Introduction to Category Theory

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Subsection 1

Definitions and Examples

The Forward Maps Functor

- Fix an object A of a category $\mathscr{A}$.
- We will consider the totality of maps out of A.
- To each $B \in \mathscr{A}$, there is assigned the set (or class) $\mathscr{A}(A, B)$ of maps from A to B.

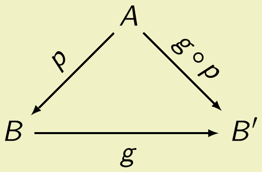
Definition

Let $\mathscr{A}$ be a locally small category and $A \in \mathscr{A}$. We define a functor

$$H^A = \mathscr{A}(A, -) : \mathscr{A} \to \mathbf{Set}$$

as follows:

- For objects $B \in \mathscr{A}$, put $H^A(B) = \mathscr{A}(A, B)$;
- For maps $B \xrightarrow{g} B'$ in $\mathscr{A}$, define $H^A(g) = \mathscr{A}(A, g) : \mathscr{A}(A, B) \to \mathscr{A}(A, B')$ by $p \mapsto g \circ p$, for all $p : A \to B$.



Remarks

(a) Recall that "locally small" means that each class $\mathscr{A}(A, B)$ is in fact a set.

This hypothesis is clearly necessary in order for the definition to make sense.

(b) Sometimes $H^A(g)$ is written as $g \circ -$ or g_*.

All three forms, as well as $\mathscr{A}(A, g)$, are in use.

Representable Functors

Definition

Let $\mathscr{A}$ be a locally small category. A functor $X : \mathscr{A} \to \mathbf{Set}$ is **representable** if

$$X \cong H^A, \text{ for some } A \in \mathscr{A}.$$

A **representation** of X is a choice of:

- An object $A \in \mathscr{A}$;
- An isomorphism between H^A and X.

- Representable functors are sometimes just called "**representables**".
- Only set valued functors (that is, functors with codomain **Set**) can be representable.

Example

- Consider $H^1 : \mathbf{Set} \to \mathbf{Set}$, where 1 is the one-element set.
- Since a map from 1 to a set B amounts to an element of B, we have

$$H^1(B) \cong B, \text{ for each } B \in \mathbf{Set}.$$

- It is easily verified that this isomorphism is natural in B.
- So H^1 is isomorphic to the identity functor $1_{\mathbf{Set}}$.
- Hence $1_{\mathbf{Set}}$ is representable.

Example

- The forgetful functor $\mathbf{Top} \to \mathbf{Set}$ is isomorphic to $H^1 = \mathbf{Top}(1, -)$.
- The forgetful functor $\mathbf{Grp} \to \mathbf{Set}$ is isomorphic to $\mathbf{Grp}(\mathbb{Z}, -)$.
- For each prime p, there is a functor $U_p : \mathbf{Grp} \to \mathbf{Set}$ defined on objects by

  $$U_p(G) = \{\text{elements of } G \text{ of order } 1 \text{ or } p\}.$$

  Then $U_p \cong \mathbf{Grp}(\mathbb{Z}/p\mathbb{Z}, -)$.

  Hence U_p is representable.

Example

- There is a functor $\mathrm{ob} : \mathbf{Cat} \to \mathbf{Set}$ sending a small category to its set of objects.
- It is representable.
- Indeed, consider the terminal category $\mathbf{1}$ (with one object and only the identity map).
- A functor from $\mathbf{1}$ to a category $\mathscr{B}$ simply picks out an object of $\mathscr{B}$.
- Thus,
$$H^{\mathbf{1}}(\mathscr{B}) \cong \mathrm{ob}\mathscr{B}.$$
- Again, it is easily verified that this isomorphism is natural in $\mathscr{B}$.
- Hence $\mathrm{ob} \cong \mathbf{Cat}(\mathbf{1}, -)$.
- It can be shown similarly that the functor $\mathbf{Cat} \to \mathbf{Set}$ sending a small category to its set of maps is representable.

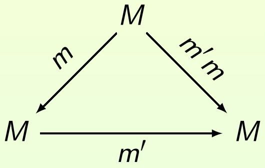
Example

- Let M be a monoid, regarded as a one-object category.
- Recall that a set-valued functor on M is just an M-set.
- Since the category M has only one object, there is only one representable functor on it (up to isomorphism).

$$M^M : M \to \mathbf{Set};$$

- As an M-set, the unique representable is the so-called **left regular representation** of M, that is, the underlying set of M acted on by multiplication on the left.

$$M^M(m') : m \mapsto m'm.$$



Example

- Fix a field k and vector spaces U and V over k.
- There is a functor

$$\mathbf{Bilin}(U, V; -) : \mathbf{Vect}_k \to \mathbf{Set}$$

  whose value $\mathbf{Bilin}(U, V; W)$ at $W \in \mathbf{Vect}_k$ is the set of bilinear maps $U \times V \to W$.
- It can be shown that this functor is representable, in other words, there is a space T with the property that

$$\mathbf{Bilin}(U, V; W) \cong \mathbf{Vect}_k(T, W)$$

  naturally in W.
- This T is the tensor product $U \otimes V$.

Adjunctions and Representables

Lemma

Let $\mathscr{A} \overset{F}{\underset{G}{\rightleftarrows}} \mathscr{B}$, with $F \dashv G$ and $\mathscr{A}, \mathscr{B}$ locally small categories, and let $A \in \mathscr{A}$.

Then the functor $\mathscr{A}(A, G(-)) : \mathscr{B} \to \mathbf{Set}$ (the composite $\mathscr{B} \overset{G}{\to} \mathscr{A} \overset{H^A}{\to} \mathbf{Set}$) is representable.

- We have

$$\mathscr{A}(A, G(B)) \cong \mathscr{B}(F(A), B),$$

  for each $B \in \mathscr{B}$.

  If we can show that this isomorphism is natural in B, then we will have proved that $\mathscr{A}(A, G(-))$ is isomorphic to $H^{F(A)}$ and is therefore representable.

  Let $B \overset{q}{\to} B'$ be a map in $\mathscr{B}$.

Adjunctions and Representables (Cont'd)

- We must show that the following square commutes

$$\begin{array}{ccc} \mathscr{A}(A, G(B)) & \longrightarrow & \mathscr{B}(F(A), B) \\ {\scriptstyle G(q)\circ -}\downarrow & & \downarrow{\scriptstyle q\circ -} \\ \mathscr{A}(A, G(B')) & \longrightarrow & \mathscr{B}(F(A), B') \end{array}$$

where the horizontal arrows are the bijections provided by the adjunction. For $f : A \to G(B)$, we have

$$\begin{array}{ccc} f & \longmapsto & \overline{f} \\ \downarrow & & \downarrow \\ G(q)\circ f & \longmapsto & \frac{q\circ \overline{f}}{G(q)\circ f} \end{array}$$

So we must prove that $q \circ \overline{f} = \overline{G(q)\circ f}$.
This follows immediately from the naturality condition in the definition of adjunction (with $g = \overline{f}$).

Set-Valued Functors with Left Adjoints

Proposition

Any set-valued functor with a left adjoint is representable.

- Let $G : \mathscr{A} \to \mathbf{Set}$ be a functor with a left adjoint F.

  Write 1 for the one-point set.

  Then
$$G(A) \cong \mathbf{Set}(1, G(A))$$
  naturally in $A \in \mathscr{A}$.

  That is, $G \cong \mathbf{Set}(1, G(-))$.

  So by the lemma, G is representable; indeed, $G \cong H^{F(1)}$.

Example

- Several of the examples of representables mentioned previously arise as in the proposition.
- For instance, $U : \mathbf{Top} \to \mathbf{Set}$ has a left adjoint D.

  $D(1) \cong 1$.

  So we recover the result that $U \cong H^1$.
- Similarly, there is a left adjoint D to the objects functor $\mathrm{ob} : \mathbf{Cat} \to \mathbf{Set}$.

  This functor D satisfies $D(1) \cong \mathbf{1}$.

  So $\mathrm{ob} \cong H^1$.

Example

- The forgetful functor $U : \mathbf{Vect}_k \to \mathbf{Set}$ is representable, since it has a left adjoint.
- Indeed, if F denotes the left adjoint, then $F(1)$ is the 1-dimensional vector space k.
- So $U \cong H^k$.
- This is also easy to see directly:

  A map from k to a vector space V is uniquely determined by the image of 1, which can be any element of V.

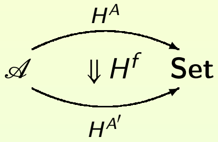
  Hence $\mathbf{Vect}_k(k, V) \cong U(V)$ naturally in V.

Example

- We stated that forgetful functors between categories of algebraic structures usually have left adjoints.
- Take the category **CRing** of commutative rings and the forgetful functor $U : \mathbf{CRing} \to \mathbf{Set}$.
- This general principle suggests that U has a left adjoint.
- Then the proposition tells us that U is representable.
- We see how this works explicitly.
  Given a set S, let $\mathbb{Z}[S]$ be the ring of polynomials over $\mathbb{Z}$ in commuting variables x_s, $s \in S$.
  Then $S \mapsto \mathbb{Z}[S]$ defines a functor $\mathbf{Set} \to \mathbf{CRing}$.
  This is left adjoint to U.
  Hence $U \cong H^{\mathbb{Z}[x]}$.
- Again, this can be verified directly:
  For any ring R, the maps $\mathbb{Z}[x] \to R$ correspond one-to-one with the elements of R.

The Natural Transformation H^f

- The family $(H^A)_{A \in \mathscr{A}}$ of "views" from various objects of a category $\mathscr{A}$ has some consistency, meaning that whenever there is a map between objects A and A', there is also a map between H^A and $H^{A'}$.
- A map $A' \xrightarrow{f} A$ induces a natural transformation



whose B-component (for $B \in \mathscr{A}$) is the function

$$\begin{array}{ccc} H^A(B) = \mathscr{A}(A, B) & \longrightarrow & H^{A'}(B) = \mathscr{A}(A', B) \\ p & \longmapsto & p \circ f. \end{array}$$

- Again, H^f goes by a variety of other names: $\mathscr{A}(f, -)$, f^*, and $- \circ f$.

The Functor $H^\bullet$

- Note that, even though each functor H^A is covariant, they come together to form a contravariant functor, as in the following definition:

Definition

Let $\mathscr{A}$ be a locally small category. The functor

$$H^\bullet : \mathscr{A}^{\text{op}} \to [\mathscr{A}, \mathbf{Set}]$$

is defined:

- On objects A by $H^\bullet(A) = H^A$;
- On maps f by $H^\bullet(f) = H^f$.

- The symbol $\bullet$ is another type of blank, like $-$.

The Functor H_A

Definition

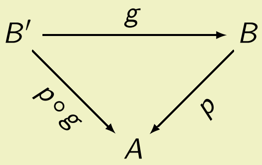
Let $\mathscr{A}$ be a locally small category and $A \in \mathscr{A}$. We define a functor

$$H_A = \mathscr{A}(-, A) : \mathscr{A}^{\text{op}} \to \textbf{Set}$$

as follows:

- For objects $B \in \mathscr{A}$, put $H_A(B) = \mathscr{A}(B, A)$;
- For maps $B' \xrightarrow{g} B$ in $\mathscr{A}$, define

$$H_A(g) = \mathscr{A}(g, A) = g^* = - \circ g : \mathscr{A}(B, A) \to \mathscr{A}(B', A)$$



by $p \mapsto p \circ g$ for all $p : B \to A$.

Representability Revisited

- We now define representability for *contravariant* set-valued functors.
- Strictly speaking, this is unnecessary, as a contravariant functor on $\mathscr{A}$ is a covariant functor on $\mathscr{A}^{\text{op}}$, and we already know what it means for a covariant set-valued functor to be representable.
- Here is a direct definition:

Definition

Let $\mathscr{A}$ be a locally small category. A functor $X : \mathscr{A}^{\text{op}} \to \textbf{Set}$ is **representable** if

$$X \cong H_A, \quad \text{for some } A \in \mathscr{A}.$$

A **representation** of X is a choice of:

- An object $A \in \mathscr{A}$;
- An isomorphism between H_A and X.

Example

- There is a functor
$$\mathscr{P} : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}$$
sending each set B to its power set $\mathscr{P}(B)$, and defined on maps $g : B' \to B$ by
$$(\mathscr{P}(g))(U) = g^{-1}U, \text{ for all } U \in \mathscr{P}(B).$$
- Here $g^{-1}U$ denotes the inverse image or preimage of U under g, defined by $g^{-1}U = \{x' \in B' : g(x') \in U\}$.
- As we saw previously, a subset amounts to a map into the two-point set 2.
- Precisely put, $\mathscr{P} \cong H_2$.

Example

- There is a functor
$$\mathscr{O} : \mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$$
defined on objects B by taking $\mathscr{O}(B)$ to be the set of open subsets of B.
- If S denotes the two-point topological space in which exactly one of the two singleton subsets is open, then continuous maps from a space B into S correspond naturally to open subsets of B.
- Hence $\mathscr{O} \cong H_S$, and $\mathscr{O}$ is representable.

Example

- In a previous example, we defined a functor $C:\mathbf{Top}^{\mathrm{op}}\to\mathbf{Ring}$, assigning to each space the ring of continuous real-valued functions on it.
- The composite functor

$$\mathbf{Top}^{\mathrm{op}}\xrightarrow{C}\mathbf{Ring}\xrightarrow{U}\mathbf{Set}$$

is representable, since by definition, $U(C(X))=\mathbf{Top}(X,\mathbb{R})$ for topological spaces X.

The Functor H_f

- Any map $A \xrightarrow{f} A'$ in $\mathscr{A}$ induces a natural transformation

$$\mathscr{A}^{\mathrm{op}} \underset{H_{A'}}{\overset{H_A}{\rightrightarrows}} \mathbf{Set} \qquad \Downarrow H_f$$

(also called $\mathscr{A}(-,f)$, f_* or $f \circ -$), whose component at an object $B \in \mathscr{A}$ is

$$\begin{array}{rcl} H_A(B) = \mathscr{A}(B,A) & \to & H_{A'}(B) = \mathscr{A}(B,A') \\ p & \mapsto & f \circ p. \end{array}$$

The Yoneda Embedding $H_\bullet$

Definition

Let $\mathscr{A}$ be a locally small category. The **Yoneda embedding** of $\mathscr{A}$ is the functor

$$H_\bullet : \mathscr{A} \to [A^{\text{op}}, \mathbf{Set}]$$

defined

- on objects A by $H_\bullet(A) = H_A$;
- on maps f by $H_\bullet(f) = H_f$.

Summary of Definitions

For each $A \in \mathscr{A}$, we have a functor	$\mathscr{A} \xrightarrow{H^A} \mathbf{Set}$;
Putting them all together gives a functor	$\mathscr{A}^{\mathrm{op}} \xrightarrow{H^\bullet} [\mathscr{A}, \mathbf{Set}]$;
For each $A \in \mathscr{A}$, we have a functor	$\mathscr{A}^{\mathrm{op}} \xrightarrow{H_A} \mathbf{Set}$
Putting them all together gives a functor	$\mathscr{A} \xrightarrow{H_\bullet} [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$.

- The second pair of functors is the dual of the first.
- In the theory of representable functors, it does not make much difference whether we work with the first or the second pair.
- Any theorem that we prove about one dualizes to give a theorem about the other.
- We choose to work with the second pair, the H_A's and $H_\bullet$.
- In a sense to be explained, $H_\bullet$ "embeds" $\mathscr{A}$ into $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$.
- This can be useful, because the category $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ has some good properties that $\mathscr{A}$ might not have.

A Functor Unifying the Dual Pairs

Definition

Let $\mathscr{A}$ be a locally small category. The functor

$$\text{Hom}_{\mathscr{A}} : \mathscr{A}^{\text{op}} \times \mathscr{A} \to \textbf{Set}$$

is defined by

$$\begin{array}{ccc} (A,B) & \mapsto & \mathscr{A}(A,B) \\ f \uparrow \; \downarrow g & \mapsto & \downarrow g \circ - \circ f \\ (A',B') & \mapsto & \mathscr{A}(A',B') \end{array}$$

In other words, $\text{Hom}_{\mathscr{A}}(A,B) = \mathscr{A}(A,B)$ and $(\text{Hom}_{\mathscr{A}}(f,g))(p) = g \circ p \circ f$, whenever

$$A' \xrightarrow{f} A \xrightarrow{p} B \xrightarrow{g} B'.$$

Remark

- We saw that for any set B, there is an adjunction $(- \times B) \dashv (-)^B$ of functors $\mathbf{Set} \to \mathbf{Set}$.
- Similarly, for any category B, there is an adjunction $(- \times B) \dashv [B, -]$ of functors $\mathbf{CAT} \to \mathbf{CAT}$.
- In other words, there is a canonical bijection

$$\mathbf{CAT}(\mathscr{A} \times \mathscr{B}, \mathscr{C}) \cong \mathbf{CAT}(\mathscr{A}, [\mathscr{B}, \mathscr{C}])$$

  for $\mathscr{A}, \mathscr{B}, \mathscr{C} \in \mathbf{CAT}$.
- Under this bijection, the functors

$$\mathrm{Hom}_{\mathscr{A}} : \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \to \mathbf{Set}, \quad H^{\bullet} : \mathscr{A}^{\mathrm{op}} \to [\mathscr{A}, \mathbf{Set}]$$

  correspond to one another.
- Thus, $\mathrm{Hom}_{\mathscr{A}}$ carries the same information as $H^{\bullet}$ (or $H_{\bullet}$), presented slightly differently.

Naturality in Definition of Adjunction (Revisited)

- Take categories and functors $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$.
- They give rise to functors

$$\begin{array}{ccc} \mathscr{A}^{\mathrm{op}} \times \mathscr{B} & \xrightarrow{1 \times G} & \mathscr{A}^{\mathrm{op}} \times \mathscr{A} \\ {\scriptstyle F^{\mathrm{op}} \times 1} \downarrow & & \downarrow {\scriptstyle \mathrm{Hom}_{\mathscr{A}}} \\ \mathscr{B}^{\mathrm{op}} \times \mathscr{B} & \xrightarrow[\mathrm{Hom}_{\mathscr{B}}]{} & \mathbf{Set} \end{array}$$

- The lower path sends (A, B) to $\mathscr{B}(F(A), B)$.
  It can be written as $\mathscr{B}(F(-), -)$.
- The upper path sends (A, B) to $\mathscr{A}(A, G(B))$.
- These two functors

$$\mathscr{B}(F(-), -), \mathscr{A}(-, G(-)) : \mathscr{A}^{\mathrm{op}} \times \mathscr{B} \to \mathbf{Set}$$

  are naturally isomorphic if and only if F and G are adjoint.

Generalized Elements

- Objects of an arbitrary category do not have elements in any obvious sense.
- However, sets certainly have elements, and we have observed that an element of a set A is the same thing as a map $1 \to A$.
- This inspires the following definition.

Definition

Let A be an object of a category. A **generalized element** of A is a map with codomain A. A map $S \to A$ is a generalized element of A of **shape** S.

- "Generalized element" is nothing more than a synonym of "map", but sometimes it is useful to think of maps as generalized elements.

Examples

- When A is a set:
  - A generalized element of A of shape 1 is an ordinary element of A;
  - A generalized element of A of shape $\mathbb{N}$ is a sequence in A.
- In the category of topological spaces:
  - The generalized elements of shape 1 (the one-point space) are the points;
  - The generalized elements of shape S^1 (the circle) are, by definition, loops.

As this suggests, in categories of geometric objects, we might equally well say "figures of shape S".

Examples (Cont'd)

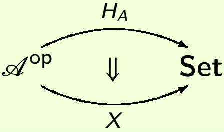
- For an object S of a category A, the functor $H^S : \mathscr{A} \to \mathbf{Set}$ sends an object to its set of generalized elements of shape S.
- The functoriality tells us that any map $A \to B$ in $\mathscr{A}$ transforms S-elements of A into S-elements of B.
- For example, taking $\mathscr{A} = \mathbf{Top}$ and $S = S^1$, any continuous map $A \to B$ transforms loops in A into loops in B.

Subsection 2

The Yoneda Lemma

Posing a Question

- Fix a locally small category $\mathscr{A}$.
- Take an object $A \in \mathscr{A}$ and a functor $X : \mathscr{A}^{\text{op}} \to \mathbf{Set}$.
- The object A gives rise to another functor $H_A = \mathscr{A}(-, A) : \mathscr{A}^{\text{op}} \to \mathbf{Set}$.
- We ask what are the maps $H_A \to X$?
- Since H_A and X are both objects of the presheaf category $[\mathscr{A}^{\text{op}}, \mathbf{Set}]$, the "maps" concerned are maps in $[\mathscr{A}^{\text{op}}, \mathbf{Set}]$.
- So, we are asking what natural transformations



  there are.
- The set of such natural transformations is called $[\mathscr{A}^{\text{op}}, \mathbf{Set}](H_A, X)$.

Content of the Yoneda Lemma

- Given as input an object $A \in \mathscr{A}$ and a presheaf X on $\mathscr{A}$, we can construct the set $[\mathscr{A}^{\text{op}}, \mathbf{Set}](H_A, X)$.
- Another way to construct a set from the same input data (A, X) is to simply take the set $X(A)$!
- The content of the Yoneda Lemma is that these two sets are the same:

$$[\mathscr{A}^{\text{op}}, \mathbf{Set}](H_A, X) \cong X(A),$$

  for all $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{\text{op}}, \mathbf{Set}]$.
- Informally, then, the Yoneda lemma says that for any $A \in \mathscr{A}$ and presheaf X on $\mathscr{A}$:

  A natural transformation $H_A \to X$ is an element of $X(A)$.

The Yoneda Lemma

Theorem (Yoneda)

Let $\mathscr{A}$ be a locally small category. Then

$$[\mathscr{A}^{op}, \mathbf{Set}](H_A, X) \cong X(A)$$

naturally in $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{op}, \mathbf{Set}]$.

- Recall that for functors $F, G : \mathscr{C} \to \mathscr{D}$, the phrase "$F(C) \cong G(C)$ naturally in C" means that there is a natural isomorphism $F \cong G$.
- So the use of this phrase in the Yoneda lemma suggests that each side is functorial in both A and X.
- This means, for instance, that a map $X \to X'$ must induce a map

  $$[\mathscr{A}^{op}, \mathbf{Set}](H_A, X) \to [\mathscr{A}^{op}, \mathbf{Set}](H_A, X'),$$

  and that not only does the Yoneda isomorphism hold for every A and X, but also, the isomorphisms can be chosen in a way that is compatible with these induced maps.

Further Explanations

- The Yoneda lemma states that the composite functor

$$\begin{array}{ccc} \mathscr{A}^{\mathrm{op}} \times [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}] & \overset{H^{\mathrm{op}} \times 1}{\longrightarrow} & [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]^{\mathrm{op}} \times [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}] \\ (A, X) & \longmapsto & (H_A, X) \\ & \overset{\mathrm{Hom}_{[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]}}{\longrightarrow} & \mathbf{Set} \\ & \longmapsto & [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) \end{array}$$

is naturally isomorphic to the evaluation functor

$$\begin{array}{ccc} \mathscr{A}^{\mathrm{op}} \times [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}] & \to & \mathbf{Set} \\ (A, X) & \mapsto & X(A). \end{array}$$

World View Without Yoneda

- If the Yoneda lemma were false then the world would look much more complex.
- Take a presheaf $X : \mathscr{A}^{\text{op}} \to \textbf{Set}$.
- Define a new presheaf X' by

$$X' = [\mathscr{A}^{\text{op}}, \textbf{Set}](H_A, X) : \mathscr{A}^{\text{op}} \to \textbf{Set},$$

  that is, $X'(A) = [\mathscr{A}^{\text{op}}, \textbf{Set}](H_A, X)$ for all $A \in \mathscr{A}$.
- Yoneda tells us that $X'(A) \cong X(A)$ naturally in A.
- In other words, $X' \cong X$.
- If Yoneda were false then starting from a single presheaf X, we could build an infinite sequence $X, X', X'', \ldots$ of new presheaves, potentially all different.
- But in reality, the situation is very simple: they are all the same.

Proof

- We have to define, for each A and X, a bijection between the sets $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X)$ and $X(A)$.
- We then have to show that our bijection is natural in A and X.
- Fix $A \in \mathscr{A}$ and $X \in [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$.
- We define functions

$$[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) \underset{\widetilde{(\ )}}{\overset{\widehat{(\ )}}{\rightleftarrows}} X(A)$$

  and show that they are mutually inverse.
- So we have to do four things:
  - Define the function $\widehat{(\ )}$;
  - Define the function $\widetilde{(\ )}$;
  - Show that $\widetilde{\widehat{(\ )}}$ is the identity;
  - Show that $\widehat{\widetilde{(\ )}}$ is the identity.

Proof (Cont'd)

- Given $\alpha : H_A \to X$, define $\widehat{\alpha} \in X(A)$ by $\widehat{\alpha} = \alpha_A(1_A)$.
- Let $x \in X(A)$.

  We have to define a natural transformation $\widetilde{x} : H_A \to X$.

  That is, we have to define for each $B \in \mathscr{A}$ a function

  $$\widetilde{x}_B : H_A(B) = \mathscr{A}(B, A) \to X(B)$$

  and show that the family $\widetilde{x} = (\widetilde{x}_B)_{B \in \mathscr{A}}$ satisfies naturality.

  Given $B \in \mathscr{A}$ and $f \in \mathscr{A}(B, A)$, define

  $$\widetilde{x}_B(f) = (X(f))(x) \in X(B).$$

  This makes sense, since $X(f)$ is a map $X(A) \to X(B)$.

Proof (Cont'd)

To prove naturality, we must show that for any map $B' \xrightarrow{g} B$ in $\mathscr{A}$, the following square commutes:

$$\begin{array}{ccc}
\mathscr{A}(B,A) & \xrightarrow{H_A(g) = - \circ g} & \mathscr{A}(B',A) \\
\widetilde{x}_B \big\downarrow & & \big\downarrow \widetilde{x}_{B'} \\
X(B) & \xrightarrow[X(g)]{} & X(A)
\end{array}$$

To reduce clutter, let us write $X(g)$ as Xg, and so on.

Now for all $f \in \mathscr{A}(B,A)$, we have

$$\begin{array}{ccc}
f & \longmapsto & f \circ g \\
\big\downarrow & & \big\downarrow \\
(Xf)(x) & \longmapsto & \begin{array}{c}(X(f \circ g))(x) \\ (Xg)((Xf)(x))\end{array}
\end{array}$$

But $X(f \circ g) = (Xg) \circ (Xf)$ by functoriality.

Proof (Cont'd)

- Given $x \in X(A)$, we have to show that $\widehat{\widetilde{x}} = x$:

$$\widehat{\widetilde{x}} = \widetilde{x}_A(1_A) = (X1_A)(x) = 1_{X(A)}(x) = x.$$

- Given $\alpha : H_A \to X$, we have to show that $\widetilde{\widehat{\alpha}} = \alpha$.

  Two natural transformations are equal if and only if all their components are equal.

  So, we have to show that $\widetilde{\widehat{\alpha}}_B = \alpha_B$, for all $B \in \mathscr{A}$.

  Each side is a function from $H_A(B) = \mathscr{A}(B, A)$ to $X(B)$.

  Two functions are equal if and only if they take equal values at every element of the domain.

  So, we have to show that $\widetilde{\widehat{\alpha}}_B(f) = \alpha_B(f)$, for all $B \in \mathscr{A}$ and $f : B \to A$ in $\mathscr{A}$.

Proof (Cont'd)

We have to show that $\widetilde{\widehat{\alpha}}_B(f) = \alpha_B(f)$, for all $B \in \mathscr{A}$ and $f : B \to A$ in $\mathscr{A}$.

The left-hand side is by definition

$$\widetilde{\widehat{\alpha}}_B(f) = (Xf)(\widehat{\alpha}) = (Xf)(\alpha_A(1_A)).$$

So it remains to prove that $(Xf)(\alpha_A(1_A)) = \alpha_B(f)$.

This follows by the naturality of α:

$$\begin{array}{ccc}
\mathscr{A}(A,A) & \xrightarrow{H_A(f) = - \circ f} & \mathscr{A}(B,A) \\
{\scriptstyle \alpha_A}\downarrow & & \downarrow{\scriptstyle \alpha_B} \\
X(A) & \xrightarrow[Xf]{} & X(B)
\end{array}$$

Proof (Cont'd)

- We now show that the bijection is natural in A and X.
- We employ two mildly labor-saving devices.
- First, in principle we have to prove naturality of both $\widehat{(\ )}$ and $\widetilde{(\ )}$.

  However, by a previous lemma, it is enough to prove naturality of just one of them.

  We prove naturality of $\widehat{(\ )}$.
- Second, naturality in two variables simultaneously is equivalent to naturality in each variable separately.

  Thus, $\widehat{(\ )}$ is natural in the pair (A, X) if and only if it is:
  - natural in A for each fixed X and
  - natural in X for each fixed A.
- So, it remains to check these two types of naturality.

Proof (Cont'd)

- Naturality in A states that for each $X \in [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ and $B \xrightarrow{f} A$ in $\mathscr{A}$, the following square commutes

$$\begin{array}{ccc} [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) & \xrightarrow{-\circ H_f} & [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_B, X) \\ \widehat{(\ )} \downarrow & & \downarrow \widehat{(\ )} \\ X(A) & \xrightarrow[Xf]{} & X(B) \end{array}$$

For $\alpha : H_A \to X$, we have

$$\begin{array}{ccc} \alpha & \longmapsto & \alpha \circ H_f \\ \downarrow & & \downarrow \\ \alpha_A(1_A) & \longmapsto & \begin{array}{c} (\alpha \circ H_f)_B(1_B) \\ (Xf)(\alpha_A(1_A)) \end{array} \end{array}$$

So we have to show that $(\alpha \circ H_f)_B(1_B) = (Xf)(\alpha_A(1_A))$.

Proof (Cont'd)

To show that

$$(\alpha \circ H_f)_B(1_B) = (Xf)(\alpha_A(1_A)).$$

We have

$$\begin{array}{rcl}(\alpha \circ H_f)_B(1_B) & = & \alpha_B((H_f)_B(1_B)) \text{ (composition in } [\mathscr{A}^{\text{op}}, \textbf{Set}]) \\ & = & \alpha_B(f \circ 1_B) \text{ (definition of } H_f) \\ & = & \alpha_B(f) \\ & = & (Xf)(\alpha_A(1_A)). \text{ (as shown above)}\end{array}$$

Proof (Cont'd)

- Naturality in X states that for each $A \in \mathscr{A}$ and map

$$\mathscr{A}^{\mathrm{op}} \underset{X'}{\overset{X}{\rightrightarrows}} \mathbf{Set} \quad (\Downarrow \theta)$$

in $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$, the following square commutes:

$$\begin{array}{ccc} [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X) & \xrightarrow{\theta \circ -} & [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}](H_A, X') \\ \widehat{(\ )} \downarrow & & \downarrow \widehat{(\ )} \\ X(A) & \xrightarrow[\theta_A]{} & X'(A) \end{array}$$

Proof (Cont'd)

$$\begin{array}{ccc}
[\mathscr{A}^{\text{op}},\mathbf{Set}](H_A,X) & \xrightarrow{\theta\circ -} & [\mathscr{A}^{\text{op}},\mathbf{Set}](H_A,X') \\
\widehat{(\ )}\downarrow & & \downarrow\widehat{(\ )} \\
X(A) & \xrightarrow[\theta_A]{} & X'(A)
\end{array}$$

For $\alpha : H_A \to X$, we have

$$\begin{array}{ccc}
\alpha & \longmapsto & \theta\circ\alpha \\
\downarrow & & \downarrow \\
\alpha_A(1_A) & \longmapsto & \begin{array}{c}(\theta\circ\alpha)_A(1_A)\\ \theta_A(\alpha_A(1_A))\end{array}
\end{array}$$

Since $(\theta\circ\alpha)_A = \theta_A\circ\alpha_A$ by definition of composition in $[\mathscr{A}^{\text{op}},\mathbf{Set}]$, we are done.

Subsection 3

Consequences of the Yoneda Lemma

Rephrasing of the Yoneda Lemma

Corollary

Let $\mathscr{A}$ be a locally small category and $X : \mathscr{A}^{\text{op}} \to \mathbf{Set}$. Then a representation of X consists of an object $A \in \mathscr{A}$ together with an element $u \in X(A)$ such that:

> For each $B \in \mathscr{A}$ and $x \in X(B)$, there is a unique map $\overline{x} : B \to A$ such that $(X\overline{x})(u) = x$.

- By definition, a representation of X is an object $A \in \mathscr{A}$ together with a natural isomorphism $\alpha : H_A \xrightarrow{\sim} X$.
- The corollary states that such pairs (A, α) are in natural bijection with pairs (A, u) satisfying the displayed condition.

Elements and Universal Elements

- Pairs (B,x) with $B \in \mathscr{A}$ and $x \in X(B)$ are sometimes called **elements** of the presheaf X.
- The Yoneda lemma tells us that x amounts to a generalized element of X of shape H_B.
- An element $u \in X(A)$ satisfying the condition

  For each $B \in \mathscr{A}$ and $x \in X(B)$, there is a unique map $\overline{x} : B \to A$ such that $(X\overline{x})(u) = x$.

  is sometimes called a **universal element** of X.
- So, the corollary says that a representation of a presheaf X amounts to a universal element of X.

Proof of the Corollary

- By the Yoneda lemma, we have only to show that for $A \in \mathscr{A}$ and $u \in X(A)$, the natural transformation $\widetilde{u} : H_A \to X$ is an isomorphism if and only if

  for each $B \in \mathscr{A}$ and $x \in X(B)$, there is a unique map $\overline{x} : B \to A$ such that $(X\overline{x})(u) = x$.

  Now, $\widetilde{u}$ is an isomorphism if and only if for all $B \in \mathscr{A}$, the function

  $$\widetilde{u}_B : H_A(B) = \mathscr{A}(B, A) \to X(B)$$

  is a bijection, if and only if for all $B \in \mathscr{A}$ and $x \in X(B)$, there is a unique $\overline{x} \in \mathscr{A}(B, A)$ such that $\widetilde{u}_B(\overline{x}) = x$.

  But $\widetilde{u}_B(\overline{x}) = (X\overline{x})(u)$.

  So this is exactly the displayed condition.

A Dual Formulation

Corollary

Let $\mathscr{A}$ be a locally small category and $X : \mathscr{A} \to \mathbf{Set}$. Then a representation of X consists of an object $A \in \mathscr{A}$ together with an element $u \in X(A)$ such that:

> For each $B \in \mathscr{A}$ and $x \in X(B)$, there is a unique map $\overline{x} : A \to B$ such that $(X\overline{x})(u) = x$.

- Follows immediately from the corollary by duality.

Example

- Fix a set S and consider the functor

$$X = \mathbf{Set}(S, U(-)) : \begin{array}{ccc} \mathbf{Vect}_k & \to & \mathbf{Set} \\ V & \mapsto & \mathbf{Set}(S, U(V)). \end{array}$$

- Here are two familiar (and true!) statements about X:
  (a) There exist a vector space $F(S)$ and an isomorphism $\mathbf{Vect}_k(F(S), V) \cong \mathbf{Set}(S, U(V))$ natural in $V \in \mathbf{Vect}_k$;
  (b) There exist a vector space $F(S)$ and a function $u : S \to U(F(S))$ such that:

  For each vector space V and function $f : S \to U(V)$, there is a unique linear map $\overline{f} : F(S) \to V$ such that the following commutes:

$$\begin{array}{ccc} S & \xrightarrow{u} & U(F(S)) \\ & {}_{f}\searrow & \downarrow{\scriptstyle U(\overline{f})} \\ & & U(V) \end{array}$$

Example (Cont'd)

- Each of these two statements says that X is representable:
  - Statement (a) says that there is an isomorphism $X(V) \cong \mathbf{Set}(F(S), V)$ natural in V. That is, an isomorphism $X \cong H^{F(S)}$.
    So X is representable, by definition of representability.
  - Statement (b) says that $u \in X(F(S))$ satisfies the condition in the preceding corollary.
    So X is representable, by that corollary.
- The first way of saying that X is representable is substantially shorter than the second.
- Indeed, it is clear that if the situation of (b) holds then there is an isomorphism
$$\mathbf{Vect}_k(F(S), V) \xrightarrow{\sim} \mathbf{Set}(S, U(V))$$
natural in V, defined by $g \mapsto U(g) \circ u$.
- Even though (b) states that the two functors are not only naturally isomorphic, but naturally isomorphic in a rather special way, both are equivalent.

Example

- The same can be said for any other adjunction $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$.
- Fix $A \in \mathscr{A}$ and put
$$X = \mathscr{A}(A, G(-)) : \mathscr{B} \to \mathbf{Set}.$$
- Then X is representable, and this can be expressed in either of the following ways:
  (a) $\mathscr{A}(A, G(B)) \cong \mathscr{B}(F(A), B)$ naturally in B.
  In other words, $X \cong H^{F(A)}$;
  (b) The unit map $\eta_A : A \to G(F(A))$ is an initial object of the comma category $(A \Rightarrow G)$;
  That is, $\eta_A \in X(F(A))$ satisfies

  For each $B \in \mathscr{B}$ and $x \in X(B)$, there is a unique map $\overline{x} : F(A) \to B$ such that $(X\overline{x})(\eta_A) = x$.

Example

- For any group G and element $x \in G$, there is a unique homomorphism $\phi : \mathbb{Z} \to G$ such that $\phi(1) = x$.
- This means that $1 \in U(Z)$ is a universal element of the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$.
- In other words, we have

  For each $B \in \mathbf{Grp}$ and $x \in U(B)$, there is a unique map $\overline{x} : \mathbb{Z} \to B$ such that $(U\overline{x})(1) = x$.
- So $1 \in U(\mathbb{Z})$ gives a representation $H^{\mathbb{Z}} \stackrel{\sim}{\to} U$ of U.
- On the other hand, the same is true with -1 in place of 1.
- The isomorphisms $H^{\mathbb{Z}} \stackrel{\sim}{\to} U$ coming from 1 and -1 are not equal, because the corollary provides a one-to-one correspondence between universal elements and representations.

The Yoneda Embedding

Corollary

For any locally small category $\mathscr{A}$, the Yoneda embedding $H_\bullet : \mathscr{A} \to [\mathscr{A}^{\text{op}}, \mathbf{Set}]$ is full and faithful.

- Informally, this says that for $A; A' \in \mathscr{A}$, a map $H_A \to H_{A'}$ of presheaves is the same thing as a map $A \to A'$ in $\mathscr{A}$.
- We have to show that for each $A, A' \in \mathscr{A}$, the function

$$\begin{array}{ccc} \mathscr{A}(A, A') & \to & [\mathscr{A}^{\text{op}}, \mathbf{Set}](H_A, H_{A'}) \\ f & \mapsto & H_f \end{array}$$

is bijective. By the Yoneda lemma (taking X to be $H_{A'}$), the function $\widetilde{(\ )} : H_{A'}(A) \to [\mathscr{A}^{\text{op}}, \mathbf{Set}](H_A, H_{A'})$ is bijective. So it is enough to prove that these functions are equal.

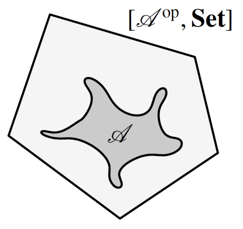
Thus, given $f : A \to A'$, we have to prove that $\widetilde{f} = H_f$, or equivalently, $\widehat{H_f} = f$. Indeed we have $\widehat{H_f} = (H_f)_A(1_A) = f \circ 1_A = f$.

Remarks on Embeddings

- In mathematics, the word "embedding" is used to mean a map $A \to B$ that makes A isomorphic to its image in B.
- For example, an injection of sets $i : A \to B$ might be called an embedding, because it provides a bijection between A and the subset iA of B.
- Similarly, a map $i : A \to B$ of topological spaces might be called an embedding if it is a homeomorphism to its image, so that $A \cong iA$.
- A previous corollary tells us that in category theory, a full and faithful functor $\mathscr{A} \to \mathscr{B}$ can reasonably be called an embedding, as it makes $\mathscr{A}$ equivalent to a full subcategory of $\mathscr{B}$.

Remarks on the Yoneda Embedding

- The Yoneda embedding $H_\bullet : \mathscr{A} \to [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ embeds $\mathscr{A}$ into its own presheaf category.
  So, $\mathscr{A}$ is equivalent to the full subcategory of $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ whose objects are the representables.



- In general, full subcategories are the easiest subcategories to handle.
- For instance, given objects A and A' of a full subcategory, we can speak unambiguously of the "maps" from A to A';
- It makes no difference whether this is understood to mean maps in the subcategory or maps in the whole category.
- Similarly, we can speak unambiguously of isomorphism of objects of the subcategory.

Isomorphisms and Full and Faithful Functors

Lemma

Let $J : \mathscr{A} \to \mathscr{B}$ be a full and faithful functor and $A, A' \in \mathscr{A}$. Then:

(a) A map f in $\mathscr{A}$ is an isomorphism if and only if the map $J(f)$ in $\mathscr{B}$ is an isomorphism;

(b) For any isomorphism $g : J(A) \to J(A')$ in $\mathscr{B}$, there is a unique isomorphism $f : A \to A'$ in $\mathscr{A}$ such that $J(f) = g$;

(c) The objects A and A' of $\mathscr{A}$ are isomorphic if and only if the objects $J(A)$ and $J(A')$ of $\mathscr{B}$ are isomorphic.

Example

- We considered the representations of the forgetful functor $U : \mathbf{Grp} \to \mathbf{Set}$, and found two different isomorphisms $H^{\mathbb{Z}} \xrightarrow{\sim} U$.
- Are there others?
- Since $H^{\mathbb{Z}} \cong U$, there are as many isomorphisms $H^{\mathbb{Z}} \xrightarrow{\sim} U$ as there are isomorphisms $H^{\mathbb{Z}} \xrightarrow{\sim} H^{\mathbb{Z}}$.
- By the preceding corollary and Part (b) of the preceding lemma, there are as many of these as there are group isomorphisms $\mathbb{Z} \xrightarrow{\sim} \mathbb{Z}$.
- There are precisely two such (corresponding to the two generators ± 1 of $\mathbb{Z}$).
- So we did indeed find all the isomorphisms $H^{\mathbb{Z}} \xrightarrow{\sim} U$.
- Differently put, there are exactly two universal elements of $U(\mathbb{Z})$.

Isomorphism of Representables

Corollary

Let $\mathscr{A}$ be a locally small category and $A, A' \in \mathscr{A}$. Then

$$H_A \cong H_{A'} \quad \Leftrightarrow \quad A \cong A' \quad \Leftrightarrow \quad H^A \cong H^{A'}.$$

- By duality, it is enough to prove the first $\Leftrightarrow$.
  This follows from the preceding corollary and Part (c) of the preceding lemma.
- Since functors always preserve isomorphism, the force of this statement is that $H_A \cong H_{A'} \Rightarrow A \cong A'$.
- In other words, if $\mathscr{A}(B, A) \cong \mathscr{A}(B, A')$ naturally in B, then $A \cong A'$.
- Thinking of $\mathscr{A}(B, A)$ as "A viewed from B", the corollary tells us that two objects are the same if and only if they look the same from all viewpoints.

Example

- Consider the case $\mathscr{A} = \mathbf{Grp}$.
- Take two groups A and A', and suppose someone tells us that A and A' "look the same from B" (meaning that $H_A(B) \cong H_{A'}(B)$) for all groups B. Then, for instance:
  - $H_A(1) \cong H_{A'}(1)$, where 1 is the trivial group.
    But $H_A(1) = \mathbf{Grp}(1, A)$ is a one-element set, as is $H_{A'}(1)$, no matter what A and A' are.
    So this tells us nothing at all.
  - $H_A(\mathbb{Z}) \cong H_{A'}(\mathbb{Z})$.
    We know that $H_A(\mathbb{Z})$ is the underlying set of A, and similarly for A'.
    So A and A' have isomorphic underlying sets.
  - $H_A(\mathbb{Z}/p\mathbb{Z}) \cong H_{A'}(\mathbb{Z}/p\mathbb{Z})$ for every prime p.
    So A and A' have the same number of elements of each prime order.
- Each of these isomorphisms gives only partial information about the similarity of A and A'.
- But if we know that $H_A(B) \cong H_{A'}(B)$ for all groups B, and naturally in B, then $A \cong A'$.

Example

- For any set A, we have

$$A \cong \mathbf{Set}(1, A) = H_A(1).$$

- So $H_A(1) \cong H_{A'}(1)$ implies $A \cong A'$.
- In other words, two objects of **Set** are the same if they look the same from the point of view of the one-element set.
- This is a familiar feature of sets: the only thing that matters about a set is its elements!
- For a general category, the preceding corollary tells us that two objects are the same if they have the same generalized elements of all shapes.
- But the category of sets has a special property:
- If we choose an object and we know only what its generalized elements of shape 1 are, then we can deduce exactly what the object must be.

Example

- Let $G : \mathscr{B} \to \mathscr{A}$ be a functor.
- Suppose that both F and F' are left adjoint to G.
- Then for each $A \in \mathscr{A}$, we have

$$\mathscr{B}(F(A), B) \cong \mathscr{A}(A, G(B)) \cong \mathscr{B}(F'(A), B)$$

  naturally in $B \in \mathscr{B}$.
- So $H^{F(A)} \cong H^{F'(A)}$.
- So $F(A) \cong F'(A)$ by the corollary.
- In fact, this isomorphism is natural in A, so that $F \cong F'$.
- This shows that left adjoints are unique.
- Dually, right adjoints are unique.

Example

- The corollary implies that if a set-valued functor is isomorphic to both H^A and $H^{A'}$ then $A \cong A'$.
- So the functor determines the representing object, if one exists.
- For instance, take the functor
$$\mathbf{Bilin}(U, V; -) : \mathbf{Vect}_k \to \mathbf{Set}.$$
- The corollary implies that up to isomorphism, there is at most one vector space T such that
$$\mathbf{Bilin}(U, V; W) \cong \mathbf{Vect}_k(T, W)$$
naturally in W.
- It can be shown that there does, in fact, exist such a vector space T.
- Since all such spaces T are isomorphic, it is legitimate to refer to any of them as *the* tensor product of U and V.