$ is the trivial homomorphism.

Hence, $B_p(X)$ is a subgroup of $Z_p(X)$, $p \geq 0$.

The quotient group

$$H_p(X) = Z_p(X)/B_p(X)$$

is the p-**dimensional singular homology group** of X.

Singular Homology and Orientation

- Many similarities in the definitions of the simplicial and singular homology groups should be obvious.
- Note, however, that no mention of orientation was made in the singular case.
- This was taken care of implicitly in the definition of the boundary operator

$$\partial(g \cdot s^n) = \sum_{i=0}^{n} (-1)^i g \cdot s_i^n.$$

- The definition in effect requires that the standard n-simplex Δ_n be assigned the orientation induced by the ordering $v_0 < v_1 < \cdots < v_n$.
- This orientation is then preserved in each singular n-simplex.

Induced Homomorphisms

Definition

Let X and Y be spaces and $f : X \to Y$ a continuous map.
If $s \in S_p(X)$, the composition fs belongs to $S_p(Y)$.
Hence f induces a homomorphism $f_p : C_p(X) \to C_p(Y)$ defined by

$$f_p\left(\sum_{i=0}^{r} g_i \cdot s(i)^p\right) = \sum_{i=0}^{r} g_i \cdot fs(i)^p, \quad \sum_{i=0}^{r} g_i \cdot s(i)^p \in C_p(X).$$

One easily observes that the diagram is commutative. So f_p maps $Z_p(X)$ into $Z_p(Y)$ and $B_p(X)$ into $B_p(Y)$. Thus f induces for each p a homomorphism $f_p^* : H_p(X) \to H_p(Y)$ defined by

$$\begin{array}{ccc} C_p(X) & \xrightarrow{f_p} & C_p(Y) \\ \partial\downarrow & & \downarrow\partial \\ C_{p-1}(X) & \xrightarrow[f_{p-1}]{} & C_{p-1}(Y) \end{array}$$

$$f_p^*(z_p + B_p(X)) = f_p(z_p) + B_p(Y), \quad (z_p + B_p(X)) \in H_p(X).$$

The sequence $\{f_p^*\}$ is the **sequence of homomorphisms induced by** f.

Advantages of Singular Homology

- Singular homology has two advantages over simplicial homology.
  (1) The singular theory applies to all topological spaces, not just polyhedra.
  (2) The induced homomorphisms are defined more easily in the singular theory.
    - In the simplicial theory a continuous map between two polyhedra must be replaced by a simplicial approximation in order to define the induced homomorphisms.
      This presents problems of uniqueness which are completely avoided by the singular approach.
- Singular and simplicial homology groups are isomorphic for polyhedra.

Subsection 5

Axioms for Homology Theory

(Abstract) Homology Theory: The Data

- Eilenberg and Steenrod defined the term "homology theory".
- The definition applies to various categories of:
  - Pairs (X, A), where X is a space with subspace A;
  - Continuous functions on such pairs.
- A **homology theory** consists of three functions H, $*$ and ∂, having the following properties:
  (1) H assigns to each pair (X, A) under consideration and each integer p an abelian group $H_p(X, A)$. This group is the p-**dimensional relative homology group of** X modulo A.
  If $A = \emptyset$, then
  $$H_p(X, \emptyset) = H_p(X)$$
  is the p-**dimensional homology group** of X.

(Abstract) Homology Theory: The Data (Cont'd)

(2) Let (X,A) and (Y,B) be pairs and $f: X \to Y$, with $f(A) \subseteq B$, an admissible map. Then the function $*$ determines, for each integer p, a homomorphism

$$f_p^*: H_p(X,A) \to H_p(Y,B)$$

called the **homomorphism induced by** f in dimension p.

(3) The function ∂ assigns, to each pair (X,A) and each integer p, a homomorphism

$$\partial: H_p(X,A) \to H_{p-1}(A),$$

called the **boundary operator** on $H_p(X,A)$.

Homology Theory: The Eilenberg-Steenrod Axioms

The functions H, $*$, and ∂ are required to satisfy the following **Eilenberg-Steenrod Axioms**:

I **(The Identity Axiom)** If $i:(X,A)\to(X,A)$ is the identity map, then the induced homomorphism

$$i_p^*:H_p(X,A)\to H_p(X,A)$$

is the identity isomorphism for each integer p.

II **(The Composition Axiom)** If $f:(X,A)\to(Y,B)$ and $g:(Y,B)\to(Z,C)$ are admissible maps, then

$$(gf)_p^*=g_p^*f_p^*:H_p(X,A)\to H_p(Z,C)$$

for each integer p.

The Eilenberg-Steenrod Axioms (Cont'd)

III **(The Commutativity Axiom)**

If $f:(X,A)\to(Y,B)$ is an admissible map and $g:A\to B$ is the restriction of f, then the diagram on the right is commutative for each integer p.

$$\begin{array}{ccc} H_p(X,A) & \xrightarrow{f_p^*} & H_p(Y,B) \\ \partial\downarrow & & \downarrow\partial \\ H_{p-1}(A) & \xrightarrow[g_p^*]{} & H_{p-1}(B) \end{array}$$

IV **(The Exactness Axiom)** If $i:A\to X$ and $j:(X,\emptyset)\to(X,A)$ are inclusion maps, then the homology sequence

$$\cdots\to H_p(A)\xrightarrow{i^*}H_p(X)\xrightarrow{j^*}H_p(X,A)\xrightarrow{\partial}H_{p-1}(A)\to\cdots$$

is exact.

The Eilenberg-Steenrod Axioms (Cont'd)

V **(The Homotopy Axiom)** If the maps $f, g : (X, A) \to (Y, B)$ are homotopic, then the induced homomorphisms f_p^* and g_p^* are equal for each integer p.

VI **(The Excision Axiom)** If U is an open subset of X, with $\overline{U} \subseteq A$, then the inclusion map $e : (X \setminus U, A \setminus U) \to (X, A)$ induces an isomorphism

$$e_p^* : H_p(X \setminus U, A \setminus U) \to H_p(X, A),$$

for each integer p. (The map e is called the **excision of** U.)

VII **(The Dimension Axiom)** If X is a space with only one point, then $H_p(X) = \{0\}$, for each nonzero value of p.

Simplicial and Singular Homologies as Homologies

- Simplicial homology theory as presented in this book applies to the category of pairs (X, A), where X and A have triangulations K and L for which L is a subcomplex of K.
- The singular homology theory applies to all pairs (X, A), where X is a topological space with subspace A.
- A survey of homology theory from the axiomatic point of view is given in the classic book *Foundations of Algebraic Topology* by Eilenberg and Steenrod.