Introduction to Analytic Number Theory

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Subsection 1

Introduction

Asymptotic Behavior of Arithmetical Functions

- We study the behavior of arithmetical functions $f(n)$ for large values of n.

  Example: Consider the function

  $$d(n) = \text{the number of divisors of } n.$$

  It takes on the value 2 infinitely often (when n is prime).

  It also takes on arbitrarily large values when n has a large number of divisors.

  Thus, the values of $d(n)$ fluctuate considerably as n increases.
- Many arithmetical functions fluctuate in this manner.
- So it is often difficult to determine their behavior for large n.

Introducing Averages

- The flactuation of many arithmetical functions makes it difficult to determine their behavior for large n.
- Sometimes it is more fruitful to study the arithmetic mean

$$\widetilde{f}(n) = \frac{1}{n}\sum_{k=1}^{n} f(k).$$

- Averages smooth out fluctuations, so it is reasonable to expect that the mean values $\widetilde{f}(n)$ might behave more regularly than $f(n)$.
  Example: We will show that the average $\widetilde{d}(n)$ grows like $\log n$ for large n,

$$\lim_{n\to\infty}\frac{\widetilde{d}(n)}{\log n} = 1.$$

  This is described by saying that

  "the average order of $d(n)$ is $\log n$".

Partial Sums

- To study the average of an arbitrary function f we need a knowledge of its partial sums
$$\sum_{k=1}^{n} f(k).$$
- Sometimes it is convenient to replace the upper index n by an arbitrary positive real number x and to consider instead sums of the form
$$\sum_{k \leq x} f(k).$$
- In the last sum, it is understood that the index k varies from 1 to $[x]$, the greatest integer $\leq x$.
- If $0 < x < 1$ the sum is empty and we assign it the value 0.
- Our goal is to determine the behavior of this sum as a function of x, especially for large x.

Example: The Divisor Function Revisited

- For the divisor function we will prove a result obtained by Dirichlet in 1849, namely

$$\sum_{k \leq x} d(k) = x \log x + (2C - 1)x + O(\sqrt{x}), x \geq 1.$$

- In this sum, C is Euler's constant, defined by the equation

$$C = \lim_{n \to \infty} \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n \right).$$

- The symbol $O(\sqrt{x})$ represents an unspecified function of x which grows no faster than some constant times $\sqrt{x}$.
- This is an example of the "big oh" notation.

Subsection 2

The Big Oh Notation. Asymptotic Equality of Functions

The Big Oh Notation

Definition

If $g(x) > 0$, for all $x \geq a$, we write

$$f(x) = O(g(x)) \quad \text{(read: "}f(x)\text{ is big oh of }g(x)\text{")}$$

to mean that the quotient $\frac{f(x)}{g(x)}$ is bounded for $x \geq a$.
That is, there exists a constant $M > 0$, such that

$$|f(x)| \leq Mg(x), \quad \text{for all } x \geq a.$$

Big Oh in Equations

- An equation of the form

$$f(x) = h(x) + O(g(x))$$

means that

$$f(x) - h(x) = O(g(x)).$$

Note: $f(t) = O(g(t))$, for $t \geq a$, implies

$$\int_a^x f(t)dt = O\left(\int_a^x g(t)dt\right), \quad \text{for } x \geq a.$$

Asymptotic Behavior of Functions

Definition

We say that $f(x)$ is **asymptotic to** $g(x)$ as $x \to \infty$, and we write

$$f(x) \sim g(x) \text{ as } x \to \infty$$

if

$$\lim_{x \to \infty} \frac{f(x)}{g(x)} = 1.$$

Example

Consider again the equation

$$\sum_{k \le x} d(k) = x \log x + (2C - 1)x + O(\sqrt{x}).$$

- It implies that
$$\sum_{k \le x} d(k) \sim x \log x \text{ as } x \to \infty.$$
- In the first equation the term $x \log x$ is called the **asymptotic value** of the sum.
- The other two terms represent the **error** made by approximating the sum by its asymptotic value.

The Error Terms

- Consider again

$$\sum_{k \leq x} d(k) = x \log x + (2C - 1)x + O(\sqrt{x}).$$

- If we denote the error by $E(x)$, then

$$E(x) = (2C - 1)x + O(\sqrt{x}).$$

- This could also be written

$$E(x) = O(x).$$

- Even though this equation is correct, it does not convey the more precise information in the displayed equation above, which tells us that the asymptotic value of $E(x)$ is $(2C - 1)x$.

Subsection 3

Euler's Summation Formula

Euler's Summation Formula

- Sometimes the asymptotic value of a partial sum can be obtained by comparing it with an integral.
- A summation formula of Euler gives an exact expression for the error made in such an approximation.

Theorem (Euler's Summation Formula)

If f has a continuous derivative f' on the interval $[y, x]$, where $0 < y < x$, then

$$\sum_{y<n\leq x} f(n) = \int_y^x f(t)dt + \int_y^x (t-[t])f'(t)dt + f(x)([x]-x) - f(y)([y]-y).$$

Euler's Summation Formula (Cont'd)

- Let $m = [y]$, $k = [x]$.
  For integers n and $n-1$ in $[y, x]$ we have
  $$\begin{aligned} \int_{n-1}^{n} [t] f'(t) dt &= \int_{n-1}^{n} (n-1) f'(t) dt \\ &= (n-1)\{f(n) - f(n-1)\} \\ &= \{nf(n) - (n-1)f(n-1)\} - f(n). \end{aligned}$$
  Summing from $n = m+1$ to $n = k$ we find
  $$\begin{aligned} \int_{m}^{k} [t] f'(t) dt &= \sum_{n=m+1}^{k} \{nf(n) - (n-1)f(n-1)\} \\ &\qquad - \sum_{y<n\leq x} f(n) \\ &= kf(k) - mf(m) - \sum_{y<n\leq x} f(n). \end{aligned}$$
  Hence,
  $$\begin{aligned} \sum_{y<n\leq x} f(n) &= -\int_{m}^{k} [t] f'(t) dt + kf(k) - mf(m) \\ &= -\int_{y}^{x} [t] f'(t) dt + kf(x) - mf(y). \end{aligned}$$

Euler's Summation Formula (Cont'd)

- We got

$$\sum_{y<n\leq x} f(n) = -\int_y^x [t]f'(t)dt + kf(x) - mf(y).$$

Integration by parts gives

$$\int_y^x f(t)dt = xf(x) - yf(y) - \int_y^x tf'(t)dt.$$

Thus, we get

$$\begin{aligned} \textstyle\sum_{y<n\leq x} f(n) &= -\textstyle\int_y^x [t]f'(t)dt + kf(x) - mf(y) \\ &= -\textstyle\int_y^x tf'(t)dt + \int_y^x (t-[t])f'(t)dt \\ &\qquad + kf(x) - mf(y) \\ &= \textstyle\int_y^x f(t)dt + \int_y^x (t-[t])f'(t)dt \\ &\qquad + (k-x)f(x) - (m-y)f(y). \end{aligned}$$

Subsection 4

Some Elementary Asymptotic Formulas

Euler's Constant and Riemann Zeta Function

- Recall Euler's constant

$$C = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right).$$

- $\zeta(s)$ denotes the **Riemann zeta function** which is defined by the equation

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \quad \text{if } s > 1,$$

and by the equation

$$\zeta(s) = \lim_{x\to\infty}\left(\sum_{n\leq x} \frac{1}{n^s} - \frac{x^{1-s}}{1-s}\right), \quad \text{if } 0 < s < 1.$$

Asymptotic Formulas

Theorem

If $x \geq 1$, we have:

(a) $\sum_{n \leq x} \frac{1}{n} = \log x + C + O(\frac{1}{x})$;

(b) $\sum_{n \leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s})$, if $s > 0$, $s \neq 1$;

(c) $\sum_{n > x} \frac{1}{n^s} = O(x^{1-s})$, if $s > 1$;

(d) $\sum_{n \leq x} n^\alpha = \frac{x^{\alpha+1}}{\alpha+1} + O(x^\alpha)$, if $\alpha \geq 0$.

(a) Take $f(t) = \frac{1}{t}$ in Euler's summation formula to obtain

$$\begin{aligned}
\sum_{n \leq x} \frac{1}{n} &= \int_1^x \frac{dt}{t} - \int_1^x \frac{t-[t]}{t^2} dt + 1 - \frac{x-[x]}{x} \\
&= \log x - \int_1^x \frac{t-[t]}{t^2} dt + 1 + O(\frac{1}{x}) \\
&= \log x + 1 - \int_1^\infty \frac{t-[t]}{t^2} dt + \int_x^\infty \frac{t-[t]}{t^2} dt + O(\frac{1}{x}).
\end{aligned}$$

Asymptotic Formulas (Part (a) Cont'd)

- We obtained

$$\sum_{n \leq x} \frac{1}{n} = \log x + 1 - \int_1^\infty \frac{t-[t]}{t^2} dt + \int_x^\infty \frac{t-[t]}{t^2} dt + O\left(\frac{1}{x}\right).$$

The improper integral $\int_1^\infty (t-[t])t^{-2}dt$ exists since it is dominated by $\int_1^\infty t^{-2}dt$. Also,

$$0 \leq \int_x^\infty \frac{t-[t]}{t^2} dt \leq \int_x^\infty \frac{1}{t^2} dt = \frac{1}{x}.$$

Asymptotic Formulas (Part (a) Cont'd)

- We have:
  - $\sum_{n\leq x}\frac{1}{n}=\log x+1-\int_1^\infty\frac{t-[t]}{t^2}dt+\int_x^\infty\frac{t-[t]}{t^2}dt+O\left(\frac{1}{x}\right)$;
  - $0\leq\int_x^\infty\frac{t-[t]}{t^2}dt\leq\int_x^\infty\frac{1}{t^2}dt=\frac{1}{x}$.

  So we get

$$\sum_{n\leq x}\frac{1}{n}=\log x+1-\int_1^\infty\frac{t-[t]}{t^2}dt+O\left(\frac{1}{x}\right).$$

  This proves (a) with $C=1-\int_1^\infty\frac{t-[t]}{t^2}dt$.

  Letting $x\to\infty$ in (a) we find that

$$\lim_{x\to\infty}\left(\sum_{n\leq x}\frac{1}{n}-\log x\right)=1-\int_1^\infty\frac{t-[t]}{t^2}dt.$$

  So C is also equal to Euler's constant.

Asymptotic Formulas (Part (b))

(b) We use a similar argument with $f(x) = x^{-s}$, where $s > 0$, $s \neq 1$. Euler's summation formula gives us

$$\begin{aligned} \sum_{n\leq x} \frac{1}{n^s} &= \int_1^x \frac{dt}{t^s} - s\int_1^x \frac{t-[t]}{t^{s+1}}dt + 1 - \frac{x-[x]}{x^s} \\ &= \frac{x^{1-s}}{1-s} - \frac{1}{1-s} + 1 - s\int_1^\infty \frac{t-[t]}{t^{s+1}}dt + O(x^{-s}). \end{aligned}$$

Therefore,

$$\sum_{n\leq x} \frac{1}{n^s} = \frac{x^{1-s}}{1-s} + C(s) + O(x^{-s}),$$

where

$$C(s) = 1 - \frac{1}{1-s} - s\int_1^\infty \frac{t-[t]}{t^{s+1}}dt.$$

Asymptotic Formulas (Part (b) Cont'd)

- We have

$$\sum_{n\leq x}\frac{1}{n^s}=\frac{x^{1-s}}{1-s}+C(s)+O(x^{-s}),$$

where $C(s)=1-\frac{1}{1-s}-s\int_1^\infty\frac{t-[t]}{t^{s+1}}dt$.

  - Suppose $s>1$.
    The left member of the equation approaches $\zeta(s)$ as $x\to\infty$.
    The terms x^{1-s} and x^{-s} both approach 0.
    Hence, $C(s)=\zeta(s)$, if $s>1$.
  - Suppose $0<s<1$.
    Then $x^{-s}\to 0$.
    The equation shows that

$$\lim_{n\to\infty}\left(\sum_{n\leq x}\frac{1}{n^s}-\frac{x^{1-s}}{1-s}\right)=C(s).$$

    Therefore, $C(s)$ is also equal to $\zeta(s)$.

Asymptotic Formulas (Part (c)-(d))

(c) We use (b) with $s > 1$ to obtain

$$\sum_{n>x} \frac{1}{n^s} = \zeta(s) - \sum_{n\leq x} \frac{1}{n^s} = \frac{x^{1-s}}{s-1} + O(x^{-s}) \overset{x^{-s} \leq x^{1-s}}{=} O(x^{1-s}).$$

(d) Using Euler's summation formula, with $f(t) = t^\alpha$, we obtain

$$\begin{aligned} \sum_{n\leq x} n^\alpha &= \int_1^x t^\alpha dt + \alpha \int_1^x t^{\alpha-1}(t-[t])dt + 1 - (x-[x])x^\alpha \\ &= \frac{x^{\alpha+1}}{\alpha+1} - \frac{1}{\alpha+1} + O(\alpha \int_1^x t^{\alpha-1}dt) + O(x^\alpha) \\ &= \frac{x^{\alpha+1}}{\alpha+1} + O(x^\alpha). \end{aligned}$$

Subsection 5

The Average Order of $d(n)$

The Average Order of $d(n)$

Theorem

For all $x \geq 1$, we have

$$\sum_{n \leq x} d(n) = x \log x + (2C - 1)x + O(\sqrt{x}),$$

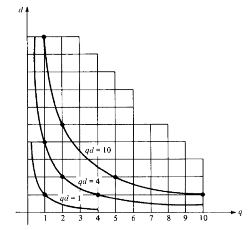
where C is Euler's constant.

- By definition, $d(n) = \sum_{d|n} 1$. So $\sum_{n \leq x} d(n) = \sum_{n \leq x} \sum_{d|n} 1$.
  This is a double sum extended over n and d.
  Since $d \mid n$, we can write $n = qd$ and extend the sum over all pairs of positive integers q, d, with $qd \leq x$. Thus,

$$\sum_{n \leq x} d(n) = \sum_{\substack{q,d \\ qd \leq x}} 1.$$

The Average Order of $d(n)$ (Cont'd)

- We wrote
$$\sum_{n\leq x} d(n) = \sum_{\substack{q,d\\ qd\leq x}} 1.$$
The sum extends over lattice points in the qd-plane that lie on hyperbolas $qd = n$, with $n \leq x$.



So it counts the number of lattice points which lie on these hyperbolas corresponding to $n = 1, 2, \ldots, [x]$.

The Average Order of $d(n)$ (Cont'd)

- For each fixed $d \leq x$, we can:
  - Count first those lattice points on the horizontal line segment $1 \leq q \leq \frac{x}{d}$;
  - Then sum over all $d \leq x$.

Thus we get

$$\sum_{n \leq x} d(n) = \sum_{d \leq x} \sum_{q \leq x/d} 1.$$

We use Part (d) of the preceding theorem with $\alpha = 0$ to obtain

$$\sum_{q \leq x/d} 1 = \frac{x}{d} + O(1).$$

The Average Order of $d(n)$ (Cont'd)

- Now we use this along with Part (a) of the preceding theorem to find

$$\begin{aligned}
\sum_{n\leq x} d(n) &= \sum_{d\leq x}\{\frac{x}{d}+O(1)\} \\
&= x\sum_{d\leq x}\frac{1}{d}+O(x) \\
&= x\{\log x + C + O(\frac{1}{x})\} + O(x) \\
&= x\log x + O(x).
\end{aligned}$$

  This is a weak version of the theorem which implies

$$\sum_{n\leq x} d(n) \sim x\log x, \quad \text{as } x\to\infty.$$

  So it gives $\log n$ as the average order of $d(n)$.

The Average Order of $d(n)$ (Cont'd)

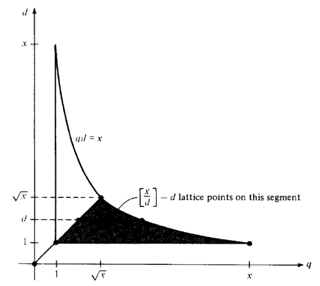
- To prove the more precise formula we return to

$$\sum_{n\leq x} d(n) = \sum_{\substack{q,d \\ qd\leq x}} 1.$$

It counts the number of lattice points in a hyperbolic region.

We take advantage of the symmetry of the region about the line $q = d$.

The total number of points equals twice the number below the line $q = d$ plus the number on the bisecting segment.



$$\sum_{n\leq x} d(n) = 2\sum_{d\leq\sqrt{x}} \left\{ \left[\frac{x}{d}\right] - d \right\} + [\sqrt{x}].$$

The Average Order of $d(n)$ (Cont'd)

- We obtained

$$\sum_{n\leq x} d(n) = 2\sum_{d\leq\sqrt{x}}\left\{\left[\frac{x}{d}\right]-d\right\}+[\sqrt{x}].$$

Now we use the relation $[y]=y+0(1)$ and Parts (a) and (d) of the preceding theorem to obtain

$$\begin{aligned}\sum_{n\leq x} d(n) &= 2\sum_{d\leq\sqrt{x}}\{\tfrac{x}{d}-d+O(1)\}+O(\sqrt{x})\\ &= 2x\sum_{d\leq\sqrt{x}}\tfrac{1}{d}-2\sum_{d\leq\sqrt{x}}d+O(\sqrt{x})\\ &= 2x\left\{\log\sqrt{x}+C+O\left(\tfrac{1}{\sqrt{x}}\right)\right\}\\ &\qquad -2\left\{\tfrac{x}{2}+O(\sqrt{x})\right\}+O(\sqrt{x})\\ &= x\log x+(2C-1)x+O(\sqrt{x}).\end{aligned}$$

Subsection 6

The Average Order of the Divisor Functions $\sigma_\alpha(n)$

The Average Order of $\sigma_1(n)$

- The case $\alpha = 0$ was considered in the preceding theorem.

Theorem

For all $x \geq 1$, we have

$$\sum_{n \leq x} \sigma_1(x) = \frac{1}{2}\zeta(2)x^2 + O(x \log x).$$

Note: It can be shown that $\zeta(2) = \frac{\pi^2}{6}$.

Therefore, the average order of $\sigma_1(n)$ is $\frac{\pi^2 n}{12}$.

The Average Order of $\sigma_1(n)$ (Cont'd)

- We have

$$\begin{aligned}
\sum_{n\leq x}\sigma_1(x) &= \sum_{n\leq x}\sum_{q|n} q \\
&= \sum_{\substack{q,d \\ qd\leq x}} q \\
&= \sum_{d\leq x}\sum_{q\leq x/d} q \\
&\overset{(d)}{=} \sum_{d\leq x}\{\tfrac{1}{2}(\tfrac{x}{d})^2 + O(\tfrac{x}{d})\} \\
&= \tfrac{x^2}{2}\sum_{d\leq x}\tfrac{1}{d^2} + O(x\sum_{d\leq x}\tfrac{1}{d}) \\
&\overset{(b)}{=} \tfrac{x^2}{2}\{-\tfrac{1}{x} + \zeta(2) + O(\tfrac{1}{x^2})\} + O(x\log x) \\
&= \tfrac{1}{2}\zeta(2)x^2 + O(x\log x).
\end{aligned}$$

The Average Order of $\sigma_\alpha(n)$, $0 < \alpha \neq 1$

Theorem

If $x \geq 1$ and $\alpha > 0$, $\alpha \neq 1$, we have

$$\sum_{n \leq x} \sigma_\alpha(n) = \frac{\zeta(\alpha+1)}{\alpha+1} x^{\alpha+1} + O(x^\beta),$$

where $\beta = \max\{1, \alpha\}$.

- We have

$$\begin{aligned}
\sum_{n \leq x} \sigma_\alpha(n) &= \sum_{n \leq x} \sum_{q \mid n} q^\alpha \\
&= \sum_{d \leq x} \sum_{q \leq x/d} q^\alpha \\
&\stackrel{(d)}{=} \sum_{d \leq x} \left\{ \frac{1}{\alpha+1} \left(\frac{x}{d}\right)^{\alpha+1} + O\left(\frac{x^\alpha}{d^\alpha}\right) \right\}
\end{aligned}$$

The Average Order of $\sigma_\alpha(n)$, $0 < \alpha \neq 1$ (Cont'd)

- Contining

$$\begin{aligned}
\sum_{n\leq x}\sigma_\alpha(n) &= \sum_{d\leq x}\left\{\frac{1}{\alpha+1}\left(\frac{x}{d}\right)^{\alpha+1}+O\left(\frac{x^\alpha}{d^\alpha}\right)\right\}\\
&= \frac{x^{\alpha+1}}{\alpha+1}\sum_{d\leq x}\frac{1}{d^{\alpha+1}}+O\left(x^\alpha\sum_{d\leq x}\frac{1}{d^\alpha}\right)\\
&\overset{(b)}{=} \frac{x^{\alpha+1}}{\alpha+1}\left\{\frac{x^{-\alpha}}{-\alpha}+\zeta(\alpha+1)+O(x^{-\alpha-1})\right\}\\
&\qquad + O\left(x^\alpha\left\{\frac{x^{-\alpha-1}}{1-\alpha}+\zeta(\alpha)+O(x^{-\alpha})\right\}\right)\\
&= \frac{\zeta(\alpha+1)}{\alpha+1}x^{\alpha+1}+O(x)+O(1)+O(x^\alpha)\\
&= \frac{\zeta(\alpha+1)}{\alpha+1}x^{\alpha+1}+O(x^{\max\{1,\alpha\}})\\
&= \frac{\zeta(\alpha+1)}{\alpha+1}x^{\alpha+1}+O(x^\beta).
\end{aligned}$$

The Average Order of $\sigma_\alpha(n)$, $\alpha < 0$

- For negative α we write $\alpha = -\beta$, where $\beta > 0$.

Theorem

If $\beta > 0$, let $\delta = \max\{0, 1-\beta\}$. Then if $x > 1$, we have

$$\sum_{n \leq x} \sigma_{-\beta}(n) = \begin{cases} \zeta(\beta+1)x + O(x^\delta), & \text{if } \beta \neq 1 \\ \zeta(2)x + O(\log x), & \text{if } \beta = 1 \end{cases}.$$

The Average Order of $\sigma_\alpha(n)$, $\alpha < 0$ (Cont'd)

- We have

$$\begin{aligned}\sum_{n\le x}\sigma_{-\beta}(n) &= \sum_{n\le x}\sum_{d|n}\frac{1}{d^\beta}\\ &= \sum_{d\le x}\frac{1}{d^\beta}\sum_{q\le x/d}1\\ &\overset{(d)}{=} \sum_{d\le x}\frac{1}{d^\beta}\{\frac{x}{d}+O(1)\}\\ &= x\sum_{d\le x}\frac{1}{d^{\beta+1}}+O(\sum_{d\le x}\frac{1}{d^\beta}).\end{aligned}$$

Moreover,

$$\begin{aligned}x\sum_{d\le x}\frac{1}{d^{\beta+1}} &\overset{(b)}{=} \frac{x^{1-\beta}}{-\beta}+\zeta(\beta+1)x+O(x^{-\beta})\\ &= \zeta(\beta+1)x+O(x^{1-\beta}).\end{aligned}$$

Finally,

$$O\left(\sum_{d\le x}\frac{1}{d^\beta}\right)=\begin{cases}O(\log x), & \text{if } \beta=1,\\ O(x^\delta), & \text{if } \beta\neq 1.\end{cases}$$

Subsection 7

The Average Order of $\varphi(n)$

Asymptotic Formula for $\sum_{n\leq x}\frac{\mu(n)}{n^2}$

- The asymptotic formula for the partial sums of Euler's totient involves the sum of the series $\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}$.
- This series converges absolutely since it is dominated by $\sum_{n=1}^{\infty}\frac{1}{n^2}$.
- We will prove later that

$$\sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}=\frac{1}{\zeta(2)}=\frac{6}{\pi^2}.$$

- Assuming this result for the time being we have

$$\begin{aligned}\sum_{n\leq x}\frac{\mu(n)}{n^2} &= \sum_{n=1}^{\infty}\frac{\mu(n)}{n^2}-\sum_{n>x}\frac{\mu(n)}{n^2}\\ &= \frac{6}{\pi^2}+O\Big(\sum_{n>x}\frac{1}{n^2}\Big)\\ &\overset{(c)}{=} \frac{6}{\pi^2}+O\Big(\frac{1}{x}\Big).\end{aligned}$$

The Average Order of $\varphi(n)$

Theorem

For $x > 1$, we have

$$\sum_{n\le x}\varphi(n)=\frac{3}{\pi^2}x^2+O(x\log x).$$

So the average order of $\varphi(n)$ is $\frac{3n}{\pi^2}$.

- We start with

$$\varphi(n)=\sum_{d|n}\mu(d)\frac{n}{d}.$$

The Average Order of $\varphi(n)$ (Cont'd)

- We obtain

$$\begin{array}{rcl}
\sum_{n\leq x}\varphi(n) & = & \sum_{n\leq x}\sum_{d|n}\mu(d)\frac{n}{d} \\
& = & \sum_{\substack{q,d \\ qd\leq x}}\mu(d)q \\
& = & \sum_{d\leq x}\mu(d)\sum_{q\leq x/d}q \\
& \overset{(d)}{=} & \sum_{d\leq x}\mu(d)\{\frac{1}{2}(\frac{x}{d})^2+O(\frac{x}{d})\} \\
& = & \frac{1}{2}x^2\sum_{d\leq x}\frac{\mu(d)}{d^2}+O(x\sum_{d\leq x}\frac{1}{d}) \\
& \overset{(a)}{=} & \frac{1}{2}x^2\{\frac{6}{\pi^2}+O(\frac{1}{x})\}+O(x\log x) \\
& = & \frac{3}{\pi^2}x^2+O(x\log x).
\end{array}$$

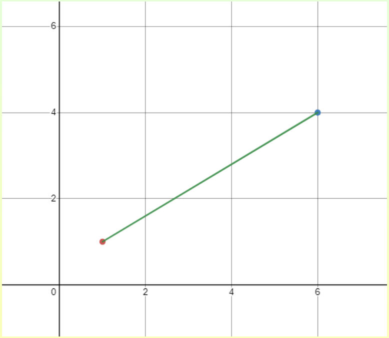
Subsection 8

Distribution of Lattice Points Visible from the Origin

Mutual Visibility of Lattice Points

Definition

Two lattice points P and Q are said to be **mutually visible** if the line segment which joins them contains no lattice points other than the endpoints P and Q.



Characterization of Mutual Visibility of Lattice Points

Theorem

Two lattice points (a, b) and (m, n) are mutually visible if, and only if, $a - m$ and $b - n$ are relatively prime.

- It is clear that (a, b) and (m, n) are mutually visible if and only if $(a - m, b - n)$ is visible from the origin.

  Hence, it suffices to prove the theorem when $(m, n) = (0, 0)$.

  Assume (a, b) is visible from the origin.

  Let $d = (a, b)$. We wish to prove that $d = 1$.

  Suppose $d > 1$.

  Then $a = da'$ and $b = db'$.

  The lattice point (a', b') is on the segment joining $(0, 0)$ to (a, b).

  This contradiction proves that $d = 1$.

Mutual Visibility of Lattice Points (Cont'd)

- Conversely, assume $(a, b) = 1$.

  Let (a', b') be a lattice point on the segment joining $(0, 0)$ to (a, b).

  Then $a' = ta$ and $b' = tb$, where $0 < t < 1$.

  Hence, t is rational.

  So $t = \frac{r}{s}$, where r, s are positive integers with $(r, s) = 1$.

  Thus,

  $$sa' = ar \quad \text{and} \quad sb' = br.$$

  So $s \mid ar$, $s \mid br$.

  But $(s, r) = 1$, so $s \mid a$, $s \mid b$.

  Hence $s = 1$, since $(a, b) = 1$.

  This contradicts the inequality $0 < t < 1$.

  Therefore, the lattice point (a, b) is visible from the origin.

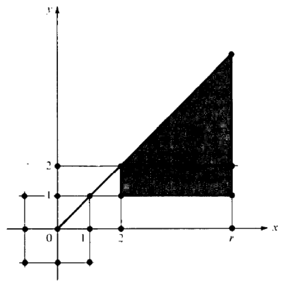
Density of Lattice Points Visible from Origin

- There are infinitely many lattice points visible from the origin.
- It is natural to ask how they are distributed in the plane.
- Consider a large square region in the xy-plane defined by the inequalities $|x| \leq r$ and $|y| \leq r$.
  - $N(r)$ denotes the number of lattice points in this square;
  - $N'(r)$ denote the number which are visible from the origin.
- The quotient $\frac{N'(r)}{N(r)}$ measures the fraction of those lattice points in the square which are visible from the origin.
- The next theorem shows that this fraction tends to a limit as $r \to \infty$.
- We call this limit the **density** of the lattice points visible from the origin.

The Density Theorem

Theorem

The set of lattice points visible from the origin has density $\frac{6}{\pi^2}$.



- We shall prove that $\lim_{r\to\infty}\frac{N'(r)}{N(r)}=\frac{6}{\pi^2}$.

  The eight lattice points nearest the origin are all visible from the origin.

  By symmetry, $N'(r)$ is equal to 8, plus 8 times the number of visible points in the region $\{(x,y):2\le x\le r,1\le y\le x\}$.

  This number is

$$N'(r)=8+8\sum_{2\le n\le r}\sum_{\substack{1\le m<n\\(m,n)=1}}1=8\sum_{1\le n\le r}\varphi(n).$$

The Density Theorem (Cont'd)

- We have $N'(r) = 8\sum_{1\leq n\leq r}\varphi(n)$.

  Using a previous theorem,

$$N'(r) = 8\left(\frac{3}{\pi^2}r^2 + O(r\log r)\right) = \frac{24}{\pi^2}r^2 + O(r\log r).$$

  But the total number of lattice points in the square is

$$N(r) = (2[r]+1)^2 = (2r+O(1))^2 = 4r^2 + O(r).$$

  So

$$\frac{N'(r)}{N(r)} = \frac{\frac{24}{\pi^2}r^2 + O(r\log r)}{4r^2+O(r)} = \frac{\frac{6}{\pi^2}+O(\frac{\log r}{r})}{1+O(\frac{1}{r})}.$$

  Hence, as $r\to\infty$, we find $\frac{N'(r)}{N(r)} \to \frac{6}{\pi^2}$.

A Probabilistic View of the Density Theorem

- We found
$$\lim_{r\to\infty} \frac{N'(r)}{N(r)} = \frac{6}{\pi^2}.$$
- This result is sometimes described by saying that a lattice point chosen at random has probability $\frac{6}{\pi^2}$ of being visible from the origin.
- Altenatively, if two integers a and b are chosen at random, the probability that they are relatively prime is $\frac{6}{\pi^2}$.

Subsection 9

The Average Order of $\mu(n)$ and of $\Lambda(n)$

On the Average Order of $\mu(n)$ and of $\Lambda(n)$

- The average orders of $\mu(n)$ and $\Lambda(n)$ are considerably more difficult to determine than those of $\varphi(n)$ and the divisor functions.
- It is known that:
  - $\mu(n)$ has average order 0, i.e.,
$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leq x}\mu(n)=0;$$
  - $\Lambda(n)$ has average order 1, i.e.,
$$\lim_{x\to\infty}\frac{1}{x}\sum_{n\leq x}\Lambda(n)=1.$$

Average Order of $\mu(n)$ and of $\Lambda(n)$ (Cont'd)

- In the next chapter we will prove that both

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} \mu(n) = 0$$

and

$$\lim_{x\to\infty} \frac{1}{x} \sum_{n\leq x} \Lambda(n) = 1$$

are equivalent to the Prime Number Theorem,

$$\lim_{x\to\infty} \frac{\pi(x)\log x}{x} = 1,$$

where $\pi(x)$ is the number of primes $\leq x$.

Subsection 10

The Partial Sums of a Dirichlet Product

Partial Sums of a Dirichlet Product

Theorem

Suppose $h = f * g$ and let $H(x) = \sum_{n \leq x} h(n)$, $F(x) = \sum_{n \leq x} f(n)$ and $G(x) = \sum_{n \leq x} g(n)$. Then we have

$$H(x) = \sum_{n \leq x} f(n) G\left(\frac{x}{n}\right) = \sum_{n \leq x} g(n) F\left(\frac{x}{n}\right).$$

- We make use of the associative law relating the operations $\circ$ and $*$. Let

$$U(x) = \begin{cases} 0, & \text{if } 0 < x < 1 \\ 1, & \text{if } x \geq 1 \end{cases}.$$

Then $F = f \circ U$, $G = g \circ U$. So we have

$$\begin{aligned} f \circ G &= f \circ (g \circ U) = (f * g) \circ U = H, \\ g \circ F &= g \circ (f \circ U) = (g * f) \circ U = H. \end{aligned}$$

Consequence

Theorem

If $F(x) = \sum_{n \leq x} f(n)$, we have

$$\sum_{n \leq x} \sum_{d|n} f(d) = \sum_{n \leq x} f(n) \left[\frac{x}{n} \right] = \sum_{n \leq x} F\left(\frac{x}{n} \right).$$

- Take g, such that $g(n) = 1$, for all n,

  Then

  $$G(x) = [x].$$

  By the theorem,

  $$H(x) = \sum_{n \leq x} f(n) G\left(\frac{x}{n} \right) = \sum_{n \leq x} g(n) F\left(\frac{x}{n} \right).$$

  This yields the conclusion.

Subsection 11

Applications to $\mu(n)$ and $\Lambda(n)$

Sums Involving $\mu(n)$ and $\Lambda(n)$

- We take $f(n) = \mu(n)$ and $\Lambda(n)$ in the preceding theorem.

Theorem

For $x \geq 1$, we have

$$\sum_{n \leq x} \mu(n) \left[\frac{x}{n}\right] = 1 \quad \text{and} \quad \sum_{n \leq x} \Lambda(n) \left[\frac{x}{n}\right] = \log [x]!.$$

- By the preceding theorem,

$$\sum_{n \leq x} \mu(n)[\tfrac{x}{n}] = \sum_{n \leq x} \sum_{d \mid n} \mu(d) = \sum_{n \leq x} [\tfrac{1}{n}] = 1;$$
$$\sum_{n \leq x} \Lambda(n)[\tfrac{x}{n}] = \sum_{n \leq x} \sum_{d \mid n} \Lambda(d) = \sum_{n \leq x} \log n = \log [x]!.$$

Note: The sums in the theorem can be regarded as weighted averages of the functions $\mu(n)$ and $\Lambda(n)$.

Bounded Partial Sums for $\sum \frac{\mu(n)}{n}$

Theorem

For all $x \geq 1$, we have

$$\left|\sum_{n \leq x} \frac{\mu(n)}{n}\right| \leq 1,$$

with equality holding only if $x < 2$.

- If $x < 2$, there is only one term in the sum, $\mu(1) = 1$.

  Assume that $x \geq 2$.

  For each real y, let $\{y\} = y - [y]$.

  Then

$$\begin{aligned} 1 &= \sum_{n \leq x} \mu(n)[\tfrac{x}{n}] \\ &= \sum_{n \leq x} \mu(n)(\tfrac{x}{n} - \{\tfrac{x}{n}\}) \\ &= x \sum_{n \leq x} \frac{\mu(n)}{n} - \sum_{n \leq x} \mu(n)\{\tfrac{x}{n}\}. \end{aligned}$$

Bounded Partial Sums for $\sum \frac{\mu(n)}{n}$ (Cont'd)

- We got

$$x\sum_{n\leq x}\frac{\mu(n)}{n}-\sum_{n\leq x}\mu(n)\left\{\frac{x}{n}\right\}=1.$$

But $0\leq\{y\}<1$.

So we get

$$\begin{array}{rcl} x|\sum_{n\leq x}\frac{\mu(n)}{n}| & = & |1+\sum_{n\leq x}\mu(n)\{\frac{x}{n}\}| \\ & \leq & 1+\sum_{n\leq x}\{\frac{x}{n}\} \\ & = & 1+\{x\}+\sum_{2\leq n\leq x}\{\frac{x}{n}\} \\ & < & 1+\{x\}+[x]-1 \\ & = & x. \end{array}$$

Dividing by x, we obtain $|\sum_{n\leq x}\frac{\mu(n)}{n}|<1$.

Legendre's Identity

Theorem (Legendre's Identity)

For every $x \geq 1$, we have

$$[x]! = \prod_{p \leq x} p^{\alpha(p)},$$

where the product is extended over all primes $\leq x$, and

$$\alpha(p) = \sum_{m=1}^{\infty} \left[\frac{x}{p^m} \right].$$

Note: The sum for $\alpha(p)$ is finite since $[\frac{x}{p^m}] = 0$, for $p > x$.

Legendre's Identity (Cont'd)

- Recall that:
  - $\Lambda(n) = 0$ unless n is a prime power;
  - $\Lambda(p^m) = \log p$.

  So we have

$$\begin{aligned} \log [x]! &= \sum_{n \le x} \Lambda(n)[\tfrac{x}{n}] \\ &= \sum_{p \le x} \sum_{m=1}^{\infty} [\tfrac{x}{p^m}] \log p \\ &= \sum_{p \le x} \alpha(p) \log p, \end{aligned}$$

  where $\alpha(p) = \sum_{m=1}^{\infty} [\frac{x}{p^m}]$.

  The last sum is also the logarithm of the product $\prod_{p \le x} p^{\alpha(p)}$.

Asymptotic Formula for $\log [x]!$

Theorem

If $x \geq 2$, we have

$$\log [x]! = x \log x - x + O(\log x).$$

Hence,

$$\sum_{n \leq x} \Lambda(n) \left[\frac{x}{n} \right] = x \log x - x + O(\log x).$$

- Recall Euler's summation formula

$$\sum_{y < n \leq x} f(n) = \int_y^x f(t) dt + \int_y^x (t - [t]) f'(t) dt + f(x)([x] - x) - f(y)([y] - y).$$

Asymptotic Formula for $\log [x]!$ (Cont'd)

- Taking $f(t) = \log t$, we obtain

$$\begin{aligned}\sum_{n\leq x} \log n &= \int_1^x \log t dt + \int_1^x \frac{t-[t]}{t} dt - (x-[x]) \log x \\ &= x \log x - x + 1 + \int_1^x \frac{t-[t]}{t} dt + O(\log x).\end{aligned}$$

Now note that

$$\int_1^x \frac{t-[t]}{t} dt = O\left(\int_1^x \frac{1}{t} dt\right) = O(\log x).$$

Thus,

$$\log [x]! = x \log x - x + O(\log x).$$

The second equation follows from

$$\sum_{n\leq x} \Lambda(n) \left[\frac{x}{n}\right] = \log [x]!.$$

Asymptotic Formula for $\sum_{p\leq x}[\frac{x}{p}]\log p$

Theorem

For $x \geq 2$, we have

$$\sum_{p\leq n}\left[\frac{x}{p}\right]\log p = x\log x + O(x),$$

where the sum is extended over all primes $\leq x$.

- Recall $\Lambda(n) = 0$, unless n is a prime power.

  Consequently, we have

$$\sum_{n\leq x}\left[\frac{x}{n}\right]\Lambda(n) = \sum_{p}\sum_{\substack{m=1\\ p^m\leq x}}^{\infty}\left[\frac{x}{p^m}\right]\Lambda(p^m).$$

Asymptotic Formula for $\sum_{p\leq x}[\frac{x}{p}]\log p$ (Cont'd)

- We obtained

$$\sum_{n\leq x}\left[\frac{x}{n}\right]\Lambda(n)=\sum_{p}\sum_{\substack{m=1\\p^m\leq x}}^{\infty}\left[\frac{x}{p^m}\right]\Lambda(p^m).$$

  Now $p^m\leq x$ implies $p\leq x$.

  Moreover, if $p>x$, $[\frac{x}{p^m}]=0$.

  So we can write the last sum as

$$\sum_{p\leq x}\sum_{m=1}^{\infty}\left[\frac{x}{p^m}\right]\log p=\sum_{p\leq x}\left[\frac{x}{p}\right]\log p+\sum_{p\leq x}\sum_{m=2}^{\infty}\left[\frac{x}{p^m}\right]\log p.$$

Asymptotic Formula for $\sum_{p\leq x}[\frac{x}{p}]\log p$

- Claim: $\sum_{p\leq x}\sum_{m=2}^{\infty}[\frac{x}{p^m}]\log p = O(x)$.

  We have

$$\begin{array}{rcl}\sum_{p\leq x}\log p\sum_{m=2}^{\infty}[\frac{x}{p^m}] & \leq & \sum_{p\leq x}\log p\sum_{m=2}^{\infty}\frac{x}{p^m} \\ & = & x\sum_{p\leq x}\log p\sum_{m=2}^{\infty}(\frac{1}{p})^m \\ & = & x\sum_{p\leq x}\log p\cdot\frac{1}{p^2}\cdot\frac{1}{1-\frac{1}{p}} \\ & = & x\sum_{p\leq x}\frac{\log p}{p(p-1)} \\ & \leq & x\sum_{n=2}^{\infty}\frac{\log n}{n(n-1)} = O(x).\end{array}$$

  Hence we have shown that $\sum_{n\leq x}[\frac{x}{n}]\Lambda(n) = \sum_{p\leq x}[\frac{x}{p}]\log p + O(x)$. Combined with $\sum_{n\leq x}\Lambda(n)[\frac{x}{n}] = x\log x - x + O(\log x)$, it proves the theorem.

Subsection 12

Another Identity for the Partial Sums of a Dirichlet Product

Partial Sums

- We generalize the formula for the partial sums of Dirichlet products.
- As before, we write

$$F(x)=\sum_{n\leq x}f(n),\quad G(x)=\sum_{n\leq x}g(n),\quad H(x)=\sum_{n\leq x}(f*g)(n).$$

- So

$$H(x)=\sum_{n\leq x}\sum_{d|n}f(d)g\left(\frac{n}{d}\right)=\sum_{\substack{q,d\\ qd\leq x}}f(d)g(q).$$

The Generalized Partial Sums Theorem

Theorem

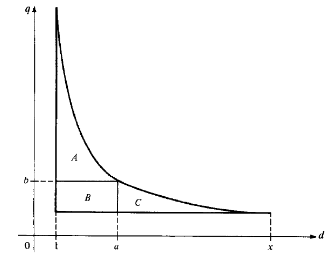
If a and b are positive real numbers, such that $ab = x$, then

$$\sum_{\substack{q,d \\ qd \leq x}} f(d)g(q) = \sum_{n \leq a} f(n) G\left(\frac{x}{n}\right) + \sum_{n \leq b} g(n) F\left(\frac{x}{n}\right) - F(a)G(b).$$

- The sum $H(x)$ on the left is extended over the lattice points in the hyperbolic region shown in the figure.

  We split the sum into two parts:
  - One over the lattice points in $A \cup B$;
  - The other over those in $B \cup C$.

  The lattice points in B are covered twice.



The Generalized Partial Sums Theorem (Cont'd)

- Consequently, we have:

$$\begin{aligned}\sum_{\substack{q,d \\ qd\le x}} f(d)g(q) &= \sum_{d\le a}\sum_{q\le\frac{x}{d}} f(d)g(q) + \sum_{q\le b}\sum_{d\le\frac{x}{q}} f(d)g(q) \\ &\quad - \sum_{d\le a}\sum_{q\le b} f(d)g(q) \\ &= \sum_{n\le a} f(n)\sum_{q\le\frac{x}{n}} g(q) + \sum_{n\le b} g(n)\sum_{d\le\frac{x}{n}} f(d) \\ &\quad - \sum_{d\le a} f(d)\sum_{q\le b} g(q) \\ &= \sum_{n\le a} f(n)G(\tfrac{x}{n}) + \sum_{n\le b} g(n)F(\tfrac{x}{n}) - F(a)G(b).\end{aligned}$$

Remark

- Consider
$$\sum_{\substack{q,d \\ qd \le x}} f(d)g(q) = \sum_{n \le a} f(n)G\left(\frac{x}{n}\right) + \sum_{n \le b} g(n)F\left(\frac{x}{n}\right) - F(a)G(b).$$
  - Take $a = 1$.
    Then
    $$\begin{aligned} H(x) &= f(1)G(x) + \textstyle\sum_{n \le x} g(n)F(\frac{x}{n}) - F(1)G(x) \\ &= \textstyle\sum_{n \le x} g(n)F(\frac{x}{n}). \end{aligned}$$
  - Take $b = 1$.
    Then
    $$\begin{aligned} H(x) &= \textstyle\sum_{n \le x} f(n)G(\frac{x}{n}) + g(1)F(x) - F(x)G(1) \\ &= \textstyle\sum_{n \le x} f(n)G(\frac{x}{n}). \end{aligned}$$
- These are the special cases proved previously.