Introduction to Category Theory

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

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1. Limits
   - Limits: Definition and Examples
   - Colimits: Definition and Examples
   - Interactions Between Functors and Limits

Subsection 1

Limits: Definition and Examples

Products in Sets

- Let X and Y be sets.
- The familiar cartesian product $X \times Y$ is characterized by the property that an element of $X \times Y$ is an element of X together with an element of Y.
- Since elements are just maps from 1, this says that a map $1 \to X \times Y$ amounts to a pair of maps $(1 \to X, 1 \to Y)$.
- A little thought reveals that the same is true when 1 is replaced throughout by any set A whatsoever:
  A generalized element of $X \times Y$ of shape A amounts to a generalized element of X of shape A together with a generalized element of Y of shape A.
- The bijection between maps $A \to X \times Y$ and pairs of maps $(A \to X, A \to Y)$ is given by composing with the projection maps

$$\begin{array}{ccccc} X & \xleftarrow{p_1} & X \times Y & \xrightarrow{p_2} & Y \\ x & \longmapsfrom & (x,y) & \longmapsto & y \end{array}$$

Products

Definition

Let $\mathscr{A}$ be a category and $X, Y \in \mathscr{A}$. A **product** of X and Y consists of an object P and maps

$$X \xleftarrow{p_1} P \xrightarrow{p_2} Y$$

with the property that for all objects and maps $X \xleftarrow{f_1} A \xrightarrow{f_2} Y$ in $\mathscr{A}$, there exists a unique map $\overline{f} : A \to P$, such that the following diagram commutes:

$$\begin{tikzcd} & A \arrow[dl, "f_1"'] \arrow[d, "\overline{f}"] \arrow[dr, "f_2"] & \\ X & P \arrow[l, "p_1"] \arrow[r, "p_2"'] & Y \end{tikzcd}$$

The maps p_1 and p_2 are called the **projections**.

On Existence, Uniqueness and Terminology

- Products do not always exist.

  For example, if $\mathscr{A}$ is the discrete two-object category

  $$X \bullet \qquad \bullet Y$$

  then X and Y do not have a product.
- When objects X and Y of a category do have a product, it is unique up to isomorphism.

  This justifies talking about *the* product of X and Y.
- Strictly speaking, the product consists of the object P together with the projections p_1 and p_2.

  But informally, we often refer to P alone as the product of X and Y.
- We write P as $X \times Y$.

Example

- Any two sets X and Y have a product in **Set**.
- It is the usual cartesian product $X \times Y$, equipped with the usual projection maps p_1 and p_2.
- Take sets and functions $X \xleftarrow{f_1} A \xrightarrow{f_2} Y$. Define $\overline{f} : A \to X \times Y$ by

$$\overline{f}(a) = (f_1(a), f_2(a)).$$

  Then $p_i \circ \overline{f} = f_i$ for $i = 1, 2$, i.e., the required diagram commutes.

  Moreover, this is the only map making that diagram commute:

  Suppose that $\widehat{f} : A \to X \times Y$, in place of $\overline{f}$, also makes the diagram commute. Let $a \in A$, and write $\widehat{f}(a)$ as (x, y). Then $f_1(a) = p_1(\widehat{f}(a)) = p_1(x, y) = x$. Similarly, $f_2(a) = y$.
  Hence, for all $a \in A$, $\widehat{f}(a) = (f_1(a), f_2(a)) = \overline{f}(a)$. So, $\widehat{f} = \overline{f}$.
- In general, in any category, the map $\overline{f}$ is usually written as (f_1, f_2).

Example

- In the category of topological spaces, any two objects X and Y have a product.
- It is the set $X \times Y$ equipped with the product topology and the standard projection maps.
- The product topology is deliberately designed so that a function

$$\begin{array}{rcl} A & \to & X \times Y \\ t & \mapsto & (x(t), y(t)) \end{array}$$

  is continuous if and only if it is continuous in each coordinate (that is to say, both functions $t \mapsto x(t)$, $t \mapsto y(t)$ are continuous.

Example (Cont'd)

- A closely related statement is that the product topology is the smallest topology on $X \times Y$ for which the projections are continuous.
- Here "smallest" means that for any other topology $\mathscr{T}$ on $X \times Y$ such that p_1 and p_2 are continuous, every subset of $X \times Y$ open in the product topology is also open in $\mathscr{T}$.
- Thus, to define the product topology, we declare just enough sets to be open that the projections are continuous.

Example

- Let X and Y be vector spaces.
- We can form their direct sum, $X \oplus Y$, whose elements can be written as either (x,y) or $x+y$ (with $x \in X$ and $y \in Y$), according to taste.
- There are linear projection maps

$$\begin{tikzcd} & X \oplus Y \arrow[dl, "p_1"'] \arrow[dr, "p_2"] & \\ X & & Y \end{tikzcd} \qquad \begin{tikzcd} & (x,y) \arrow[dl, mapsto] \arrow[dr, mapsto] & \\ x & & y \end{tikzcd}$$

- It can be shown that $X \oplus Y$, together with p_1 and p_2, is the product of X and Y in the category of vector spaces.

The Reals as an Ordered Set

- Let $x, y \in \mathbb{R}$.
- Their minimum $\min\{x,y\}$ satisfies

$$\min\{x,y\} \leq x, \quad \min\{x,y\} \leq y.$$

- It has the further property that whenever $a \in \mathbb{R}$ with

$$a \leq x, \quad a \leq y,$$

  we have $a \leq \min\{x,y\}$.
- This means exactly that when the poset $(\mathbb{R}, \leq)$ is viewed as a category, the product of $x, y \in \mathbb{R}$ is $\min\{x,y\}$.
- The definition of product simplifies when interpreted in a poset, since all diagrams commute.

Power Sets as Ordered Sets

- Fix a set S.
- Let $X, Y \in \mathscr{P}(S)$.
- Then $X \cap Y$ satisfies
$$X \cap Y \subseteq X, \quad X \cap Y \subseteq Y.$$
- It has the further property that whenever $A \in \mathscr{P}(S)$ with
$$A \subseteq X, \quad A \subseteq Y,$$
we have $A \subseteq X \cap Y$.
- This means that $X \cap Y$ is the product of X and Y in the poset $(\mathscr{P}(S), \subseteq)$ regarded as a category.

Natural Numbers with Divisibility as Ordered Sets

- Let $x, y \in \mathbb{N}$.
- Their greatest common divisor $\gcd(x, y)$ satisfies

$$\gcd(x, y) \mid x, \quad \gcd(x, y) \mid y.$$

- It has the further property that whenever $a \in \mathbb{N}$ with

$$a \mid x, \quad a \mid y,$$

  we have $a \mid \gcd(x, y)$.
- This means that $\gcd(x, y)$ is the product of x and y in the poset $(\mathbb{N}, \mid)$ regarded as a category.

Products in Partially Ordered Sets

- Let $(A, \leq)$ be a poset and $x, y \in A$.
- A **lower bound** for x and y is an element $a \in A$ such that $a \leq x$ and $a \leq y$.
- A **greatest lower bound** or **meet** of x and y is a lower bound z for x and y with the further property that whenever a is a lower bound for x and y, we have $a \leq z$.
- When a poset is regarded as a category, meets are exactly products.
- They do not always exist, but when they do, they are unique.
- The meet of x and y is usually written as $x \wedge y$ rather than $x \times y$.
- Thus, in the three examples above,

$$x \wedge y = \min\{x, y\}, \quad X \wedge Y = X \cap Y, \quad x \wedge y = \gcd(x, y),$$

the second example being the origin of the notation.

General Products

- We have been discussing products $X \times Y$ of two objects, so-called **binary products**.
- We can talk about products of an arbitrary number of objects.

Definition

Let $\mathscr{A}$ be a category, I a set, and $(X_i)_{i \in I}$ a family of objects of $\mathscr{A}$. A **product** of $(X_i)_{i \in I}$ consists of an object P and a family of maps $(P \xrightarrow{p_i} X_i)_{i \in I}$ with the property that for all objects A and families of maps $(A \xrightarrow{f_i} X_i)_{i \in I}$ there exists a unique map $\overline{f} : A \to P$ such that $p_i \circ \overline{f} = f_i$ for all $i \in I$.

$$\begin{tikzcd} A \arrow[d, "\overline{f}"'] \arrow[dr, "f_i"] & \\ P \arrow[r, "p_i"'] & X_i \end{tikzcd}$$

- Products do not always exist but, when the product P exists, we write P as $\prod_{i \in I} X_i$ and the map $\overline{f}$ as $(f_i)_{i \in I}$.
- We call the maps f_i the **components** of the map $(f_i)_{i \in I}$.
- With I a two-element set, we recover binary products.

Example

- In ordered sets, the extension from binary to arbitrary products works in the obvious way:
- Given an ordered set $(A, \leq)$, a **lower bound** for a family $(x_i)_{i \in I}$ of elements is an element $a \in A$ such that $a \leq x_i$ for all i.
- A **greatest lower bound** or **meet** of the family is a lower bound greater than any other, written as $\bigwedge_{i \in I} x_i$.
- These are the products in $(A, \leq)$.
- For example, in $\mathbb{R}$ with its usual ordering, the meet of a family $(x_i)_{i \in I}$ is $\inf \{x_i : i \in I\}$ (and one exists if and only if the other does).

The Case of Empty Index Set

- Let $\mathscr{A}$ be a category.
- In general, an I-indexed family $(X_i)_{i\in I}$ of objects of $\mathscr{A}$ is a function $I \to \mathrm{ob}(\mathscr{A})$.
- When I is empty, there is exactly one such function, i.e., there is exactly one family $(X_i)_{i\in\emptyset}$, the empty family.
- Similarly, when I is empty, there is exactly one family $(A \xrightarrow{f_i} X_i)_{i\in\emptyset}$ for any given object A.
- A product of the empty family therefore consists of an object P of $\mathscr{A}$ such that for each object A of $\mathscr{A}$, there exists a unique map $\overline{f} : A \to P$ (the condition $p_i \circ \overline{f} = f_i$ for all $i \in I$ holds trivially).
- In other words, a product of the empty family is exactly a terminal object.

The Case of Empty Index Set (Cont'd)

- We have been writing 1 for terminal objects, which was justified by the fact that in categories such as **Set**, **Top**, **Ring** and **Grp**, the terminal object has one element.
- But we have just seen that the terminal object is the product of no things, which in the context of elementary arithmetic is the number 1.
- This is a second, related, reason for the notation.

Powers

- Take an object X of a category $\mathscr{A}$, and a set I.
- There is a constant family $(X)_{i \in I}$.
- Its product $\prod_{i \in I} X$, if it exists, is written as X^I and called a **power** of X.
- We met powers in **Set**:

  When X is a set, X^I is the set of functions from I to X, also written as $\mathbf{Set}(I, X)$.

Equalizers

- A **fork** in a category consists of objects and maps $A \xrightarrow{f} X \underset{t}{\overset{s}{\rightrightarrows}} Y$, such that $sf = tf$.

Definition

Let $\mathscr{A}$ be a category and let $X \underset{t}{\overset{s}{\rightrightarrows}} Y$ be objects and maps in $\mathscr{A}$. An **equalizer** of s and t is an object E together with a map $E \xrightarrow{i} X$ such that

$$\begin{tikzcd} E \arrow[r, "i"] & X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y \end{tikzcd}$$

is a fork, and with the property that for any fork $A \xrightarrow{f} X \underset{t}{\overset{s}{\rightrightarrows}} Y$, there exists a unique map $\overline{f} : A \to E$ such that the following triangle commutes:

$$\begin{tikzcd} A \arrow[d, "\overline{f}"'] \arrow[dr, "f"] & \\ E \arrow[r, "i"'] & X \end{tikzcd}$$

Example

- Equalizers in **Set**, as defined previously, are equalizers in the sense of the general definition.
- Indeed, take sets and functions $X \overset{s}{\underset{t}{\rightrightarrows}} Y$.
- Write
$$E = \{x \in X : s(x) = t(x)\}.$$
- Write $i : E \to X$ for the inclusion.
- Then $si = ti$, so we have a fork.
- We can check that it is universal among all forks on s and t.

Example (Cont'd)

- Suppose $A \xrightarrow{f} X \overset{s}{\underset{t}{\rightrightarrows}} Y$ is another fork in **Set**.

$$
\begin{tikzcd}
A \arrow[d, dotted, "\overline{f}"'] \arrow[dr, "f"] & & \\
E \arrow[r, "i"'] & X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y
\end{tikzcd}
$$

Define, for all $a \in A$,

$$\overline{f}(a) = f(a).$$

This makes sense, because, as $s(f(a)) = t(f(a))$, $f(a) \in E$.
Moreover, for all $a \in A$, $i(\overline{f}(a)) = i(f(a)) = f(a)$.
Thus, the requisite triangle commutes.
Suppose that $\widehat{f} : A \to E$ also made the triangle commute.
Then, we would have $i\widehat{f} = f = i\overline{f}$. As i is injective, we get that $\widehat{f} = \overline{f}$.
Thus, the function $\overline{f}$ is unique.

Combining Equalizers

- An equalizer describes the set of solutions of a single equation.
- By combining equalizers with products, we can also describe the solution-set of any system of simultaneous equations.
- Take a set Λ and a family $(X \overset{s_\lambda}{\underset{t_\lambda}{\rightrightarrows}} Y_\lambda)_{\lambda \in \Lambda}$ of pairs of maps in **Set**.
- Then the solution-set

$$\{x \in X : s_\lambda(x) = t_\lambda(x), \text{ for all } \lambda \in \Lambda\}$$

is the equalizer of the functions $X \overset{(s_\lambda)_{\lambda \in \Lambda}}{\underset{(t_\lambda)_{\lambda \in \Lambda}}{\rightrightarrows}} \prod_{\lambda \in \Lambda} X_\lambda$.

- To see this, observe that for $x \in X$,

$$\begin{aligned} (s_\lambda)_{\lambda \in \Lambda}(x) = (t_\lambda)_{\lambda \in \Lambda}(x) \quad &\Leftrightarrow \quad (s_\lambda(x))_{\lambda \in \Lambda} = (t_\lambda(x))_{\lambda \in \Lambda} \\ &\Leftrightarrow \quad s_\lambda(x) = t_\lambda(x), \text{ for all } \lambda \in \Lambda. \end{aligned}$$

Example

- Take continuous maps $X \overset{s}{\underset{t}{\rightrightarrows}} Y$ between topological spaces.
- We can form their equalizer E in the category of sets, with inclusion map $i : E \to X$, say.
- Since E is a subset of the space X, it acquires the subspace topology from X, and i is then continuous.
- This space E, together with i, is the equalizer of s and t.
- Showing this amounts to showing that for any fork $A \xrightarrow{f} X \overset{s}{\underset{t}{\rightrightarrows}} Y$ in **Top**, the induced function $\overline{f}$ is continuous.
- This follows from the definition of the subspace topology, which is the smallest topology such that the inclusion map is continuous.

Example: Kernels in **Grp**

- Let $\theta : G \to H$ be a homomorphism of groups.
- The homomorphism θ gives rise to a fork

$$\ker\theta \overset{\iota}{\hookrightarrow} G \underset{\varepsilon}{\overset{\theta}{\rightrightarrows}} H,$$

where ι is the inclusion and ε is the trivial homomorphism.

- This is an equalizer in **Grp**.
- Showing this amounts to showing that the map that we have been calling $\overline{f}$ is a homomorphism.

However, this follows from the fact that it value is identical with the value of f, as we saw previously.

- Thus, kernels are a special case of equalizers.

Example

- Let $V \overset{s}{\underset{t}{\rightrightarrows}} W$ be linear maps between vector spaces.
- There is a linear map $t - s : V \to W$, and the equalizer of s and t in the category of vector spaces is the space $\ker(t-s)$ together with the inclusion map $\ker(t-s) \hookrightarrow V$.

Pullbacks

Definition

Let $\mathscr{A}$ be a category and take objects and maps in $\mathscr{A}$:

$$\begin{tikzcd} & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

A **pullback** of this diagram is an object $P \in \mathscr{A}$ together with maps $p_1 : P \to X$ and $p_2 : P \to Y$, such that the following commutes

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

and...

Pullbacks (Cont'd)

Definition (Cont'd)

... with the property that for any commutative square

$$\begin{tikzcd} A \arrow[r, "f_2"] \arrow[d, "f_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

in $\mathscr{A}$, there is a unique map $\overline{f} : A \to P$, such that the following commutes:

$$\begin{tikzcd} A \arrow[dr, "\overline{f}"] \arrow[drr, bend left, "f_2"] \arrow[ddr, bend right, "f_1"'] & & \\ & P \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \arrow[d, "t"] \\ & X \arrow[r, "s"'] & Z \end{tikzcd}$$

Terminology

- We call the following diagram a **pullback square**:

$$\begin{tikzcd} P \arrow[r, "p_2"] \arrow[d, "p_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

- Another name for pullback is **fibered product**.
- This name is partially explained by the following fact:
  When Z is a terminal object (and s and t are the only maps they can possibly be), a pullback of the diagram

$$\begin{tikzcd} & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

  is simply a product of X and Y.

Example

- The pullback of a diagram

$$\begin{tikzcd} & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

in **Set** is

$$P = \{(x, y) \in X \times Y : s(x) = t(y)\}$$

with projections p_1 and p_2 given by

$$p_1(x, y) = x \quad \text{and} \quad p_2(x, y) = y.$$

Example

- A basic construction with sets and functions is the formation of inverse images.
- They are an instance of pullbacks.
- Indeed, given a function $f : X \to Y$ and a subset $Y' \subseteq Y$, we obtain a new set, the inverse image

$$f^{-1}Y' = \{x \in X : f(x) \in Y'\} \subseteq X,$$

and a new function,

$$\begin{array}{rccc} f' : & f^{-1}Y' & \to & Y' \\ & x & \mapsto & f(x) \end{array}$$

Example (Cont'd)

- We also have the inclusion functions $j : Y' \hookrightarrow Y$ and $i : f^{-1}Y' \hookrightarrow X$.
- Putting everything together gives a commutative square

$$\begin{tikzcd} f^{-1}Y' \arrow[r, "f'"] \arrow[d, hook, "i"'] & Y' \arrow[d, hook, "j"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

- The data we started with was the lower-right part of this square (X, Y, Y', f and j), and from it we constructed the rest of the square ($f^{-1}Y'$, f' and i).
- This square is a pullback which we verify next.

Example (Cont'd)

- Take any commutative square as on the outside of the following

$$\begin{tikzcd}
A \arrow[dr, "k"] \arrow[drr, "h"] \arrow[ddr, "g"'] & & \\
& f^{-1}Y' \arrow[r, "f'"'] \arrow[d, hook, "i"] & Y' \arrow[d, hook, "j"] \\
& X \arrow[r, "f"'] & Y
\end{tikzcd}$$

- We must show that there is a unique map $k : A \to f^{-1}Y'$, as shown, such that the diagram commutes.

Example (Cont'd)

- For uniqueness, let k be a map making the diagram commute.
  Then for all $a \in A$, we have $i(k(a)) = g(a)$.
  So $k(a) = g(a)$, and this determines k uniquely.
- For existence, note that, for all $a \in A$, $f(g(a)) = j(h(a)) \in Y'$.
  So $g(a) \in f^{-1}Y'$.
  Hence we may define $k : A \to f^{-1}Y'$ by setting, for all $a \in A$,
  $$k(a) = g(a).$$
  Then for all $a \in A$, we have $i(k(a)) = k(a) = g(a)$.
  Also $f'(k(a)) = f(k(a)) = f(g(a)) = j(h(a)) = h(a)$.
  Hence $i \circ k = g$ and $f' \circ k = h$, as required.

Example

- Intersection of subsets provides another example of pullbacks.
- Indeed, let X and Y be subsets of a set Z.
- Then

$$\begin{tikzcd} X \cap Y \arrow[r, hook] \arrow[d, hook] & Y \arrow[d, hook] \\ X \arrow[r, hook] & Z \end{tikzcd}$$

  is a pullback square, where all the arrows are inclusions of subsets.
- In fact, this is a special case of the inverse image construction, since $X \cap Y$ is the inverse image of $Y \subseteq Z$ under the inclusion map $X \hookrightarrow Z$.
- In the situation of inverse images, where we have a map $f : X \to Y$ and a subset Y' of Y, people sometimes say that $f^{-1}Y'$ is obtained by "pulling Y' back" along f: hence the name.

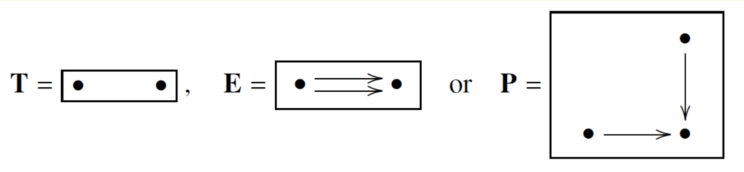
Diagrams

- We use typeface **A**, **B**, **C**, ... to denote small categories, and typeface $\mathscr{A}, \mathscr{B}, \mathscr{C}, \ldots$ to denote arbitrary categories.

Definition

Let $\mathscr{A}$ be a category and **I** a small category. A functor $\mathbf{I} \to \mathscr{A}$ is called a **diagram in $\mathscr{A}$ of shape I**.

- The following are the categories **T**, **E**, **P**, respectively:



Diagrams (Cont'd)

- The following are diagrams of shape **T**, **E**, **P** in a category $\mathscr{A}$:

$$X \qquad Y \qquad\qquad X \overset{s}{\underset{t}{\rightrightarrows}} Y \qquad\qquad \begin{array}{ccc} & & Y \\ & & \downarrow t \\ X & \xrightarrow[s]{} & Z \end{array}$$

- We already have the definitions of product of a diagram of shape **T**, equalizer of a diagram of shape **E**, and pullback of a diagram of shape **P**.

Cones and Limits

Definition

Let $\mathscr{A}$ be a category, $\mathbf{I}$ a small category, and $D : \mathbf{I} \to \mathscr{A}$ a diagram in $\mathscr{A}$.

(a) A **cone** on D is an object $A \in \mathscr{A}$ (the **vertex** of the cone) together with a family

$$(A \xrightarrow{f_I} D(I))_{I \in \mathbf{I}}$$

of maps in $\mathscr{A}$ such that for all maps $I \xrightarrow{u} J$ in $\mathbf{I}$, the triangle commutes:

$$\begin{tikzcd} & A \arrow[dl, "f_I"'] \arrow[dr, "f_J"] & \\ D(I) \arrow[rr, "Du"'] & & D(J) \end{tikzcd}$$

(b) A **limit** of D is a cone $(L \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ with the property that for any cone $(A \xrightarrow{f_I} D(I))_{I \in \mathbf{I}}$ on D, there exists a unique map $\overline{f} : A \to L$ such that $p_I \circ \overline{f} = f_I$ for all $I \in \mathbf{I}$. The maps p_I are called the **projections**.

Remarks

(a) Loosely, the universal property says that for any $A \in \mathscr{A}$, maps $A \to L$ correspond one-to-one with cones on D with vertex A.

Any map $g : A \to L$ gives rise to a cone $(A \stackrel{p_I g}{\to} D(I))_{I \in \mathbf{I}}$, and the definition of limit is that for each A, this process is bijective.

We will use this thought to rephrase the definition of limit in terms of representability.

From this it will follow that limits are unique up to canonical isomorphism, when they exist.

Alternatively, uniqueness can be proved by the usual kind of direct argument.

Remarks (Cont'd)

(b) If $(L \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ is a limit of D, we sometimes abuse language slightly by referring to L (rather than the whole cone) as the limit of D.

For emphasis, we sometimes call $(L \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ a **limit cone**.

We write $L = \lim\limits_{\leftarrow \mathbf{I}} D$.

Remark (a) can then be stated as:

$$\text{A map into } \lim_{\leftarrow \mathbf{I}} D \text{ is a cone on } D.$$

(c) By assuming from the outset that the shape category $\mathbf{I}$ is small, we are restricting ourselves to what are officially called small **limits**.

Example

- A diagram D of shape $\mathbf{T}$ in a category $\mathscr{A}$ is a pair (X, Y) of objects of $\mathscr{A}$.

  A cone on D is an object A together with maps $f_1 : A \to X$ and $f_2 : A \to Y$.

  A limit of D is a product of X and Y.
- More generally, let I be a set and write $\mathbf{I}$ for the discrete category on I.

  A functor $D : \mathbf{I} \to \mathscr{A}$ is an I-indexed family $(X_i)_{i \in I}$ of objects of $\mathscr{A}$.

  A limit of D is exactly a product of the family $(X_i)_{i \in I}$.
- In particular, a limit of the unique functor $\emptyset \to \mathscr{A}$ is a terminal object of $\mathscr{A}$, where $\emptyset$ denotes the empty category.

Example

- A diagram D of shape $\mathbf{E}$ in a category $\mathscr{A}$ is a parallel pair $X \overset{s}{\underset{t}{\rightrightarrows}} Y$ of maps in $\mathscr{A}$.
- A cone on D consists of objects and maps

$$\begin{tikzcd} & A \arrow[dl, "f"'] \arrow[dr, "g"] & \\ X \arrow[rr, shift left, "s"] \arrow[rr, shift right, "t"'] & & Y \end{tikzcd}$$

  such that $s \circ f = g$ and $t \circ f = g$.
- But since g is determined by f, it is equivalent to say that a cone on D consists of an object A and a map $f : A \to X$ such that $A \xrightarrow{f} X \overset{s}{\underset{t}{\rightrightarrows}} Y$ is a fork.
- A limit of D is a universal fork on s and t, that is, an equalizer of s and t.

Example

- A diagram D of shape $\mathbf{P}$ in a category $\mathscr{A}$ consists of objects and maps

$$\begin{tikzcd} & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

- Performing a simplification, we see that a cone on D is a commutative square

$$\begin{tikzcd} A \arrow[r, "f_2"] \arrow[d, "f_1"'] & Y \arrow[d, "t"] \\ X \arrow[r, "s"'] & Z \end{tikzcd}$$

- A limit of D is a pullback.

Example

- Let $\mathbf{I} = (\mathbb{N}, \leq)^{\mathrm{op}}$.
- A diagram $D : \mathbf{I} \to \mathscr{A}$ consists of objects and maps

$$\cdots \xrightarrow{s_3} X_2 \xrightarrow{s_2} X_1 \xrightarrow{s_1} X_0.$$

- For example, suppose that we have a set X_0 and a chain of subsets

$$\cdots \subseteq X_2 \subseteq X_1 \subseteq X_0.$$

- The inclusion maps form a diagram in **Set** of the type above.
- Its limit is $\bigcap_{i \in \mathbb{N}} X_i$.
- In this and similar contexts, limits are sometimes referred to as **inverse limits**, although this usage may be regarded as old-fashioned.

Example

- Let $D : \mathbf{I} \to \mathbf{Set}$ and, as a kind of thought experiment, let us ask ourselves what $\varprojlim_{\mathbf{I}} D$ would have to be if it existed.
- We would have

$$\begin{array}{rcl} \varprojlim_{\mathbf{I}} D & \cong & \mathbf{Set}(1, \varprojlim_{\mathbf{I}} D) \\ & \cong & \{\text{cones on } D \text{ with vertex } 1\} \\ & \cong & \{(x_I)_{I \in \mathbf{I}} : x_I \in D(I) \text{ for all } I \in \mathbf{I} \text{ and} \\ & & (Du)(x_I) = x_J \text{ for all } I \xrightarrow{u} J \text{ in } \mathbf{I}\}, \end{array}$$

  where the second isomorphism is due to the limit property and the third is by definition of cone.
- In fact, this equation really is the limit of D in **Set**, with projections $p_J : \varprojlim_{\mathbf{I}} D \to D(J)$ given by $p_J((x_I)_{I \in \mathbf{I}}) = x_J$.
- So in **Set**, all limits exist.

Example

- The same formula gives limits in categories of algebras such as **Grp**, **Ring**, $\mathbf{Vect}_k$, ...
- Of course, we also have to say what the group/ring/... structure on the set above is, but this works in the most straightforward way imaginable.
- For instance, in $\mathbf{Vect}_k$, if $(x_I)_{I \in \mathbf{I}}, (y_I)_{I \in \mathbf{I}} \in \varprojlim_{\mathbf{I}} D$ then
$$(x_I)_{I \in \mathbf{I}} + (y_I)_{I \in \mathbf{I}} = (x_I + y_I)_{I \in \mathbf{I}}.$$
- The same formula also gives limits in **Top**.

  The topology on the set resulting is the smallest for which the projection maps are continuous.

Having Limits

Definition

(a) Let $\mathbf{I}$ be a small category. A category $\mathscr{A}$ **has limits of shape I** if for every diagram D of shape $\mathbf{I}$ in $\mathscr{A}$, a limit of D exists.

(b) A category **has all limits** (or properly, **has small limits**) if it has limits of shape $\mathbf{I}$ for all small categories $\mathbf{I}$.

- Thus, **Set**, **Top**, **Grp**, **Ring**, $\mathbf{Vect}_k$, ... all have all limits.
- Similar terminology can be applied to special classes of limits (for instance, "has pullbacks").
- A category is **finite** if it contains only finitely many maps (in which case it also contains only finitely many objects).
- A **finite limit** is a limit of shape $\mathbf{I}$ for some finite category $\mathbf{I}$.
- For instance, binary products, terminal objects, equalizers and pullbacks are all finite limits.

Constructing Limits Using Products and Equalizers

Proposition

Let $\mathscr{A}$ be a category.

(a) If $\mathscr{A}$ has all products and equalizers then $\mathscr{A}$ has all limits.

(b) If $\mathscr{A}$ has binary products, a terminal object and equalizers then $\mathscr{A}$ has finite limits.

- To understand the idea, consider the formula derived for limits in **Set**:

$$\begin{array}{rcl} \varprojlim_{\mathbf{I}} D & \cong & \{(x_I)_{I\in\mathbf{I}} : x_I \in D(I) \text{ for all } I \in \mathbf{I} \text{ and} \\ & & (Du)(x_I) = x_J \text{ for all } I \xrightarrow{u} J \text{ in } \mathbf{I}\}. \end{array}$$

- The limit of D is described as the subset of the product $\prod_{I\in\mathbf{I}} D(I)$ consisting of those elements for which certain equations hold.
- But we saw that the set of solutions to any system of simultaneous equations can be described via products and equalizers.
- Thus, we can describe any limit in **Set** in terms of products and equalizers and this is valid in any category.

Proof in **Set**

- Equation

$$\lim_{\longleftarrow \mathbf{I}} D \cong \{(x_I)_{I \in \mathbf{I}} : x_I \in D(I) \text{ for all } I \in \mathbf{I} \text{ and } (Du)(x_I) = x_J \text{ for all } I \xrightarrow{u} J \text{ in } \mathbf{I}\},$$

  states that in **Set**, the limit of a diagram $D : \mathbf{I} \to \mathbf{Set}$ consists of the elements $(x_I)_{I \in \mathbf{I}} \in \prod_{I \in \mathbf{I}} D(I)$ such that

$$(Du)(x_J) = x_K,$$

  for each map $J \xrightarrow{u} K$ in $\mathbf{I}$.
- For each such map u, define maps $\prod_{I \in \mathbf{I}} D(I) \overset{s_u}{\underset{t_u}{\rightrightarrows}} D(K)$ by

$$s_u((x_I)_{I \in \mathbf{I}}) = (Du)(x_J); \qquad t_u((x_I)_{I \in \mathbf{I}}) = x_K.$$

- Then $\lim_{\longleftarrow \mathbf{I}} D$ is the set of families $x = (x_I)_{I \in \mathbf{I}}$ satisfying the equation $s_u(x) = t_u(x)$ for each map u in $\mathbf{I}$.

Proof in **Set** (Cont'd)

- It follows that $\varprojlim_{\mathbf{I}} D$ is the equalizer of

$$\prod_{I \in \mathbf{I}} D(I) \underset{t}{\overset{s}{\rightrightarrows}} \prod_{J \xrightarrow{u} K \text{ in } \mathbf{I}} D(K)$$

where s and t are the maps with components s_u and t_u, respectively.
- We have described any limit in **Set** in terms of products and equalizers.
- A similar argument can be carried out in an arbitrary category.

Example

- Let **CptHff** denote the category of compact Hausdorff spaces and continuous maps.
- It is a classic exercise in topology to show that given continuous maps s and t from a topological space X to a Hausdorff space Y, the subset $\{x \in X : s(x) = t(x)\}$ of X is closed.
- From this it follows that **CptHff** has equalizers.
- Also, Tychonoff's theorem states that any product (in **Top**) of compact spaces is compact.
- Moreover, it is easy to show that any product (in **Top**) of Hausdorff spaces is Hausdorff.
- From this it follows that **CptHff** has all products.
- Hence by the proposition, **CptHff** has all limits

Example

- Recall that kernels provide equalizers in $\mathbf{Vect}_k$.
- By the proposition, finite limits in $\mathbf{Vect}_k$ can always be expressed in terms of $\oplus$ (binary direct sum), $\{0\}$, and kernels.
- The same is true in $\mathbf{Ab}$.

Monics

Definition

Let $\mathscr{A}$ be a category. A map $X \xrightarrow{f} Y$ in $\mathscr{A}$ is **monic** (or a **monomorphism**) if for all objects A and maps $A \overset{x}{\underset{x'}{\rightrightarrows}} X$,

$$f \circ x = f \circ x' \quad \Rightarrow \quad x = x'.$$

- This can be rephrased suggestively in terms of generalized elements:

  f is monic if for all generalized elements x and x' of X (of the same shape), $fx = fx' \;\Rightarrow\; x = x'$.

- Being monic is, therefore, the generalized-element analogue of injectivity.

Example

- In **Set**, a map is monic if and only if it is injective.

  Indeed, if f is injective then certainly f is monic, and for the converse, take $A = 1$.

- In categories of algebras such as **Grp**, $\mathbf{Vect}_k$, **Ring**, etc., it is also true that the monic maps are exactly the injections.

  Again, it is easy to show that injections are monic.

  For the converse, take $A = F(1)$ where F is the free functor.

Monics as Pullback

Lemma

A map $X \xrightarrow{f} Y$ is monic if and only if the following square is a pullback:

$$\begin{tikzcd} X \arrow[r, "1"] \arrow[d, "1"'] & X \arrow[d, "f"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

- The significance of this lemma is that whenever we prove a result about limits, a result about monics will follow.
- For example, we will soon show that the forgetful functors from **Grp**, $\mathbf{Vect}_k$, etc., to **Set** preserve limits (in a sense to be defined), from which it will follow immediately that they also preserve monics.
- This in turn gives an alternative proof that monics in these categories are injective.

Subsection 2

Colimits: Definition and Examples

Colimits

Definition

Let $\mathscr{A}$ be a category and $\mathbf{I}$ a small category. Let $D : \mathbf{I} \to \mathscr{A}$ be a diagram in $\mathscr{A}$, and write D^{op} for the corresponding functor $\mathbf{I}^{\mathrm{op}} \to \mathscr{A}^{\mathrm{op}}$. A **cocone** on D is a cone on D^{op}. A **colimit** of D is a limit of D^{op}.

- Explicitly, a cocone on D is an object $A \in \mathscr{A}$ (the **vertex** of the cocone) together with a family $(D(I) \xrightarrow{f_I} A)_{I \in \mathbf{I}}$ of maps in $\mathscr{A}$ such that for all maps $I \xrightarrow{u} J$ in $\mathbf{I}$, the following triangle commutes:

$$\begin{tikzcd} D(I) \arrow[rr, "Du"] \arrow[dr, "f_I"'] & & D(J) \arrow[dl, "f_J"] \\ & A & \end{tikzcd}$$

Colimits (Cont'd)

- A colimit of D is a cocone $(D(I) \xrightarrow{p_I} C)_{I \in \mathbf{I}}$ with the property that for any cocone $(D(I) \xrightarrow{f_I} A)_{I \in \mathbf{I}}$ on D, there is a unique map $\overline{f} : C \to A$ such that $\overline{f} \circ p_I = f_I$ for all $I \in \mathbf{I}$.

$$\begin{tikzcd}
D(I) \arrow[rr, "Du"] \arrow[dr, "p_I"] \arrow[ddr, "f_I"'] & & D(J) \arrow[dl, "p_J"'] \arrow[ddl, "f_J"] \\
& C \arrow[d, dashed, "\overline{f}"] & \\
& A &
\end{tikzcd}$$

- We write (the vertex of) the colimit as $\varinjlim_{\mathbf{I}} D$, and call the maps p_I **coprojections**.

Sums or Coproducts

Definition

A **sum** or **coproduct** is a colimit over a discrete category. That is, it is a colimit of shape $\mathbf{I}$ for some discrete category $\mathbf{I}$.

- Let $(X_i)_{i \in I}$ be a family of objects of a category.

  Their sum (if it exists) is written as $\sum_{i \in I} X_i$ or $\coprod_{i \in I} X_i$.
- When I is a finite set $\{1, \ldots, n\}$, we write $\sum_{i \in I} X_i$ as $X_1 + \cdots + X_n$, or as 0 if $n = 0$.

Initial Objects

- Let $\mathscr{A}$ be a category.
- In general, an I-indexed family $(X_i)_{i \in I}$ of objects of $\mathscr{A}$ is a function $I \to \mathrm{ob}(\mathscr{A})$.
- When I is empty, there is exactly one such function, i.e., there is exactly one family $(X_i)_{i \in \varnothing}$, the empty family.
- Similarly, when I is empty, there is exactly one family $(X_i \xrightarrow{f_i} A)_{i \in \varnothing}$ for any given object A.
- A coproduct of the empty family therefore consists of an object C of $\mathscr{A}$ such that for each object A of $\mathscr{A}$, there exists a unique map $\overline{f} : C \to A$ (the condition $\overline{f} \circ p_i = f_i$ for all $i \in I$ holds trivially).
- In other words, a coproduct of the empty family is exactly an initial object.

Binary Sums in **Set**

- Take two sets, X_1 and X_2.
- Form their sum, $X_1 + X_2$, and consider the inclusions $X_1 \xrightarrow{p_1} X_1 + X_2 \xleftarrow{p_2} X_2$.
- This is a colimit cocone.
- To prove this, we have to prove the following universal property:

  For any diagram $X_1 \xrightarrow{f_1} A \xleftarrow{f_2} X_2$ of sets and functions, there is a unique function $\overline{f} : X_1 + X_2 \to A$ making the following commute:

$$\begin{tikzcd}
X_1 \arrow[dr, "p_1"'] \arrow[drr, "f_1"] & & \\
& X_1 + X_2 \arrow[r, "\overline{f}"] & A \\
X_2 \arrow[ur, "p_2"] \arrow[urr, "f_2"'] & &
\end{tikzcd}$$

Binary Sums in **Set** (Cont'd)

$$
\begin{tikzcd}
X_1 \arrow[dr, "p_1"'] \arrow[drr, "f_1"] & & \\
& X_1 + X_2 \arrow[r, "\overline{f}"] & A \\
X_2 \arrow[ur, "p_2"] \arrow[urr, "f_2"'] & &
\end{tikzcd}
$$

- p_1 and p_2 are injections whose images partition $X_1 + X_2$.
  So every element x of $X_1 + X_2$ is
  - either equal to $p_1(x_1)$ for a unique $x_1 \in X_1$
  - or equal to $p_2(x_2)$ for a unique $x_2 \in X_2$.

  So we may define $\overline{f}(x)$ to be equal to $f_1(x_1)$ in the first case and $f_2(x_2)$ in the second.

  This defines a function $\overline{f}$ making the diagram commute.

  It is clearly the unique function that does so.

Example

- Let X_1 and X_2 be vector spaces.
- There are linear maps

$$X_1 \xrightarrow{i_1} X_1 \oplus X_2 \xleftarrow{i_2} X_2$$

defined by $i_1(x_1) = (x_1, 0)$ and $i_2(x_2) = (0, x_2)$.
- It can be checked that the diagram is a colimit cocone in $\mathbf{Vect}_k$.
- Hence binary direct sums are sums in the categorical sense.
- This is remarkable, since we saw previously that $X_1 \oplus X_2$ is also the product of X_1 and X_2!
- Contrast this with the category of sets (or almost any other category), where sums and products are very different.

The Reals as an Ordered Set (Covisited)

- Let $x, y \in \mathbb{R}$.
- Their maximum $\max\{x, y\}$ satisfies

$$x \le \max\{x, y\}, \quad y \le \max\{x, y\}.$$

- It has the further property that whenever $a \in \mathbb{R}$ with

$$x \le a, \quad y \le a,$$

  we have $\max\{x, y\} \le a$.
- This means exactly that when the poset $(\mathbb{R}, \le)$ is viewed as a category, the coproduct of $x, y \in \mathbb{R}$ is $\max\{x, y\}$.
- The definition of coproduct simplifies when interpreted in a poset, since all diagrams commute.

Power Sets as Ordered Sets (Covisited)

- Fix a set S.
- Let $X, Y \in \mathscr{P}(S)$.
- Then $X \cup Y$ satisfies

$$X \subseteq X \cup Y, \quad Y \subseteq X \cup Y.$$

- It has the further property that whenever $A \in \mathscr{P}(S)$ with

$$X \subseteq A, \quad Y \subseteq A,$$

  we have $X \cup Y \subseteq A$.
- This means that $X \cup Y$ is the coproduct of X and Y in the poset $(\mathscr{P}(S), \subseteq)$ regarded as a category.

Natural Numbers with Divisibility (Covisited)

- Let $x, y \in \mathbb{N}$.
- Their least common multiple $\mathrm{lcm}(x, y)$ satisfies

$$x \mid \mathrm{lcm}(x, y), \quad y \mid \mathrm{lcm}(x, y).$$

- It has the further property that whenever $a \in \mathbb{N}$ with

$$x \mid a, \quad y \mid a,$$

  we have $\mathrm{lcm}(x, y) \mid a$.
- This means that $\mathrm{lcm}(x, y)$ is the coproduct of x and y in the poset $(\mathbb{N}, \mid)$ regarded as a category.

Sums in Partially Ordered Sets

- Let $(A, \leq)$ be a poset and $x, y \in A$.
- An **upper bound** for x and y is an element $a \in A$ such that $x \leq a$ and $y \leq a$.
- A **least upper bound** or **join** of x and y is an upper bound z for x and y with the further property that whenever a is an upper bound for x and y, we have $z \leq a$.
- When a poset is regarded as a category, joins are exactly coproducts.
- They do not always exist, but when they do, they are unique.
- The join of x and y is usually written as $x \vee y$ rather than $x + y$.
- Thus, in the three examples above,

$$x \vee y = \max\{x, y\}, \quad X \vee Y = X \cup Y, \quad x \vee y = \mathrm{lcm}(x, y),$$

the second example being the origin of the notation.

Empty Joins

- Let $(A, \leq)$ be an ordered set.
- A join of the empty family (where $I = \emptyset$) is an initial object of the category A.
- Equivalently, it is a **least element** of A, i.e., an element $0 \in A$ such that $0 \leq a$ for all $a \in A$.

Examples of Least Upper Bounds and Least Elements

- In $(\mathbb{R}, \leq)$,
  - join is supremum;
  - a least element does not exist.
- In a power set $(\mathscr{P}(S), \subseteq)$,
  - join is union;
  - the least element is $\emptyset$.
- In $(\mathbb{N}, |)$,
  - join is least common multiple;
  - the least element is 1 (since 1 divides everything).

Coequalizers

- Recall that $\mathbf{E}$ is the category $\bullet \rightrightarrows \bullet$.

Definition

A **coequalizer** is a colimit of shape $\mathbf{E}$.

- In other words, given a diagram $X \overset{s}{\underset{t}{\rightrightarrows}} Y$, a coequalizer of s and t is a map $Y \xrightarrow{p} C$ satisfying $p \circ s = p \circ t$ and universal with this property.

$$\begin{tikzcd} X \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & Y \arrow[r, "p"] \arrow[dr, "f"'] & C \arrow[d, "\overline{f}"] \\ & & A \end{tikzcd}$$

Equivalence Relations

- A binary relation R on a set A can be viewed as a subset $R \subseteq A \times A$.
- Think of $(a, a') \in R$ as meaning "a and a' are related".
- We can speak of one relation S on A "containing" another such relation, R.

  This means that $R \subseteq S$: whenever a and a' are R-related, they are also S-related.
- We will need to use the fact that for any binary relation R on a set A, there is a smallest equivalence relation $\sim$ containing R.
- This is called the equivalence relation **generated by** R.
- "Smallest" means that any equivalence relation containing R also contains $\sim$.

Construction of $\sim$

- We can construct $\sim$ as the intersection of all equivalence relations on A containing R, since the intersection of any family of equivalence relations is again an equivalence relation.
- Roughly speaking, writing $x \to y$ to mean $(x, y) \in R$, we should have $a \sim a'$ if and only if there is a zigzag such as $a \to b \leftarrow c \leftarrow d \to e \leftarrow a'$ between a and a'.
- To make this precise, we first define a relation S on A by
$$S = \{(a, a') \in A \times A : (a, a') \in R \text{ or } (a', a) \in R\}$$
(which enlarges R to a symmetric relation).
- Then define $\sim$ by declaring that $a \sim a'$ if and only if there exist $n \geq 0$ and $a_0, \ldots, a_n \in A$ such that
$$a = a_0,\ (a_0, a_1) \in S,\ (a_1, a_2) \in S, \ldots,\ (a_{n-1}, a_n) \in S,\ a_n = a'$$
(which forces reflexivity and transitivity, while preserving the symmetry).

Quotients and Universal Property

- Recall that, given any equivalence relation $\sim$ on a set A, we can construct the set $A/{\sim}$ of equivalence classes and the quotient map $p : A \to A/{\sim}$.
- This quotient map p is surjective and has the property that

$$p(a) = p(a') \;\Leftrightarrow\; a \sim a', \text{ for } a, a' \in A.$$

- We saw that for any set B, the maps $A/{\sim} \to B$ correspond one-to-one (via composition with p) with the maps $f : A \to B$ such that

$$\forall a, a' \in A, \quad a \sim a' \;\Rightarrow\; f(a) = f(a').$$

Generated Equivalences and Universal Property

- We now consider this universal property in the case where $\sim$ is the equivalence relation generated by some relation R.
- The condition
$$\forall a, a' \in A, \quad a \sim a' \Rightarrow f(a) = f(a').$$
is then equivalent to:
$$\forall a, a' \in A, \quad (a, a') \in R \Rightarrow f(a) = f(a').$$
- To see this, define an equivalence relation $\approx$ on A by $a \approx a' \Leftrightarrow f(a) = f(a')$.
- Then, the former condition says that $\sim \subseteq \approx$.
- The latter condition says that $R \subseteq \approx$.
- But $\sim$ is the smallest equivalence relation containing R.
- So these statements are equivalent.
- In conclusion, for any set B, the maps $A/\sim \to B$ correspond one-to-one with the maps $f : A \to B$ satisfying the condition above.

Coequalizers in **Set**

- Take sets and functions $X \overset{s}{\underset{t}{\rightrightarrows}} Y$.
- To find the coequalizer of s and t, we must construct in some canonical way a set C and a function $p : Y \to C$ such that $p(s(x)) = p(t(x))$ for all $x \in X$.
- So, let $\sim$ be the equivalence relation on Y generated by $s(x) \sim t(x)$ for all $x \in X$, i.e., $\sim$ is generated by the relation

$$R = \{(s(x), t(x)) : x \in X\}$$

  on Y.
- Take the quotient map $p : Y \to Y/\sim$.
- By the correspondence described in the preceding remarks, this is indeed the coequalizer of s and t.

Example

- For each pair of homomorphisms $A \underset{t}{\overset{s}{\rightrightarrows}} B$ in **Ab**, there is a homomorphism $t - s : A \to B$, which gives rise to a subgroup $\mathrm{im}(t-s)$ of B.
- The coequalizer of s and t is the canonical homomorphism $B \to B/\mathrm{im}(t-s)$.

Pushouts

- A **pushout** is a colimit of shape

$$\mathbf{P}^{\mathrm{op}} = \begin{tikzcd} \bullet \arrow[r] \arrow[d] & \bullet \\ \bullet & \end{tikzcd}$$

- In other words, the pushout of a diagram as on the left

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \\ Z & \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \arrow[d] \\ Z \arrow[r] & \cdot \end{tikzcd}$$

  is (if it exists) a commutative square as on the right that is universal as such.
- In other words still, a pushout in a category $\mathscr{A}$ is a pullback in $\mathscr{A}^{\mathrm{op}}$.

Pushouts in **Set**

- Take a diagram in **Set**:

$$\begin{tikzcd} X \arrow[r, "s"] \arrow[d, "t"'] & Y \\ Z & \end{tikzcd}$$

- Its pushout P is $(Y+Z)/\sim$, where $\sim$ is the equivalence relation on $Y+Z$ generated by $s(x)\sim t(x)$ for all $x\in X$.
- The coprojection $Y\to P$ sends $y\in Y$ to its equivalence class in P, and similarly for the coprojection $Z\to P$.

Example

- For example, let Y and Z be subsets of some set A.
- Then

$$\begin{tikzcd} Y \cap Z \arrow[r, hook] \arrow[d, hook] & Y \arrow[d, hook] \\ Z \arrow[r, hook] & Y \cup Z \end{tikzcd}$$

  is a pushout square in **Set**.
- It is also a pullback square!
- This coincidence is a special property of the category of sets.
- You can check this by verifying the universal property or by using the formula just stated.
- In this case, the formula takes the two sets Y and Z, places them side by side (giving $Y + Z$), then glues the subset $Y \cap Z$ of Y to the subset $Y \cap Z$ of Z (giving $(Y + Z)/\sim = Y \cup Z$).

Sums as Pushouts

- If $\mathscr{A}$ is a category with an initial object 0, and if $Y, Z \in \mathscr{A}$, then a pushout of the unique diagram

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & Y \\ Z & \end{tikzcd}$$

is exactly a sum of Y and Z.

Direct Limits

- A diagram $D:(\mathbb{N},\leq)\to\mathscr{A}$ consists of objects and maps
$$X_0 \xrightarrow{s_1} X_1 \xrightarrow{s_2} X_2 \xrightarrow{s_3} \cdots$$
in $\mathscr{A}$.
- Colimits of such diagrams are traditionally called **direct limits**.
- Although the old terms "inverse limit" and "direct limit" are made redundant by the general categorical terms "limit" and "colimit" respectively, it is worth being aware of them.

Construction of Colimits in **Set**

- The colimit of a diagram $D : \mathbf{I} \to \mathbf{Set}$ is given by
$$\varinjlim_{\mathbf{I}} D = (\sum_{I \in \mathbf{I}} D(I))/\sim,$$
where $\sim$ is the equivalence relation on $\sum D(I)$ generated by $x \sim (Du)(x)$ for all $I \xrightarrow{u} J$ in $\mathbf{I}$ and $x \in D(I)$.
- To see this, note that for any set A, the maps
$$(\sum D(I))/\sim \to A$$
correspond bijectively with the maps $f : \sum D(I) \to A$ such that $f(x) = f((Du)(x))$, for all u and x.
- These in turn correspond to families of maps $(D(I) \xrightarrow{f_I} A)_{I \in \mathbf{I}}$ such that $f_I(x) = f_J((Du)(x))$ for all u and x.
- But these are exactly the cocones on D with vertex A.
- Note that whereas the limit is constructed as a subset of a product, the colimit is a quotient of a sum.

Example

- In elementary texts, surfaces are almost always seen as subsets of Euclidean space $\mathbb{R}^3$, with the sphere S^2 typically defined as

$$\{(x,y,z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = 1\}.$$

- This is a subspace of the product space $\mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$, which suggests that it is a limit.
- Indeed, the sphere is the equalizer

$$\begin{tikzcd} S^2 \arrow[r, hook] & \mathbb{R}^3 \arrow[r, shift left, "s"] \arrow[r, shift right, "t"'] & \mathbb{R} \end{tikzcd}$$

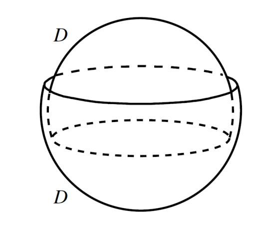
  where the maps $s, t : \mathbb{R}^3 \to \mathbb{R}$ are given by

$$s(x,y,z) = x^2 + y^2 + z^2, \quad t(x,y,z) = 1.$$

- An equation is captured by an equalizer.

Example (Cont'd)

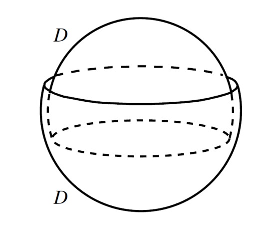
- In more advanced mathematics, a surface can instead be thought of as the gluing-together of lots of little patches, each isomorphic to the open unit disk D.
- For example, we could in principle construct an entire bicycle inner tube by gluing together a large number of puncture-repair patches.
- The figure shows the simpler example of a sphere made up of two disks glued together. This realizes the sphere as a quotient (gluing) of the sum (disjoint union) of the two copies of D, suggesting that we have constructed the sphere as a colimit.



Example (Cont'd)

- Indeed, the sphere is the coequalizer

$$S^1 \times (0,1) \rightrightarrows D + D \longrightarrow S^2$$



where S^1 is the circle, the cylinder $S^1 \times (0,1)$ is the intersection of the two copies of D, and the two maps into $D + D$ are the inclusions of the cylinder into the first and second copies of D.

Example (Cont'd)

- One disadvantage of the limit point of view is that it makes an arbitrary choice of coordinate system.

  It is generally best to think of spaces as freestanding objects, existing independently of any particular embedding into Euclidean space.
- One disadvantage of the colimit point of view is that it makes an arbitrary choice of decomposition.

  For example, we could decompose the sphere into three patches rather than two, or use a different two patches from those shown.
- The colimit point of view has the upper hand in modern geometry.
- If you are familiar with the definition of manifold, you will recognize that an atlas is essentially a way of viewing a manifold as a colimit of Euclidean balls.
- One reason for this is that we are often concerned with maps out of spaces X, such as maps $X \to \mathbb{R}$ and we know what maps out of a colimit are by the the very definition of colimit.

Epics

Definition

Let $\mathscr{A}$ be a category. A map $X \xrightarrow{f} Y$ in $\mathscr{A}$ is **epic** (or an **epimorphism**) if for all objects Z and maps $Y \underset{g'}{\overset{g}{\rightrightarrows}} Z$,

$$g \circ f = g' \circ f \;\Rightarrow\; g = g'.$$

- This is the formal dual of the definition of monic, i.e., an epic in $\mathscr{A}$ is a monic in $\mathscr{A}^{\mathrm{op}}$.
- It is in some sense the categorical version of surjectivity.
- But whereas the definition of monic closely resembles the definition of injective, the definition of epic does not look much like the definition of surjective.
- The following examples confirm that in categories where surjectivity makes sense, it is only sometimes equivalent to being epic.

Epics in **Set**

- In **Set**, a map is epic if and only if it is surjective.
  - If f is surjective then certainly f is epic.
  - To see the converse, take Z to be a two-element set {`true`,`false`}.
    Take g to be the characteristic function of the image of f.
    Take g' to be the function with constant value `true`.
- Any isomorphism in any category is both monic and epic.
- In **Set**, the converse also holds, since any injective surjective function is invertible.

Example

- In categories of algebras, any surjective map is certainly epic.
- In some such categories, including **Ab**, $\mathbf{Vect}_k$ and **Grp**, the converse also holds.
- The proof is straightforward for **Ab** and $\mathbf{Vect}_k$, but much harder for **Grp**.
- However, there are other categories of algebras where it fails.
- For instance, in **Ring**, the inclusion $\mathbb{Z} \hookrightarrow \mathbb{Q}$ is epic but not surjective.

  This is also an example of a map that is monic and epic but not an isomorphism.

Example

- In the category of Hausdorff topological spaces and continuous maps, any map with dense image is epic.

Epics as Pushouts

Lemma

A map $X \xrightarrow{f} Y$ is epic if and only if the following square is a pushout:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "f"'] & Y \arrow[d, "1"] \\ Y \arrow[r, "1"'] & Y \end{tikzcd}$$

- The significance of this lemma is that whenever we prove a result about colimits, a result about epics will follow.
- For example, if a functor preserves colimits then, it also preserves epics.

Subsection 3

Interactions Between Functors and Limits

Preservation of Limits

Definition

(a) Let $\mathbf{I}$ be a small category. A functor $F : \mathscr{A} \to \mathscr{B}$ **preserves limits of shape I** if for all diagrams $D : \mathbf{I} \to \mathscr{A}$ and all cones $(A \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ on D,

$$\begin{aligned} &(A \xrightarrow{p_I} D(I))_{I \in \mathbf{I}} \text{ is a limit cone on } D \text{ in } \mathscr{A} \\ \Longrightarrow \quad &(F(A) \xrightarrow{Fp_I} FD(I))_{I \in \mathbf{I}} \text{ is a limit cone on } F \circ D \text{ in } \mathscr{B}. \end{aligned}$$

(b) A functor $F : \mathscr{A} \to \mathscr{B}$ **preserves limits** if it preserves limits of shape $\mathbf{I}$ for all small categories $\mathbf{I}$.

(c) **Reflection** of limits is defined as in (a), but with $\Longleftarrow$ in place of $\Longrightarrow$.

- Of course, the same terminology applies to colimits.

An Alternative Point of View

- A functor $F : \mathscr{A} \to \mathscr{B}$ preserves limits if and only if it has the following property:

  Whenever $D : \mathbf{I} \to \mathscr{A}$ is a diagram that has a limit, the composite $F \circ D : \mathbf{I} \to \mathscr{B}$ also has a limit, and the canonical map

  $$F(\varprojlim_{\mathbf{I}} D) \to \varprojlim_{\mathbf{I}} (F \circ D)$$

  is an isomorphism.

  Here the "canonical map" has I-component $F(\varprojlim_{\mathbf{I}} D) \xrightarrow{F(p_I)} F(D(I))$, where p_I is the Ith projection of the limit cone on D.

- In particular, if F preserves limits then $F(\varprojlim_{\mathbf{I}} D) \cong \varprojlim_{\mathbf{I}} (F \circ D)$ whenever D is a diagram with a limit.

Remark

- Preservation of limits says not only that the left- and right-hand sides of
$$F(\varprojlim_{\mathbf{I}} D) \cong \varprojlim_{\mathbf{I}} (F \circ D)$$
are required to be isomorphic, but isomorphic in a particular way:
- The canonical map
$$F(\varprojlim_{\mathbf{I}} D) \stackrel{F(p_I)}{\to} F(D(I)),$$
where p_I is the Ith projection of the limit cone on D, must be an isomorphism.
- Nevertheless, we will sometimes omit this check, acting as if preservation means only the first isomorphism.

Example

- The forgetful functor $U: \mathbf{Top} \to \mathbf{Set}$ preserves both limits and colimits.
- As we will see, this follows from the fact that U has adjoints on both sides.
- It does not reflect all limits or all colimits.
- For instance, choose any non-discrete spaces X and Y, and let Z be the set $U(X) \times U(Y)$ equipped with the discrete topology.
- All that matters here is that the topology on Z is strictly larger than the product topology.
- Then we have a cone
$$X \leftarrow Z \rightarrow Y$$
in **Top** whose image in **Set** is the product cone
$$U(X) \leftarrow U(X) \times U(Y) \rightarrow U(Y).$$
- But the cone in **Top** is not a product cone in **Top**, since the discrete topology on $U(X) \times U(Y)$ is not the product topology.

Example

- We observed that:
  - The forgetful functor $\mathbf{Grp} \to \mathbf{Set}$ does not preserve initial objects;
  - The forgetful functor $\mathbf{Vect}_k \to \mathbf{Set}$ does not preserve binary sums.
- Forgetful functors out of categories of algebras very seldom preserve all colimits.

Forgetful Functors on Categories of Algebras

- Take groups X_1 and X_2.
- We can form the product set $U(X_1) \times U(X_2)$, which comes equipped with projections
$$U(X_1) \overset{p_1}{\leftarrow} U(X_1) \times U(X_2) \overset{p_2}{\rightarrow} U(X_2).$$
- There is exactly one group structure on the set $U(X_1) \times U(X_2)$ with the property that p_1 and p_2 are homomorphisms.
- To prove uniqueness, suppose that we have a group structure on $U(X_1) \times U(X_2)$ with this property.
  Take elements (x_1, x_2) and (x_1', x_2') of $U(X_1) \times U(X_2)$ and write $(x_1, x_2) \cdot (x_1', x_2') = (y_1, y_2)$. Since p_1 is a homomorphism,
  $y_1 = p_1(y_1, y_2) = p_1((x_1, x_2) \cdot (x_1', x_2')) = p_1(x_1, x_2) \cdot p_1(x_1', x_2') = x_1 \cdot x_1'$;
  Similarly $y_2 = x_2 \cdot x_2'$. Hence $(x_1, x_2) \cdot (x_1', x_2') = (x_1 x_1', x_2 x_2')$.
  A similar argument shows that $(x_1, x_2)^{-1} = (x_1^{-1}, x_2^{-1})$.
  And also that the identity element 1 of the group is $(1, 1)$.

Forgetful Functors on Categories of Algebras (Cont'd)

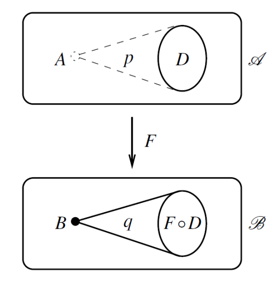
- For existence, define $\cdot, (\ )^{-1}$ and 1 by the formulas just given.
  It can then be checked that:
  - the group axioms are satisfied;
  - p_1 and p_2 are group homomorphisms.

  This proves the claim.
- Write L for the set $U(X_1) \times U(X_2)$ equipped with this group structure.
- Then we have a cone $X_1 \xleftarrow{p_1} L \xrightarrow{p_2} X_2$ in **Grp**.
- It is easy to check that this is, in fact, a product cone in **Grp**.
- Summarizing in language that is not tied to group theory, given objects X_1 and X_2 of **Grp**:
  - For any product cone on $(U(X_1), U(X_2))$ in **Set**, there is a unique cone on (X_1, X_2) in **Grp** whose image under U is the cone we started with;
  - This cone on (X_1, X_2) is a product cone.

Creation of Limits

Definition

A functor $F:\mathscr{A}\to\mathscr{B}$ **creates limits (of shape I)** if whenever $D:\mathbf{I}\to\mathscr{A}$ is a diagram in $\mathscr{A}$:



- For any limit cone $(B \xrightarrow{q_I} FD(I))_{I\in\mathbf{I}}$ on the diagram $F\circ D$, there is a unique cone $(A \xrightarrow{p_I} D(I))_{I\in\mathbf{I}}$ on D such that $F(A)=B$ and $F(p_I)=q_I$ for all $I\in\mathbf{I}$;
- This cone $(A \xrightarrow{p_I} D(I))_{I\in\mathbf{I}}$ is a limit cone on D.

- The forgetful functors from **Grp**, **Ring**,... to **Set** all create limits.

Creation, Availability and Preservation of Limits

Lemma

Let $F : \mathscr{A} \to \mathscr{B}$ be a functor and $\mathbf{I}$ a small category. Suppose that $\mathscr{B}$ has, and F creates, limits of shape $\mathbf{I}$. Then $\mathscr{A}$ has, and F preserves, limits of shape $\mathbf{I}$.

- Since **Set** has all limits, it follows that all our categories of algebras have all limits, and that the forgetful functors preserve them.

Remark

- The preceding definition refers to equality of objects of a category.
- It is almost always better to replace equality by isomorphism.
- If we replace equality by isomorphism throughout the definition of "creates limits", we obtain a more healthy and inclusive notion.
- We ask that if $F \circ D$ has a limit then there exists a cone on D whose image under F is a limit cone, and that every such cone is itself a limit cone.
- What we are calling creation of limits should really be called strict creation of limits, with "creation of limits" reserved for the more inclusive notion, and that is how "creates" is used in most of the literature.