Introduction to Category Theory

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1. Adjoints
   - Definition and Examples
   - Adjunctions via Units and Counits
   - Adjunctions via Initial Objects

Subsection 1

Definition and Examples

Adjoints

- Consider a pair of functors in opposite directions, $F:\mathscr{A}\to\mathscr{B}$ and $G:\mathscr{B}\to\mathscr{A}$.
- Roughly speaking, F is said to be *left adjoint* to G if, whenever $A\in\mathscr{A}$ and $B\in\mathscr{B}$, maps $F(A)\to B$ are essentially the same thing as maps $A\to G(B)$.

Definition

Let $\mathscr{A}\underset{G}{\overset{F}{\rightleftarrows}}\mathscr{B}$ be categories and functors. We say that F is **left adjoint** to G, and G is **right adjoint** to F, and write $F\dashv G$, if

$$\mathscr{B}(F(A),B)\cong\mathscr{A}(A,G(B))$$

naturally in $A\in\mathscr{A}$ and $B\in\mathscr{B}$, "naturally" being defined below. An **adjunction** between F and G is a choice of natural isomorphism as above.

Definition of Transpose

- Given objects $A \in \mathscr{A}$ and $B \in \mathscr{B}$, the correspondence above between maps $F(A) \to B$ and $A \to G(B)$ is denoted by a horizontal bar, in both directions:

$$\begin{array}{rcl}(F(A) \xrightarrow{g} B) & \mapsto & (A \xrightarrow{\overline{g}} G(B)), \\ (F(A) \xrightarrow{\overline{f}} B) & \leftarrow\!\shortmid & (A \xrightarrow{f} G(B)).\end{array}$$

- So $\overline{\overline{f}} = f$ and $\overline{\overline{g}} = g$.
- We call $\overline{f}$ the **transpose** of f, and similarly for g.

Definition of Naturality

- "Naturally in $A \in \mathscr{A}$ and $B \in \mathscr{B}$" means that there is a specified bijection
$$\mathscr{B}(F(A), B) \cong \mathscr{A}(A, G(B))$$
for each $A \in \mathscr{A}$ and $B \in \mathscr{B}$, and that it satisfies a naturality axiom.
- The naturality axiom has two parts:
  - $$\overline{(F(A) \xrightarrow{g} B \xrightarrow{q} B')} = (A \xrightarrow{\overline{g}} G(B) \xrightarrow{G(q)} G(B'))$$
  (that is, $\overline{q \circ g} = G(q) \circ \overline{g}$) for all g and q;
  - $$\overline{(A' \xrightarrow{p} A \xrightarrow{f} G(B))} = (F(A') \xrightarrow{F(p)} F(A) \xrightarrow{\overline{f}} B)$$
  (that is, $\overline{f \circ p} = \overline{f} \circ F(p)$) for all p and f.

Remarks

(a) The naturality axiom might seem ad hoc, but we will see that it simply says that two particular functors are naturally isomorphic.

For now, think of "naturality" as formalizing the idea of something defined without making any arbitrary choices.

(b) The naturality axiom implies that from each array of maps

$$A_0 \to \cdots \to A_n, \quad F(A_n) \to B_0, \quad B_0 \to \cdots \to B_m,$$

it is possible to construct exactly one map

$$A_0 \to G(B_m).$$

(c) Adjoint functors arise everywhere.

In addition, whenever you see a pair of functors $\mathscr{A} \rightleftarrows \mathscr{B}$, there is an excellent chance that they are adjoint (one way round or the other).

Remarks (Cont'd)

(d) A given functor G may or may not have a left adjoint, but if it does, it is unique up to isomorphism.

So we may speak of "the left adjoint of G".

The same goes for right adjoints.

Example (**Algebra: Free ⊣ Forgetful**)

- Let k be a field.
- There is an adjunction $\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Vect}_k$, where U is the forgetful functor and F is the free functor.
- Adjointness says that given a set S and a vector space V, a linear map $F(S) \to V$ is essentially the same thing as a function $S \to U(V)$.
- Fix a set S and a vector space V.
- Given a linear map $g : F(S) \to V$, we may define a map of sets $\overline{g} : S \to U(V)$ by
$$\overline{g}(s) = g(s), \quad \text{for all } s \in S.$$
- This gives a function
$$\begin{array}{ccc} \mathbf{Vect}_k(F(S), V) & \to & \mathbf{Set}(S, U(V)) \\ g & \mapsto & \overline{g} \end{array}$$

Example (Cont'd)

- In the other direction, given a map of sets $f : S \to U(V)$, we may define a linear map $\overline{f} : F(S) \to V$ by

$$\overline{f}(\textstyle\sum_{s\in S}\lambda_s s) = \textstyle\sum_{s\in S}\lambda_s f(s), \qquad \text{for all formal linear combinations } \textstyle\sum \lambda_s s \in F(S).$$

- This gives a function

$$\begin{array}{ccc} \mathbf{Set}(S, U(V)) & \to & \mathbf{Vect}_k(F(S), V) \\ f & \mapsto & \overline{f} \end{array}$$

Example (Cont'd)

- These two functions "bar" are mutually inverse:
- For any linear map $g : F(S) \to V$, we have, for all $\sum \lambda_s s \in F(S)$,

$$\overline{\overline{g}}(\sum_{s\in S} \lambda_s s) = \sum_{s\in S} \lambda_s \overline{g}(s) = \sum_{s\in S} \lambda_s g(s) = g(\sum_{s\in S} \lambda_s s).$$

So $\overline{\overline{g}} = g$.

- Also, for any map of sets $f : S \to U(V)$, we have, for all $s \in S$,

$$\overline{\overline{f}}(s) = \overline{f}(s) = f(s).$$

So $\overline{\overline{f}} = f$.

- We therefore have a canonical bijection between $\mathbf{Vect}_k(F(S), V)$ and $\mathbf{Set}(S, U(V))$ for each $S \in \mathbf{Set}$ and $V \in \mathbf{Vect}_k$.

Example

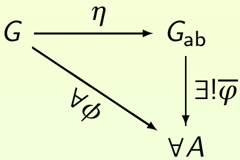
- In the same way, there is an adjunction

$$\mathbf{Set} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{Grp},$$

where F and U are the free and forgetful functors.
- The free group functor is tricky to construct explicitly.
- Later we will prove a result (the *general adjoint functor theorem*) guaranteeing that U and many functors like it all have left adjoints.

Example

- There is an adjunction $\mathbf{Grp} \overset{F}{\underset{U}{\rightleftarrows}} \mathbf{Ab}$, where U is the inclusion functor.
- If G is a group then $F(G)$ is the **abelianization** G_{ab} of G.
- This is an abelian quotient group of G, with the property that every map from G to an abelian group factorizes uniquely through G_{ab}:



- Here η is the natural map from G to its quotient G_{ab}, and A is any abelian group.
- The bijection $\mathbf{Ab}(G_{\text{ab}}, A) \cong \mathbf{Grp}(G, U(A))$ is given in the left-to-right direction by $\psi \mapsto \psi \circ \eta$ and in the right-to-left direction by $\phi \mapsto \overline{\phi}$.

Example

- There are adjunctions

$$\mathbf{Mon} \underset{R}{\overset{F}{\underset{\longleftarrow U \longrightarrow}{\longrightarrow}}} \mathbf{Grp}$$

  between the categories of groups and monoids.
- The middle functor U is inclusion.
- The left adjoint F is, again, tricky to describe explicitly.
- Informally, $F(M)$ is obtained from M by throwing in an inverse to every element (e.g., if M is the additive monoid of natural numbers then $F(M)$ is the group of integers).
- The general adjoint functor theorem guarantees the existence of F.
- This example is unusual in that forgetful functors do not usually have right adjoints.
- Here, given a monoid M, the group $R(M)$ is the submonoid of M consisting of all the invertible elements.

Reflective and Coreflective Subcategories

- Let $\mathscr{A}$ be a category and $\mathscr{B}$ a subcategory of $\mathscr{A}$.
- Denote by $I : \mathscr{B} \hookrightarrow \mathscr{A}$ the inclusion functor.
- $\mathscr{B}$ is a **reflective subcategory of** $\mathscr{A}$ if I has a left adjoint.
- $\mathscr{B}$ is a **coreflective subcategory of** $\mathscr{A}$ if I has a right adjoint.
- We saw the following examples:
  - The category **Grp** is both a reflective and a coreflective subcategory of **Mon**.
  - The category **Ab** is a reflective subcategory of **Grp**.

Example

- Let **Field** be the category of fields, with ring homomorphisms as the maps.
- The forgetful functor $\mathbf{Field} \to \mathbf{Set}$ does not have a left adjoint.
- The theory of fields is unlike the theories of groups, rings, and so on, because the operation $x \mapsto x^{-1}$ is not defined for all x (only for $x \neq 0$).

Remark on Algebraic Theories

- The theory of groups, the theory of rings, the theory of vector spaces over $\mathbb{R}$, the theory of vector spaces over $\mathbb{C}$, the theory of monoids, and (rather trivially) the theory of sets are all examples of **algebraic theories**.
- We will not define "algebraic theory", but only give the general idea.
- A group can be defined as a set X equipped with a function $\cdot : X \times X \to X$ (multiplication), another function $(\ )^{-1} : X \to X$ (inverse), and an element $e \in X$ (the identity), satisfying a familiar list of equations.
- More systematically, the three pieces of structure on X can be seen as maps of sets

$$\cdot : X^2 \to X; \quad (\ )^{-1} : X^1 \to X; \quad e : X^0 \to X,$$

where X^0 is the one-element set 1 and a map $1 \to X$ of sets is essentially the same thing as an element of X.

Remark on Algebraic Theories (Cont'd)

- An algebraic theory consists of two things:
  - A collection of operations, each with a specified arity (number of inputs);
  - A collection of equations.
- For example, the theory of groups has one operation of arity 2, one of arity 1, and one of arity 0.
- An **algebra** or **model** for an algebraic theory consists of a set X together with a specified map $X^n \to X$ for each operation of arity n, such that the equations hold everywhere.
- For example, an algebra for the theory of groups is exactly a group.

Remark on Algebraic Theories (Cont'd)

- A more subtle example is the theory of vector spaces over $\mathbb{R}$.
- This is an algebraic theory with, among other things, an infinite number of operations of arity 1:

  For each $\lambda \in \mathbb{R}$, we have the operation $\lambda \cdot - : X \to X$ of scalar multiplication by λ (for any vector space X).
- The theory of vector spaces over $\mathbb{R}$ is different from the theory of vector spaces over $\mathbb{C}$, because they have different operations of arity 1.
- In a nutshell, the main property of algebras for an algebraic theory is that the operations are defined everywhere on the set, and the equations hold everywhere too.
- For example, every element of a group has a specified inverse, and every element x satisfies the equation $x \cdot x^{-1} = 1$.
- This is why the theories of groups, rings, and so on, are algebraic theories, but the theory of fields is not.

Example

- There are adjunctions

$$D \dashv U \dashv I, \qquad D, I : \mathbf{Set} \to \mathbf{Top}, \quad U : \mathbf{Top} \to \mathbf{Set}$$

where:

  - U sends a space to its set of points;
  - D equips a set with the discrete topology;
  - I equips a set with the indiscrete topology.

Example

- Given sets A and B, we can form their (cartesian) product $A \times B$.
- We can also form the set B^A of functions from A to B.
- This is the same as the set $\mathbf{Set}(A, B)$, but we tend to use the notation B^A when we want to emphasize that it is an object of the same category as A and B.
- Now fix a set B.
- Taking the product with B defines a functor

$$\begin{array}{rccc} - \times B: & \mathbf{Set} & \to & \mathbf{Set} \\ & A & \mapsto & A \times B. \end{array}$$

- There is also a functor

$$\begin{array}{rccc} (-)^B: & \mathbf{Set} & \to & \mathbf{Set} \\ & C & \mapsto & C^B. \end{array}$$

Example (Cont'd)

- Moreover, there is a canonical bijection
$$\mathbf{Set}(A \times B, C) \cong \mathbf{Set}(A, C^B)$$
for any sets A and C.
- Given a map $g : A \times B \to C$, define $\overline{g} : A \to C^B$ by
$$(\overline{g}(a))(b) = g(a,b), \quad \text{for all } a \in A, b \in B.$$
- In the other direction, given $f : A \to C^B$, define $\overline{f} : A \times B \to C$ by
$$\overline{f}(a,b) = (f(a))(b), \quad \text{for all } a \in A,\ b \in B.$$
- Putting all this together, we obtain an adjunction
$$\begin{array}{c} \mathbf{Set} \\ - \times B \big\uparrow \dashv \big\downarrow (-)^B \\ \mathbf{Set} \end{array}$$
for every set B.

Initial and Terminal Objects

Definition

Let $\mathscr{A}$ be a category.
An object $I \in \mathscr{A}$ is **initial** if for every $A \in \mathscr{A}$, there is exactly one map $I \to A$.
An object $T \in \mathscr{A}$ is **terminal** if for every $A \in \mathscr{A}$, there is exactly one map $A \to T$.

- The empty set is initial in **Set**, the trivial group is initial in **Grp**, and $\mathbb{Z}$ is initial in **Ring**.
- The one-element set is terminal in **Set**, the trivial group is terminal (as well as initial) in **Grp**, and the trivial (one-element) ring is terminal in **Ring**.
- The terminal object of **CAT** is the category **1** containing just one object and one map (necessarily the identity on that object).

Essential Uniqueness of Initial Objects

Lemma

Let I and I' be initial objects of a category. Then there is a unique isomorphism $I \to I'$. In particular, $I \cong I'$.

- Since I is initial, there is a unique map $f : I \to I'$. Since I' is initial, there is a unique map $f' : I' \to I$. Now $f' \circ f$ and 1_I are both maps $I \to I$. Since I is initial, $f' \circ f = 1_I$. Similarly, $f \circ f' = 1_{I'}$. Hence f is an isomorphism, as required.
- The concept of terminal object is dual to the concept of initial object.
- Since any two initial objects of a category are uniquely isomorphic, the *principle of duality* implies that the same is true of terminal objects.

Initial and Terminal Objects as Adjoints

- Initial and terminal objects can be described as adjoints.
- Let $\mathscr{A}$ be a category.
- There is precisely one functor $\mathscr{A} \to \mathbf{1}$.
- Also, a functor $\mathbf{1} \to \mathscr{A}$ is essentially just an object of $\mathscr{A}$ (namely, the object to which the unique object of $\mathbf{1}$ is mapped).
- Viewing functors $\mathbf{1} \to \mathscr{A}$ as objects of $\mathscr{A}$, a left adjoint to $\mathscr{A} \to \mathbf{1}$ is exactly an initial object of $\mathscr{A}$.
- Similarly, a right adjoint to the unique functor $\mathscr{A} \to \mathbf{1}$ is exactly a terminal object of $\mathscr{A}$.

Composability of Adjunctions

- Adjunctions can be composed.
- Take adjunctions
$$\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{A}' \underset{G'}{\overset{F'}{\rightleftarrows}} \mathscr{A}''.$$
- Then we obtain an adjunction
$$\mathscr{A} \underset{G \circ G'}{\overset{F' \circ F}{\rightleftarrows}} \mathscr{A}''.$$
- To see this note that
$$\mathscr{A}''(F'(F(A)), A'') \cong \mathscr{A}'(F(A), G'(A'')) \cong \mathscr{A}(A, G(G'(A''))),$$
naturally in A and A''.

Subsection 2

Adjunctions via Units and Counits

Naturality Revisited

- Take an adjunction $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$.
- Intuitively, naturality says that, as A varies in $\mathscr{A}$ and B varies in $\mathscr{B}$, the isomorphism between $\mathscr{B}(F(A), B)$ and $\mathscr{A}(A, G(B))$ varies in a way that is compatible with all the structure already in place.
- That is, the isomorphism is compatible with composition in the categories $\mathscr{A}$ and $\mathscr{B}$ and the action of the functors F and G.
- Suppose, for example, that we have maps $F(A) \xrightarrow{g} B \xrightarrow{q} B'$ in $\mathscr{B}$. There are two things we can do with this data:
  - Compose and then take the transpose, which produces a map $\overline{q \circ g} : A \to G(B')$;
  - Take the transpose of g and then compose it with $G(q)$, which produces a potentially different map $G(q) \circ \overline{g} : A \to G(B')$.

  Compatibility means that they are equal and that is the first naturality equation.
- The second is its dual, and can be explained in a similar way.

Unit and Count of the Adjunction

- For each $A \in \mathscr{A}$, we have a map

$$(A \xrightarrow{\eta_A} GF(A)) = \overline{(F(A) \xrightarrow{1} F(A))}.$$

- Dually, for each $B \in \mathscr{B}$, we have a map

$$(FG(B) \xrightarrow{\varepsilon_B} B) = \overline{(G(B) \xrightarrow{1} G(B))}.$$

- These define natural transformations

$$\eta : 1_{\mathscr{A}} \to G \circ F, \qquad \varepsilon : F \circ G \to 1_{\mathscr{B}}.$$

- They are called the **unit** and **counit** of the adjunction, respectively.

Example

- Consider the usual adjunction $\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Vect}_k$.
- Its unit $\eta : 1_{\mathbf{Set}} \to U \circ F$ has components

$$\begin{array}{rccl} \eta_S : & S & \to & UF(S) \quad = \{\text{formal } k\text{-linear sums } \sum_{s \in S} \lambda_s s\} \\ & s & \mapsto & s, \end{array}$$

  $S \in \mathbf{Set}$.
- The component of the counit ε at a vector space V is the linear map

$$\varepsilon_V : FU(V) \to V$$

  that sends a formal linear sum $\sum_{v \in V} \lambda_v v$ to its actual value in V.

Example (Cont'd)

- The vector space $FU(V)$ is enormous.
- E.g., if $k = \mathbb{R}$ and V is the vector space $\mathbb{R}^2$, then:
    - $U(V)$ is the set $\mathbb{R}^2$;
    - $FU(V)$ is a vector space with one basis element for every element of $\mathbb{R}^2$.

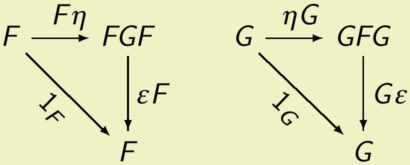
  Thus, $FU(V)$ is uncountably infinite-dimensional.

  Then $\varepsilon_V : FU(V) \to V$ is a map from this infinite-dimensional space to the 2-dimensional space V.

The Triangle Identities

Lemma

Given an adjunction $F \dashv G$ with unit η and counit ε, the following triangles commute:



- These are called the **triangle identities**.
- They are commutative diagrams in the functor categories $[\mathscr{A},\mathscr{B}]$ and $[\mathscr{B},\mathscr{A}]$, respectively.

The Triangle Identities (Cont'd)

- An equivalent statement is that the following triangles commute for all $A \in \mathscr{A}$ and all $B \in \mathscr{B}$,

$$\begin{array}{ccc} F(A) & \xrightarrow{F(\eta_A)} & FGF(A) \\ & {\scriptstyle 1_{F(A)}} \searrow & \downarrow {\scriptstyle \varepsilon_{F(A)}} \\ & & F(A) \end{array} \qquad \begin{array}{ccc} G(B) & \xrightarrow{\eta_{G(B)}} & GFG(B) \\ & {\scriptstyle 1_{G(B)}} \searrow & \downarrow {\scriptstyle G(\varepsilon_B)} \\ & & G(B) \end{array}$$

Proof: Let $A \in \mathscr{A}$. By definition of the counit, $\overline{1_{GF(A)}} = \varepsilon_{F(A)}$. By naturality, $\overline{f \circ p} = \overline{f} \circ F(p)$. Hence,

$$\overline{(A \xrightarrow{\eta_A} GF(A) \xrightarrow{1} GF(A))} = (F(A) \xrightarrow{F(\eta_A)} FGF(A) \xrightarrow{\varepsilon_{F(A)}} F(A)).$$

The left-hand side is $\overline{\eta_A} = \overline{\overline{1_{F(A)}}} = 1_{F(A)}$, proving the first identity. The second follows by duality.

Transposes via Units and Counits

Lemma

Let $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ be an adjunction, with unit η and counit ε.

- $\overline{g} = G(g) \circ \eta_A$, for any $g : F(A) \to B$;
- $\overline{f} = \varepsilon_B \circ F(f)$, for any $f : A \to G(B)$.

- For any map $g : F(A) \to B$, we have

$$\begin{aligned} \overline{(F(A) \stackrel{g}{\to} B)} &= \overline{(F(A) \stackrel{1}{\to} F(A) \stackrel{g}{\to} B)} \\ &= (A \stackrel{\eta_A}{\to} GF(A) \stackrel{G(g)}{\to} G(B)). \end{aligned}$$

This gives the first statement.

The second follows by duality.

Adjunctions in terms of Units and Counits

Theorem

Take categories and functors $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$. There is a one-to-one correspondence between:

(a) Adjunctions between F and G (with F on the left and G on the right), i.e., choices of isomorphisms $\mathscr{B}(F(A), B) \cong \mathscr{A}(A, G(B))$ satisfying the naturality conditions, for all g, q, p and f,

$$\begin{array}{rcl} \overline{(F(A) \xrightarrow{g} B \xrightarrow{q} B')} & = & (A \xrightarrow{\overline{g}} G(B) \xrightarrow{G(q)} G(B')), \\ \overline{(A' \xrightarrow{p} A \xrightarrow{f} G(B))} & = & (F(A') \xrightarrow{F(p)} F(A) \xrightarrow{\overline{f}} B); \end{array}$$

(b) Pairs $(1_{\mathscr{A}} \xrightarrow{\eta} GF, FG \xrightarrow{\varepsilon} 1_{\mathscr{B}})$ of natural transformations satisfying the triangle identities.

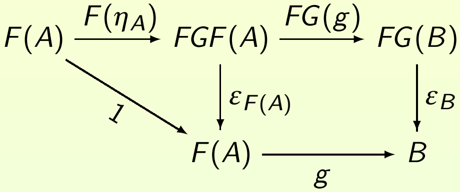
Proof

- We have shown that every adjunction between F and G gives rise to a pair (η,ε) satisfying the triangle identities.
  We now have to show that this process is bijective.
  So, take a pair (η,ε) of natural transformations satisfying the triangle identities.
  We must show that there is a unique adjunction between F and G with unit η and counit ε.
  Uniqueness follows from the preceding lemma.
  For existence, take natural transformations $\eta : 1_{\mathscr{A}} \to GF$ and $\varepsilon : FG \to 1_{\mathscr{B}}$, satisfying the triangle identities.
  For each A and B, define functions
  $$\mathscr{B}(F(A),B) \rightleftarrows \mathscr{A}(A,G(B))$$
  both denoted by a bar, as follows:
  - Given $g \in \mathscr{B}(F(A),B)$, put $\overline{g} = G(g) \circ \eta_A \in \mathscr{A}(A,G(B))$.
  - Given $f \in \mathscr{A}(A,G(B))$, put $\overline{f} = \varepsilon_B \circ F(f) \in \mathscr{B}(F(A),B)$.

Proof (Cont'd)

Claim: For each A and B, the two functions $g \mapsto \overline{g}$ and $f \mapsto \overline{f}$ are mutually inverse.

Given a map $g : F(A) \to B$ in $\mathscr{B}$, we have a commutative diagram



The composite map from $F(A)$ to B by one route around the outside of the diagram is $\varepsilon_B \circ FG(g) \circ F(\eta_A) = \varepsilon_B \circ F(G(g) \circ \eta_A) = \varepsilon_B \circ F(\overline{g}) = \overline{\overline{g}}$ and by the other is $g \circ 1 = g$. So $\overline{\overline{g}} = g$.

Dually, $\overline{\overline{f}} = f$ for any map $f : A \to G(B)$ in $\mathscr{A}$.

Proof (Cont'd)

- It is straightforward to check the naturality equations:

$$F(A) \xrightarrow{g} B \xrightarrow{q} B'$$

$$\overline{q \circ g} = G(q \circ g) \circ \eta_A = G(q) \circ G(g) \circ \eta_A = G(q) \circ \overline{g}.$$

The bar functions therefore define an adjunction.

Finally, its unit and counit are η and ε, since the component of the unit at A is

$$\overline{1_{F(A)}} = G(1_{F(A)}) \circ \eta_A = 1 \circ \eta_A = \eta_A.$$

Dually for the counit.

Adjunctions via Unit and Counit

Corollary

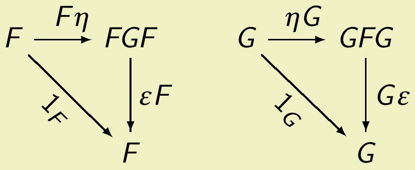
Take categories and functors

$$\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}.$$

Then $F \dashv G$ if and only if there exist natural transformations

$$1 \xrightarrow{\eta} GF \quad \text{and} \quad FG \xrightarrow{\varepsilon} 1$$

satisfying the triangle identities:



Example

- An adjunction between ordered sets consists of order-preserving maps $A \underset{g}{\overset{f}{\rightleftarrows}} B$ such that, for all $a \in A$ and all $b \in B$,

$$f(a) \leq b \iff a \leq g(b).$$

- This is because both sides of the isomorphism in the definition of adjunction are sets with at most one element.
- So they are isomorphic if and only if they are both empty or both nonempty.
- The naturality requirements hold automatically, since in an ordered set, any two maps with the same domain and codomain are equal.

Example (Cont'd)

- Recall that if $C \overset{p}{\underset{q}{\rightrightarrows}} D$ are order-preserving maps of ordered sets then there is at most one natural transformation from p to q, and there is one if and only if $p(c) \leq q(c)$, for all $c \in C$.
- The unit of the adjunction above is the statement that $a \leq gf(a)$ for all $a \in A$.
- The counit is the statement that $fg(b) \leq b$ for all $b \in B$.
- The triangle identities say nothing, since they assert the equality of two maps in an ordered set with the same domain and codomain.
- In the case of ordered sets, the preceding corollary states that the following two conditions are equivalent:
  - (a) $f(a) \leq b \Leftrightarrow a \leq g(b)$, for all $a \in A$, $b \in B$;
  - (b) $a \leq gf(a)$ and $fg(b) \leq b$, for all $a \in A$, $b \in B$.

Example (Cont'd)

- Let X be a topological space.
- Take the set $\mathscr{C}(X)$ of closed subsets of X and the set $\mathscr{P}(X)$ of all subsets of X, both ordered by $\subseteq$.
- There are order-preserving maps

$$\mathscr{P}(X) \underset{i}{\overset{\mathrm{Cl}}{\rightleftarrows}} \mathscr{C}(X)$$

  where i is the inclusion map and Cl is closure.
- This is an adjunction, with Cl left adjoint to i, as witnessed by the fact that, for all $A \subseteq X$ and $B \in \mathscr{C}(X)$,

$$\mathrm{Cl}(A) \subseteq B \quad \Longleftrightarrow \quad A \subseteq B.$$

- An equivalent statement is that $A \subseteq \mathrm{Cl}(A)$ for all $A \subseteq X$ and $\mathrm{Cl}(B) \subseteq B$, for all closed $B \subseteq X$.
- The topological operation of closure arises as an adjoint functor.

Adjunctions versus Equivalences

- The preceding theorem states that an adjunction may be regarded as a quadruple $(F, G, \eta, \varepsilon)$ of functors and natural transformations satisfying the triangle identities.
- An equivalence $(F, G, \eta, \varepsilon)$ of categories is not necessarily an adjunction.
- It is true that F is left adjoint to G, but η and ε are not necessarily the unit and counit (because there is no reason why they should satisfy the triangle identities).

Subsection 3

Adjunctions via Initial Objects

Example

- Consider once more the adjunction $\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Vect}_k$.
- Let S be a set.
- The universal property of $F(S)$, the vector space whose basis is S, is most commonly stated like this:

  > Given a vector space V, any function $f : S \to V$ extends uniquely to a linear map $\overline{f} : F(S) \to V$.

- In this statement, "$f : S \to V$" should strictly speaking be "$f : S \to U(V)$".
- Also, the word "extends" refers implicitly to the embedding

$$\begin{array}{cccc} \eta_S : & S & \to & UF(S) \\ & s & \mapsto & s. \end{array}$$

Example (Cont'd)

- So in precise language, the statement reads:

  For any $V \in \mathbf{Vect}_k$ and $f \in \mathbf{Set}(S, U(V))$, there is a unique $\overline{f} \in \mathbf{Vect}_k(F(S), V)$, such that the diagram

$$\begin{array}{ccc} S & \xrightarrow{\eta_S} & U(F(S)) \\ & {\scriptstyle f}\searrow & \downarrow {\scriptstyle U(\overline{f})} \\ & & U(V) \end{array}$$

  commutes.

- We show that this statement is equivalent to the statement that F is left adjoint to U with unit η.

The Comma Category

Definition

Given categories and functors $\mathscr{A} \xrightarrow{P} \mathscr{C}$, $\mathscr{B} \xrightarrow{Q} \mathscr{C}$, the **comma category** $(P \Rightarrow Q)$ (often written as $(P \downarrow Q)$) is the category defined as follows:

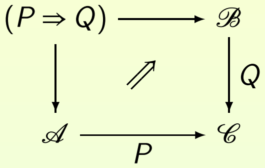
- Objects are triples (A, h, B) with $A \in \mathscr{A}$, $B \in \mathscr{B}$ and $h : P(A) \to Q(B)$ in $\mathscr{C}$;
- Maps $(A, h, B) \to (A', h', B')$ are pairs $(f : A \to A', g : B \to B')$ of maps such that the square

$$\begin{array}{ccc} P(A) & \xrightarrow{P(f)} & P(A') \\ {\scriptstyle h}\downarrow & & \downarrow{\scriptstyle h'} \\ Q(B) & \xrightarrow[Q(g)]{} & Q(B') \end{array}$$

commutes.

Functors Involving the Comma Category

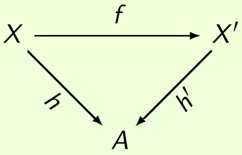
- Given $P : \mathscr{A} \to \mathscr{C}$ and $Q : \mathscr{B} \to \mathscr{C}$, there are canonical functors and a canonical natural transformation as shown:



- In a suitable 2-categorical sense, $(P \Rightarrow Q)$ is universal with this property.

The Slice Category

- Let $\mathscr{A}$ be a category and $A \in \mathscr{A}$.
- The **slice category** of $\mathscr{A}$ over A, denoted by $\mathscr{A}/A$, is the category whose objects are maps into A and whose maps are commutative triangles.
- More precisely:
  - An object is a pair (X, h) with $X \in \mathscr{A}$ and $h : X \to A$ in $\mathscr{A}$;
  - A map $(X, h) \to (X', h')$ in $\mathscr{A}/A$ is a map $f : X \to X'$ in $\mathscr{A}$ making the following triangle commute:



Comma Categories and Slice Categories

- Slice categories are a special case of comma categories.
- Recall that functors $\mathbf{1} \to \mathscr{A}$ are just objects of $\mathscr{A}$.
- Given an object A of $\mathscr{A}$, consider the comma category $(1_{\mathscr{A}} \Rightarrow A)$, as in the diagram

$$\begin{array}{ccc} & & \mathbf{1} \\ & & \big\downarrow{\scriptstyle A} \\ \mathscr{A} & \xrightarrow[1_{\mathscr{A}}]{} & \mathscr{A} \end{array}$$

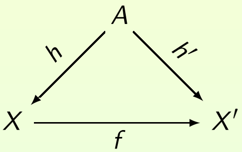
- An object of $(1_{\mathscr{A}} \Rightarrow A)$ is in principle a triple (X, h, B) with $X \in \mathscr{A}$, $B \in \mathbf{1}$ and $h : X \to A$ in $\mathscr{A}$.
- But $\mathbf{1}$ has only one object, so it is essentially just a pair (X, h).
- Hence the comma category $(1_{\mathscr{A}} \Rightarrow A)$ has the same objects as the slice category $\mathscr{A}/A$.
- One can check that it has the same maps too, so that $\mathscr{A}/A \cong (1_{\mathscr{A}} \Rightarrow A)$.

The Coslice Category

- Dually (reversing all the arrows), there is a **coslice category** $A/\mathscr{A} \cong (A \Rightarrow 1_{\mathscr{A}})$, whose objects are the maps out of A.
  - An object is a pair (X, h), with

$$A \xrightarrow{h} X$$

  - A map $f : (X, h) \to (X', h')$ is a morphism $f : X \to X'$ in $\mathscr{A}$, that makes the following triangle commute



Example

- Let $G : \mathscr{B} \to \mathscr{A}$ be a functor and let $A \in \mathscr{A}$.
- We can form the comma category $(A \Rightarrow G)$, as in the diagram

$$\begin{array}{ccc} & & \mathscr{B} \\ & & \downarrow{\scriptstyle G} \\ \mathbf{1} & \xrightarrow[A]{} & \mathscr{A} \end{array}$$

- Its objects are pairs $(B \in \mathscr{B}, f : A \to G(B))$.
- A map $(B, f) \to (B', f')$ in $(A \Rightarrow G)$ is a map $q : B \to B'$ in $\mathscr{B}$ making the triangle commute:

$$\begin{array}{ccc} A & \xrightarrow{f} & G(B) \\ & \searrow^{f'} & \downarrow{\scriptstyle G(q)} \\ & & G(B') \end{array}$$

Example (Cont'd)

- Notice how this diagram resembles the diagram

$$\begin{array}{ccc} S & \xrightarrow{\eta_S} & U(F(S)) \\ & {}_{f}\searrow & \downarrow{\scriptstyle U(\overline{f})} \\ & & U(V) \end{array}$$

  in the vector space example.
- We will use comma categories $(A \Rightarrow G)$ to capture the kind of universal property discussed there.
- Speaking casually, we say that $f : A \to G(B)$ is an object of $(A \Rightarrow G)$, when what we should really say is that the pair (B, f) is an object of $(A \Rightarrow G)$.
- There is potential for confusion here, since there may be different objects B, B' of $\mathscr{B}$ with $G(B) = G(B')$.
- But we will often use this convention.

Units of Adjunctions and Comma Categories

Lemma

Take an adjunction $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ and an object $A \in \mathscr{A}$. Then the unit map $\eta_A : A \to GF(A)$ is an initial object of $(A \Rightarrow G)$.

- Let $(B, f : A \to G(B))$ be an object of $(A \Rightarrow G)$.
  We show that there is exactly one map from $(F(A), \eta_A)$ to (B, f).
  A map $(F(A), \eta_A) \to (B, f)$ in $(A \Rightarrow G)$ is a map $q : F(A) \to B$ in $\mathscr{B}$ such that the triangle commutes:

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GF(A) \\ & \underset{f}{\searrow} & \downarrow G(q) \\ & & G(B) \end{array}$$

  But $G(q) \circ \eta_A = \overline{q}$ by a previous lemma.
  So the triangle commutes if and only if $f = \overline{q}$, if and only if $q = \overline{f}$.
  Hence $\overline{f}$ is the unique map $(F(A), \eta_A) \to (B, f)$ in $(A \Rightarrow G)$.

Adjunctions and Comma Categories

Theorem

Take categories and functors $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$. There is a one-to-one correspondence between:

(a) Adjunctions between F and G (with F on the left and G on the right);

(b) Natural transformations $\eta : 1_{\mathscr{A}} \to GF$ such that $\eta_A : A \to GF(A)$ is initial in $(A \Rightarrow G)$ for every $A \in \mathscr{A}$.

- We have just shown that every adjunction between F and G gives rise to a natural transformation η with the property stated in (b).

  To prove the theorem, we have to show that every η with the property in (b) is the unit of exactly one adjunction between F and G.

  By a previous theorem, an adjunction between F and G amounts to a pair (η, ε) of natural transformations satisfying the triangle identities.

Adjunctions and Comma Categories (Cont'd)

- So it is enough to prove that for every η with the property in (b), there exists a unique natural transformation $\varepsilon : FG \to 1_{\mathscr{B}}$ such that the pair (η, ε) satisfies the triangle identities.
- Let $\eta : 1_{\mathscr{A}} \to GF$ be a natural transformation with the property in (b).

  Uniqueness: Suppose that $\varepsilon ; \varepsilon' : FG \to 1_{\mathscr{B}}$ are natural transformations such that both (η, ε) and (η, ε') satisfy the triangle identities.

  One of the triangle identities states that for all $B \in \mathscr{B}$, the following triangle commutes

$$\begin{array}{ccc} G(B) & \xrightarrow{\eta_{G(B)}} & G(FG(B)) \\ & \searrow^{1} & \downarrow G(\varepsilon_B) \\ & & G(B) \end{array}$$

Adjunctions and Comma Categories (Cont'd)

- Thus, ε_B is a map

$$(FG(B), G(B) \stackrel{\eta_{G(B)}}{\rightarrow} G(FG(B))) \rightarrow (B, G(B) \stackrel{1}{\rightarrow} G(B))$$

in $(G(B) \Rightarrow G)$.
The same is true of ε'_B.
But $\eta_{G(B)}$ is initial.
So there is only one such map and $\varepsilon_B = \varepsilon'_B$.
This holds for all B. So $\varepsilon = \varepsilon'$.

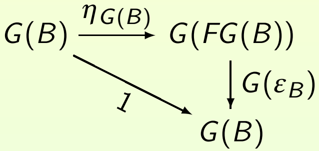
Adjunctions and Comma Categories (Cont'd)

- Existence: For $B \in \mathscr{B}$, define $\varepsilon_B : FG(B) \to B$ to be the unique map

$$(FG(B), \eta_{G(B)}) \to (B, 1_{G(B)})$$

in $(G(B) \Rightarrow G)$.

So by definition of ε_B, the following triangle commutes.

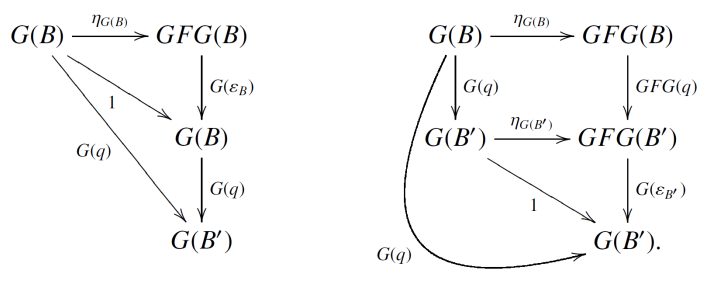


We show that $(\varepsilon_B)_{B \in \mathscr{B}}$ is a natural transformation $FG \to 1$ such that η and ε satisfy the triangle identities.

Adjunctions and Comma Categories (Cont'd)

- To prove naturality, take $B \xrightarrow{q} B'$ in $\mathscr{B}$.

  We have commutative diagrams

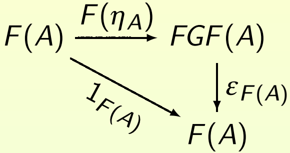


  $q \circ \varepsilon_B$ and $\varepsilon_{B'} \circ FG(q)$ are maps $(GFG(B), \eta_{G(B)}) \to (G(B'), G(q))$ in $(G(B) \Rightarrow G)$. Since $\eta_{G(B)}$ is initial, they must be equal. This shows naturality of ε with respect to q.

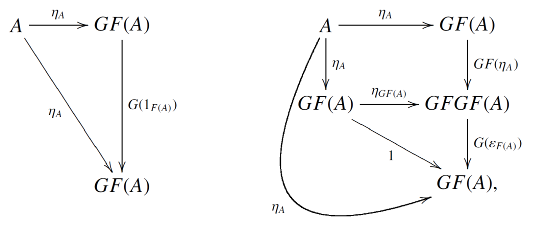
  Hence ε is a natural transformation.

Adjunctions and Comma Categories (Cont'd)

- We have already observed that one of the triangle identities holds. The other states that for $A \in \mathscr{A}$, the following triangle commutes.



There are commutative diagrams



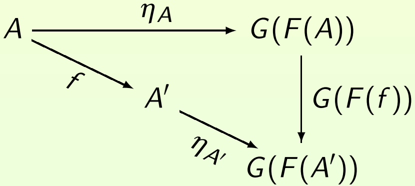
So by initiality of η_A, we have $\varepsilon_{F(A)} \circ F(\eta_A) = 1_{F(A)}$.

Left Adjoints and Initial Objects in Comma Categories

Corollary

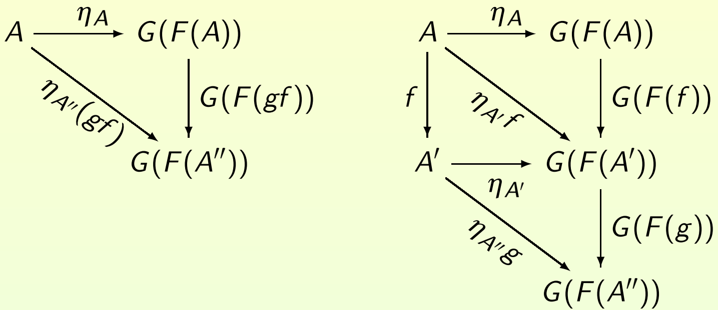
Let $G : \mathscr{B} \to \mathscr{A}$ be a functor. Then G has a left adjoint if and only if for each $A \in \mathscr{A}$, the category $(A \Rightarrow G)$ has an initial object.

- The preceding lemma proves "only if".
  To prove "if", let us choose for each $A \in \mathscr{A}$ an initial object of $(A \Rightarrow G)$ and call it $(F(A), \eta_A : A \to GF(A))$.
  For each map $f : A \to A'$ in $\mathscr{A}$, let $F(f) : F(A) \to F(A')$ be the unique map such that the following commutes



Proof (Cont'd)

- It is easily checked that F is a functor $\mathscr{A} \to \mathscr{B}$:



Since η_A is initial in $(A \Rightarrow G)$, $F(gf) = F(g)F(f)$.

The previous diagram tells us that η is a natural transformation $1 \to GF$.

So by the theorem, F is left adjoint to G.