Introduction to Analytic Number Theory

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Subsection 1

Introduction

Introducing Dirichlet's Theorem

- The arithmetic progression of odd numbers $1, 3, 5, \ldots, 2n+1, \ldots$ contains infinitely many primes.
- It is natural to ask whether other arithmetic progressions have this property.
- An arithmetic progression with first term h and common difference k consists of all numbers of the form $kn + h$, $n = 0, 1, 2, \ldots$.
- A necessary condition for the existence of infinitely many primes in the arithmetic progression is that $(h, k) = 1$.

  Suppose h and k have a common factor $d > 1$.

  Then each term of the progression is divisible by d.

  So there can be no more than one prime in the progression.
- Dirichlet proved that the condition is also sufficient.

  If $(h, k) = 1$ the arithmetic progression $kn + h$ contains infinitely many primes.

Subsection 2

Dirichlet's Theorem for Primes of the Form $4n-1$ and $4n+1$

Dirichlet's Theorem for Primes of the Form $4n-1$

Theorem

There are infinitely many primes of the form $4n-1$.

- Assume there are only a finite number of such primes.

  Let p be the largest, and set $N = 2^2 \cdot 3 \cdot 5 \cdots p - 1$.

  The product $3 \cdot 5 \cdots p$ contains all the odd primes $\leq p$ as factors.

  Since N is of the form $4n-1$, it cannot be prime because $N > p$.

  No prime $\leq p$ divides N.

  So all the prime factors of N must exceed p.

  But all of the prime factors of N cannot be of the form $4n+1$ because the product of two such numbers is again of the same form.

  Hence, some prime factor of N must be of the form $4n-1$.

  This is a contradiction.

Dirichlet's Theorem for Primes of the Form $4n+1$

Theorem

There are infinitely many primes of the form $4n+1$.

- Let N be any integer >1.

  We show that there is a prime $p>N$, such that $p \equiv 1 \pmod 4$.

  Let
  $$m=(N!)^2+1.$$
  Then m is odd, $m>1$.

  Let p be the smallest prime factor of m.

  As none of the numbers $2,3,\ldots,N$ divides m, $p>N$.

  Also, we have
  $$(N!)^2 \equiv -1 \pmod p.$$

Dirichlet's Theorem for Primes $4n+1$ (Cont'd)

- Raise both members to the $\frac{p-1}{2}$ power,

$$(N!)^{p-1} \equiv (-1)^{\frac{p-1}{2}} \pmod{p}.$$

By the Euler-Fermat Theorem, $(N!)^{p-1} \equiv 1 \pmod{p}$.
This gives

$$(-1)^{\frac{p-1}{2}} \equiv 1 \pmod{p}.$$

Now the difference $(-1)^{\frac{p-1}{2}} - 1$ is either 0 or -2.
It cannot be -2, because it is divisible by p. So it must be 0.
That is,

$$(-1)^{\frac{p-1}{2}} = 1.$$

This means that $\frac{p-1}{2}$ is even. So $p \equiv 1 \pmod{4}$.
So, for each $N > 1$, there is a prime $p > N$, such that $p \equiv 1 \pmod{4}$.
Therefore, there are infinitely many primes of the form $4n+1$.

Subsection 3

The Plan of the Proof of Dirichlet's Theorem

The End Theorem

- Dirichlet's Theorem follows from an asymptotic formula.

Theorem

If $k > 0$ and $(h, k) = 1$, we have, for all $x > 1$,

$$\sum_{\substack{p \le x \\ p \equiv h \pmod{k}}} \frac{\log p}{p} = \frac{1}{\varphi(k)} \log x + O(1),$$

where the sum is extended over those primes $p \le x$ which are congruent to $h \mod k$.

- Since $\log x \to \infty$ as $x \to \infty$, this relation implies that there are infinitely many primes $p \equiv h \pmod{k}$.
- So, there are infinitely many primes in $nk + h$, $n = 0, 1, 2, \ldots$.

Remarks

- Consider again the sum

$$\sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} \frac{\log p}{p} = \frac{1}{\varphi(k)} \log x + O(1),$$

  extended over those primes $p \leq x$ which are congruent to $h \mod k$.
- The principal term on the right of the equation is independent of h.
- Therefore, the theorem not only implies Dirichlet's Theorem but also shows that the primes in each of the $\varphi(k)$ reduced residue classes mod k make the same contribution to the principal term in

$$\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1).$$

Notation

- The positive integer k represents a fixed modulus.
- h is a fixed integer relatively prime to k.
- The $\varphi(k)$ Dirichlet characters mod k are denoted by $\chi_1, \chi_2, \ldots, \chi_{\varphi(k)}$, with χ_1 denoting the principal character.
- For $\chi \neq \chi_1$, we write $L(1, \chi)$ and $L'(1, \chi)$

$$L(1, \chi) = \sum_{n=1}^{\infty} \frac{\chi(n)}{n}, \quad L'(1, \chi) = -\sum_{n=1}^{\infty} \frac{\chi(n) \log n}{n}.$$

- The convergence of each of these series was shown in the previous set.
- Moreover, we proved that $L(1, \chi) \neq 0$, if χ is real-valued.
- The symbol p denotes a prime, and $\sum_{p \leq x}$ denotes a sum extended over all primes $p \leq x$.

Lemma 1

Lemma 1

For $x > 1$, we have

$$\sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} \frac{\log p}{p} = \frac{1}{\varphi(k)} \log x + \frac{1}{\varphi(k)} \sum_{r=2}^{\varphi(k)} \overline{\chi}_r(h) \sum_{p \leq x} \frac{\chi_r(p) \log p}{p} + O(1).$$

- It is clear that Lemma 1 will imply the theorem if we show that

$$\sum_{p \leq x} \frac{\chi(p) \log p}{p} = O(1),$$

for each $\chi \neq \chi_1$.

Lemma 2

- The next lemma expresses $\sum_{p\leq x}\frac{\chi(p)\log p}{p}$ in a form which is not extended over primes.

Lemma 2

For $x>1$ and $\chi\neq\chi_1$, we have

$$\sum_{p\leq x}\frac{\chi(p)\log p}{p}=-L'(1,\chi)\sum_{n\leq x}\frac{\mu(n)\chi(n)}{n}+O(1).$$

- So Lemma 2 will imply $\sum_{p\leq x}\frac{\chi(p)\log p}{p}=O(1)$, for each $\chi\neq\chi_1$, if we show that

$$\sum_{n\leq x}\frac{\mu(n)\chi(n)}{n}=O(1).$$

Lemma 3

- $\sum_{n\leq x}\frac{\mu(n)\chi(n)}{n}=O(1)$ will be deduced from the following lemma.

Lemma 3

For $x>1$ and $\chi\neq\chi_1$, we have

$$L(1,\chi)\sum_{n\leq x}\frac{\mu(n)\chi(n)}{n}=O(1).$$

- If $L(1,\chi)\neq 0$, we cancel $L(1,\chi)$ to obtain $\sum_{n\leq x}\frac{\mu(n)\chi(n)}{n}=O(1)$.
- Ultimately, we must show $L(1,\chi)\neq 0$, for all $\chi\neq\chi_1$.
- This was proved for real $\chi\neq\chi_1$ in a previous theorem.
- So it remains to prove that $L(1,\chi)\neq 0$, for all $\chi\neq\chi_1$ which take complex as well as real values.

Remark on Complex Valued Characters

- We let $N(k)$ denote the number of nonprincipal characters χ mod k, such that
$$L(1, \chi) = 0.$$
- If $L(1, \chi) = 0$, then
$$L(1, \overline{\chi}) = \sum_{n=1}^{\infty} \frac{\overline{\chi}(n)}{n} = 0.$$
- Moreover, if $L(1, \chi) = 0$, $\chi \neq \overline{\chi}$, since χ is not real.
- So the characters χ for which $L(1, \chi) = 0$ occur in conjugate pairs.
- It follows that $N(k)$ is even.
- Our goal is to prove that $N(k) = 0$.

Lemma 4

- We deduce $N(k) = 0$ from the following asymptotic formula.

Lemma 4

For $x > 1$, we have

$$\sum_{\substack{p \leq x \\ p \equiv 1 \pmod{k}}} \frac{\log p}{p} = \frac{1 - N(k)}{\varphi(k)} \log x + O(1).$$

- If $N(k) \neq 0$, then $N(k) \geq 2$, since $N(k)$ is even.

  Hence, the coefficient of $\log x$ is negative.

  So the right member $\to -\infty$ as $x \to \infty$.

  This is a contradiction, since all the terms on the left are positive.

  Therefore, Lemma 4 implies that $N(k) = 0$.

Lemma 5

- The proof of Lemma 4 will be based on the following asymptotic formula.

Lemma 5

If $\chi \neq \chi_1$ and $L(1,\chi) = 0$, we have

$$L'(1,\chi) \sum_{n \leq x} \frac{\mu(n)\chi(n)}{n} = \log x + O(1).$$

Subsection 4

Proof of Lemma 1

Proof of Lemma 1

- We begin with the asymptotic formula mentioned earlier,

$$\sum_{p \le x} \frac{\log p}{p} = \log x + O(1).$$

Extract those terms arising from primes $p \equiv h \pmod{k}$.

For the extraction use the orthogonality relation for Dirichlet characters

$$\sum_{r=1}^{\varphi(k)} \chi_r(m)\overline{\chi}_r(n) = \begin{cases} \varphi(k), & \text{if } m \equiv n \pmod{k} \\ 0, & \text{if } m \not\equiv n \pmod{k} \end{cases}$$

This is valid for $(n, k) = 1$.

Take $m = p$ and $n = h$, where $(h, k) = 1$,

$$\sum_{r=1}^{\varphi(k)} \chi_r(p)\overline{\chi}_r(h) = \begin{cases} \varphi(k), & \text{if } p \equiv h \pmod{k} \\ 0, & \text{if } p \not\equiv h \pmod{k} \end{cases}$$

Proof of Lemma 1 (Cont'd)

- Multiply both members by $\frac{\log p}{p}$ and sum over all $p \leq x$,

$$\sum_{p \leq x} \sum_{r=1}^{\varphi(k)} \chi_r(p)\overline{\chi}_r(h)\frac{\log p}{p} = \varphi(k) \sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} \frac{\log p}{p}.$$

Isolate those terms involving only the principal character χ_1,

$$\varphi(k) \sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} \frac{\log p}{p} = \overline{\chi}_1(h) \sum_{p \leq x} \frac{\chi_1(p) \log p}{p} + \sum_{r=2}^{\varphi(k)} \overline{\chi}_r(h) \sum_{p \leq x} \frac{\chi_r(p) \log p}{p}.$$

Proof of Lemma 1 (Cont'd)

- We have:
  - $\chi_1(h) = 1$;
  - $\chi_1(p) = 0$, unless $(p,k) = 1$, in which case $\chi_1(p) = 1$.

Hence, the first term on the right is given by

$$\begin{aligned}\sum_{\substack{p \leq x \\ (p,k)=1}} \frac{\log p}{p} &= \sum_{p \leq x} \frac{\log p}{p} - \sum_{\substack{p \leq x \\ p \mid k}} \frac{\log p}{p} \\ &= \sum_{p \leq x} \frac{\log p}{p} + O(1),\end{aligned}$$

since there are only a finite number of primes which divide k.

So

$$\varphi(k) \sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} \frac{\log p}{p} = \sum_{p \leq x} \frac{\log p}{p} + \sum_{r=2}^{\varphi(k)} \overline{\chi}_r(h) \sum_{p \leq x} \frac{\chi_r(p) \log p}{p} + O(1).$$

Finally, use $\sum_{p \leq x} \frac{\log p}{p} = \log x + O(1)$ and divide by $\varphi(k)$.

Subsection 5

Proof of Lemma 2

Proof of Lemma 2

- We express the sum

$$p\sum_{n\leq x}\frac{\chi(n)\Lambda(n)}{n},$$

where $\Lambda(n)$ is Mangoldt's function, in two ways.

First we note that the definition of $\Lambda(n)$ gives us

$$\sum_{n\leq x}\frac{\chi(n)\Lambda(n)}{n}=\sum_{p\leq x}\sum_{\substack{a=1\\p^a\leq x}}^{\infty}\frac{\chi(p^a)\log p}{p^a}.$$

We separate the terms with $a=1$ and write

$$\sum_{n\leq x}\frac{\chi(n)\Lambda(n)}{n}=\sum_{p\leq x}\frac{\chi(p)\log p}{p}+\sum_{p\leq a}\sum_{\substack{a=2\\p^a\leq x}}^{\infty}\frac{\chi(p^a)\log p}{p^a}.$$

Proof of Lemma 2 (Cont'd)

- The sum

$$\sum_{p \leq a} \sum_{\substack{a=2 \\ p^a \leq x}}^{\infty} \frac{\chi(p^a) \log p}{p^a}$$

is majorized by

$$\sum_p \log p \sum_{a=2}^{\infty} \frac{1}{p^a} = \sum_p \frac{\log p}{p(p-1)} < \sum_{n=2}^{\infty} \frac{\log n}{n(n-1)} = O(1).$$

So we get

$$\sum_{p \leq x} \frac{\chi(p) \log p}{p} = \sum_{n \leq x} \frac{\chi(n) \Lambda(n)}{n} + O(1).$$

Proof of Lemma 2 (Cont'd)

- Now we recall that

$$\Lambda(n) = \sum_{d|n} \mu(d) \log \frac{n}{d}.$$

Hence,

$$\sum_{n \le x} \frac{\chi(n)\Lambda(n)}{n} = \sum_{n \le x} \frac{\chi(n)}{n} \sum_{d|n} \mu(d) \log \frac{n}{d}.$$

In the last sum we write $n = cd$ and use the multiplicative property of χ to obtain

$$\sum_{n \le x} \frac{\chi(n)\Lambda(n)}{n} = \sum_{d \le x} \frac{\mu(d)\chi(d)}{d} \sum_{c \le x/d} \frac{\chi(c) \log c}{c}.$$

Proof of Lemma 2 (Cont'd)

- Since $\frac{x}{d} > 1$, in the sum

$$\sum_{c \leq x/d} \frac{\chi(c) \log c}{c}$$

we may use a previous theorem to obtain

$$\sum_{c \leq \frac{x}{d}} \frac{\chi(c) \log c}{c} = -L'(1, \chi) + O\left(\frac{\log \frac{x}{d}}{\frac{x}{d}}\right).$$

So

$$\sum_{n \leq x} \frac{\chi(n)\Lambda(n)}{n} = -L'(1, \chi) \sum_{d \leq x} \frac{\mu(d)\chi(d)}{d} + O\left(\sum_{d \leq x} \frac{1}{d} \frac{\log \frac{x}{d}}{\frac{x}{d}}\right).$$

Proof of Lemma 2 (Conclusion)

- Note
$$\sum_{d\le x}\log d=\log[x]!=x\log x+O(x).$$
So the sum in the O-term above is
$$\frac{1}{x}\sum_{d\le x}(\log x-\log d)=\frac{1}{x}\left([x]\log x-\sum_{d\le x}\log d\right)=O(1).$$
Therefore, we get
$$\sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n}=-L'(1,\chi)\sum_{d\le x}\frac{\mu(d)\chi(d)}{d}+O(1).$$
But we have shown that $\sum_{p\le x}\frac{\chi(p)\log p}{p}=\sum_{n\le x}\frac{\chi(n)\Lambda(n)}{n}+O(1)$.
So we have the conclusion.

Subsection 6

Proof of Lemma 3

Proof of Lemma 3

- We use the generalized Möbius Inversion Formula.

  It states that, if α is completely multiplicative, then

$$G(x) = \sum_{n \leq x} \alpha(n) F\left(\frac{x}{n}\right) \quad \text{iff} \quad F(x) = \sum_{n \leq x} \mu(n)\alpha(n) G\left(\frac{x}{n}\right).$$

  Take $\alpha(n) = \chi(n)$ and $F(x) = x$.

  Then

$$G(x) = \sum_{n \leq x} \chi(n) \frac{x}{n} = x \sum_{n \leq x} \frac{\chi(n)}{n}.$$

  Moreover, by the Inversion Formula,

$$x = \sum_{n \leq x} \mu(n)\chi(n) G\left(\frac{x}{n}\right).$$

Proof of Lemma 3 (Cont'd)

- Now recall that

$$\sum_{n\leq x}\frac{\chi(n)}{n}=\sum_{n=1}^{\infty}\frac{\chi(n)}{n}+O\left(\frac{1}{x}\right).$$

So we can write

$$G(x)=xL(1,\chi)+O(1).$$

Therefore,

$$\begin{aligned} x &= \sum_{n\leq x}\mu(n)\chi(n)\{\tfrac{x}{n}L(1,\chi)+O(1)\} \\ &= xL(1,\chi)\sum_{n\leq x}\frac{\mu(n)\chi(n)}{n}+O(x). \end{aligned}$$

Finally, divide by x to obtain the conclusion.

Subsection 7

Proof of Lemma 5

Proof of Lemma 5

- Consider again the Generalized Möbius Inversion Formula

$$G(x) = \sum_{n \leq x} \alpha(n) F\left(\frac{x}{n}\right) \quad \text{iff} \quad F(x) = \sum_{n \leq x} \mu(n)\alpha(n) G\left(\frac{x}{n}\right).$$

Take $\alpha(n) = \chi(n)$ and $F(x) = x \log x$.

Then

$$\begin{aligned} G(x) &= \sum_{n \leq x} \chi(n) \frac{x}{n} \log \frac{x}{n} \\ &= x \log x \sum_{n \leq x} \frac{\chi(n)}{n} - x \sum_{n \leq x} \frac{\chi(n) \log n}{n}. \end{aligned}$$

Moreover, by the Inversion Formula,

$$x \log x = \sum_{n \leq x} \mu(n)\chi(n) G\left(\frac{x}{n}\right).$$

Proof of Lemma 5 (Cont'd)

- By a previous theorem:
  - $\sum_{n\le x}\frac{\chi(n)}{n}=\sum_{n=1}^{\infty}\frac{\chi(n)}{n}+O\left(\frac{1}{x}\right)$;
  - $\sum_{n\le x}\frac{\chi(n)\log n}{n}=\sum_{n=1}^{\infty}\frac{\chi(n)\log n}{n}+O\left(\frac{\log x}{x}\right)$.

  We get

$$\begin{aligned} G(x) &= x\log x\sum_{n\le x}\frac{\chi(n)}{n}-x\sum_{n\le x}\frac{\chi(n)\log n}{n} \\ &= x\log x\{L(1,\chi)+O(\tfrac{1}{x})\}+x\{L'(1,\chi)+O(\tfrac{\log x}{x})\} \\ &= xL'(1,\chi)+O(\log x), \end{aligned}$$

  since we are assuming that $L(1,\chi)=0$.

Proof of Lemma 5 (Cont'd)

- Hence, we get

$$\begin{aligned} x \log x &= \sum_{n \le x} \mu(n)\chi(n) G\left(\frac{x}{n}\right) \\ &= \sum_{n \le x} \mu(n)\chi(n) \{\frac{x}{n} L'(1,\chi) + O(\log \frac{x}{n})\} \\ &= x L'(1,\chi) \sum_{n \le x} \frac{\mu(n)\chi(n)}{n} + O(\sum_{n \le x} (\log x - \log n)). \end{aligned}$$

We know the O-term is $O(x)$.

Hence we have

$$x \log x = x L'(1,\chi) \sum_{n \le x} \frac{\mu(n)\chi(n)}{n} + O(x).$$

Finally, divide by x.

Subsection 8

Proof of Lemma 4

Proof of Lemma 4

- Lemma 1 gives

$$\sum_{\substack{p\le x \\ p\equiv h \pmod{k}}} \frac{\log p}{p} = \frac{1}{\varphi(k)}\log x + \frac{1}{\varphi(k)}\sum_{r=2}^{\varphi(k)} \overline{\chi}_r(h)\sum_{p\le x}\frac{\chi_r(p)\log p}{p} + O(1).$$

Setting $h = 1$, we get

$$\sum_{\substack{p\le x \\ p\equiv 1 \pmod{k}}} \frac{\log p}{p} = \frac{1}{\varphi(k)}\log x + \frac{1}{\varphi(k)}\sum_{r=2}^{\varphi(k)}\sum_{p\le x}\frac{\chi_r(p)\log p}{p} + O(1).$$

Proof of Lemma 4 (Cont'd)

- Lemma 2 says, for $x > 1$ and $\chi \neq \chi_1$,

$$\sum_{p \leq x} \frac{\chi(p) \log p}{p} = -L'(1, \chi) \sum_{n \leq x} \frac{\mu(n)\chi(n)}{n} + O(1).$$

Substitute this in on the right in the main formula

$$\sum_{\substack{p \leq x \\ p \equiv 1 \pmod{k}}} \frac{\log p}{p} = \frac{1}{\varphi(k)} \log x + \frac{1}{\varphi(k)} \sum_{r=2}^{\varphi(k)} \sum_{p \leq x} \frac{\chi_r(p) \log p}{p} + O(1).$$

We get

$$\begin{aligned} \sum_{\substack{p \leq x \\ p \equiv 1 \pmod{k}}} \frac{\log p}{p} &= \frac{1}{\varphi(k)} \log x \\ &+ \frac{1}{\varphi(k)} \sum_{r=2}^{\varphi(k)} \left[-L'(1, \chi_r) \sum_{n \leq x} \frac{\mu(n)\chi_r(n)}{n} + O(1) \right] + O(1). \end{aligned}$$

Proof of Lemma 4 (Cont'd)

- Now we have

$$\sum_{\substack{p\le x \\ p\equiv 1 \pmod{k}}} \frac{\log p}{p} = \frac{1}{\varphi(k)}\log x + \frac{1}{\varphi(k)}\sum_{r=2}^{\varphi(k)}\left[-L'(1,\chi_r)\sum_{n\le x}\frac{\mu(n)\chi_r(n)}{n}\right] + O(1).$$

By Lemma 3, for $x > 1$ and $\chi \neq \chi_1$, we have

$$L(1,\chi)\sum_{n\le x}\frac{\mu(n)\chi(n)}{n} = O(1).$$

So, if $L(1,\chi_r) \neq 0$, the contribution to the sum on the right is $O(1)$.

Proof of Lemma 4 (Cont'd)

- By Lemma 5, if $\chi \neq \chi_1$ and $L(1,\chi) = 0$, we have

$$L'(1,\chi) \sum_{n \leq x} \frac{\mu(n)\chi(n)}{n} = \log x + O(1).$$

So, if $L(1,\chi_r) = 0$,

$$-L'(1,\chi_r) \sum_{n \leq x} \frac{\mu(n)\chi_r(n)}{n} = -\log x + O(1).$$

Therefore the sum on the right is $\frac{1}{\varphi(k)}\{-N(k)\log x + O(1)\}$. Thus, the equation becomes

$$\sum_{\substack{p \leq x \\ p \equiv 1 \pmod{k}}} \frac{\log p}{p} = \frac{1 - N(k)}{\varphi(k)} \log x + O(1).$$

Subsection 9

Distribution of Primes in Arithmetic Progressions

Distribution of Primes in Arithmetic Progressions

Theorem (Dirichlet's Theorem)

If $k > 0$ and $(h, k) = 1$, there are infinitely many primes in the arithmetic progression $nk + h$, $n = 0, 1, 2, \ldots$.

- It follows from the main theorem of the preceding section.
- If $k > 0$ and $(a, k) = 1$, let

$$\pi_a(x) = \sum_{\substack{p \le x \\ p \equiv a \pmod{k}}} 1.$$

- The function $\pi_a(x)$ counts the number of primes $\le x$ in the progression $nk + a$, $n = 0, 1, 2, \ldots$.
- Dirichlet's Theorem shows that $\pi_a(x) \to \infty$ as $x \to \infty$.

Prime Number Theorem for Arithmetic Progressions

- There is also a prime number theorem for arithmetic progressions.
- It states that, if $(a, k) = 1$, then, as $x \to \infty$,

$$\pi_a(x) \sim \frac{\pi(x)}{\varphi(k)} \sim \frac{1}{\varphi(k)} \frac{x}{\log x}.$$

- The prime number theorem for progressions is suggested by the formula

$$\sum_{\substack{p \leq x \\ p \equiv h \pmod{k}}} \frac{\log p}{p} = \frac{1}{\varphi(k)} \log x + O(1).$$

- Note that the principal term is independent of h.
- Thus, the primes seem to be equally distributed among the $\varphi(k)$ reduced residue classes mod k.

Prime Number Theorem for Progressions

Theorem

If the relation

$$\pi_a(x) \sim \frac{\pi(x)}{\varphi(k)} \text{ as } x \to \infty$$

holds for every integer a relatively prime to k, then

$$\pi_a(x) \sim \pi_b(x) \text{ as } x \to \infty,$$

whenever $(a, k) = (b, k) = 1$.
Conversely, the latter implies the former.

- Only the "only if" needs proof.

Prime Number Theorem for Progressions (Cont'd)

- Let $A(k)$ denote the number of primes that divide k.

  If $x > k$, we have

$$\begin{aligned}
\pi(x) &= \sum_{p \le x} 1 \\
&= A(k) + \sum_{\substack{p \le x \\ p \nmid k}} 1 \\
&= A(k) + \sum_{\substack{a=1 \\ (a,k)=1}}^{k} \sum_{\substack{p \le x \\ p \equiv a \pmod{k}}} 1 \\
&= A(k) + \sum_{\substack{a=1 \\ (a,k)=1}}^{k} \pi_a(x).
\end{aligned}$$

  Therefore,

$$\frac{\pi(x) - A(k)}{\pi_b(x)} = \sum_{\substack{a=1 \\ (a,k)=1}}^{k} \frac{\pi_a(x)}{\pi_b(x)}.$$

Prime Number Theorem for Progressions (Cont'd)

- We got
$$\frac{\pi(x) - A(k)}{\pi_b(x)} = \sum_{\substack{a=1 \\ (a,k)=1}}^{k} \frac{\pi_a(x)}{\pi_b(x)}.$$
By hypothesis, each term in the sum tends to 1 as $x \to \infty$.
So the sum tends to $\varphi(k)$.
Hence,
$$\frac{\pi(x)}{\pi_b(x)} - \frac{A(k)}{\pi_b(x)} \to \varphi(k), \text{ as } x \to \infty.$$
But, since $\pi_b(x) \to \infty$ as $x \to \infty$, $\frac{A(k)}{\pi_b(x)} \to 0$.
So
$$\frac{\pi(x)}{\pi_b(x)} \to \varphi(k).$$