Introduction to Category Theory

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LSSU Math 400

1. Adjoints, Representables and Limits
   - Limits In Terms of Representables and Adjoints
   - Limits and Colimits of Presheaves
   - Interactions Between Adjoint Functors and Limits

Subsection 1

Limits In Terms of Representables and Adjoints

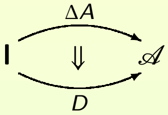
The Diagonal Functor

- Given categories $\mathbf{I}$ and $\mathscr{A}$ and an object $A \in \mathscr{A}$, there is a functor $\Delta A : \mathbf{I} \to \mathscr{A}$ with constant value A on objects and 1_A on maps.
- This defines, for each $\mathbf{I}$ and $\mathscr{A}$, the **diagonal functor** $\Delta : \mathscr{A} \to [\mathbf{I}, \mathscr{A}]$.
- The name can be understood by considering the case in which $\mathbf{I}$ is the discrete category with two objects.

  Then $[\mathbf{I}, \mathscr{A}] = \mathscr{A} \times \mathscr{A}$ and $\Delta(A) = (A, A)$.

Cones as Natural Transformations

- Given a diagram $D : \mathbf{I} \to \mathscr{A}$ and an object $A \in \mathscr{A}$, a cone on D with vertex A is simply a natural transformation



- Writing $\mathrm{Cone}(A, D)$ for the set of cones on D with vertex A, we therefore have
$$\mathrm{Cone}(A, D) = [\mathbf{I}, \mathscr{A}](\Delta A, D).$$
- Thus, $\mathrm{Cone}(A, D)$ is functorial in A (contravariantly) and D (covariantly).

Limits as Representables

Proposition

Let $\mathbf{I}$ be a small category, $\mathscr{A}$ a category, and $D : \mathbf{I} \to \mathscr{A}$ a diagram. Then there is a one-to-one correspondence between limit cones on D and representations of the functor

$$\text{Cone}(-, D) : \mathscr{A}^{\text{op}} \to \mathbf{Set},$$

with the representing objects of $\text{Cone}(-, D)$ being the limit objects (that is, the vertices of the limit cones) of D.

- Briefly, a limit of D is a representation of $[\mathbf{I}, \mathscr{A}](\Delta -, D)$.
- By a previous corollary, a representation of $\text{Cone}(-, D)$ consists of a cone on D with a certain universal property.

  This is exactly the universal property in the definition of limit cone.

A Correspondence

- The proposition formalizes the thought that cones on a diagram D correspond one-to-one with maps into $\varprojlim_{\mathbf{I}} D$.
- It implies that if D has a limit then

$$\mathrm{Cone}(A,D) \cong \mathscr{A}(A, \varprojlim_{\mathbf{I}} D)$$

  naturally in A.
- The correspondence is given:
  - From left to right by $(f_I)_{I \in \mathbf{I}} \mapsto \overline{f}$;
  - From right to left by $(p_I \circ g)_{I \in \mathbf{I}} \leftarrow\!\shortmid g$, where $p_I : \varprojlim_{\mathbf{I}} D \to D(I)$ are the projections.
- From the proposition and a previous corollary,

Corollary

Limits are unique up to isomorphism.

Varying Diagrams

Lemma

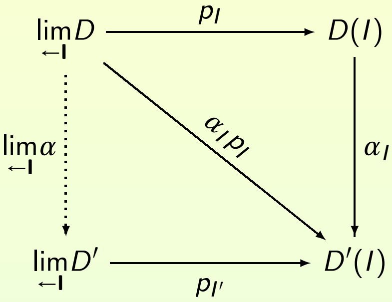
Let $\mathbf{I}$ be a small category and $(\mathbf{I} \xrightarrow{D} \mathscr{A}) \overset{\alpha}{\Rightarrow} (\mathbf{I} \xrightarrow{D'} \mathscr{A})$ a natural transformation. Let $(\varprojlim_{\mathbf{I}} D \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ and $(\varprojlim_{\mathbf{I}} D' \xrightarrow{p'_I} D'(I))_{I \in \mathbf{I}}$ be limit cones. Then:

(a) There is a unique map $\varprojlim_{\mathbf{I}} \alpha : \varprojlim_{\mathbf{I}} D \to \varprojlim_{\mathbf{I}} D'$, such that for all $I \in \mathbf{I}$, the following square commutes:

$$\begin{array}{ccc} \varprojlim_{\mathbf{I}} D & \xrightarrow{p_I} & D(I) \\ {\scriptstyle \varprojlim_{\mathbf{I}} \alpha} \downarrow & & \downarrow {\scriptstyle \alpha_I} \\ \varprojlim_{\mathbf{I}} D' & \xrightarrow[p'_I]{} & D'(I) \end{array}$$

- This follows immediately from the fact that $(\varprojlim_{\mathbf{I}} D \xrightarrow{\alpha_I p_I} D'(I))_{I \in \mathbf{I}}$ is a cone on D'.

Varying Diagrams (Illustation)



Varying Diagrams (Cont'd)

Lemma (Cont'd)

(b) Given cones $(A \xrightarrow{f_I} D(I))_{I \in \mathbf{I}}$ and $(A' \xrightarrow{f'_I} D'(I))_{I \in \mathbf{I}}$ and a map $s : A \to A'$ such that the left rectangle commutes, for all $I \in \mathbf{I}$,

$$\begin{array}{ccc} A & \xrightarrow{f_I} & D(I) \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \alpha_I} \\ A' & \xrightarrow[f'_I]{} & D'(I) \end{array} \qquad \begin{array}{ccc} A & \xrightarrow{\overline{f}} & \varprojlim_{\mathbf{I}} D \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle \varprojlim_{\mathbf{I}} \alpha} \\ A' & \xrightarrow[\overline{f'}]{} & \varprojlim_{\mathbf{I}} D' \end{array}$$

the square on the right also commutes.

- Note that for each $I \in \mathbf{I}$, we have $p'_I \circ (\varprojlim_{\mathbf{I}} \alpha) \circ \overline{f} = \alpha_I \circ p_I \circ \overline{f} = \alpha_I \circ f_I = f'_I \circ s = p'_I \circ \overline{f'} \circ s.$

  So we get $(\varprojlim_{\mathbf{I}} \alpha) \circ \overline{f} = \overline{f'} \circ s.$

Limits as Adjoints

Proposition

Let $\mathbf{I}$ be a small category and $\mathscr{A}$ a category with all limits of shape $\mathbf{I}$. Then $\varprojlim_{\mathbf{I}}$ defines a functor $[\mathbf{I},\mathscr{A}] \to \mathscr{A}$, and this functor is right adjoint to the diagonal functor.

- Choose for each $D \in [\mathbf{I},\mathscr{A}]$ a limit cone on D, and call its vertex $\varprojlim_{\mathbf{I}} D$.

  For each map $\alpha : D \to D'$ in $[\mathbf{I},\mathscr{A}]$, we have a canonical map $\varprojlim_{\mathbf{I}} \alpha : \varprojlim_{\mathbf{I}} D \to \varprojlim_{\mathbf{I}} D'$, defined as in Part (a) of the lemma. This makes $\varprojlim_{\mathbf{I}}$ into a functor.

  The preceding proposition implies that $[\mathbf{I},\mathscr{A}](\Delta A, D) = \mathrm{Cone}(A, D) \cong \mathscr{A}(A, \varprojlim_{\mathbf{I}} D)$ naturally in A. Taking $s = 1_A$ in Part (b) of the lemma tells us that the isomorphism is also natural in D.

Remarks

- To define the functor $\varprojlim$ we had to choose for each D a limit cone on D.
- This is a non-canonical choice.
- Nevertheless, different choices only affect the functor $\varprojlim$ up to natural isomorphism, by uniqueness of adjoints.

Subsection 2

Limits and Colimits of Presheaves

Natural Correspondence Between Maps: Products

- Recall that, by definition of product, a map $A \to X \times Y$ amounts to a pair of maps $(A \to X, A \to Y)$.
- Here A, X and Y are objects of a category $\mathscr{A}$ with binary products.
- There is, therefore, a bijection

$$\mathscr{A}(A, X \times Y) \cong \mathscr{A}(A, X) \times \mathscr{A}(A, Y)$$

natural in $A, X, Y \in \mathscr{A}$.

Natural Correspondence Between Maps: Equalizers

- Suppose that $\mathscr{A}$ has equalizers.
- Write $\mathrm{Eq}(X \overset{s}{\underset{t}{\rightrightarrows}} Y)$ for the equalizer of maps s and t.
- By definition of equalizer, maps $A \to \mathrm{Eq}(X \overset{s}{\underset{t}{\rightrightarrows}} Y)$ correspond one-to-one with maps $f : A \to X$ such that $s \circ f = t \circ f$.
- Recall that s and t induce maps

$$\begin{aligned} s_* &= \mathscr{A}(A,s) : \mathscr{A}(A,X) \to \mathscr{A}(A,Y), \\ t_* &= \mathscr{A}(A,s) : \mathscr{A}(A,X) \to \mathscr{A}(A,Y). \end{aligned}$$

- In this notation, what we have just said is that maps $A \to \mathrm{Eq}(X \overset{s}{\underset{t}{\rightrightarrows}} Y)$ correspond one-to-one with elements $f \in \mathscr{A}(A,X)$ such that $(\mathscr{A}(A,s))(f) = (\mathscr{A}(A,t))(f)$.

Equalizers (Cont'd)

- By the explicit formula for equalizers in **Set**, a map $f \in \mathscr{A}(A, X)$ such that $(\mathscr{A}(A, s))(f) = (\mathscr{A}(A, t))(f)$ is exactly an element of the equalizer of $\mathscr{A}(A, s)$ and $\mathscr{A}(A, t)$.
- So, we have a canonical bijection

$$\mathscr{A}(A, \mathrm{Eq}(X \overset{s}{\underset{t}{\rightrightarrows}} Y)) \cong \mathrm{Eq}(\mathscr{A}(A, X) \overset{\mathscr{A}(A,s)}{\underset{\mathscr{A}(A,t)}{\rightrightarrows}} \mathscr{A}(A, Y)).$$

- This looks something like our isomorphism

$$\mathscr{A}(A, X \times Y) \cong \mathscr{A}(A, X) \times \mathscr{A}(A, Y)$$

we saw for products.

A Conjectured Natural Isomorphism

- The preceding isomorphisms suggest we might have

$$\mathscr{A}(A, \lim_{\leftarrow \mathbf{I}} D) \cong \lim_{\leftarrow \mathbf{I}} \mathscr{A}(A, D)$$

naturally in $A \in \mathscr{A}$ and $D \in [\mathbf{I}, \mathscr{A}]$, whenever $\mathscr{A}$ is a category with limits of shape $\mathbf{I}$.

- Here $\mathscr{A}(A, D)$ is the functor

$$\begin{array}{rccc} \mathscr{A}(A, D): & \mathbf{I} & \to & \mathbf{Set} \\ & I & \mapsto & \mathscr{A}(A, D(I)). \end{array}$$

- This functor could also be written as $\mathscr{A}(A, D(-))$, and is the composite

$$\mathbf{I} \xrightarrow{D} \mathscr{A} \xrightarrow{\mathscr{A}(A,-)} \mathbf{Set}.$$

- The conjectured isomorphism states, essentially, that representables preserve limits.

Cones and Limits

Lemma

Let $\mathbf{I}$ be a small category, $\mathscr{A}$ a locally small category, $D : \mathbf{I} \to \mathscr{A}$ a diagram, and $A \in \mathscr{A}$. Then

$$\text{Cone}(A, D) \cong \lim_{\leftarrow \mathbf{I}} \mathscr{A}(A, D)$$

naturally in A and D.

- Like all functors from a small category into **Set**, the functor $\mathscr{A}(A, D)$ does have a limit, given by the explicit formula

$$\begin{aligned}\lim_{\leftarrow \mathbf{I}} D \quad \cong \quad & \{(x_I)_{I \in \mathbf{I}} : x_I \in D(I) \text{ for all } I \in \mathbf{I} \text{ and} \\ & (Du)(x_I) = x_J \text{ for all } I \xrightarrow{u} J \text{ in } \mathbf{I}\}.\end{aligned}$$

Cones and Limits (Cont'd)

- According to this formula, $\varprojlim_{\mathbf{I}} \mathscr{A}(A, D)$ is the set of all families $(f_I)_{I \in \mathbf{I}}$ such that $f_I \in \mathscr{A}(A, D(I))$ for all $I \in \mathbf{I}$ and

$$(\mathscr{A}(A, Du))(f_I) = f_J,$$

for all $I \xrightarrow{u} J$ in $\mathbf{I}$.

But this equation just says that $(Du) \circ f_I = f_J$.

So an element of $\varprojlim_{\mathbf{I}} \mathscr{A}(A, D)$ is nothing but a cone on D with vertex A.

Representables Preserve Limits

Proposition (Representables Preserve Limits)

Let $\mathscr{A}$ be a locally small category and $A \in \mathscr{A}$. Then $\mathscr{A}(A, -) : \mathscr{A} \to \mathbf{Set}$ preserves limits.

- Let $\mathbf{I}$ be a small category and let $D : \mathbf{I} \to \mathscr{A}$ be a diagram that has a limit. Then

$$\mathscr{A}(A, \lim_{\leftarrow \mathbf{I}} D) \cong \mathrm{Cone}(A, D) \cong \lim_{\leftarrow \mathbf{I}} \mathscr{A}(A, D)$$

naturally in A.

Here the first isomorphism is from a previous proposition and the second from the preceding lemma.

The Dual Statement

- The preceding proposition tells us that

$$\mathscr{A}(A, \lim_{\leftarrow \mathbf{I}} D) \cong \lim_{\leftarrow \mathbf{I}} \mathscr{A}(A, D).$$

- To dualize, we replace $\mathscr{A}$ by $\mathscr{A}^{\text{op}}$.
- Thus, $\mathscr{A}(-, A) : \mathscr{A}^{\text{op}} \to \mathbf{Set}$ preserves limits.
- A limit in $\mathscr{A}^{\text{op}}$ is a colimit in $\mathscr{A}$, so $\mathscr{A}(-, A)$ transforms colimits in $\mathscr{A}$ into limits in **Set**:

$$\mathscr{A}(\lim_{\rightarrow \mathbf{I}} D, A) \cong \lim_{\leftarrow \mathbf{I}} \mathscr{A}(D, A).$$

- The right-hand side is a limit, not a colimit!
- So even though the two preceding natural isomorphisms are dual statements, there are, in total, more limits than colimits involved.

Example

- Let X, Y and A be objects of a category $\mathscr{A}$.
- Suppose that the sum $X+Y$ exists.
- By definition of sum, a map $X+Y \to A$ amounts to a pair of maps $(X \to A, Y \to A)$.
- In other words, there is a canonical isomorphism
$$\mathscr{A}(X+Y,A) \cong \mathscr{A}(X,A) \times \mathscr{A}(Y,A).$$
- This is the isomorphism exhibited in the preceding slide in the case where $\mathbf{I}$ is the discrete category with two objects.

Functor Categories

- Consider a small category $\mathbf{A}$ and a locally small category $\mathscr{S}$.
- The the functor category $[\mathbf{A},\mathscr{S}]$ is locally small.
- The most important cases for us will be $\mathscr{S} = \mathbf{Set}$ and $\mathscr{S} = \mathbf{Set}^{\mathrm{op}}$.
- For that reason, we will assume whenever necessary that $\mathscr{S}$ has all limits and colimits.
- We show that limits and colimits in $[\mathbf{A},\mathscr{S}]$ work in the simplest way imaginable.
- For instance, if $\mathscr{S}$ has binary products then so does $[\mathbf{A},\mathscr{S}]$, and the product of two functors $X, Y : \mathbf{A} \to \mathscr{S}$ is the functor $X \times Y : \mathbf{A} \to \mathscr{S}$ given by

  $$(X \times Y)(A) = X(A) \times Y(A),$$

  for all $A \in \mathbf{A}$.

Notation

- Let $\mathbf{A}$ and $\mathscr{S}$ be categories.
- For each $A \in \mathbf{A}$, there is a functor

$$\begin{array}{rccc} \mathrm{ev}_A : & [\mathbf{A}, \mathscr{S}] & \to & \mathscr{S} \\ & X & \mapsto & X(A), \end{array}$$

  called **evaluation at** A.
- We will be working with diagrams in $[\mathbf{A}, \mathscr{S}]$, and given such a diagram $D : \mathbf{I} \to [\mathbf{A}, \mathscr{S}]$, we have for each $A \in \mathbf{A}$ a functor

$$\begin{array}{rccc} \mathrm{ev}_A \circ D : & \mathbf{I} & \to & \mathscr{S} \\ & \mathbf{I} & \mapsto & D(I)(A). \end{array}$$

- We write $\mathrm{ev}_A \circ D$ as $D(-)(A)$.

Limits in Functor Categories

Theorem (Limits in Functor Categories)

Let $\mathbf{A}$ and $\mathbf{I}$ be small categories and $\mathscr{S}$ a locally small category. Let $D : \mathbf{I} \to [\mathbf{A}, \mathscr{S}]$ be a diagram, and suppose that for each $A \in \mathbf{A}$, the diagram $D(-)(A) : \mathbf{I} \to \mathscr{S}$ has a limit. Then there is a cone on D whose image under ev_A is a limit cone on $D(-)(A)$ for each $A \in \mathbf{A}$. Moreover, any such cone on D is a limit cone.

- The statement is often expressed as a slogan:

  Limits in a functor category are computed pointwise.

- The "points" in the word "pointwise" are the objects of $\mathbf{A}$.

Remarks

- The slogan means, for example, that given two functors $X, Y \in [\mathbf{A}, \mathscr{S}]$, their product can be computed by:
  - First taking the product $X(A) \times Y(A)$ in $\mathscr{S}$ for each "point" A;
  - Then assembling them to form a functor $X \times Y$.
- Of course, the theorem has a dual, stating that colimits in a functor category are also computed pointwise.

Proof of the Theorem

- Take for each $A \in \mathbf{A}$ a limit cone

$$(L(A) \xrightarrow{p_{I,A}} D(I)(A))_{I \in \mathbf{I}}$$

on the diagram $D(-)(A) : \mathbf{I} \to \mathscr{S}$.

We prove two statements:

(a) There is exactly one way of extending L to a functor on $\mathbf{A}$ with the property that $(L \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ is a cone on D;

(b) This cone $(L \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ is a limit cone.

The theorem will follow immediately.

Proof of the Theorem (Cont'd)

(a) Take a map $f : A \to A'$ in $\mathbf{A}$.
By a previous lemma applied to the natural transformation

$$\mathbf{I} \underset{D(-)(A')}{\overset{D(-)(A)}{\Downarrow D(-)(f)}} \mathscr{S}$$

there is a unique map $L(f) : L(A) \to L(A')$ such that for all $I \in \mathbf{I}$, the square commutes:

$$\begin{array}{ccc} L(A) & \xrightarrow{p_{I,A}} & D(I)(A) \\ {\scriptstyle L(f)}\downarrow & & \downarrow{\scriptstyle D(I)(f)} \\ L(A') & \xrightarrow[p_{I,A'}]{} & D(I)(A') \end{array}$$

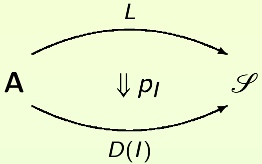
This is our definition of $L(f)$.

Proof of the Theorem (Cont'd)

- We have now defined L on objects and maps of A.

  It is easy to check that L preserves composition and identities, and is therefore a functor $L : \mathbf{A} \to \mathscr{S}$.

  Moreover, the commutativity of the square above says exactly that for each $I \in \mathbf{I}$, the family $(L(A) \xrightarrow{p_{I,A}} D(I)(A))_{A \in \mathbf{A}}$ is a natural transformation



  So we have a family $(L \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ of maps in $[\mathbf{A}, \mathscr{S}]$.

  From the fact that $(L(A) \xrightarrow{p_{I,A}} D(I)(A))_{I \in \mathbf{I}}$ is a cone on $D(-)(A)$ for each $A \in \mathbf{A}$, it follows immediately that $(L \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ is a cone on D.

Proof of the Theorem (Cont'd)

(b) Let $X \in [\mathbf{A}, \mathscr{S}]$

Let $(X \xrightarrow{q_I} D(I))_{I \in \mathbf{I}}$ be a cone on D in $[\mathbf{A}, \mathscr{S}]$.

For each $A \in \mathbf{A}$, we have a cone

$$(X(A) \xrightarrow{q_{I,A}} D(I)(A))_{I \in \mathbf{I}}$$

on $D(-)(A)$ in $\mathscr{S}$.

So there is a unique map $\overline{q}_A : X(A) \to L(A)$ such that $p_{I,A} \circ \overline{q}_A = q_{I,A}$ for all $I \in \mathbf{I}$.

It only remains to prove that $\overline{q}_A$ is natural in A.

But that follows from a previous lemma.

A Consequence and a Warning

Corollary

Let $\mathbf{I}$ and $\mathbf{A}$ be small categories, and $\mathscr{S}$ a locally small category. If $\mathscr{S}$ has all limits (respectively, colimits) of shape $\mathbf{I}$ then so does $[\mathbf{A},\mathscr{S}]$, and for each $A \in \mathbf{A}$, the evaluation functor $\mathrm{ev}_A : [\mathbf{A},\mathscr{S}] \to \mathscr{S}$ preserves them.

- If $\mathscr{S}$ does not have all limits of shape $\mathbf{I}$ then $[\mathbf{A},\mathscr{S}]$ may contain limits of shape $\mathbf{I}$ that are not computed pointwise, that is, are not preserved by all the evaluation functors.

On Commutation of Limits

- Take categories $\mathbf{I}$, $\mathbf{J}$ and $\mathscr{S}$.
- There are isomorphisms of categories

$$[\mathbf{I},[\mathbf{J},\mathscr{S}]] \cong [\mathbf{I}\times\mathbf{J},\mathscr{S}] \cong [\mathbf{J},[\mathbf{I},\mathscr{S}]].$$

- Under these isomorphisms, a functor $D:\mathbf{I}\times\mathbf{J}\to\mathscr{S}$ corresponds to the functors

$$\begin{array}{rccc} D^{\bullet}: & \mathbf{I} & \to & [\mathbf{J},\mathscr{S}] \\ & I & \mapsto & D(I,-) \end{array} \quad \text{and} \quad \begin{array}{rccc} D_{\bullet}: & \mathbf{J} & \to & [\mathbf{I},\mathscr{S}] \\ & J & \mapsto & D(-,J). \end{array}$$

- Supposing that $\mathscr{S}$ has all limits, so do the various functor categories, by the preceding corollary.

On Commutation of Limits (Cont'd)

- In particular, there is an object $\varprojlim_{\mathbf{I}} D^{\bullet}$ of $[\mathbf{J}, \mathscr{S}]$.
- This is itself a diagram in $\mathscr{S}$, so we obtain in turn an object $\varprojlim_{\mathbf{J}} \varprojlim_{\mathbf{I}} D^{\bullet}$ of $\mathscr{S}$.
- Alternatively, we can take limits in the other order, producing an object $\varprojlim_{\mathbf{I}} \varprojlim_{\mathbf{J}} D_{\bullet}$ of $\mathscr{S}$.
- And there is a third possibility, i.e., taking the limit of D itself, we obtain another object $\varprojlim_{\mathbf{I} \times \mathbf{J}} D$ of $\mathscr{S}$.
- The next result states that these three objects are the same.

Limits Commute With Limits

Proposition (Limits Commute With Limits)

Let $\mathbf{I}$ and $\mathbf{J}$ be small categories. Let $\mathscr{S}$ be a locally small category with limits of shape $\mathbf{I}$ and of shape $\mathbf{J}$. Then for all $D : \mathbf{I} \times \mathbf{J} \to \mathscr{S}$, we have

$$\lim_{\leftarrow \mathbf{J}} \lim_{\leftarrow \mathbf{I}} D^{\bullet} \cong \lim_{\leftarrow \mathbf{I} \times \mathbf{J}} D \cong \lim_{\leftarrow \mathbf{I}} \lim_{\leftarrow \mathbf{J}} D_{\bullet},$$

and all these limits exist. In particular, $\mathscr{S}$ has limits of shape $\mathbf{I} \times \mathbf{J}$.

- By symmetry, it is enough to prove the first isomorphism.

  Since $\mathscr{S}$ has limits of shape $\mathbf{I}$, so does $[\mathbf{J}, \mathscr{S}]$.

  So $\lim_{\leftarrow \mathbf{I}} D^{\bullet}$ exists. and is an object of $[\mathbf{J}, \mathscr{S}]$.

  Since $\mathscr{S}$ has limits of shape $\mathbf{J}$, $\lim_{\leftarrow \mathbf{J}} \lim_{\leftarrow \mathbf{I}} D^{\bullet}$ exists and is an object of $\mathscr{S}$.

Limits Commute With Limits (Cont'd)

- Then for $S \in \mathscr{S}$,

$$\begin{array}{rcl}\mathscr{S}(S, \varprojlim_{\mathbf{J}} \varprojlim_{\mathbf{I}} D^\bullet) & \cong & [\mathbf{J}, \mathscr{S}](\Delta S, \varprojlim_{\mathbf{I}} D^\bullet) \\ & \cong & [\mathbf{I}, [\mathbf{J}, \mathscr{S}]](\Delta(\Delta S), D^\bullet) \\ & \cong & [\mathbf{I} \times \mathbf{J}, \mathscr{S}](\Delta S, D)\end{array}$$

  naturally in S.

  The first two steps each follow from a previous proposition.

  The third uses the isomorphism $[\mathbf{I}, [\mathbf{J}, \mathscr{S}]] \cong [\mathbf{I} \times \mathbf{J}, \mathscr{S}]$, under which $\Delta(\Delta S)$ corresponds to ΔS and $D^\bullet$ corresponds to D.

  Hence $\varprojlim_{\mathbf{J}} \varprojlim_{\mathbf{I}} D^\bullet$ is a representing object for the functor $[\mathbf{I} \times \mathbf{J}, \mathscr{S}](\Delta -, D)$.

  By the same proposition, this says that $\varprojlim_{\mathbf{I} \times \mathbf{J}} D$ exists and is isomorphic to $\varprojlim_{\mathbf{J}} \varprojlim_{\mathbf{I}} D^\bullet$.

Example

- When $\mathbf{I} = \mathbf{J} = \bullet \quad \bullet$, the proposition says that binary products commute with binary products.
- If $\mathscr{S}$ has binary products and $S_{11}, S_{12}, S_{21}, S_{22} \in \mathscr{S}$, then the 4-fold product $\prod_{i,j \in \{1,2\}} S_{ij}$ exists and satisfies
$$(S_{11} \times S_{21}) \times (S_{12} \times S_{22}) \cong \prod_{i,j \in \{1,2\}} S_{ij} \cong (S_{11} \times S_{12}) \times (S_{21} \times S_{22}).$$
- More generally, it makes no difference what order we write products in or where we put the brackets:
There are canonical isomorphisms
$$\begin{array}{rcl} S \times T & \cong & T \times S, \\ (S \times T) \times U & \cong & S \times (T \times U) \end{array}$$
in any category with binary products.
- If there is also a terminal object 1, there are further canonical isomorphisms $S \times 1 \cong S \cong 1 \times S$.

The Dual Proposition

- The dual of the proposition states that colimits commute with colimits.
- For instance,

$$(S_{11}+S_{21})+(S_{12}+S_{22})\cong(S_{11}+S_{12})+(S_{21}+S_{22})$$

  in any category $\mathscr{S}$ with binary sums.
- But limits do not in general commute with colimits.
- For instance, in general,

$$(S_{11}+S_{21})\times(S_{12}+S_{22})\ncong(S_{11}\times S_{12})+(S_{21}\times S_{22}).$$

- A counterexample is given by taking $\mathscr{S}=\mathbf{Set}$ and each S_{ij} to be a one-element set.

  Then the left-hand side has $(1+1)\times(1+1)=4$ elements, whereas the right-hand side has $(1\times1)+(1\times1)=2$ elements.

Limits and Colimits in Presheaf Categories

Corollary

Let $\mathscr{A}$ be a small category. Then $[\mathscr{A}^{\mathrm{op}}, \mathbf{Set}]$ has all limits and colimits, and for each $A \in \mathscr{A}$, the evaluation functor $\mathrm{ev}_A : [\mathscr{A}^{\mathrm{op}}, \mathbf{Set}] \to \mathbf{Set}$ preserves them.

- Since **Set** has all limits and colimits, this is immediate from a preceding corollary.

Limits and the Yoneda Embedding

Corollary

The Yoneda embedding $H_\bullet : \mathbf{A} \to [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$ preserves limits, for any small category $\mathbf{A}$.

- Let $D : \mathbf{I} \to \mathbf{A}$ be a diagram in $\mathbf{A}$. Let $(\varprojlim_{\mathbf{I}} D \xrightarrow{p_I} D(I))_{I \in \mathbf{I}}$ be a limit cone.

  For each $A \in \mathbf{A}$, the composite functor $\mathbf{A} \xrightarrow{H_\bullet} [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}] \xrightarrow{\mathrm{ev}_A} \mathbf{Set}$ is H^A, which we know preserves limits. So for each $A \in \mathbf{A}$,

  $$(\mathrm{ev}_A H_\bullet(\varprojlim_{\mathbf{I}} D) \xrightarrow{\mathrm{ev}_A H_\bullet(p_I)} \mathrm{ev}_A H_\bullet(D(I)))_{I \in \mathbf{I}}$$

  is a limit cone. But then, by a previous theorem applied to the diagram $H_\bullet \circ D$ in $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$, the cone

  $$(H_\bullet(\varprojlim_{\mathbf{I}} D) \xrightarrow{H_\bullet(p_I)} H_\bullet(D(I)))_{I \in \mathbf{I}}$$

  is also a limit, as required.

Example

- Let $\mathbf{A}$ be a category with binary products.
- By the corollary, for all $X, Y \in \mathbf{A}$, $H_{X \times Y} \cong H_X \times H_Y$ in $[\mathbf{A}^{\text{op}}, \mathbf{Set}]$.
- When evaluated at a particular object A, this says that

$$\mathbf{A}(A, X \times Y) \cong \mathbf{A}(A, X) \times \mathbf{A}(A, Y)$$

  (using the fact that products are computed pointwise).
- This is the isomorphism that we met at the beginning of this section.
- Suppose that we view $\mathbf{A}$ as a subcategory of $[\mathbf{A}^{\text{op}}, \mathbf{Set}]$, identifying $A \in \mathbf{A}$ with the representable $H_A \in [\mathbf{A}^{\text{op}}, \mathbf{Set}]$.
- Then the isomorphism above means that given two objects of $\mathbf{A}$ whose product we want to form, it makes no difference whether we think of the product as taking place in $\mathbf{A}$ or $[\mathbf{A}^{\text{op}}, \mathbf{Set}]$.
- Similarly, if $\mathbf{A}$ has all limits, taking limits does not help us to escape from $\mathbf{A}$ into the rest of $[\mathbf{A}^{\text{op}}, \mathbf{Set}]$:
  Any limit of representable presheaves is again representable.

Remark on Colimits and the Yoneda Embedding

- The Yoneda embedding does not preserve colimits.
- For example, if **A** has an initial object 0, then H_0 is not initial:
  - $H_0(0) = \mathbf{A}(0,0)$ is a one-element set;
  - The initial object of $[\mathbf{A}^{\text{op}}, \mathbf{Set}]$ is the presheaf with constant value $\emptyset$.
- We investigate colimits of representables next.

Introduction to the "Power" of Colimits

- We know that the Yoneda embedding preserves limits but not colimits.
- The situation for colimits is at the opposite extreme from the situation for limits:

  > By taking colimits of representable presheaves, we can obtain any presheaf we like!

- Every positive integer can be expressed as a product of primes in an essentially unique way.
- Somewhat similarly, every presheaf can be expressed as a colimit of representables in a canonical (though not unique) way.
- The representables are the building blocks of presheaves.
- By analogy, recalling that any complex function holomorphic in a neighborhood of 0 has a power series expansion, such as $e^z = 1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots$, the power functions $z \mapsto z^n$ are the building blocks of holomorphic functions.
- Taking the analogy further, $(\ \ )^n$ is like a representable $\mathrm{Hom}(n, -)$, and in the categorical context, quotients and sums are types of colimit.

Example

- Let $\mathbf{A}$ be the discrete category with two objects, K and L.
- A presheaf X on $\mathbf{A}$ is just a pair $(X(K), X(L))$ of sets, and $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}] \cong \mathbf{Set} \times \mathbf{Set}$.
- There are two representables, H_K and H_L, given, for $A, B \in \{K, L\}$, by

$$H_A(B) = \mathbf{A}(B, A) \cong \begin{cases} 1, & \text{if } A = B \\ \varnothing, & \text{if } A \neq B \end{cases}.$$

- Identifying $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$ with $\mathbf{Set} \times \mathbf{Set}$, we have $H_K \cong (1, \varnothing)$ and $H_L \cong (\varnothing, 1)$.
- Every object of $\mathbf{Set} \times \mathbf{Set}$ is a sum of copies of $(1, \varnothing)$ and $(\varnothing, 1)$.

Example (Cont'd)

- Suppose, for instance, that $X(K)$ has three elements and $X(L)$ has two elements.
- Then

$$(X(K),X(L)) \cong (1,\varnothing)+(1,\varnothing)+(1,\varnothing)+(\varnothing,1)+(\varnothing,1)$$

  in $\mathbf{Set}\times\mathbf{Set}$.
- Equivalently,

$$X \cong H_K + H_K + H_K + H_L + H_L$$

  in $[\mathbf{A}^{\mathrm{op}},\mathbf{Set}]$, exhibiting X as a sum of representables.

Category of Elements

- In the example, X is expressed as a sum of five representables, that is, a sum indexed by the set $X(K)+X(L)$ of "elements" of X.
- A sum is a colimit over a discrete category.
- In the general case, a presheaf X on a category $\mathbf{A}$ is expressed as a colimit over a category whose objects can be thought of as the "elements" of X.

Definition

Let $\mathbf{A}$ be a category and X a presheaf on $\mathbf{A}$. The **category of elements** $\mathbf{E}(X)$ of X is the category in which:

- Objects are pairs (A, x) with $A \in \mathbf{A}$ and $x \in X(A)$;
- Maps $(A', x') \to (A, x)$ are maps $f : A' \to A$ in $\mathbf{A}$ such that $(Xf)(x) = x'$.

There is a projection functor $P : \mathbf{E}(X) \to \mathbf{A}$ defined by $P(A, x) = A$ and $P(f) = f$.

Density Theorem

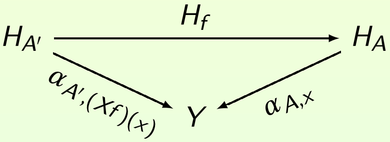
Theorem (Density)

Let $\mathbf{A}$ be a small category and X a presheaf on $\mathbf{A}$. Then X is the colimit of the diagram

$$\mathbf{E}(X) \xrightarrow{P} \mathbf{A} \xrightarrow{H_\bullet} [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$$

in $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$. That is, $X \cong \varinjlim_{\mathbf{I}} (H_\bullet \circ P)$.

- First note that since $\mathbf{A}$ is small, so too is $\mathbf{E}(X)$.
  Hence $H_\bullet \circ P$ really is a diagram in our customary sense.
  Now let $Y \in [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$. A cocone on $H_\bullet \circ P$ with vertex Y is a family $(H_A \xrightarrow{\alpha_{A,x}} Y)_{A \in \mathbf{A}, x \in X(A)}$ of natural transformations with the property that for all maps $A' \xrightarrow{f} A$ in $\mathbf{A}$ and all $x \in X(A)$, the following diagram commutes:



Density Theorem

- Equivalently (by the Yoneda lemma), a cocone on $H_\bullet \circ P$ with vertex Y is a family $(y_{A,x})_{A\in\mathbf{A},x\in X(A)}$, with $y_{A,x}\in Y(A)$, such that for all maps $A'\xrightarrow{f}A$ in $\mathbf{A}$ and all $x\in X(A)$, $(Yf)(y_{A,x})=y_{A',(Xf)(x)}$.
  To see this, note that if $\alpha_{A,x}\in[\mathbf{A}^{\mathrm{op}},\mathbf{Set}](H_A,Y)$ corresponds to $y_{A,x}\in Y(A)$, then $\alpha_{A,x}\circ H_f\in[\mathbf{A}^{\mathrm{op}},\mathbf{Set}](H_{A'},Y)$ corresponds to $(Yf)(y_{A,x})\in Y(A')$.

  Equivalently (writing $y_{A,x}$ as $\overline{\alpha}_A(x)$), it is a family $(X(A)\xrightarrow{\overline{\alpha}_A}Y(A))_{A\in\mathbf{A}}$ of functions with the property that for all maps $A'\xrightarrow{f}A$ in $\mathbf{A}$ and all $x\in X(A)$, $(Yf)(\overline{\alpha}_A(x))=\overline{\alpha}_{A'}((Xf)(x))$.
  But this is simply a natural transformation $\overline{\alpha}:X\to Y$.
  So we have, for each $Y\in[\mathbf{A}^{\mathrm{op}},\mathbf{Set}]$, a canonical bijection

$$[\mathbf{E}(X),[\mathbf{A}^{\mathrm{op}},\mathbf{Set}]](H_\bullet\circ P,\Delta Y)\cong[\mathbf{A}^{\mathrm{op}},\mathbf{Set}](X,Y).$$

  Hence X is the colimit of $H_\bullet\circ P$.

Example

- In the previous example we expressed a particular presheaf X as a sum of representables.
- Let us check that the way we did this is a special case of the general construction in the density theorem.
- Since $\mathbf{A}$ is discrete, the category of elements $\mathbf{E}(X)$ is also discrete;
  It is the set $X(K)+X(L)$ with five elements.
- The projection $P:\mathbf{E}(X)\to\mathbf{A}$ sends three of the elements to K and the other two to L.
- So the diagram $H_\bullet\circ P:\mathbf{E}(X)\to[\mathbf{A}^{\mathrm{op}},\mathbf{Set}]$ sends three of the elements to H_K and two to H_L.
- The colimit of $H_\bullet\circ P$ is the sum of these five representables, which is X.

Category of Elements Versus Generalized Elements

- The term "category of elements" is compatible with the generalized element terminology.
- A generalized element of an object X is just a map into X, say $Z \to X$.
- As explained after that definition, we often focus on certain special shapes Z.
- Now suppose that we are working in a presheaf category $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$.
- Among all presheaves, the representables have a special status, so we might be especially interested in generalized elements of representable shape.
- The Yoneda lemma implies that for a presheaf X, the generalized elements of X of representable shape correspond to pairs (A, x) with $A \in \mathbf{A}$ and $x \in X(A)$.
- In other words, they are the objects of the category of elements.

On the Term "Density"

- In topology, a subspace A of a space B is called dense if every point in B can be obtained as a limit of points in A.
- This provides some explanation for the name of the theorem,
- The category $\mathbf{A}$ is "dense" in $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$ because every object of $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$ can be obtained as a colimit of objects of $\mathbf{A}$.

Subsection 3

Interactions Between Adjoint Functors and Limits

Adjoints, Limits and Colimits

- We saw that any set-valued functor with a left adjoint is representable.
- We also saw that any representable preserves limits.
- Hence, any set-valued functor with a left adjoint preserves limits.
- This conclusion holds not only for set-valued functors:

Theorem

Let $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ be an adjunction. Then F preserves colimits and G preserves limits.

- By duality, it is enough to prove that G preserves limits.

  Let $D : \mathbf{I} \to \mathscr{B}$ be a diagram for which a limit exists.

Adjoints, Limits and Colimits (Cont'd)

- Then

$$\begin{array}{rcl}\mathscr{A}(A, G(\varprojlim D)) & \cong & \mathscr{B}(F(A), \varprojlim D) \\ & \cong & \varprojlim \mathscr{B}(F(A), D) \\ & \cong & \varprojlim \mathscr{A}(A, G \circ D) \\ & \cong & \mathrm{Cone}(A, G \circ D)\end{array}$$

naturally in $A \in \mathscr{A}$.

The first isomorphism is by adjointness.

The second is because representables preserve limits.

The third is by adjointness again

The last is by a previous lemma.

So $G(\varprojlim D)$ represents $\mathrm{Cone}(-, G \circ D)$.

That is, it is a limit of $G \circ D$.

Example

- Forgetful functors from categories of algebras to **Set** have left adjoints, but hardly ever right adjoints.
- Correspondingly, they preserve all limits, but rarely all colimits.

Example

- Every set B gives rise to an adjunction $(- \times B) \dashv (-)^B$ of functors from **Set** to **Set**.
- So $- \times B$ preserves colimits and $(-)^B$ preserves limits.
- In particular, $- \times B$ preserves finite sums and $(-)^B$ preserves finite products.
- This gives isomorphisms

$$\begin{array}{rclcrcl} 0 \times B & \cong & 0 & \qquad & (A_1 + A_2) \times B & \cong & (A_1 \times B) + (A_2 \times B) \\ 1^B & \cong & 1 & & (A_1 \times A_2)^B & \cong & A_1^B \times A_2^B. \end{array}$$

- These are the analogues of standard rules of arithmetic.
- Indeed, if we know these for just finite sets then by taking cardinality on both sides, we obtain exactly these standard rules.

Example

- Given a category $\mathscr{A}$ with all limits of shape $\mathbf{I}$, we have the adjunction
$$\mathscr{A} \underset{\varprojlim_{\mathbf{I}}}{\overset{\Delta}{\rightleftarrows}} [\mathbf{I}, \mathscr{A}].$$
- Hence $\varprojlim_{\mathbf{I}}$ preserves limits, or equivalently, limits of shape $\mathbf{I}$ commute with (all) limits.
- This gives another proof that limits commute with limits, at least in the case where the category has all limits of one of the shapes concerned.

Example

- The theorem is often used to prove that a functor does not have an adjoint.
- For instance, it was claimed in a previous example that the forgetful functor $U : \mathbf{Field} \to \mathbf{Set}$ does not have a left adjoint.
- We can now prove this.
- If U had a left adjoint $F : \mathbf{Set} \to \mathbf{Field}$, then F would preserve colimits, and in particular, initial objects.
- Hence $F(\emptyset)$ would be an initial object of **Field**.
- But **Field** has no initial object, since there are no maps between fields of different characteristic.

Completeness

- Every functor with a left adjoint preserves limits.
- But limit-preservation alone does not guarantee the existence of a left adjoint.
- For example, let $\mathscr{B}$ be any category.

  The unique functor $\mathscr{B} \to \mathbf{1}$ always preserves limits.

  But, by a previous example, it only has a left adjoint if $\mathscr{B}$ has an initial object.
- On the other hand, if we have a limit-preserving functor $G : \mathscr{B} \to \mathscr{A}$ and $\mathscr{B}$ has all limits, then there is an excellent chance that G has a left adjoint.
- It is still not always true, but counterexamples are harder to find.
- The condition of having all limits has its own word:

Definition

A category is **complete** (or properly, **small complete**) if it has all limits.

Introducing Adjoint Functor Theorems

- There are various results called adjoint functor theorems, all of the following form:

  Let $\mathscr{A}$ be a category, $\mathscr{B}$ a complete category, and $G : \mathscr{B} \to \mathscr{A}$ a functor. Suppose that $\mathscr{A}$, $\mathscr{B}$ and G satisfy certain further conditions. Then

  $$G \text{ has a left adjoint } \Leftrightarrow G \text{ preserves limits.}$$

- The forwards implication is immediate from a previous theorem.
- It is the backwards implication that concerns us here.
- Typically, the "further conditions" involve the distinction between small and large collections.
- But in the special case where $\mathscr{A}$ and $\mathscr{B}$ are ordered sets these complications disappear.
- We use this to explain the main idea behind the proofs of the adjoint functor theorems.

Completeness and Preservation of Limits in Posets

- Recall that limits in ordered sets are meets.
- More precisely, if $D : \mathbf{I} \to \mathbf{B}$ is a diagram in an ordered set $\mathbf{B}$, then

$$\lim_{\leftarrow \mathbf{I}} D = \bigwedge_{I \in \mathbf{I}} D(I),$$

  with one side defined if and only if the other is.
- So an ordered set is complete if and only if every subset has a meet.
- Similarly, a map $G : \mathbf{B} \to \mathbf{A}$ of ordered sets preserves limits if and only if

$$G(\bigwedge_{i \in I} B_i) = \bigwedge_{i \in I} G(B_i),$$

  whenever $(B_i)_{i \in I}$ is a family of elements of $\mathbf{B}$ for which a meet exists.
- We now show that for ordered sets, there is an adjoint functor theorem of the simplest possible kind, i.e., in which there are no "further conditions" at all.

Adjoint Functor Theorem for Ordered Sets

Proposition (Adjoint Functor Theorem for Ordered Sets)

Let $\mathbf{A}$ be an ordered set, $\mathbf{B}$ a complete ordered set, and $G : \mathbf{B} \to \mathbf{A}$ an order-preserving map. Then

$$G \text{ has a left adjoint } \Leftrightarrow G \text{ preserves meets.}$$

- Suppose that G preserves meets.

  By a previous corollary, it is enough to show that for each $A \in \mathbf{A}$, the comma category $(A \Rightarrow G)$ has an initial object.

  Let $A \in \mathbf{A}$. Then $(A \Rightarrow G)$ is an ordered set, namely, $\{B \in \mathbf{B} : A \leq G(B)\}$ with the order inherited from $\mathbf{B}$.

  We have to show that $(A \Rightarrow G)$ has a least element.

Adjoint Functor Theorem for Ordered Sets (Cont'd)

- Since **B** is complete, the meet $\bigwedge_{B \in \mathbf{B}: A \le G(B)} B$ exists in **B**.
  This is the meet of all the elements of $(A \Rightarrow G)$.
  So it suffices to show that the meet is itself an element of $(A \Rightarrow G)$.
  And indeed, since G preserves meets, we have
  $$G\left(\bigwedge_{B \in \mathbf{B}: A \le G(B)} B\right) = \bigwedge_{B \in \mathbf{B}: A \le G(B)} G(B) \ge A.$$
- In the general setting, the initial object of $(A \Rightarrow G)$ is the pair $(F(A), A \stackrel{\eta_A}{\to} GF(A))$, where F is the left adjoint and η is the unit map.
  So in the proposition, the left adjoint F is given by
  $$F(A) = \bigwedge_{B \in \mathbf{B}: A \le G(B)} B.$$

Example

- Consider the proposition in the case $\mathbf{A} = \mathbf{1}$.
- The unique functor $G : \mathbf{B} \to \mathbf{1}$ automatically preserves meets.
- Also, as observed above, a left adjoint to G is an initial object of $\mathbf{B}$.
- So in the case $\mathbf{A} = \mathbf{1}$, the proposition states that a complete ordered set has a least element.
- This is not quite trivial, since completeness means the existence of all meets, whereas a least element is an empty join.
- By the formula $F(A) = \bigwedge_{B \in \mathbf{B} : A \leq G(B)} B$, the least element of $\mathbf{B}$ is $\bigwedge_{B \in \mathbf{B}} B$.
- Thus, a least element is not only a colimit of the functor $\varnothing \to \mathbf{B}$, it is also a limit of the identity functor $\mathbf{B} \to \mathbf{B}$.
- The synonym "least upper bound" for "join" suggests a theorem:

  A poset with all meets also has all joins.
- Indeed, given a poset $\mathbf{B}$ with all meets, the join of a subset of $\mathbf{B}$ is simply the meet of its upper bounds (its least upper bound).

From Ordered-Sets to Categories

- Start with a limit-preserving functor G from a complete category $\mathscr{B}$ to a category $\mathscr{A}$.
- In the case of ordered sets, we had for each $A \in \mathscr{A}$ an inclusion map $P_A : (A \Rightarrow G) \hookrightarrow \mathbf{B}$, and we showed that the left adjoint F was given by $F(A) = \lim_{\leftarrow (A \Rightarrow G)} P_A$.
- In the general case, the analogue of the inclusion functor is the projection functor

$$\begin{array}{rccc} P_A : & (A \Rightarrow G) & \to & \mathscr{B} \\ & (B, A \xrightarrow{f} G(B)) & \mapsto & B. \end{array}$$

- The case of ordered sets suggests that in general, the preceding equation might define a left adjoint F to G.
- And indeed, it can be shown that if this limit in $\mathscr{B}$ exists and is preserved by G, then the formula does really give a left adjoint.

From Ordered-Sets to Categories (Cont'd)

- This might seem to suggest that our adjoint functor theorem generalizes smoothly from ordered sets to arbitrary categories, with no need for further conditions.
- But it does not, for reasons that are quite subtle.
- Those reasons are more easily explained if we relax our terminology slightly.
- When we defined limits, we built in the condition that the shape category $\mathbf{I}$ was small.
- However, the definition of limit makes sense for an arbitrary category $\mathbf{I}$.
- In this discussion, we will need to refer to this more inclusive notion of limit, so we temporarily suspend the convention that the shape categories $\mathbf{I}$ of limits are always small.

From Ordered-Sets to Categories (Cont'd)

- Now, in the template for adjoint functor theorems, it was only required that $\mathscr{B}$ has, and G preserves, small limits.
- But if $\mathscr{B}$ is a large category then $(A \Rightarrow G)$ might also be large, since to specify an object or map in $(A \Rightarrow G)$, we have to specify (among other things) an object or map in $\mathscr{B}$.
- So, the limit defining the left adjoint is not guaranteed to be small.
- Hence there is no guarantee that this limit exists in $\mathscr{B}$, nor that it is preserved by G.
- It follows that the functor F "defined" by the formula above might not be defined at all, let alone a left adjoint.
- For difficulties with reasoning about small and large collections, it might be useful to compare finite and infinite collections.

  For instance, if $\mathscr{B}$ is a finite category and $\mathscr{A}$ has finite hom-sets then $(A \Rightarrow G)$ is also finite, but otherwise $(A \Rightarrow G)$ might be infinite.

From Ordered-Sets to Categories (Cont'd)

- The preceding proposition still stands, since there we were dealing with ordered sets, which as categories are small.
- We might hope to extend it from posets to arbitrary small categories, since the problem just described affects only large categories.
- This turns out not to be very fruitful, since in fact, complete posets are the only complete small categories.
- Alternatively, we could try to salvage the argument by assuming that $\mathscr{B}$ has, and G preserves, all (possibly large) limits.
- But again, this is unhelpful: there are almost no such categories $\mathscr{B}$.
- The situation therefore becomes more complicated.
- Each of the best-known adjoint functor theorems imposes further conditions implying that the large limit $\lim\limits_{\leftarrow (A \Rightarrow G)} P_A$ can be replaced by a small limit in some clever way.
- This allows one to proceed with the argument above.

Weakly Initial Sets

Definition

Let $\mathscr{C}$ be a category. A **weakly initial set** in $\mathscr{C}$ is a set $\mathbf{S}$ of objects with the property that for each $C \in \mathscr{C}$, there exist an element $S \in \mathbf{S}$ and a map $S \to C$.

- Note that $\mathbf{S}$ must be a set, that is, small.
- So, the existence of a weakly initial set is some kind of size restriction.
- Such size restrictions are comparable to finiteness conditions in algebra.

The General Adjoint Functor Theorem (GAFT)

Theorem (General Adjoint Functor Theorem)

Let $\mathscr{A}$ be a category, $\mathscr{B}$ a complete category, and $G : \mathscr{B} \to \mathscr{A}$ a functor. Suppose that $\mathscr{B}$ is locally small and that for each $A \in \mathscr{A}$, the category $(A \Rightarrow G)$ has a weakly initial set. Then

$$G \text{ has a left adjoint } \Leftrightarrow G \text{ preserves limits.}$$

- The heart of the proof is the case $\mathscr{A} = \mathbf{1}$, where GAFT asserts that a complete locally small category with a weakly initial set has an initial object.

Weakly Initial Sets and Initial Objects

Lemma

Let $\mathscr{C}$ be a complete locally small category with a weakly initial set. Then $\mathscr{C}$ has an initial object.

- Let $\mathbf{S}$ be a weakly initial set in $\mathscr{C}$.

  Regard $\mathbf{S}$ as a full subcategory of $\mathscr{C}$.

  Then $\mathbf{S}$ is small, since $\mathscr{C}$ is locally small.

  We may therefore take a limit cone

  $$(0 \xrightarrow{p_S} S)_{S \in \mathbf{S}}$$

  of the inclusion $\mathbf{S} \hookrightarrow \mathscr{C}$.

  We prove that 0 is initial.

Weakly Initial Sets and Initial Objects (Cont'd)

- Let $C \in \mathscr{C}$.

  We have to show that there is exactly one map $0 \to C$.

  Certainly there is at least one, since we may choose some $S \in \mathbf{S}$ and map $j : S \to C$, and we then have the composite $jp_S : 0 \to C$.

  To prove uniqueness, let $f, g : 0 \to C$.

  Form the equalizer $E \xrightarrow{i} 0 \underset{g}{\overset{f}{\rightrightarrows}} C$.

  Since $\mathbf{S}$ is weakly initial, we may choose $S \in \mathbf{S}$ and $h : S \to E$.

  We then have maps $0 \xrightarrow{p_S} S \xrightarrow{h} E \xrightarrow{i} 0$ with the property that for all $S' \in \mathbf{S}$,

  $$p_{S'}(ihp_S) = (p_{S'}ih)p_S = p_{S'} = p_{S'}1_0.$$

  By a property of limits, $ihp_S = 1_0$.

  Hence $f = fihp_S = gihp_S = g$.

Projections of Comma Categories and Creation of Limits

Lemma

Let $\mathscr{A}$ and $\mathscr{B}$ be categories. Let $G:\mathscr{B}\to\mathscr{A}$ be a functor that preserves limits. Then the projection functor $P_A:(A\Rightarrow G)\to\mathscr{B}$ creates limits, for each $A\in\mathscr{A}$. In particular, if $\mathscr{B}$ is complete then so is each comma category $(A\Rightarrow G)$.

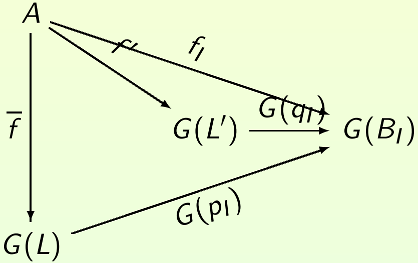
- We show the first statement; the second holds by a previous lemma.
  Suppose $\mathbf{I}$ is a small category and let $D:\mathbf{I}\to(A\Rightarrow G)$ be a diagram in $A\Rightarrow G$, with $D(I)=(A\stackrel{f_I}{\to}G(B_I))$, such that the diagram $P_AD:\mathbf{I}\to\mathscr{B}$ has a limit $(L\stackrel{p_I}{\to}B_I)_{I\in\mathbf{I}}$ in $\mathscr{B}$.
  Since $G:\mathscr{B}\to\mathscr{A}$ preserves limits, $(G(L)\stackrel{G(p_I)}{\to}G(G(B_I)))_{I\in\mathbf{I}}$ is a limit cone in $\mathscr{A}$.
  Consider the cone D in $A\Rightarrow G$.
  Since $(G(L)\stackrel{G(p_I)}{\to}G(B_I))_{I\in\mathbf{I}}$ is a limiting cone, there exists unique $\overline{f}:A\to G(L)$, such that $G(p_I)\overline{f}=f_I$, for all $I\in\mathbf{I}$.

Projections of Comma Categories and Creation of Limits

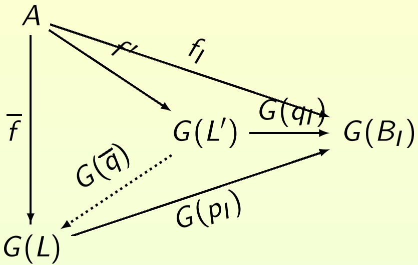
- We show that the cone $((L,\overline{f}),(p_I)_{I\in\mathbf{I}})$ is a unique cone such that $P_A((L,\overline{f}))=L$ and $P_A(p_I)=p_I$ and, moreover, it is a limiting cone in $A\Rightarrow G$.

  The relations $P_A((L,\overline{f}))=L$ and $P_A(p_I)=p_I$ are straightforward and ensure uniqueness.

  For the limiting property, suppose $((L',f'),(q_I)_{I\in\mathbf{I}})$ is another cone in $A\Rightarrow G$, i.e., such that $G(q_I)f'=f_I$, for all $I\in\mathbf{I}$.



Projections of Comma Categories and Creation of Limits



- Since $(L \xrightarrow{p_I} B_I)_{I \in \mathbf{I}}$ is a limit cone in $\mathscr{B}$, we get unique $\overline{q} : L' \to L$ in $\mathscr{B}$, such that $p_I q = q_I$, for all $I \in \mathbf{I}$.

  Now we are almost done, because we get, for all $I \in \mathbf{I}$,

$$G(p_I)G(\overline{q})f' = G(q_I)f' = f_I = G(p_I)\overline{f}.$$

  Thus, by the uniqueness of limit maps in $\mathscr{A}$, $G(\overline{q})f' = \overline{f}$.

  Thus $((L, \overline{f}), (p_I)_{I \in \mathbf{I}})$ is a limit cone of D in $A \Rightarrow G$.

Proof of GAFT

- By a previous corollary, it is enough to show that $(A \Rightarrow G)$ has an initial object for each $A \in \mathscr{A}$.

  Let $A \in \mathscr{A}$.

  By the preceding lemma, $(A \Rightarrow G)$ is complete.

  By hypothesis, it has a weakly initial set.

  It is also locally small, since $\mathscr{B}$ is.

  Hence by the previous lemma, it has an initial object, as required.

Example

- The general adjoint functor theorem (GAFT) implies that for any category $\mathscr{B}$ of algebras (**Grp**, $\mathbf{Vect}_k$, ...), the forgetful functor $U : \mathscr{B} \to \mathbf{Set}$ has a left adjoint.
- Indeed, we saw in a previous example that $\mathscr{B}$ has all limits.
- Moreover we saw that U preserves them.
- Also, $\mathscr{B}$ is locally small.
- To apply GAFT, we now just have to check that for each $A \in \mathbf{Set}$, the comma category $(A \Rightarrow U)$ has a weakly initial set.
- This requires a little cardinal arithmetic, omitted here.

Example (Cont'd)

- So GAFT tells us that, for instance, the free group functor exists.
- In previous examples, we began to see the trickiness of explicitly constructing the free group on a generating set A:
  - Define the set of "formal expressions" (such as $x^{-1}yx^2zy^{-3}$, with $x, y, z \in A$);
  - Define what it means for two such expressions to be equivalent (so that $x^{-2}x^5y$ is equivalent to x^3y);
  - Define $F(A)$ to be the set of all equivalence classes;
  - Define the group structure;
  - Check the group axioms;
  - Prove that the resulting group has the universal property required.
- Using GAFT, we can avoid these complications entirely.

Example (Cont'd)

- The price to be paid is that GAFT does not give us an explicit description of free groups (or left adjoints more generally).
- When people speak of knowing some object "explicitly", they usually mean knowing its elements.
- An element of an object is a map into it, and we have no handle on maps into $F(A)$:

  Since F is a left adjoint, it is maps out of $F(A)$ that we know about.
- This is why explicit descriptions of left adjoints are often hard to come by.

Example

- More generally, GAFT guarantees that forgetful functors between categories of algebras, such as

$$\mathbf{Ab} \to \mathbf{Grp}, \quad \mathbf{Grp} \to \mathbf{Mon}, \quad \mathbf{Ring} \to \mathbf{Mon}, \quad \mathbf{Vect}_{\mathbb{C}} \to \mathbf{Vect}_{\mathbb{R}}$$

  have left adjoints.
- This is "more generally" because **Set** can be seen as a degenerate example of a category of algebras:

  A group, ring, etc., is a set equipped with some operations satisfying some equations, and a set is a set equipped with no operations satisfying no equations.

Cartesian Closed Categories

- We saw that for every set B, there is an adjunction $(- \times B) \dashv (-)^B$.
- Moreover, for every category $\mathscr{B}$, there is an adjunction $(- \times \mathscr{B}) \dashv [\mathscr{B}, -]$.

Definition

A category $\mathscr{A}$ is **cartesian closed** if it has finite products and for each $B \in \mathscr{A}$, the functor $- \times B : \mathscr{A} \to \mathscr{A}$ has a right adjoint.

- We write the right adjoint as $(-)^B$, and, for $C \in \mathscr{A}$, call C^B an **exponential**.
- We may think of C^B as the space of maps from B to C.
- Adjointness says that for all $A, B, C \in \mathscr{A}$,

$$\mathscr{A}(A \times B, C) \cong \mathscr{A}(A, C^B)$$

  naturally in A and C.
- The isomorphism is natural in B too (that comes for free).

Examples

- **Set** is cartesian closed.
  C^B is the function set **Set**(B, C).
- **CAT** is cartesian closed.
  $\mathscr{C}^{\mathscr{B}}$ is the functor category $[\mathscr{B}, \mathscr{C}]$.

Arithmetic-like Properties

- In any cartesian closed category with finite sums, the isomorphisms

$$\begin{array}{rclcrcl} 0 \times B & \cong & 0 & \quad & (A_1 + A_2) \times B & \cong & (A_1 \times B) + (A_2 \times B) \\ 1^B & \cong & 1 & \quad & (A_1 \times A_2)^B & \cong & A_1^B \times A_2^B. \end{array}$$

  hold.
- The objects of a cartesian closed category therefore possess an arithmetic like that of the natural numbers.
- These isomorphisms provide a way of proving that a category is not cartesian closed.

Example

- $\mathbf{Vect}_k$ is not cartesian closed, for any field k.
- It does have finite products, as we saw in a previous example:
- Binary product is direct sum;
- The terminal object is the trivial vector space $\{0\}$, which is also initial.
- But if $\mathbf{Vect}_k$ were cartesian closed then the arithmetic equations would hold.
- So $\{0\} \oplus B \cong \{0\}$ for all vector spaces B.
- This is plainly false.

Preasheaves and Exponentials

- For any set I, the product category $\mathbf{Set}^I$ is cartesian closed, just because $\mathbf{Set}$ is.
- Exponentials in $\mathbf{Set}^I$, as well as products, are computed pointwise.
- Put another way, $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$ is cartesian closed whenever $\mathbf{A}$ is discrete.
- We now show that, in fact, $[\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$ is cartesian closed for any small category $\mathbf{A}$ whatsoever.
- Write $\widehat{\mathbf{A}} = [\mathbf{A}^{\mathrm{op}}, \mathbf{Set}]$.
- If $\widehat{\mathbf{A}}$ is cartesian closed, what must exponentials in $\widehat{\mathbf{A}}$ be?
- In other words, given presheaves Y and Z, what must Z^Y be in order that
$$\widehat{\mathbf{A}}(X, Z^Y) \cong \widehat{\mathbf{A}}(X \times Y, Z)$$
for all presheaves X?

Preasheaves and Exponentials (Cont'd)

- If this is true for all presheaves X, then in particular it is true when X is representable.
- So
$$Z^Y(A) \cong \widehat{\mathbf{A}}(H_A, Z^Y) \cong \widehat{\mathbf{A}}(H_A \times Y, Z)$$
for all $A \in \mathbf{A}$, the first step by Yoneda.
- This tells us what Z^Y must be.
- Notice that $Z^Y(A)$ is not simply $Z(A)^{Y(A)}$, as one might at first guess:

  Exponentials in a presheaf category are not generally computed pointwise.

Presheaf Categories and Cartesian Closedness

Theorem

For any small category $\mathbf{A}$, the presheaf category $\widehat{\mathbf{A}}$ is cartesian closed.

- The argument in the thought experiment gives us the isomorphism $\widehat{\mathbf{A}}(X, Z^Y) \cong \widehat{\mathbf{A}}(X \times Y, Z)$, whenever X is representable.

  A general presheaf X is not representable, but it is a colimit of representables, and this allows us to bootstrap our way up.

  We know that $\widehat{\mathbf{A}}$ has all limits, and in particular, finite products.

  It remains to show that $\widehat{\mathbf{A}}$ has exponentials.

  Fix $Y \in \widehat{\mathbf{A}}$. First we prove that $- \times Y : \widehat{\mathbf{A}} \to \widehat{\mathbf{A}}$ preserves colimits.

  Indeed, since products and colimits in $\widehat{\mathbf{A}}$ are computed pointwise, it is enough to prove that for any set S, the functor $- \times S : \mathbf{Set} \to \mathbf{Set}$ preserves colimits.

  This follows from the fact that $\mathbf{Set}$ is cartesian closed.

Presheaf Categories and Cartesian Closedness (Cont'd)

- For each presheaf Z on $\mathbf{A}$, let Z^Y be the presheaf defined by
$$Z^Y(A) = \widehat{\mathbf{A}}(H_A \times Y, Z),$$
for all $A \in \mathbf{A}$.
This defines a functor
$$(-)^Y : \widehat{\mathbf{A}} \to \widehat{\mathbf{A}}.$$
Claim: $(- \times Y) \dashv (-)^Y$.

Presheaf Categories and Cartesian Closedness (Cont'd)

- Let $X, Z \in \widehat{\mathbf{A}}$.
  Write $P : \mathbf{E}(X) \to \mathbf{A}$ for the projection, and write $H_P = H_\bullet \circ P$.
  Then

$$\begin{array}{rcll}
\widehat{\mathbf{A}}(X, Z^Y) & \cong & \widehat{\mathbf{A}}(\varinjlim_{\mathbf{E}(X)} H_P, Z^Y) & \text{(previous theorem)} \\
& \cong & \varprojlim_{\mathbf{E}(X)} \widehat{\mathbf{A}}(H_P, Z^Y) & \text{(repres's preserve limits)} \\
& \cong & \varprojlim_{\mathbf{E}(X)} Z^Y(P) & \text{(Yoneda)} \\
& \cong & \varprojlim_{\mathbf{E}(X)} \widehat{\mathbf{A}}(H_P \times Y, Z) & \text{(definition)} \\
& \cong & \widehat{\mathbf{A}}(\varinjlim_{\mathbf{E}(X)} (H_P \times Y), Z) & \text{(repres's preserve limits)} \\
& \cong & \widehat{\mathbf{A}}((\varinjlim_{\mathbf{E}(X)} H_P) \times Y, Z) & (- \times Y \text{ preserves colimits}) \\
& \cong & \widehat{\mathbf{A}}(X \times Y, Z), & \text{(previous theorem)}
\end{array}$$

  naturally in X and Z.