Introduction to Category Theory

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

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Subsection 1

Introduction to Category Theory

Example

- Let 1 denote a set with one element. (It does not matter what this element is called.)
- Then 1 has the following property:

  For all sets X, there exists a unique map from X to 1.
- Indeed, let X be a set.
- There exists a map $X \to 1$, because we can define $f : X \to 1$ by taking $f(x)$ to be the single element of 1 for each $x \in X$.
- This is the unique map $X \to 1$, because there is no choice in the matter: Any map $X \to 1$ must send each element of X to the single element of 1.

Universal Properties

- Phrases of the form "there exists a unique such-and-such satisfying some condition" are common in category theory.
- The phrase means that there is one and only one such-and-such satisfying the condition.
  - To prove the existence part, we have to show that there is at least one.
  - To prove the uniqueness part, we have to show that there is at most one.

    In other words, any two such-and-suches satisfying the condition are equal.
- Properties such as this are called "universal" because they state how the object being described (in this case, the set 1) relates to the entire universe in which it lives (in this case, the universe of sets).
- The property begins with the words "for all sets X", and therefore says something about the relationship between 1 and every set X:

  namely, that there is a unique map from X to 1.

Example

- This example involves rings, which in this book are always taken to have a multiplicative identity, called 1.
- Similarly, homomorphisms of rings are understood to preserve multiplicative identities.
- The ring $\mathbb{Z}$ has the following property:

  For all rings R, there exists a unique homomorphism $\mathbb{Z} \to R$.
- To prove existence, let R be a ring.
- Define a function $\phi : \mathbb{Z} \to R$ by

$$\phi(n) = \begin{cases} \underbrace{1 + \cdots + 1}_{n}, & \text{if } n > 0 \\ 0, & \text{if } n = 0 \\ -\phi(-n), & \text{if } n < 0 \end{cases}, \quad n \in \mathbb{Z}.$$

- A series of elementary checks confirms that ϕ is a homomorphism.

Example (Cont'd)

- To prove uniqueness, let R be a ring and let $\psi : \mathbb{Z} \to R$ be a homomorphism.
- We show that ψ is equal to the homomorphism ϕ just defined.
- Since homomorphisms preserve multiplicative identities, $\psi(1) = 1$.
- Since homomorphisms preserve addition,

$$\begin{aligned} \psi(n) &= \psi(\underbrace{1 + \cdots + 1}_{n}) = \underbrace{\psi(1) + \cdots + \psi(1)}_{n} \\ &= \underbrace{1 + \cdots + 1}_{n} = \phi(n). \end{aligned}$$

- Since homomorphisms preserve zero, $\psi(0) = 0 = \phi(0)$.
- Finally, since homomorphisms preserve negatives, $\psi(n) = -\psi(-n) = -\phi(-n) = \phi(n)$, whenever $n < 0$.

Uniqueness of the Universal Object

Lemma

Let A be a ring with the following property:

For all rings R, there exists a unique homomorphism $A \to R$.

Then $A \cong \mathbb{Z}$.

- Let us call a ring with this property "initial".
  We are given that A is initial, and we proved that $\mathbb{Z}$ is initial.
  Since A is initial, there is a unique homomorphism $\phi : A \to \mathbb{Z}$.
  Since $\mathbb{Z}$ is initial, there is a unique homomorphism $\phi' : \mathbb{Z} \to A$.
  Now $\phi' \circ \phi : A \to A$ is a homomorphism, but so too is the identity map $1_A : A \to A$. Hence, since A is initial, $\phi' \circ \phi = 1_A$.
  Similarly, $\phi \circ \phi' : \mathbb{Z} \to \mathbb{Z}$ is a homomorphism, but so too is the identity map $1_{\mathbb{Z}} : \mathbb{Z} \to \mathbb{Z}$. Hence, since $\mathbb{Z}$ is initial, $\phi \circ \phi' = 1_{\mathbb{Z}}$.
  So ϕ and ϕ' are mutually inverse, and therefore define an isomorphism between A and $\mathbb{Z}$.

Example: Vector Spaces

- Let V be a vector space with a basis $(v_s)_{s\in S}$.
- For example, if V is finite-dimensional then we might take $S = \{1, \ldots, n\}$.
- If W is another vector space, we can specify a linear map from V to W simply by saying where the basis elements go.
- Thus, for any W, there is a natural one-to-one correspondence between

$$\text{linear maps } V \to W$$

  and

$$\text{functions } S \to W.$$

- This is because any function defined on the basis elements extends uniquely to a linear map on V.

Example: Vector Spaces (Cont'd)

- We rephrase this last statement.
- Define a function $i : S \to V$ by $i(s) = v_s$, $s \in S$.
- Then V, together with i, has the following universal property:

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr, "\forall \text{functions} f"'] & V \arrow[d, "\exists ! \text{linear} \overline{f}"] \\ & \forall W \end{tikzcd}$$

- This diagram means that for all vector spaces W and all functions $f : S \to W$, there exists a unique linear map $\overline{f} : V \to W$ such that $\overline{f} \circ i = f$.
- The symbol $\forall$ means "for all".
- The symbols $\exists!$ mean "there exists a unique".

Example: Vector Spaces (Cont'd)

- Another way to say "$\overline{f} \circ i = f$" is "$\overline{f}(v_s) = f(s)$ for all $s \in S$".
- So, the diagram asserts that every function f defined on the basis elements extends uniquely to a linear map $\overline{f}$ defined on the whole of V.
- In other words still, the function

$$\begin{array}{ccc} \{\text{linear maps } V \to W\} & \to & \{\text{functions } S \to W\} \\ \overline{f} & \mapsto & \overline{f} \circ i \end{array}$$

is bijective.

Example

- Given a set S, we can build a topological space $D(S)$ by equipping S with the **discrete topology**, i.e., all subsets are open.
- With this topology, any map from S to a space X is continuous.
- We rephrase this:
- Define a function $i: S \to D(S)$ by $i(s) = s$, $s \in S$.
- Then $D(S)$ together with i has the following universal property:

$$\begin{tikzcd} S \arrow[r, "i"] \arrow[dr, "\forall \text{functions} f"'] & D(S) \arrow[d, "\exists ! \text{continuous} \overline{f}"] \\ & \forall X \end{tikzcd}$$

- In other words, for all topological spaces X and all functions $f: S \to X$, there exists a unique continuous map $\overline{f}: D(S) \to X$ such that $\overline{f} \circ i = f$.
- The continuous map $\overline{f}$ is the same thing as the function f, regarded as a continuous map between topological spaces.

Example: Bilinear Maps

- Given vector spaces U, V and W, a **bilinear map** $f : U \times V \to W$ is a function f that is linear in each variable:

$$\begin{array}{rcl} f(u, v_1 + \lambda v_2) & = & f(u, v_1) + \lambda f(u, v_2), \\ f(u_1 + \lambda u_2, v) & = & f(u_1, v) + \lambda f(u_2, v), \end{array}$$

  for all $u, u_1, u_2 \in U$, $v, v_1, v_2 \in V$, and scalars λ.
- A good example is the scalar product (dot product), which is a bilinear map

$$\begin{array}{rcl} \mathbb{R}^n \times \mathbb{R}^n & \to & \mathbb{R} \\ (\boldsymbol{u}, \boldsymbol{v}) & \mapsto & \boldsymbol{u} \cdot \boldsymbol{v}, \end{array}$$

  of real vector spaces.
- The vector product (cross product) $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ is also bilinear.

Example: Bilinear Maps (Cont'd)

- Let U and V be vector spaces.
- It is a fact that there is a "universal bilinear map out of $U \times V$".
- In other words, there exist a certain vector space T and a certain bilinear map $b: U \times V \to T$ with the following universal property:

$$\begin{tikzcd} U \times V \arrow[r, "b"] \arrow[dr, "\forall \text{bilinear} f"'] & T \arrow[d, "\exists! \text{linear} \overline{f}"] \\ & \forall W \end{tikzcd}$$

- Roughly speaking, this property says that bilinear maps out of $U \times V$ correspond one-to-one with linear maps out of T.

Example: Bilinear Maps (Cont'd)

Lemma

Let U and V be vector spaces. Suppose that $b : U \times V \to T$ and $b' : U \times V \to T'$ are both universal bilinear maps out of $U \times V$. Then $T \cong T'$. More precisely, there exists a unique isomorphism $j : T \to T'$ such that $j \circ b = b'$.

- In the universal diagram, take $(U \times V \xrightarrow{f} W)$ to be $(U \times V \xrightarrow{b'} T')$.

  This gives a linear map $j : T \to T'$ satisfying $j \circ b = b'$. Similarly, using the universality of b', we obtain a linear map $j' : T' \to T$ satisfying $j' \circ b' = b$. Now $j' \circ j : T \to T$ is a linear map satisfying $(j' \circ j) \circ b = b$. But also, the identity map $1_T : T \to T$ is linear and satisfies $1_T \circ b = b$. So, by the uniqueness part of the universal property of b, $j' \circ j = 1_T$. Similarly, $j \circ j' = 1_{T'}$. So j is an isomorphism.

$$\begin{tikzcd}
& T \arrow[d, "j"] \\
U \times V \arrow[ur, "b"] \arrow[r, "b'"] \arrow[dr, "b"'] & T' \arrow[d, "j'"] \\
& T
\end{tikzcd}$$

Tensor Products

- We saw that given vector spaces U and V, there exists a pair (T, b) with the universal property

$$\begin{tikzcd} U \times V \arrow[r, "b"] \arrow[dr, "\forall \text{bilinear} f"'] & T \arrow[d, "\exists ! \text{linear} \overline{f}"] \\ & \forall W \end{tikzcd}$$

- We proved that there is essentially only one such pair (T, b).
- The vector space T is called the **tensor product** of U and V, and is written as $U \otimes V$.
- Tensor products are very important in algebra because they reduce the study of bilinear maps to the study of linear maps, since a bilinear map out of $U \times V$ is really the same thing as a linear map out of $U \otimes V$.

Kernels

- Let $\theta : G \to H$ be a homomorphism of groups.
- Associated with θ is a diagram

$$\ker(\theta) \overset{\iota}{\hookrightarrow} G \underset{\varepsilon}{\overset{\theta}{\rightrightarrows}} H$$

  where ι is the inclusion of $\ker(\theta)$ into G and ε is the trivial homomorphism.
- "Inclusion" means that $\iota(x) = x$ for all $x \in \ker(\theta)$.
- "Trivial" means that $\varepsilon(g) = 1$ for all $g \in G$.
- The symbol $\hookrightarrow$ is often used for inclusions.

  It is a combination of a subset symbol $\subset$ and an arrow.
- The map ι into G satisfies $\theta \circ \iota = \varepsilon \circ \iota$ and is universal as such.

Kernels (Cont'd)

- The map ι into G satisfies $\theta \circ \iota = \varepsilon \circ \iota$ and is universal as such.

$$\begin{tikzcd} \ker(\theta) \arrow[r, hook, "\iota"] & G \arrow[r, shift left, "\theta"] \arrow[r, shift right, "\varepsilon"'] & H \\ J \arrow[u, "\exists ! \mathrm{hom} \overline{j}"] \arrow[ur, "\forall \mathrm{hom} j"'] & & \end{tikzcd}$$

- Suppose $x \in \ker(\theta)$. Then $\theta(\iota(x)) = \theta(x) = 1 = \varepsilon(x) = \varepsilon(i(x))$. So, $\theta \circ \iota = \varepsilon \circ \iota$.
- Suppose $j : J \to G$ is such that $\theta \circ j = \varepsilon \circ j$. Define $\overline{j} : J \to \ker(\theta)$, by $\overline{j}(x) = j(x)$.

  This is well defined: If $x \in J$, then $\theta(j(x)) = \varepsilon(j(x)) = 1$. Hence, $j(x) \in \ker(\theta)$.

  Moreover, if $x \in J$, then $\iota(\overline{j}(x)) = \iota(j(x)) = j(x)$. Hence, $\iota \circ \overline{j} = j$.

  Uniqueness of $\overline{j}$ is shown as in the bilinear case.

Example: Topological Spaces

- Take a topological space covered by two open subsets: $X = U \cup V$.
- The diagram of inclusion maps on the left has a universal property in the world of topological spaces and continuous maps, as on the right.

$$
\begin{tikzcd}
U \cap V \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & U \arrow[d, hook, "j'"] \\
V \arrow[r, hook, "i'"'] & X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
U \cap V \arrow[r, hook, "i"] \arrow[d, hook, "j"'] & U \arrow[d, hook, "j'"'] \arrow[ddr, "\forall f"] & \\
V \arrow[r, hook, "i'"] \arrow[drr, "\forall g"'] & X \arrow[dr, dashed, "\exists! h"] & \\
& & \forall Y.
\end{tikzcd}
$$

- The diagram means that given Y, f and g such that $f \circ i = g \circ j$, there is exactly one continuous map $h : X \to Y$ such that $h \circ j' = f$ and $h \circ i' = g$.

Subsection 2

Categories

Categories

Definition

A **category** $\mathscr{A}$ consists of:

- a collection $\mathrm{ob}(\mathscr{A})$ of **objects**;
- for each $A, B \in \mathrm{ob}(\mathscr{A})$, a collection $\mathscr{A}(A,B)$ of **maps** or **arrows** or **morphisms** from A to B;
- for each $A, B, C \in \mathrm{ob}(\mathscr{A})$, a function $\mathscr{A}(B,C) \times \mathscr{A}(A,B) \to \mathscr{A}(A,C)$; $(g,f) \mapsto g \circ f$, called **composition**;
- for each $A \in \mathrm{ob}(\mathscr{A})$, an element 1_A of $\mathscr{A}(A,A)$, called the **identity** on A,

satisfying the following axioms:

- **associativity**: for each $f \in \mathscr{A}(A,B)$, $g \in \mathscr{A}(B,C)$ and $h \in \mathscr{A}(C,D)$, we have $(h \circ g) \circ f = h \circ (g \circ f)$;
- **identity laws**: for each $f \in \mathscr{A}(A,B)$, we have $f \circ 1_A = f = 1_B \circ f$.

Common Conventions

- We often write:

$$\begin{array}{rcl} A \in \mathscr{A} & \text{to mean} & A \in \mathrm{ob}(\mathscr{A}); \\ f : A \to B \text{ or } A \xrightarrow{f} B & \text{to mean} & f \in \mathscr{A}(A,B); \\ gf & \text{to mean} & g \circ f. \end{array}$$

- People also write:
  - $\mathrm{ob}(\mathscr{A})$ as $|\mathscr{A}|$;
  - $\mathscr{A}(A,B)$ as $\mathrm{Hom}_{\mathscr{A}}(A,B)$ or $\mathrm{Hom}(A,B)$.

  The notation "Hom" stands for homomorphism, from one of the earliest examples of a category.

Composite of Multiple Maps

- The definition of category is set up so that in general, from each string
$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} A_{n-1} \xrightarrow{f_n} A_n$$
of maps in $\mathscr{A}$, it is possible to construct exactly one map $A_0 \to A_n$ (namely, $f_n f_{n-1} \cdots f_2 f_1$).
- If we are given extra information then we may be able to construct other maps $A_0 \to A_n$.
- In the absence of extra information, this is the only map.
- For example, a string like the one above, with $n = 4$, gives rise to maps
$$A_0 \underset{(f_4(1_{A_3} f_3))((f_2 f_1) 1_{A_0})}{\overset{((f_4 f_3) f_2) f_1}{\rightrightarrows}} A_4.$$
- The axioms imply that these maps are equal.
- It is safe to omit the brackets and write both as $f_4 f_3 f_2 f_1$.

Composite of Zero Maps

- In the case $n=0$, the statement is that for each object A_0 of a category, it is possible to construct exactly one map $A_0 \to A_0$ (namely, the identity 1_{A_0}).
- An identity map can be thought of as a zero-fold composite, in much the same way that the number 1 can be thought of as the product of zero numbers.

Commutative Diagrams

- We often speak of **commutative diagrams**.
- For instance, given objects and maps

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[d, "h"'] & & B \arrow[d, "g"] \\ C \arrow[r, "i"'] & D \arrow[r, "j"'] & E \end{tikzcd}$$

  in a category, we say that the diagram **commutes** if $gf = jih$.
- Generally, a diagram is said to **commute** if whenever there are two paths from an object X to an object Y, the map from X to Y obtained by composing along one path is equal to the map obtained by composing along the other.

Domains and Codomains

- If $f \in \mathscr{A}(A,B)$, we call A the **domain** and B the **codomain** of f.
- Every map in every category has a definite domain and a definite codomain.
- If you believe it makes sense to form the intersection of an arbitrary pair of abstract sets, you should add to the definition of category the condition that $\mathscr{A}(A,B) \cap \mathscr{A}(A',B') = \emptyset$ unless $A = A'$ and $B = B'$.

Set: The Category of Sets

- There is a category **Set** described as follows.
- Its objects are sets.
- Given sets A and B, a map from A to B in the category **Set** is exactly what is ordinarily called a map (or mapping, or function) from A to B.
- Composition in the category is ordinary composition of functions.
- The identity maps are again what you would expect.
- In situations such as this, we often do not bother to specify the composition and identities.
- We write "the category of sets and functions", leaving the reader to guess the rest.
- We sometimes just say "the category of sets".

Grp: The Category of Groups

- There is a category **Grp** of groups, whose objects are groups and whose maps are group homomorphisms.

Ring: The Category of Rings

- Similarly, there is a category **Ring** of rings and ring homomorphisms.

$\mathbf{Vect}_k$: The Category of Vector Spaces

- For each field k, there is a category $\mathbf{Vect}_k$ of vector spaces over k and linear maps between them.

Top: The Category of Topological Spaces

- There is a category **Top** of topological spaces and continuous maps.

Isomorphisms

Definition

A map $f : A \to B$ in a category $\mathscr{A}$ is an **isomorphism** if there exists a map $g : B \to A$ in $\mathscr{A}$ such that

$$gf = 1_A \quad \text{and} \quad fg = 1_B,$$

i.e., such that the following triangles commute:

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[dr, "1_A"'] & B \arrow[d, "g"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} B \arrow[r, "g"] \arrow[dr, "1_B"'] & A \arrow[d, "f"] \\ & B \end{tikzcd}$$

- In this situation we call g the **inverse** of f and write $g = f^{-1}$.
- If there exists an isomorphism from A to B, we say that A and B are **isomorphic** and write $A \cong B$.

Isomorphisms in **Set**

- The isomorphisms in **Set** are exactly the bijections.
- This statement amounts to the assertion that:

  a function has a two-sided inverse
  if and only if
  it is injective and surjective.

Isomorphisms in **Grp** and **Ring**

- The isomorphisms in **Grp** are exactly the isomorphisms of groups.
- By definition, a group isomorphism is a "bijective homomorphism".
- In order to show that this is equivalent to being an isomorphism in **Grp**, we have to prove that

  the inverse of a bijective homomorphism
  is also a homomorphism.

- Similarly, the isomorphisms in **Ring** are exactly the isomorphisms of rings.

Isomorphisms in **Top**

- The isomorphisms in **Top** are exactly the homeomorphisms.
- Note that, in contrast to the situation in **Grp** and **Ring**, a bijective map in **Top** is not necessarily an isomorphism.
- A classic example is the map

$$\begin{array}{ccc} [0,1) & \to & \{z \in \mathbb{C} : |z| = 1\}; \\ t & \mapsto & e^{2\pi i t}, \end{array}$$

which is a continuous bijection but not a homeomorphism.

Categories as Mathematical Structures

- A category can be specified by saying directly what its objects, maps, composition and identities are.
- There is a category $\varnothing$ with no objects or maps at all.
- There is a category $\mathbf{1}$ with one object and only the identity map.

  It can be drawn like this: $\bullet$
- There is another category that can be drawn as $\bullet \to \bullet$ or $A \xrightarrow{f} B$ with two objects and one non-identity map, from the first object to the second.

  Composition is defined in the only possible way.
- The point is that:

  The objects of a category need not be like sets.
  The maps in a category need not be like functions.

More Examples

$$
\begin{tikzcd}
\bullet \arrow[r, shift left] \arrow[r, shift right] & \bullet
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & B \arrow[dr, "g"] & \\
A \arrow[ur, "f"] \arrow[rr, "gf"'] & & C
\end{tikzcd}
$$

$$
\begin{tikzcd}
 & \bullet \arrow[r, "f"] \arrow[dl, "kj"'] \arrow[d, "j"'] \arrow[dr, "hj=gf"] & \bullet \arrow[d, "g"] \\
\bullet & \bullet \arrow[l, "k"] \arrow[r, "h"'] & \bullet
\end{tikzcd}
$$

Discrete Categories

- Some categories contain no maps at all apart from identities (which, as categories, they are obliged to have).
- These are called **discrete categories**.
- A discrete category amounts to just a class of objects.

Groups as Categories

- A group is essentially the same thing as a category that has only one object and in which all the maps are isomorphisms.
- Consider a category $\mathscr{A}$ with just one object, call it A.

  Then $\mathscr{A}$ consists of:
  - A set (or class) $\mathscr{A}(A,A)$;
  - An associative composition function

$$\circ : \mathscr{A}(A,A) \times \mathscr{A}(A,A) \to \mathscr{A}(A,A).$$

  - A two-sided unit $1_A \in \mathscr{A}(A,A)$.

  This would make $\mathscr{A}(A,A)$ into a group, except for inverses.

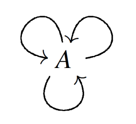
  However, to say that every map in $\mathscr{A}$ is an isomorphism is exactly to say that every element of $\mathscr{A}(A,A)$ has an inverse with respect to $\circ$.

Groups as Categories (Cont'd)

- If we write G for the group $\mathscr{A}(A,A)$, then the situation is as follows:

category $\mathscr{A}$ with single object A	group G
maps in $\mathscr{A}$	elements of G
$\circ$ in $\mathscr{A}$	$\cdot$ in G
1_A	$1 \in G$.

- The category $\mathscr{A}$ looks something like this:



- The arrows represent different maps $A \to A$, that is, different elements of the group G.

Monoids as Categories

- A **monoid** is a set equipped with an associative binary operation and a two-sided unit element.
- Groups describe the reversible transformations, or symmetries, that can be applied to an object, whereas monoids describe the not necessarily reversible transformations.
- Given any set X, there is a group consisting of all bijections $X \to X$, and there is a monoid consisting of all functions $X \to X$.

  In both cases, the binary operation is composition and the unit is the identity function on X.
- Another example of a monoid is the set $\mathbb{N} = \{0, 1, 2, \ldots\}$ of natural numbers, with $+$ as the operation and 0 as the unit.
- Alternatively, we could take the set $\mathbb{N}$ with $\cdot$ as the operation and 1 as the unit.
- A category with one object is essentially the same thing as a monoid, by the same argument as for groups.

Preorders and Preordered Sets

- A **preorder** is a reflexive transitive binary relation.
- A **preordered set** $(S, \leq)$ is a set S together with a preorder $\leq$ on it.

  Examples:
  - $S = \mathbb{R}$ and $\leq$ has its usual meaning;
  - S is the set of subsets of $\{1, \ldots, 10\}$ and $\leq$ is $\subseteq$ (inclusion);
  - $S = \mathbb{Z}$ and $a \leq b$ means that a divides b.

Preorders as Categories

- A preordered set $(S, \leq)$ can be regarded as a category $\mathscr{A}$ in which, for each $A, B \in \mathscr{A}$, there is at most one map from A to B.
- Suppose $(S, \leq)$ is a preordered set. Define the category $\mathscr{A}$ as follows:
  - $\mathrm{ob}(\mathscr{A}) = S$;
  - For all $x, y \in \mathrm{ob}(\mathscr{A})$, $\mathscr{A}(x,y) = \begin{cases} \{(x,y)\}, & \text{if } x \leq y \\ \emptyset, & \text{if } x \nleq y \end{cases}$

  Further, for all $x, y, z \in \mathrm{ob}(\mathscr{A})$, write:
  - $1_x = (x,x)$ (exists, since $x \leq x$ in S);
  - $(y,z) \circ (x,y) = (x,z)$ (exists, since, if $x \leq y$ and $y \leq z$, then $x \leq z$ in S).
- The identity and associative laws are easy to verify. We have, for all $x, y, z, w \in \mathrm{ob}(\mathscr{A})$,
  - $((z,w) \circ (y,z)) \circ (x,y) = (y,w) \cdot (x,y) = (x,w) = (z,w) \circ (x,z) = (z,w) \circ ((y,z) \circ (x,y))$;
  - $(x,y) \circ (x,x) = (x,y) = (y,y) \circ (x,y)$.

  Thus, $\mathscr{A}$ is indeed a category, in which, for each $A, B \in \mathscr{A}$, there is at most one map from A to B.

Preorders as Categories (Converse)

- Suppose $\mathscr{A}$ is a category in which, for each $A, B \in \mathscr{A}$, there is at most one map from A to B.

  Define
  - $S = \mathrm{ob}(\mathscr{A})$;
  - For all $A, B \in S$, $A \leq B$ iff $\mathscr{A}(A, B) \neq \emptyset$.

  Now we show that $(S, \leq)$ is a preordered set.
  - Since, for all $A \in S$, $\mathscr{A}(A, A) = \{1_A\} \neq \emptyset$, we get, by definition, $A \leq A$. Thus, $\leq$ is reflexive;
  - Suppose $A \leq B$ and $B \leq C$. Then $\mathscr{A}(A, B) \neq \emptyset$ and $\mathscr{A}(B, C) \neq \emptyset$. Since in $\mathscr{A}$ composition is defined, $\mathscr{A}(A, C) \neq \emptyset$. Therefore, $A \leq C$. Hence, $\leq$ is transitive.

  Since $\leq$ is reflexive and transitive, $(S, \leq)$ is a preordered set.

Order and Posets

- An **order** on a set is a preorder $\leq$ with the property that if $A \leq B$ and $B \leq A$ then $A = B$.
- Equivalently, if $A \cong B$ in the corresponding category then $A = B$.
- Ordered sets are also called **partially ordered sets** or **posets**.
- An example of a preorder that is not an order is the divisibility relation $|$ on $\mathbb{Z}$:

  E.g., we have $2 \mid -2$ and $-2 \mid 2$ but $2 \neq -2$.

The Opposite or Dual Category

- Every category $\mathscr{A}$ has an **opposite** or **dual** category $\mathscr{A}^{\mathrm{op}}$, defined by reversing the arrows:
  - $\mathrm{ob}(\mathscr{A}^{\mathrm{op}}) = \mathrm{ob}(\mathscr{A})$;
  - $\mathscr{A}^{\mathrm{op}}(B,A) = \mathscr{A}(A,B)$ for all objects A and B.
  - Identities in $\mathscr{A}^{\mathrm{op}}$ are the same as in $\mathscr{A}$.
  - Composition in $\mathscr{A}^{\mathrm{op}}$ is the same as in $\mathscr{A}$, but with the arguments reversed:
    If $A \xrightarrow{f} B \xrightarrow{g} C$ are maps in $\mathscr{A}^{\mathrm{op}}$ then $A \xleftarrow{f} B \xleftarrow{g} C$ are maps in $\mathscr{A}$.
    These give rise to a map $A \xleftarrow{f \circ g} C$ in $\mathscr{A}$;
    The composite of the original pair of maps is the corresponding map $A \to C$ in $\mathscr{A}^{\mathrm{op}}$.
- If $f : A \to B$ is an arrow in $\mathscr{A}$ then the corresponding arrow $B \to A$ in $\mathscr{A}^{\mathrm{op}}$ is also called f.

  Some people prefer to give it a different name, such as f^{op}.

Principle of Duality

- The **principle of duality** is fundamental to category theory.
- Informally, it states that every categorical definition, theorem and proof has a **dual**, obtained by reversing all the arrows.
- Invoking the principle of duality can save work:

  Given any theorem, reversing the arrows throughout its statement and proof produces a dual theorem.

Product Category

- Given categories $\mathscr{A}$ and $\mathscr{B}$, there is a product category $\mathscr{A} \times \mathscr{B}$, in which

$$\begin{array}{rcl} \mathrm{ob}(\mathscr{A} \times \mathscr{B}) & = & \mathrm{ob}(\mathscr{A}) \times \mathrm{ob}(\mathscr{B}); \\ (\mathscr{A} \times \mathscr{B})((A,B),(A',B')) & = & \mathscr{A}(A,A') \times \mathscr{B}(B,B'). \end{array}$$

- Put another way, an object of the product category $\mathscr{A} \times \mathscr{B}$ is a pair (A,B) where $A \in \mathscr{A}$ and $B \in \mathscr{B}$.
- A map $(A,B) \to (A',B')$ in $\mathscr{A} \times \mathscr{B}$ is a pair (f,g) where $f : A \to A'$ in $\mathscr{A}$ and $g : B \to B'$ in $\mathscr{B}$.
- Composition and identities in $\mathscr{A} \times \mathscr{B}$ are defined "component-wise".

Subsection 3

Functors

Functors

Definition

Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A **functor** $F : \mathscr{A} \to \mathscr{B}$ consists of:

- A function $\mathrm{ob}(\mathscr{A}) \to \mathrm{ob}(\mathscr{B})$, written as $A \mapsto F(A)$;
- For each $A, A' \in \mathscr{A}$, a function $\mathscr{A}(A, A') \to \mathscr{B}(F(A), F(A'))$, written as $f \mapsto F(f)$,

satisfying the following axioms:

- $F(f' \circ f) = F(f') \circ F(f)$ whenever $A \xrightarrow{f} A' \xrightarrow{f'} A''$ in $\mathscr{A}$;
- $F(1_A) = 1_{F(A)}$ whenever $A \in \mathscr{A}$.

Series of Maps

- The definition of functor is set up so that from each string

$$A_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} A_n$$

of maps in $\mathscr{A}$ (with $n \geq 0$), it is possible to construct exactly one map $F(A_0) \to F(A_n)$ in $\mathscr{B}$.

- For example, given maps

$$A_0 \xrightarrow{f_1} A_1 \xrightarrow{f_2} A_2 \xrightarrow{f_3} A_3 \xrightarrow{f_4} A_4$$

in $\mathscr{A}$, we can construct maps

$$F(A_0) \overset{F(f_4 f_3)F(f_2 f_1)}{\underset{F(1_{A_4})F(f_4)F(f_3 f_2)F(f_1)}{\rightrightarrows}} F(A_4)$$

in $\mathscr{B}$.

The axioms imply that they are equal.

CAT: The Category of Categories

- Structures and the structure-preserving maps between them form a category (such as **Grp**, **Ring**, etc.).
- In particular, this applies to categories and functors:
  There is a category **CAT** whose objects are categories and whose maps are functors.
  - One part of this statement is that functors can be composed:
    Given functors $\mathscr{A} \xrightarrow{F} \mathscr{B} \xrightarrow{G} \mathscr{C}$, there arises a new functor $\mathscr{A} \xrightarrow{G \circ F} \mathscr{C}$, defined in the obvious way.
  - Another is that for every category $\mathscr{A}$, there is an identity functor $1_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$.

Forgetful Functors from Algebras to Sets

- There is a functor $U : \mathbf{Grp} \to \mathbf{Set}$ defined as follows:
  - If G is a group then $U(G)$ is the underlying set of G (that is, its set of elements);
  - If $f : G \to H$ is a group homomorphism then $U(f)$ is the function f itself.

  So U forgets the group structure of groups and forgets that group homomorphisms are homomorphisms.
- Similarly, there is a functor $\mathbf{Ring} \to \mathbf{Set}$ forgetting the ring structure on rings.
- Similarly, (for any field k) there is a functor $\mathbf{Vect}_k \to \mathbf{Set}$ forgetting the vector space structure on vector spaces.

Forgetful Functors from Algebras to Algebras

- Forgetful functors do not have to forget all the structure.
- Let **Ab** be the category of abelian groups.

  There is a functor $\mathbf{Ring} \to \mathbf{Ab}$ that forgets the multiplicative structure, remembering just the underlying additive group.
- Let **Mon** be the category of monoids.

  There is a functor $U : \mathbf{Ring} \to \mathbf{Mon}$ that forgets the additive structure, remembering just the underlying multiplicative monoid.

  That is, if R is a ring then $U(R)$ is the set R made into a monoid via $\cdot$ and 1.

Forgetting Property Instead of Structure

- There is an inclusion functor $U : \mathbf{Ab} \to \mathbf{Grp}$ defined by $U(A) = A$, for any abelian group A and $U(f) = f$ for any homomorphism f of abelian groups.
- It forgets that abelian groups are abelian.

Free Groups

- Given any set S, one can build the **free group** $F(S)$ on S.
- This is a group containing S as a subset and with no further properties other than those it is forced to have.
- Intuitively, the group $F(S)$ is obtained from the set S by adding just enough new elements that it becomes a group, but without imposing any equations other than those forced by the definition of group.
- A little more precisely, the elements of $F(S)$ are formal expressions or words such as $x^{-4}yx^2zy^{-3}$ (where $x, y, z \in S$).
- Two such words are seen as equal if one can be obtained from the other by the usual cancelation rules.

  For example, x^3xy, x^4y and $x^2y^{-1}yx^2y$ all represent the same element of $F(S)$.
- To multiply two words, just write one followed by the other.

  For instance, $x^{-4}yx$ times xzy^{-3} is $x^{-4}yx^2zy^{-3}$.

The Free Functor of Groups

- This construction assigns to each set S a group $F(S)$.
- In fact, F is a functor:

  Any map of sets $f : S \to S'$ gives rise to a homomorphism of groups $F(f) : F(S) \to F(S')$.
- For instance, take the map of sets $f : \{w, x, y, z\} \to \{u, v\}$ defined by $f(w) = f(x) = f(y) = u$ and $f(z) = v$.

  This gives rise to a homomorphism $F(f) : F(\{w, x, y, z\}) \to F(\{u, v\})$, which maps
  $$x^{-4}yx^2zy^{-3} \in F(\{w, x, y, z\})$$
  to
  $$F(f)(x^{-4}yx^2zy^{-3}) = u^{-4}uu^2vu^{-3} = u^{-1}vu^{-3} \in F(\{u, v\}).$$

Free Commutative Rings

- Similarly, we can construct the free commutative ring $F(S)$ on a set S giving a functor F from **Set** to the category **CRing** of commutative rings.
- In fact, $F(S)$ is something familiar, namely, the ring of polynomials over $\mathbb{Z}$ in commuting variables x_s $(s \in S)$.
- A polynomial is, after all, just a formal expression built from the variables using the ring operations $+$, $-$ and $\cdot$.
- For example, if S is a two-element set then $F(S) \cong \mathbb{Z}[x, y]$.

Free Vector Spaces

- Fix a field k.
- The free functor $F : \mathbf{Set} \to \mathbf{Vect}_k$ is defined on objects by taking $F(S)$ to be a vector space with basis S.
- Loosely, $F(S)$ is the set of all formal k-linear combinations of elements of S, that is, expressions $\sum_{s\in S} \lambda_s s$, where each λ_s is a scalar and there are only finitely many values of s such that $\lambda_s \neq 0$.
- Elements of $F(S)$ can be added:

$$\sum_{s\in S} \lambda_s s + \sum_{s\in S} \mu_s s = \sum_{s\in S} (\lambda_s + \mu_s) s.$$

- There is also a scalar multiplication on $F(S)$:

$$c \cdot \sum_{s\in S} \lambda_s s = \sum_{s\in S} (c\lambda_s) s, \quad c \in k.$$

- In this way, $F(S)$ becomes a vector space.

Free Vector Spaces (Cont'd)

- To be completely precise and avoid talking about "expressions", we can define $F(S)$ to be the set of all functions $\lambda : S \to k$, such that $\{s \in S : \lambda(s) \neq 0\}$ is finite.
- We think of such a function λ as corresponding to the expression $\sum_{s \in S} \lambda(s)s$.
- To define addition on $F(S)$, we must define for each $\lambda, \mu \in F(S)$ a sum $\lambda + \mu \in F(S)$:

  It is given by

$$(\lambda + \mu)(s) = \lambda(s) + \mu(s), \quad s \in S.$$

- Similarly, the scalar multiplication is given by

$$(c \cdot \lambda)(s) = c \cdot \lambda(s), \quad c \in k, \lambda \in F(S), s \in S.$$

Monoid Homomorphisms as Functors

- Let G and H be monoids (or groups, if you prefer), regarded as one-object categories $\mathscr{G}$ and $\mathscr{H}$.
- A functor $F:\mathscr{G}\to\mathscr{H}$ must send the unique object of $\mathscr{G}$ to the unique object of $\mathscr{H}$.
- So it is determined by its effect on maps.
- Hence, the functor $F:\mathscr{G}\to\mathscr{H}$ amounts to a function $F:G\to H$ such that:
  - $F(g'g)=F(g')F(g)$, for all $g',g\in G$;
  - $F(1)=1$.
- In other words, a functor $\mathscr{G}\to\mathscr{H}$ is just a homomorphism $G\to H$.

Group Actions as Functors

- Let G be a monoid, regarded as a one-object category $\mathscr{G}$.
- A functor $F : \mathscr{G} \to \mathbf{Set}$ consists of a set S (the value of F at the unique object of $\mathscr{G}$) together with, for each $g \in G$, a function $F(g) : S \to S$, satisfying the functoriality axioms.
- Writing $(F(g))(s) = g \cdot s$, we see that the functor F amounts to: (a) a set S, together with (b) a function $G \times S \to S$; $(g, s) \mapsto g \cdot s$, satisfying, for all $g, g' \in G$ and $s \in S$:
  - $(g'g) \cdot s = g' \cdot (g \cdot s)$;
  - $1 \cdot s = s$.
- In other words, a functor $\mathscr{G} \to \mathbf{Set}$ is a set equipped with a **left action** by G, a **left** G**-set**, for short.
- Similarly, a functor $\mathscr{G} \to \mathbf{Vect}_k$ is exactly a k-linear representation of G, in the sense of representation theory.

Order Preserving Maps as Functors

- When A and B are (pre)ordered sets, a functor between the corresponding categories is exactly an **order-preserving map**, that is, a function $f : A \to B$ such that

$$a \leq a' \quad \Rightarrow \quad f(a) \leq f(a').$$

Contravariant Functors

- Sometimes we meet functor-like operations that reverse the arrows, with a map $A \to A'$ in $\mathscr{A}$ giving rise to a map $F(A) \leftarrow F(A')$ in $\mathscr{B}$.
- Such operations are called *contravariant functors*.

Definition

Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A **contravariant functor** from $\mathscr{A}$ to $\mathscr{B}$ is a functor $\mathscr{A}^{\mathrm{op}} \to \mathscr{B}$.

- To avoid confusion, we write "a contravariant functor from $\mathscr{A}$ to $\mathscr{B}$" rather than "a contravariant functor $\mathscr{A} \to \mathscr{B}$".
- Functors $\mathscr{C} \to \mathscr{D}$ correspond one-to-one with functors $\mathscr{C}^{\mathrm{op}} \to \mathscr{D}^{\mathrm{op}}$.
- Moreover $(\mathscr{A}^{\mathrm{op}})^{\mathrm{op}} = \mathscr{A}$.
- So a contravariant functor from $\mathscr{A}$ to $\mathscr{B}$ can also be described as a functor $\mathscr{A} \to \mathscr{B}^{\mathrm{op}}$.
- An ordinary functor $\mathscr{A} \to \mathscr{B}$ is sometimes called a **covariant functor** from $\mathscr{A}$ to $\mathscr{B}$, for emphasis.

Ring of Functions on a Topological Space

- Given a topological space X, let $C(X)$ be the ring of continuous real-valued functions on X.
- The ring operations are defined "pointwise":
  For instance, if $p_1, p_2 : X \to \mathbb{R}$ are continuous maps then the map $p_1 + p_2 : X \to \mathbb{R}$ is defined by
  $$(p_1 + p_2)(x) = p_1(x) + p_2(x), \quad x \in X.$$
- A continuous map $f : X \to Y$ induces a ring homomorphism $C(f) : C(Y) \to C(X)$, defined at $q \in C(Y)$ by taking $(C(f))(q)$ to be the composite map
  $$X \xrightarrow{f} Y \xrightarrow{q} \mathbb{R}.$$
- Note that $C(f)$ goes in the opposite direction from f.
- After checking some axioms, we conclude that C is a contravariant functor from **Top** to **Ring**.

Hom Functors

- Let k be a field.
- For any two vector spaces V and W over k, there is a vector space

$$\mathbf{Hom}(V,W) = \{\text{linear maps } V \to W\}.$$

- The elements of this vector space are themselves maps, and the vector space operations (addition and scalar multiplication) are defined pointwise.
- Now fix a vector space W.
- Any linear map $f : V \to V'$ induces a linear map

$$f^* : \mathbf{Hom}(V',W) \to \mathbf{Hom}(V,W),$$

defined at $q \in \mathbf{Hom}(V',W)$ by taking $f^*(q)$ to be the composite map

$$V \xrightarrow{f} V' \xrightarrow{q} W.$$

Hom Functors (Cont'd)

- This defines a functor
$$\mathbf{Hom}(-,W) : \mathbf{Vect}_k^{\mathrm{op}} \to \mathbf{Vect}_k.$$
- The symbol "$-$" is a blank or placeholder, into which arguments can be inserted.
- Thus, the value of $\mathbf{Hom}(-,W)$ at V is $\mathbf{Hom}(V,W)$.
- Sometimes we use a blank space instead of $-$, as in $\mathbf{Hom}(\quad,W)$.

The Dual Vector Space

- An important special case is where W is k, seen as a one-dimensional vector space over itself.
- The vector space $\mathbf{Hom}(V,k)$ is called the **dual** of V, and is written as V^*.
- So there is a contravariant functor

$$(\ )^* = \mathbf{Hom}(-,k) : \mathbf{Vect}_k^{\mathrm{op}} \to \mathbf{Vect}_k$$

sending each vector space to its dual.

Right Group Actions as Functors

- Let G be a monoid, regarded as a one-object category $\mathscr{G}$.
- A functor $\mathscr{G}^{\mathrm{op}} \to \mathbf{Set}$ is a **right** G-**set**, for essentially the same reasons as as covariant functors are essentially left G-sets.

Presheafs

- Contravariant functors whose codomain is **Set** are important enough to have their own special name.

Definition

Let $\mathscr{A}$ be a category. A **presheaf on** $\mathscr{A}$ is a functor $\mathscr{A}^{\mathrm{op}} \to \mathbf{Set}$.

- The name comes from the following special case.

Presheafs (Cont'd)

- Let X be a topological space.
- Write $\mathscr{O}(X)$ for the poset of open subsets of X, ordered by inclusion.
- View $\mathscr{O}(X)$ as a category:
  - The objects of $\mathscr{O}(X)$ are the open subsets of X;
  - For $U, U' \in \mathscr{O}(X)$, there is one map $U \to U'$ if $U \subseteq U'$, and there are none otherwise.
- A **presheaf on the space** X is a presheaf on the category $\mathscr{O}(X)$.
- For example, given any space X, there is a presheaf F on X defined by:
  - For all $U \in \mathscr{O}(X)$,
  $$F(U) = \{\text{continuous functions } U \to \mathbb{R}\};$$
  - If $U \subseteq U'$ are open subsets of X, we define the map $F(U') \to F(U)$ to be restriction.

Faithful and Full Functors

Definition

A functor $F : \mathscr{A} \to \mathscr{B}$ is **faithful** (respectively, **full**) if for each $A, A' \in \mathscr{A}$, the function

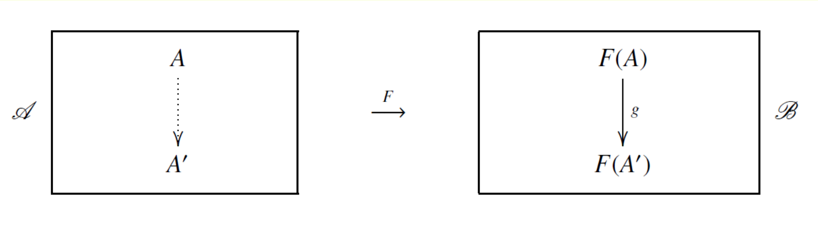
$$\begin{array}{ccc} \mathscr{A}(A, A') & \to & \mathscr{B}(F(A), F(A')) \\ f & \mapsto & F(f) \end{array}$$

is injective (respectively, surjective).

- Faithfulness *does not say* that if f_1 and f_2 are distinct maps in $\mathscr{A}$ then $F(f_1) \neq F(f_2)$.

Faithful and Full Functors (Cont'd)

- F is faithful if for each A, A' and g as shown, there is at most one dotted arrow that F sends to g.



- It is full if for each such A, A' and g, there is at least one dotted arrow that F sends to g.

Subcategories

Definition

Let $\mathscr{A}$ be a category. A **subcategory** $\mathscr{S}$ of $\mathscr{A}$ consists of a subclass ob(S) of ob($\mathscr{A}$) together with, for each $S, S' \in \text{ob}(\mathscr{S})$, a subclass $\mathscr{S}(S,S')$ of $\mathscr{A}(S,S')$, such that $\mathscr{S}$ is closed under composition and identities.
It is a **full subcategory** if $\mathscr{S}(S,S') = \mathscr{A}(S,S')$, for all $S, S' \in \text{ob}(\mathscr{S})$.

- A full subcategory therefore consists of a selection of the objects, with all of the maps between them.
- So, a full subcategory can be specified simply by saying what its objects are.
- For example, **Ab** is the full subcategory of **Grp** consisting of the groups that are abelian.

Inclusion Functors

- Whenever $\mathscr{S}$ is a subcategory of a category $\mathscr{A}$, there is an inclusion functor $I : \mathscr{S} \to \mathscr{A}$ defined by $I(S) = S$ and $I(f) = f$.
- It is automatically faithful, and it is full if and only if $\mathscr{S}$ is a full subcategory.

Image of a Functor

- The image of a functor need not be a subcategory.
- Consider the functor

$$\left(A \xrightarrow{f} B \qquad B' \xrightarrow{g} C \right) \xrightarrow{F} \left(\begin{tikzcd} & Y \arrow[dr, "q"] & \\ X \arrow[ur, "p"] \arrow[rr, "qp"'] & & Z \end{tikzcd} \right)$$

defined by

$$F(A) = X, \quad F(B) = F(B') = Y, \quad F(C) = Z,$$
$$F(f) = p, \quad F(g) = q.$$

Then p and q are in the image of F, but qp is not.

Subsection 4

Natural Transformations

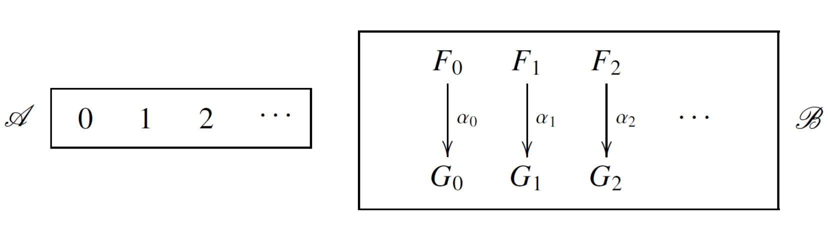
Example

- Let $\mathscr{A}$ be the discrete category whose objects are the natural numbers $0, 1, 2, \ldots$.
- A functor F from $\mathscr{A}$ to another category $\mathscr{B}$ is simply a sequence $(F_0, F_1, F_2, \ldots)$ of objects of $\mathscr{B}$.
- Let G be another functor from $\mathscr{A}$ to $\mathscr{B}$, consisting of another sequence $(G_0, G_1, G_2, \ldots)$ of objects of $\mathscr{B}$.
- It would be reasonable to define a "map" from F to G to be a sequence

$$(F_0 \xrightarrow{\alpha_0} G_0,\ F_1 \xrightarrow{\alpha_1} G_1,\ F_2 \xrightarrow{\alpha_2} G_2, \ldots)$$

  of maps in $\mathscr{B}$.

Example (Illustration)



- Some of the objects F_i or G_i might be equal, and there might be much else in $\mathscr{B}$ besides what is shown.

Natural Transformations

Definition

Let $\mathscr{A}$ and $\mathscr{B}$ be categories and let $\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}$ be functors. A **natural transformation** $\alpha : F \to G$ is a family $(F(A) \xrightarrow{\alpha_A} G(A))_{A \in \mathscr{A}}$ of maps in $\mathscr{B}$ such that for every map $A \xrightarrow{f} A'$ in $\mathscr{A}$, the square

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha_{A'}"] \\ G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd}$$

commutes. The maps α_A are called the **components** of α.

Remarks

(a) The definition of natural transformation is set up so that from each map $A \xrightarrow{f} A'$ in $\mathscr{A}$, it is possible to construct exactly one map $F(A) \to G(A')$ in $\mathscr{B}$.

When $f = 1_A$, this map is α_A.

For a general f, it is the diagonal of the square above, and "exactly one" implies that the square commutes.

(b) We write

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathscr{B} \end{tikzcd}$$

to mean that α is a natural transformation from F to G.

Example

- Let $\mathscr{A}$ be a discrete category, and let $F, G : \mathscr{A} \to \mathscr{B}$ be functors.
- Then F and G are just families $(F(A))_{A \in \mathscr{A}}$ and $(G(A))_{A \in \mathscr{A}}$ of objects of $\mathscr{B}$.
- A natural transformation $\alpha : F \to G$ is just a family

$$(F(A) \xrightarrow{\alpha_A} G(A))_{A \in \mathscr{A}}$$

  of maps in $\mathscr{B}$, as claimed previously in the case $\mathrm{ob}\mathscr{A} = \mathbb{N}$.
- In principle, this family must satisfy the naturality axiom for every map f in $\mathscr{A}$.
- But the only maps in $\mathscr{A}$ are the identities, and when f is an identity, this axiom holds automatically.

Example

- Recall that a group (or more generally, a monoid) G can be regarded as a one-object category.
- Also recall that a functor from the category G (instead of $\mathscr{G}$) to **Set** is nothing but a left G-set.
- Take two G-sets, S and T.
- Since S and T can be regarded as functors $G \to \mathbf{Set}$, we can ask what is a natural transformation

$$\begin{tikzcd} G \arrow[r, bend left, "S"] \arrow[r, bend right, "T"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{Set} \end{tikzcd}$$

- Such a natural transformation consists of a single map in Set (since G has just one object), satisfying some axioms.
- It is a function $\alpha : S \to T$ such that $\alpha(g \cdot s) = g \cdot \alpha(s)$, $s \in S$, $g \in G$.
- In other words, it is just a map of G-sets, sometimes called a G-**equivariant map**.

Determinants as Natural Transformations

- Fix a natural number n.
- For any commutative ring R, the $n \times n$ matrices with entries in R form a monoid $M_n(R)$ under multiplication.
- Moreover, any ring homomorphism $R \to S$ induces a monoid homomorphism $M_n(R) \to M_n(S)$.
- This defines a functor

$$M_n : \mathbf{CRing} \to \mathbf{Mon}$$

  from the category of commutative rings to the category of monoids.
- Also, the elements of any ring R form a monoid $U(R)$ under multiplication, giving another functor

$$U : \mathbf{CRing} \to \mathbf{Mon}.$$

Determinants as Natural Transformations (Cont'd)

- Now, every $n \times n$ matrix X over a commutative ring R has a determinant $\det_R(X)$, which is an element of R.
- Familiar properties of determinant, namely

$$\det_R(XY) = \det_R(X)\det_R(Y), \quad \det_R(I) = 1,$$

  tell us that for each R, the function $det_R : M_n(R) \to U(R)$ is a monoid homomorphism.
- So, we have a family of maps

$$(M_n(R) \xrightarrow{\det_R} U(R))_{R \in \mathbf{CRing}}.$$

- These maps define a natural transformation

$$\begin{tikzcd} \mathbf{CRing} \arrow[r, bend left, "M_n"] \arrow[r, bend right, "U"'] \arrow[r, phantom, "\Downarrow \det"] & \mathbf{Mon} \end{tikzcd}$$

Composition of Natural Transformations

- Given Natural transformations

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{A} \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathscr{B} \end{tikzcd}$$

  there is a composite natural transformation

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & \mathscr{B} \end{tikzcd}$$

  defined by

$$(\beta\circ\alpha)_A = \beta_A\circ\alpha_A, \quad A\in\mathscr{A}.$$

The Functor Category

- We saw that natural transformations from $\mathscr{A}$ to $\mathscr{B}$ compose.
- There is also an identity natural transformation

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "F"'] \arrow[r, phantom, "\Downarrow 1_F"] & \mathscr{B} \end{tikzcd}$$

  on any functor F, defined by

$$(1_F)_A = 1_{F(A)}.$$

- So for any two categories $\mathscr{A}$ and $\mathscr{B}$, there is a category whose objects are the functors from $\mathscr{A}$ to $\mathscr{B}$ and whose maps are the natural transformations between them.
- This is called the **functor category** from $\mathscr{A}$ to $\mathscr{B}$, and written as $[\mathscr{A}, \mathscr{B}]$ or $\mathscr{B}^{\mathscr{A}}$.

Example

- Let 2 be the discrete category with two objects.
- A functor from 2 to a category $\mathscr{B}$ is a pair of objects of $\mathscr{B}$.
- A natural transformation is a pair of maps.
- The functor category $[2, \mathscr{B}]$ is therefore isomorphic to the product category $\mathscr{B} \times \mathscr{B}$.
- This fits well with the alternative notation $\mathscr{B}^2$ for the functor category.

Example

- Let G be a monoid.
- $[G, \mathbf{Set}]$ is the category of left G-sets.
- $[G^{\mathrm{op}}, \mathbf{Set}]$ is the category of right G-sets

Example

- Take ordered sets A and B, viewed as categories.
- Given order-preserving maps $A \underset{g}{\overset{f}{\rightrightarrows}} B$ viewed as functors, there is at most one natural transformation

$$\begin{tikzcd} A \arrow[r, bend left, "f"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow"] & B \end{tikzcd}$$

- There is one such if and only if $f(a) \le g(a)$ for all $a \in A$.
- The naturality axiom holds automatically, because in an ordered set, all diagrams commute.
- So $[A, B]$ is an ordered set too, its elements being the order-preserving maps from A to B, and $f \le g$ if and only if $f(a) \le g(a)$ for all $a \in A$.

Natural Isomorphisms

- Everyday phrases such as "the cyclic group of order 6" and "the product of two spaces" reflect the fact that given two isomorphic objects of a category, we usually neither know nor care whether they are actually equal.
- When the category concerned is a functor category, given two functors $F, G : \mathscr{A} \to \mathscr{B}$, we usually do not care whether they are literally equal.
- Equality would imply that the objects $F(A)$ and $G(A)$ of $\mathscr{B}$ were equal for all $A \in \mathscr{A}$, a level of detail in which we are uninterested.
- What really matters is whether they are *naturally isomorphic*.

Definition

Let $\mathscr{A}$ and $\mathscr{B}$ be categories. A **natural isomorphism** between functors from $\mathscr{A}$ to $\mathscr{B}$ is an isomorphism in $[\mathscr{A}, \mathscr{B}]$.

Naturally Isomorphic Functors

Lemma

Consider a natural transformation

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \end{tikzcd}$$

Then α is a natural isomorphism if and only if $\alpha_A : F(A) \to G(A)$ is an isomorphism for all $A \in \mathscr{A}$.

- Of course, we say that functors F and G are **naturally isomorphic** if there exists a natural isomorphism from F to G.
- Since natural isomorphism is just isomorphism in a particular category (namely, $[\mathscr{A}, \mathscr{B}]$), we already have notation for this: $F \cong G$.

Naturally Isomorphic Objects

Definition

Given functors $\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}$, we say that

$$F(A) \cong G(A) \textbf{ naturally in } A$$

if F and G are naturally isomorphic.

- This alternative terminology can be understood as follows:
  - If $F(A) \cong G(A)$ naturally in A, then certainly $F(A) \cong G(A)$, for each individual A.
  - Moreover, we can choose isomorphisms $\alpha_A : F(A) \to G(A)$ in such a way that the naturality axiom is satisfied.

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\alpha_A"'] & F(A') \arrow[d, "\alpha_{A'}"] \\ G(A) \arrow[r, "G(f)"'] & G(A') \end{tikzcd}$$

Example

- Let $F, G : \mathscr{A} \to \mathscr{B}$ be functors from a discrete category $\mathscr{A}$ to a category $\mathscr{B}$.
- Then $F \cong G$ if and only if $F(A) \cong G(A)$ for all $A \in \mathscr{A}$.
- So in this case, $F(A) \cong G(A)$ naturally in A if and only if $F(A) \cong G(A)$ for all A.
- But this is only true because $\mathscr{A}$ is discrete.
- In general, it is emphatically false.
- There are many examples of categories and functors $\mathscr{A} \underset{G}{\overset{F}{\rightrightarrows}} \mathscr{B}$ such that $F(A) \cong G(A)$ for all $A \in \mathscr{A}$, but not naturally in A.

Example

- Let **FDVect** be the category of finite-dimensional vector spaces over some field k.
- The dual vector space construction defines a contravariant functor from **FDVect** to itself.
- The double dual construction therefore defines a covariant functor from **FDVect** to itself.
- Moreover, we have for each $V \in \mathbf{FDVect}$ a canonical isomorphism $\alpha_V : V \to V^{**}$.
- Given $v \in V$, the element $\alpha_V(v)$ of V^{**} is "evaluation at v", i.e., $\alpha_V(v) : V^* \to k$ maps $\phi \in V^*$ to $\phi(v) \in k$.
- That α_V is an isomorphism is a standard result in the theory of finite-dimensional vector spaces.

Example (Cont'd)

- This defines a natural transformation

$$\begin{tikzcd} \mathbf{FDVect} \arrow[r, bend left, "I_{\mathbf{FDVect}}"] \arrow[r, bend right, "(\ )^{**}"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathbf{FDVect} \end{tikzcd}$$

  from the identity functor to the double dual functor.
- By the preceding lemma, α is a natural isomorphism.
- So $1_{\mathbf{FDVect}} \cong (\ )^{**}$.
- Equivalently, $V \cong V^{**}$ naturally in V.

Equality versus Isomorphism

- Two elements of a set are either equal or not.
- Two objects of a category can be equal, not equal but isomorphic, or not even isomorphic.
- The notion of equality between two objects of a category is unreasonably strict; it is usually isomorphism that we care about.
  - The right notion of sameness of two elements of a set is equality;
  - The right notion of sameness of two objects of a category is isomorphism.
- When applied to a functor category $[\mathscr{A},\mathscr{B}]$, the second point tells us that:

  The right notion of sameness of two functors $\mathscr{A} \rightrightarrows \mathscr{B}$ is natural isomorphism.

Equivalence of Categories

- What is the right notion of sameness of two categories?
- Isomorphism is unreasonably strict, as if $\mathscr{A} \cong \mathscr{B}$ then there are functors $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ such that

$$G \circ F = 1_{\mathscr{A}} \quad \text{and} \quad F \circ G = 1_{\mathscr{B}},$$

  and we have just seen that the notion of equality between functors is too strict.
- The most useful notion of sameness of categories, called "equivalence", is looser than isomorphism.
- To obtain the definition, we simply replace the unreasonably strict equalities by isomorphisms, i.e., we stipulate

$$G \circ F \cong 1_{\mathscr{A}} \quad \text{and} \quad F \circ G \cong 1_{\mathscr{B}}.$$

Equivalence of Categories

Definition

An **equivalence** between categories $\mathscr{A}$ and $\mathscr{B}$ consists of a pair $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ of functors together with natural isomorphisms

$$\eta : 1_{\mathscr{A}} \to G \circ F, \quad \varepsilon : F \circ G \to 1_{\mathscr{B}}.$$

If there exists an equivalence between $\mathscr{A}$ and $\mathscr{B}$, we say that $\mathscr{A}$ and $\mathscr{B}$ are **equivalent**, and write $\mathscr{A} \simeq \mathscr{B}$. We also say that the functors F and G are **equivalences**.

- The symbol $\cong$ is used for isomorphism of objects of a category, and in particular for isomorphism of categories (which are objects of **CAT**).
- The symbol $\simeq$ is used for equivalence of categories.

Characterization of Equivalence Functors

Definition

A functor $F:\mathscr{A}\to\mathscr{B}$ is **essentially surjective on objects** if for all $B\in\mathscr{B}$, there exists $A\in\mathscr{A}$ such that $F(A)\cong B$.

Proposition

A functor is an equivalence if and only if it is full, faithful and essentially surjective on objects.

- Suppose, first, that $F:\mathscr{A}\to\mathscr{B}$ is an equivalence. By definition, there exist $G:\mathscr{B}\to\mathscr{A}$ and natural isomorphisms $\eta:1_{\mathscr{A}}\to GF$ and $\varepsilon:1_{\mathscr{B}}\to FG$.
  - Let $B\in\mathscr{B}$. By definition $B\cong F(G(B))$. Thus, there exists $A=G(B)\in\mathscr{A}$, such that $B\cong F(A)$. Hence, F is essentially surjective on objects.

Characterization of Equivalence Functors (Cont'd)

- Let $A, A' \in \mathscr{A}$ and suppose $f, g : A \to A'$, such that $F(f) = F(g)$. Then $G(F(f)) = G(F(g))$. Thus, we have $\eta_{A'}^{-1} G(F(f)) \eta_A = \eta_{A'}^{-1} G(F(g)) \eta_A$. Hence, by the diagram, with f (g) at the top, $f = g$. Thus, F is faithful. By symmetry, G is also faithful.

$$
\begin{tikzcd}
A \arrow[r, "?"] \arrow[d, "\eta_A"'] & A' \arrow[d, "\eta_{A'}"] \\
G(F(A)) \arrow[r, "G(F(?))"'] & G(F(A'))
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "\eta_{A'}^{-1} G(g) \eta_A"] \arrow[d, "\eta_A"'] & A' \arrow[d, "\eta_{A'}"] \\
G(F(A)) \arrow[r, "G(g)"'] & G(F(A'))
\end{tikzcd}
$$

- Let $A, A' \in \mathscr{A}$ and $g : F(A) \to F(A')$. Consider the diagram on the right. By the left diagram, with $\eta_{A'}^{-1} G(g) \eta_A$ at the top, we get $G(g) = G(F(\eta_{A'}^{-1} G(g) \eta_A))$. Since G is faithful, $g = F(\eta_{A'}^{-1} G(g) \eta_A)$. Thus, F is also full.

Characterization of Equivalence Functors (Converse)

- Suppose F is essentially surjective on objects, full and faithful.

  By essential surjectivity, for all $B \in \mathscr{B}$, we can choose $G(B) \in \mathscr{A}$ together with an isomorphism $\varepsilon_B : B \to F(G(B))$. Let $h : B \to B'$ in $\mathscr{B}$. Consider the rectangle

$$\begin{tikzcd} B \arrow[r, "h"] \arrow[d, "\varepsilon_B"'] & B' \arrow[d, "\varepsilon_{B'}"] \\ F(G(B)) \arrow[r, "\varepsilon_{B'} h \varepsilon_B^{-1}"'] & F(G(B')) \end{tikzcd}$$

  Since F is full and faithful, there exists unique $G(h) : G(B) \to G(B')$, such that $F(G(h)) = \varepsilon_{B'} h \varepsilon_B^{-1}$. G, thus defined, is a functor and $\varepsilon : 1_{\mathscr{B}} \to FG$ a natural isomorphism.

Characterization of Equivalence Functors (Converse Cont'd)

- Let $A \in \mathscr{A}$. Consider $\varepsilon_{F(A)} : F(A) \to F(G(F(A)))$. Since F is full and faithful, there exists unique $\eta_A : A \to G(F(A))$, such that $F(\eta_A) = \varepsilon_{F(A)}$. Since $\varepsilon_{F(A)}$ is an isomorphism, so is $F(\eta_A)$ and, since F is full and faithful, so is η_A. Finally, for $f : A \to A'$ in $\mathscr{A}$, we have

$$
\begin{tikzcd}
F(A) \arrow[r, "F(f)"] \arrow[d, "\varepsilon_{F(A)}"'] & F(A') \arrow[d, "\varepsilon_{F(A')}"] \\
F(G(F(A))) \arrow[r, "F(G(F(f)))"'] & F(G(F(A')))
\end{tikzcd}
\qquad
\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\eta_A"'] & A' \arrow[d, "\eta_{A'}"] \\
G(F(A)) \arrow[r, "G(F(f))"'] & G(F(A'))
\end{tikzcd}
$$

$$
\begin{aligned}
F(\eta_{A'} f) &= F(\eta_{A'})F(f) = \varepsilon_{F(A')}F(f) = F(G(F(f)))\varepsilon_{F(A)} \\
&= F(G(F(f)))F(\eta_A) = F(G(F(f))\eta_A).
\end{aligned}
$$

Hence, by faithfulness, $\eta_{A'} f = G(F(f))\eta_A$.

Full and Faithful Functors and Subcategories

Corollary

Let $F : \mathscr{C} \to \mathscr{D}$ be a full and faithful functor. Then $\mathscr{C}$ is equivalent to the full subcategory $\mathscr{C}'$ of $\mathscr{D}$ whose objects are those of the form $F(C)$ for some $C \in \mathscr{C}$.

- The functor $F' : \mathscr{C} \to \mathscr{C}'$ defined by $F'(C) = F(C)$ is full and faithful (since F is) and essentially surjective on objects (by definition of $\mathscr{C}'$).
- This result is true, with the same proof, whether we interpret "of the form $F(C)$" to mean "equal to $F(C)$" or "isomorphic to $F(C)$".

Example

- Let $\mathscr{A}$ be any category, and let $\mathscr{B}$ be any full subcategory containing at least one object from each isomorphism class of $\mathscr{A}$.
- Then the inclusion functor $\mathscr{B} \hookrightarrow \mathscr{A}$ is faithful (like any inclusion of subcategories), full, and essentially surjective on objects.
- Hence $\mathscr{B} \simeq \mathscr{A}$.
- So if we take a category and remove some (but not all) of the objects in each isomorphism class, the slimmed-down version is equivalent to the original.
- Conversely, if we take a category and throw in some more objects, each of them isomorphic to one of the existing objects, it makes no difference: The new, bigger, category is equivalent to the old one.

Example (Cont'd)

- For example, let **FinSet** be the category of finite sets and functions between them.
- For each natural number n, choose a set $\mathbf{n}$ with n elements.
- Let $\mathscr{B}$ be the full subcategory of **FinSet** with objects $\mathbf{0}, \mathbf{1}, \ldots$.
- Then $\mathscr{B} \simeq$ **FinSet**, even though $\mathscr{B}$ is in some sense much smaller than **FinSet**.

Example

- In a previous example, we saw that monoids are essentially the same thing as one-object categories.
- Let $\mathscr{C}$ be the full subcategory of **CAT** whose objects are the one-object categories.
- Let **Mon** be the category of monoids.
- Then $\mathscr{C} \simeq \mathbf{Mon}$.
- To see this, first note that given any object A of any category, the maps $A \to A$ form a monoid under composition (at least, subject to some set-theoretic restrictions).
- There is, therefore, a canonical functor $F : \mathscr{C} \to \mathbf{Mon}$ sending a one-object category to the monoid of maps from the single object to itself.
- This functor F is full and faithful and essentially surjective on objects.
- Hence F is an equivalence.

Duality

- An equivalence of the form $\mathscr{A}^{\mathrm{op}} \simeq \mathscr{B}$ is sometimes called a **duality** between $\mathscr{A}$ and $\mathscr{B}$.
- One says that $\mathscr{A}$ is **dual** to $\mathscr{B}$.
- There are many famous dualities in which $\mathscr{A}$ is a category of algebras and $\mathscr{B}$ is a category of spaces.
- Some quite advanced examples are:
  - **Stone duality**: The category of Boolean algebras is dual to the category of totally disconnected compact Hausdorff spaces.
  - **Gelfand-Naimark duality**: The category of commutative unital C^*-algebras is dual to the category of compact Hausdorff spaces. (C^*-algebras are certain algebraic structures important in functional analysis.)

Vertical Composition

- The composition of natural transformations

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{B} \end{tikzcd} \qquad \begin{tikzcd} \mathscr{A} \arrow[r, bend left, "G"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\beta"] & \mathscr{B} \end{tikzcd}$$

yielding

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "H"'] \arrow[r, phantom, "\Downarrow\beta\circ\alpha"] & \mathscr{B} \end{tikzcd}$$

is sometimes called **vertical composition**.

Horizontal Composition

- There is also **horizontal composition**, which takes natural transformations

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathscr{A}' \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow\alpha'"] & \mathscr{A}'' \end{tikzcd}$$

and produces a natural transformation

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F'\circ F"] \arrow[r, bend right, "G'\circ G"'] \arrow[r, phantom, "\Downarrow"] & \mathscr{A}'' \end{tikzcd}$$

traditionally written as $\alpha' * \alpha$.

Horizontal Composition (Cont'd)

- The component of $\alpha' * \alpha$ at $A \in \mathscr{A}$ is defined as follows:
  Use the first oval to produde $\alpha_A : F(A) \to G(A)$ in $\mathscr{A}'$.
  Use this map to produce the naturality square, based on the second oval,

$$\begin{tikzcd} F'(F(A)) \arrow[r, "F'(\alpha_A)"] \arrow[d, "\alpha'_{F(A)}"'] & F'(G(A)) \arrow[d, "\alpha'_{G(A)}"] \\ G'(F(A)) \arrow[r, "G'(\alpha_A)"'] & G'(G(A)) \end{tikzcd}$$

  $(\alpha' * \alpha)_A$ is the diagonal map of the square, i.e., $(\alpha' * \alpha)_A$ can be defined as either $\alpha'_{G(A)} \circ F'(\alpha_A)$ or $G'(\alpha_A) \circ \alpha'_{F(A)}$.

Special Cases

- The special cases of horizontal composition where either α or α' is an identity are especially important, and have their own notation.
- $$\begin{tikzcd} \mathscr{A} \arrow[r, "F"] & \mathscr{A}' \arrow[r, bend left, "F'"] \arrow[r, bend right, "G'"'] \arrow[r, phantom, "\Downarrow \alpha'"] & \mathscr{A}'' \end{tikzcd} \quad \text{gives} \quad \begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F' \circ F"] \arrow[r, bend right, "G' \circ F"'] \arrow[r, phantom, "\Downarrow \alpha' F"] & \mathscr{A}'' \end{tikzcd}$$

  where $(\alpha' F)_A = \alpha'_{F(A)}$.
- $$\begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow \alpha"] & \mathscr{A}' \arrow[r, "F'"] & \mathscr{A}'' \end{tikzcd} \quad \text{gives} \quad \begin{tikzcd} \mathscr{A} \arrow[r, bend left, "F' \circ F"] \arrow[r, bend right, "F' \circ G"'] \arrow[r, phantom, "\Downarrow F' \alpha"] & \mathscr{A}'' \end{tikzcd}$$

  where $(F' \circ \alpha)_A = F'(\alpha_A)$.

The Interchange Law

- Vertical and horizontal composition interact well: natural transformations

$$\begin{tikzcd} \mathscr{A} \arrow[r, bend left=50, "F"] \arrow[r, "G" description] \arrow[r, bend right=50, "H"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha"] \arrow[r, phantom, shift right=3, "\Downarrow\beta"] & \mathscr{A}' \arrow[r, bend left=50, "F'"] \arrow[r, "G'" description] \arrow[r, bend right=50, "H'"'] \arrow[r, phantom, shift left=3, "\Downarrow\alpha'"] \arrow[r, phantom, shift right=3, "\Downarrow\beta'"] & \mathscr{A}'' \end{tikzcd}$$

  obey the **interchange law**,

$$(\beta' \circ \alpha') * (\beta \circ \alpha) = (\beta' * \beta) \circ (\alpha' * \alpha) : F' \circ F \to H' \circ H.$$

- As usual, a statement on composition is accompanied by a statement on identities:

$$1_{F'} * 1_F = 1_{F' \circ F}.$$

Composition Functor Between Functor Categories

- All of this enables us to construct, for any categories $\mathscr{A}$, $\mathscr{A}'$ and $\mathscr{A}''$, a functor
$$[\mathscr{A}',\mathscr{A}''] \times [\mathscr{A},\mathscr{A}''] \to [\mathscr{A},\mathscr{A}'']$$
given on objects by
$$(F',F) \mapsto F' \circ F$$
and on maps by
$$(\alpha',\alpha) \mapsto \alpha' * \alpha.$$
- In particular, if $F' \cong G'$ and $F \cong G$, then $F' \circ F \cong G' \circ G$, since functors preserve isomorphism.