Introduction to Category Theory

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Subsection 1

Constructions With Sets

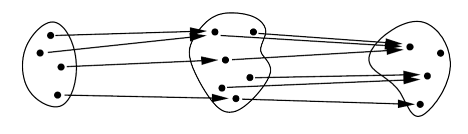
Sets

- Intuitively, a set is a bag of points, of which there may be infinitely many.
- These points, or elements, are not related to one another in any way.
  - They are not in any order;
  - They do not come with any algebraic structure (for instance, there is no specified way of multiplying elements together);
  - There is no sense of what it means for one point to be close to another.
- In particular examples, we might have some extra structure in mind.
- For instance, we often equip the set of real numbers with an order, a field structure and a metric.
- But to view $\mathbb{R}$ as a mere set is to ignore all that structure and regard it as no more than a bag of featureless points.

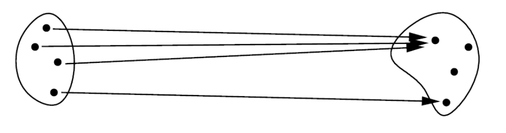
The Category of Sets

- Intuitively, a function $A \to B$ is an assignment of a point in bag B to each point in bag A.
- We can do one function after another:

  Given functions



  we obtain a composite function



- This composition of functions is associative: $h \circ (g \circ f) = (h \circ g) \circ f$.
- There is also an identity function on every set.
- Hence: Sets and functions form a category, denoted by **Set**.

The Empty Set

The empty set: There is a set $\emptyset$ with no elements.

- Suppose that someone hands you a pair of sets, A and B, and tells you to specify a function from A to B.
- Then your task is to specify for each element of A an element of B.
- The larger A is, the longer the task.
- The smaller A is, the shorter the task.
- In particular, if A is empty then the task takes no time at all; we have nothing to do.
- So there is a function from $\emptyset$ to B specified by doing nothing.
- On the other hand, there cannot be two different ways to do nothing.
- So there is only one function from $\emptyset$ to B.
- Hence: $\emptyset$ is an initial object of **Set**.

An Alternative Argument

- Suppose that we have a set A with disjoint subsets A_1 and A_2 such that $A_1 \cup A_2 = A$.
- Then a function from A to B amounts to a function from A_1 to B together with a function from A_2 to B.
- So if all the sets are finite, we should have the rule

$$\begin{aligned}&(\text{number of functions from } A \text{ to } B)\\ &\quad = (\text{number of functions from } A_1 \text{ to } B)\\ &\qquad \times (\text{number of functions from } A_2 \text{ to } B).\end{aligned}$$

- In particular, we could take $A_1 = A$ and $A_2 = \emptyset$.
- This would force the number of functions from $\emptyset$ to B to be 1.
- So if we want this rule to hold (and surely we do!), we had better say that there is exactly one function from $\emptyset$ to B.

Functions Into $\emptyset$

- What about functions into $\emptyset$?
- There is exactly one function $\emptyset \to \emptyset$, namely, the identity.
- This is a special case of the initiality of $\emptyset$.
- On the other hand, for a set A that is not empty, there are no functions $A \to \emptyset$ because there is nowhere for elements of A to go.

The One-Element Set

The one-element set: There is a set 1 with exactly one element.

- For any set A, there is exactly one function from A to 1, since every element of A must be mapped to the unique element of 1.
- That is: 1 is a terminal object of **Set**.
- A function from 1 to a set B is just a choice of an element of B.
- In short, the functions $1 \to B$ are the elements of B.
- Hence: The concept of element is a special case of the concept of function.

Products

Products: Any two sets A and B have a product, $A \times B$.

- Its elements are the ordered pairs (a, b) with $a \in A$ and $b \in B$.
- All that matters about ordered pairs is that for $a, a' \in A$ and $b, b' \in B$,

$$(a, b) = (a', b') \quad \Leftrightarrow \quad a = a' \text{ and } b = b'.$$

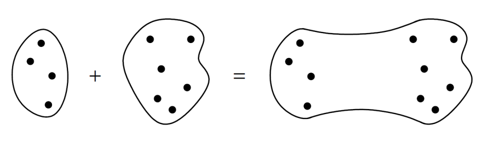
- More generally, take any set I and any family $(A_i)_{i \in I}$ of sets.
- There is a product set $\prod_{i \in I} A_i$, whose elements are families $(a_i)_{i \in I}$ with $a_i \in A_i$ for each i.
- Just as for ordered pairs,

$$(a_i)_{i \in I} = (a'_i)_{i \in I} \quad \Leftrightarrow \quad a_i = a'_i, \text{ for all } i \in I.$$

Sums

Sums: Any two sets A and B have a sum $A + B$.

- Thinking of sets as bags of points, the sum of two sets is obtained by putting all the points into one big bag:



- If A and B are finite sets with m and n elements respectively, then $A + B$ always has $m + n$ elements.
- It makes no difference what the elements of $A + B$ are called; as usual, we only care what $A + B$ is up to isomorphism.
- There are inclusion functions $A \xrightarrow{i} A + B \xleftarrow{j} B$ such that the union of the images of i and j is all of $A + B$ and the intersection of the images is empty.

Sums (Cont'd)

- Sum is sometimes called **disjoint union** and written as $\coprod$.
- It is not to be confused with (ordinary) union $\cup$.
  - We can take the sum of any two sets A and B;
  - $A \cup B$ only really makes sense when A and B come as subsets of some larger set (to say what $A \cup B$ is, we need to know which elements of A are equal to which elements of B);
  - Even if A and B do come as subsets of some larger set, $A + B$ and $A \cup B$ can be different.
- For example, take the subsets $A = \{1, 2, 3\}$ and $B = \{3, 4\}$ of $\mathbb{N}$.

  Then $A \cup B$ has 4 elements, but $A + B$ has $3 + 2 = 5$ elements.
- More generally, any family $(A_i)_{i \in I}$ of sets has a sum $\sum_{i \in I} A_i$.
- If I is finite and each A_i is finite, say with m_i elements, then $\sum_{i \in I} A_i$ has $\sum_{i \in I} m_i$ elements.

Sets of Functions

- **Sets of functions**: For any two sets A and B, we can form the set A^B of functions from B to A.
- This is a special case of the product construction:

$$A^B \text{ is the product } \prod_{b \in B} A \text{ of the constant family } (A)_{b \in B}.$$

- Indeed, an element of $\prod_{b \in B} A$ is a family $(a_b)_{b \in B}$ consisting of one element $a_b \in A$ for each $b \in B$.

  In other words, it is a function $B \to A$.

Digression on Arithmetic

- We are using notation reminiscent of arithmetic: $A \times B$, $A + B$ and A^B.
- There is good reason for this:
- If A is a finite set with m elements and B a finite set with n elements, then:
  - $A \times B$ has $m \times n$ elements;
  - $A + B$ has $m + n$ elements;
  - A^B has m^n elements.
- Our notation 1 for a one-element set and the alternative notation 0 for the empty set $\emptyset$ also follow this pattern.
- All the usual laws of arithmetic have their counterparts for sets:

$$\begin{array}{rcl} A \times (B + C) & \cong & (A \times B) + (A \times C); \\ A^{B+C} & \cong & A^B \times A^C; \\ (A^B)^C & \cong & A^{B \times C}; \end{array}$$

  and so on, where $\cong$ is isomorphism in the category of sets.
- These isomorphisms hold for all sets, not just finite ones.

The Two-Element Set

The two-element set: Let 2 be the set $1+1$ (a set with two elements!).

- We write the elements of 2 as `true` and `false`.
- Let A be a set.
- Given a subset S of A, we obtain a function $\chi_S : A \to 2$ (the **characteristic function** of $S \subseteq A$), where, for all $a \in A$,

$$\chi_S(a) = \begin{cases} \texttt{true}, & \text{if } a \in S \\ \texttt{false}, & \text{if } a \notin S \end{cases}$$

- Conversely, given a function $f : A \to 2$, we obtain a subset of A,

$$f^{-1}(\{\texttt{true}\}) = \{a \in A : f(a) = \texttt{true}\}.$$

- These two processes are mutually inverse:
  χ_S is the unique function $f : A \to 2$ such that $f^{-1}\{\texttt{true}\} = S$.
- Hence: Subsets of A correspond one-to-one with functions $A \to 2$.

Power Set

- We just saw that:

  Subsets of A correspond one-to-one with functions $A \to 2$.

- We already know that the functions from A to 2 form a set, 2^A.
- When we are thinking of 2^A as the set of all subsets of A, we call it the **power set** of A and write it as $\mathscr{P}(A)$.

Equalizers

- It would be nice if, given a set A, we could define a subset S of A by specifying a property that the elements of S are to satisfy:

$$S = \{a \in A : \text{some property of } a \text{ holds}\}.$$

- It is hard to give a general definition of "property".
- There is, however, a special type of property that is easy to handle: equality of two functions.
- Precisely, given sets and functions $A \underset{g}{\overset{f}{\rightrightarrows}} B$, there is a set

$$\{a \in A : f(a) = g(a)\}.$$

- This set is called the **equalizer** of f and g, since it is the part of A on which the two functions are equal.

Quotients

- Let A be a set and $\sim$ an equivalence relation on A.
- There is a set $A/\sim$, the **quotient of** A by $\sim$, whose elements are the equivalence classes.
- For example, given a group G and a normal subgroup N, define an equivalence relation $\sim$ on G by $g \sim h \Leftrightarrow gh^{-1} \in N$.

  Then $G/\sim = G/N$.
- There is also a **canonical map**

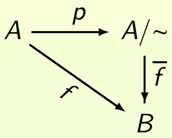
$$p : A \to A/\sim,$$

  sending an element of A to its equivalence class.
  - p is surjective;
  - It has the property

$$p(a) = p(a') \Leftrightarrow a \sim a'.$$

Quotients (Cont'd)

- This map has a universal property:
  Any function $f : A \to B$ such that for all $a, a' \in A$,
  $$a \sim a' \Rightarrow f(a) = f(a')$$
  factorizes uniquely through p, as in the diagram



- Thus, for any set B, the functions $A/\sim \to B$ correspond one-to-one with the functions $f : A \to B$ satisfying the condition, for all $a, a' \in A$,
  $$a \sim a' \Rightarrow f(a) = f(a').$$

Natural Numbers

- A function with domain $\mathbb{N}$ is usually called a **sequence**.
- A crucial property of $\mathbb{N}$ is that sequences can be defined recursively:
- Given a set X, an element $a \in X$, and a function $r : X \to X$, there is a unique sequence $(x_n)_{n=0}^{\infty}$ of elements of X such that

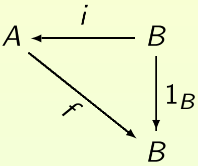
$$x_0 = a, \quad x_{n+1} = r(x_n), \text{ for all } n \in \mathbb{N}.$$

- This property refers to two pieces of structure on $\mathbb{N}$:
  - The element 0;
  - The function $s : \mathbb{N} \to \mathbb{N}$ defined by $s(n) = n + 1$.
- Reformulated in terms of functions, and writing $x_n = x(n)$, the property is this:

  For any set X, element $a \in X$, and function $r : X \to X$, there is a unique function $x : \mathbb{N} \to X$ such that $x(0) = a$ and $x \circ s = r \circ x$.
- This is a universal property of $\mathbb{N}$, 0 and s.

Choice

- Let $f : A \to B$ be a map in a category $\mathscr{A}$.
- A **section** (or **right inverse**) of f is a map $i : B \to A$ in $\mathscr{A}$ such that $f \circ i = 1_B$.



- In the category of sets, any map with a section is certainly surjective.
- The converse statement is called the **axiom of choice**:

Every surjection has a section.

- It is called "choice" because specifying a section of $f : A \to B$ amounts to choosing, for each $b \in B$, an element of $\{a \in A : f(a) = b\} \neq \emptyset$.

Subsection 2

Small and Large Categories

Comparing Cardinalities

- Given sets A and B, write $|A| \leq |B|$ (or $|B| \geq |A|$) if there exists an injection $A \to B$.
- We give no meaning to the expression "$|A|$" or "$|B|$" in isolation.
- In the case of finite sets, $|A| \leq |B|$ just means that the number of elements of A is less than or equal to the number of elements of B.
- Since identity maps are injective, $|A| \leq |A|$, for all sets A.
- Since the composite of two injections is an injection,

$$|A| \leq |B| \leq |C| \quad \Rightarrow \quad |A| \leq |C|.$$

- Also, if $A \cong B$ then $|A| \leq |B| \leq |A|$.

Theorem (Cantor-Bernstein)

Let A and B be sets. If $|A| \leq |B| \leq |A|$, then $A \cong B$.

Comparing Cardinalities (Cont'd)

- These observations tell us that $\leq$ is a preorder on the collection of all sets.
- It is not a genuine order, since $|A| \leq |B| \leq |A|$ only implies that $A \cong B$, not $A = B$.
- We write $|A| = |B|$, and say that A and B **have the same cardinality**, if $A \cong B$, or equivalently if $|A| \leq |B| \leq |A|$.
- As long as we do not confuse equality with isomorphism, the sign $\leq$ behaves as we might imagine.
- For example, write $|A| < |B|$ if $|A| \leq |B|$ and $|A| \neq |B|$.

  Then $|A| \leq |B| < |C| \Rightarrow |A| < |C|$, for sets A, B and C.
- Indeed, we have already established that $|A| \leq |C|$, and the strict inequality follows from the Cantor-Bernstein Theorem.

Cantor's Theorem

- Recall that $\mathscr{P}(A)$ is the power set of A.

Theorem (Cantor)

Let A be a set. Then $|A| < |\mathscr{P}(A)|$.

- The lemma is easy for finite sets, since if A has n elements then $\mathscr{P}(A)$ has 2^n elements, and $n < 2^n$.

Corollary

For every set A, there is a set B such that $|A| < |B|$.

- In other words, there is no biggest set.

Set-Indexed Family of Sets

Proposition

Let I be a set, and let $(A_i)_{i\in I}$ be a family of sets. Then there exists a set not isomorphic to any of the sets A_i.

- Put $A = \mathscr{P}(\sum_{i\in I} A_i)$ the power set of the sum of the sets A_i.

  For each $j \in I$, we have the inclusion function $A_j \to \sum_{i\in I} A_i$.

  So by Cantor's Theorem,

  $$|A_j| \leq |\sum_{i\in I} A_i| < |A|.$$

  Hence $|A_j| < |A|$. In particular, $A_j \ncong A$.

Classes

- We use the word **class** informally to mean any collection of mathematical objects.
- All sets are classes, but some classes (such as the class of all sets) are too big to be sets.
- A class will be called **small** if it is a set, and **large** otherwise.
- For example, the preceding proposition states that the class of isomorphism classes of sets is large.
- The crucial point is:

  Any individual set is small, but the class of sets is large.

- This is even true if we pretend that isomorphic sets are equal.
- Although the "definition" of class is not precise, it will do for our purposes.
- We make a naive distinction between small and large collections, and implicitly use some intuitively plausible principles (for example, that any subcollection of a small collection is small).

Small and Locally Small Categories

- A category $\mathscr{A}$ is **small** if the class or collection of all maps in $\mathscr{A}$ is small, and **large** otherwise.
- If $\mathscr{A}$ is small then the class of objects of $\mathscr{A}$ is small too, since objects correspond one-to-one with identity maps.
- A category $\mathscr{A}$ is **locally small** if for each $A, B \in \mathscr{A}$, the class $\mathscr{A}(A, B)$ is small.
- Clearly, small implies locally small.
- Many authors take local smallness to be part of the definition of category.
- The class $\mathscr{A}(A, B)$ is often called the **hom-set** from A to B, although strictly speaking, we should only call it this when $\mathscr{A}$ is locally small.

Examples

- **Set** is locally small, because for any two sets A and B, the functions from A to B form a set.

  This was one of the properties of sets stated in the previous section.
- $\mathbf{Vect}_k$, **Grp**, **Ab**, **Ring** and **Top** are all locally small.

  For example, given rings A and B, a homomorphism from A to B is a function from A to B with certain properties.

  The collection of all functions from A to B is small.

  So the collection of homomorphisms from A to B is certainly small.

Characterization of Smallness

- A category is small if and only if it is locally small and its class of objects is small.
- Again, it may help to consider a similar fact about finiteness:

  > A category $\mathscr{A}$ is finite (that is, the class of all maps in $\mathscr{A}$ is finite) if and only if it is locally finite (that is, each class $\mathscr{A}(A, B)$ is finite) and its class of objects is finite.

  Example: Consider the category $\mathscr{B}$ whose objects correspond to the natural numbers.

  The objects form a set, so the class of objects of $\mathscr{B}$ is small.

  Each hom-set $\mathscr{B}(m, n)$ is a set (indeed, a finite set).

  So $\mathscr{B}$ is locally small.

  Hence $\mathscr{B}$ is small.

Essential Smallness

- A category is **essentially small** if it is equivalent to some small category.
- For example, the category of finite sets is essentially small since it is equivalent to the small category $\mathscr{B}$ just mentioned.
- If two categories $\mathscr{A}$ and $\mathscr{B}$ are equivalent, the class of isomorphism classes of objects of $\mathscr{A}$ is in bijection with that of $\mathscr{B}$.
- In a small category, the class of objects is small, so the class of isomorphism classes of objects is certainly small.
- Hence in an essentially small category, the class of isomorphism classes of objects is small:

Proposition

Set is not essentially small.

- A previous proposition states that the class of isomorphism classes of sets is large. The result follows.

Example

- For any field k, the category $\mathbf{Vect}_k$ of vector spaces over k is not essentially small.
- As in the proof of the proposition, it is enough to prove that the class of isomorphism classes of vector spaces is large.
- In other words, it is enough to prove that for any set I and family $(V_i)_{i\in I}$ of vector spaces, there exists a vector space not isomorphic to any of the spaces V_i.
- To show this, write $\mathbf{Set} \underset{U}{\overset{F}{\rightleftarrows}} \mathbf{Vect}_k$ for the free and forgetful functors.

  The set $S = \mathscr{P}(\sum_{i\in I} U(V_i))$ has the property that $|U(V_i)| < |S|$ for all $i \in I$. The free vector space $F(S)$ on S contains a copy of S as a basis. So $|S| \leq |UF(S)|$. Hence $|U(V_i)| < |UF(S)|$, for all i. So $F(S) \not\cong V_i$ for all i.
- Similarly, none of the categories **Grp**, **Ab**, **Ring** and **Top** is essentially small.

The Category of Small Categories

- Recall that the category of all categories and functors is written as **CAT**.

Definition

We denote by **Cat** the category of small categories and functors between them.

Example: Monoids are by definition sets equipped with certain structure.

So the one-object categories that they correspond to are small.

Let $\mathscr{M}$ be the full subcategory of **Cat** consisting of the one-object categories.

Then there is an equivalence of categories $\mathbf{Mon} \simeq \mathscr{M}$.

Note that each object of $\mathscr{M}$ is a small one-object category. Hence, the collection of maps from the single object to itself really is a set.