College Algebra

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LSSU Math 111

1. Functions
   - Functions and Function Notation
   - Domain and Range
   - Rates of Change and Behavior of Graphs
   - Composition of Functions
   - Transformation of Functions
   - Absolute Value Functions
   - Inverse Functions

Subsection 1

Functions and Function Notation

We Will Learn How To:

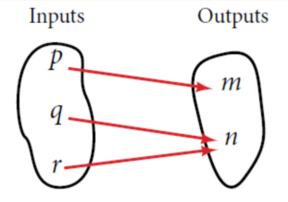
- Determine whether a relation represents a function;
- Find the value of a function;
- Determine whether a function is one-to-one;
- Use the vertical line test to identify functions;
- Use the horizontal line test to tell whether a function is one-to-one.

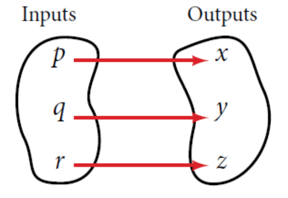
Relation, Domain and Range

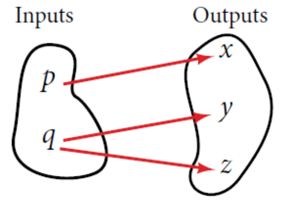
- A **relation** is a set of ordered pairs.
- E.g., $\{(1,2),(2,4),(3,6),(4,8),(5,10)\}$ is a relation.
- The set consisting of the first components of each ordered pair is called the **domain**.
- E.g., The relation above has domain $\{1,2,3,4,5\}$.
- The set consisting of the second components of each ordered pair is called the **range**.
- E.g., The relation above has range $\{2,4,6,8,10\}$.

Functions

- A **function** is a relation in which each possible input value leads to exactly one output value.
- We say "**the output is a function of the input**".
- The **input** values make up the **domain**.
- The **output** values make up the **range**.







Only the first and second diagrams depict functions.

Functional Notation

- The notation $y = f(x)$ defines a function named f.
- This is read as "y **is a function of** x."
- The letter x represents the **input value**, or **independent variable**.
- The letter y, or $f(x)$, represents the **output value**, or **dependent variable**.
- E.g., Assume that f represents a function whose input is the name of a month and output is the number of days in that month.
  (a) Use functional notation to express the fact that March has 31 days.
  $f(\text{March}) = 31$
  (b) What does the equation $f(\text{September}) = 30$ express?
  It expresses the fact that September has 30 days.

Find the Value of a Function

- Given the formula for a function, to evaluate:
  1. Replace the input variable in the formula with the value provided.
  2. Calculate the result.
- E.g., consider $f(x) = x^2 + 3x - 4$.

  Evaluate $y = f(x)$ at: a. 2 b. a c. $a + h$ d. $\frac{f(a+h)-f(a)}{h}$.

  a. $f(2) = 2^2 + 3 \cdot 2 - 4 = 4 + 6 - 4 = 6$

  b. $f(a) = a^2 + 3a - 4$

  c. $f(a+h) = (a+h)^2 + 3(a+h) - 4 = a^2 + 2ah + h^2 + 3a + 3h - 4$

  d. $$\frac{f(a+h)-f(a)}{h} = \frac{(a^2+2ah+h^2+3a+3h-4)-(a^2+3a-4)}{h} =$$
  $$\frac{a^2+2ah+h^2+3a+3h-4-a^2-3a+4}{h} = \frac{2ah+h^2+3h}{h} =$$
  $$\frac{h(2a+h+3)}{h} = 2a+h+3$$

Solving for the Input

- Given the function $h(p) = p^2 + 2p$, solve $h(p) = 3$ for p.

$$h(p) = 3$$
$$p^2 + 2p = 3$$
$$p^2 + 2p - 3 = 0$$
$$(p+3)(p-1) = 0$$
$$p + 3 = 0 \text{ or } p - 1 = 0$$
$$p = -3 \text{ or } p = 1.$$

Find an Equation for a Function

- Express the relationship $2n + 6p = 12$ as a function $p = f(n)$, if possible.

$$2n + 6p = 12$$
$$6p = 12 - 2n$$
$$p = \frac{12-2n}{6}$$
$$p = \frac{12}{6} - \frac{2n}{6}$$
$$p = 2 - \frac{1}{3}n.$$

Find the Value of a Function in Tabular Form

- Consider the function g specified by the table

n	1	2	3	4	5
$g(n)$	8	6	7	6	8

a. Evaluate $g(3)$ b. Solve $g(n) = 6$.

a.

$$g(3) = 7.$$

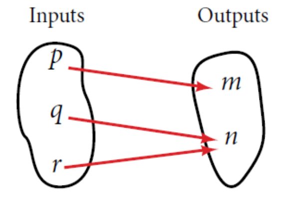
b.

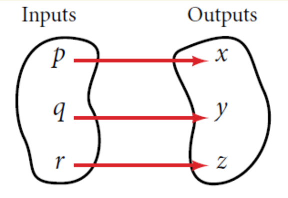
$$g(n) = 6 \quad \text{implies} \quad n = 2 \text{ or } n = 4.$$

Determine Whether a Function is One-To-One

- A **one-to-one function** is a function in which each output value corresponds to exactly one input value.

  In a one-to-one function, there are no repeated x- or y-values.
- E.g., which of the following functions is one-to-one?



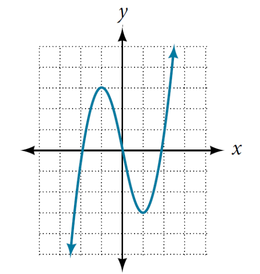


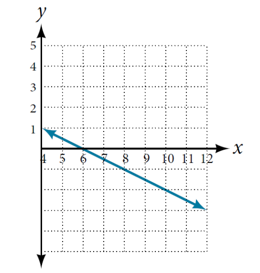
The function on the left is not one-to-one because to $y = n$ map two different x values.

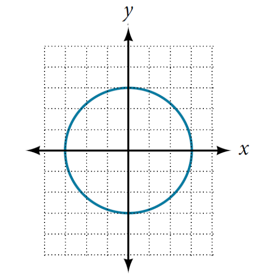
The function on the right is one-to-one.

Use the Vertical Line Test to Identify Functions

- Given a graph, use the vertical line test to determine if the graph represents a function:
  1. Inspect the graph to see if any vertical line drawn would intersect the curve more than once.
  2. If there is any such line, determine that the graph does not represent a function.
- E.g., which of the graphs represent(s) a function $y = f(x)$?

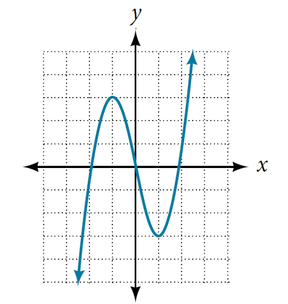


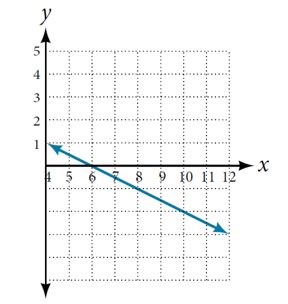




Use the Horizontal Line Test for One-to-One Property

- Given a graph of a function, use the horizontal line test to determine if the graph represents a one-to-one function.
  1. Inspect the graph to see if any horizontal line drawn would intersect the curve more than once.
  2. If there is any such line, determine that the function is not one-to-one.
- E.g., which of the graphs represent(s) a one-to-one function $y = f(x)$?





Subsection 2

Domain and Range

We Will Learn How To:

- Find the domain of a function defined by an equation;
- Find the range of a function presented through a graph;
- Create models using piece-wise defined functions;
- Graph piece-wise defined functions.

Find the Domain of a Function Specified by Ordered Pairs

- Find the domain of the following function:

$$\{(2,10),(3,10),(4,20),(5,30),(6,40)\}.$$

The input value is the first coordinate in an ordered pair.
The **domain** is the set of possible input values,
i.e., the set of the first coordinates of the ordered pairs:

$$\{2,3,4,5,6\}.$$

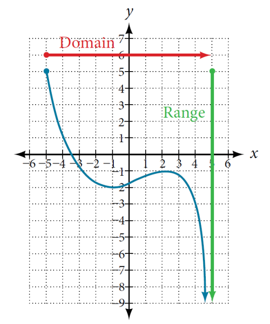
Find the Domain of a Function Defined by an Equation

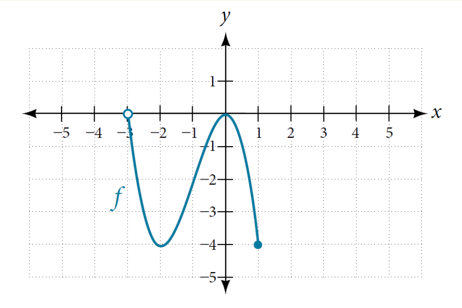
- Find the domain of the following functions
  a. $f(x) = \frac{x+1}{2-x}$;
  b. $f(x) = \sqrt{7-5x}$;
  c. $f(x) = \ln(3x-7)$;
  d. $f(x) = x^2 - 1$.

  a. We must have $2 - x \neq 0$. Set $2 - x = 0$. Solve for x: $x = 2$. So $\mathrm{Dom}(f) = \mathbb{R} - \{2\}$.
  a. We must have $7 - 5x \geq 0$. Solve for x: $7 \geq 5x$ So $x \leq \frac{7}{5}$. We get $\mathrm{Dom}(f) = (-\infty, \frac{7}{5}]$.
  c. We must have $3x - 7 > 0$. Solve for x: $3x > 7$ So $x > \frac{7}{3}$. We get $\mathrm{Dom}(f) = (\frac{7}{3}, +\infty)$.
  d. No denominators or even-indexed roots or logarithms appear. So no restrictions apply. We get $\mathrm{Dom}(f) = \mathbb{R} = (-\infty, +\infty)$.

Find the Domain and Range of a Function by a Graph

- The main idea is given in the figure below.





- E.g., find the domain and range of the function f whose graph is shown on the right.

  $\mathrm{Dom}(f) = (-3, 1]$

  $\mathrm{Ran}(f) = [-4, 0]$

Piecewise-Defined Functions

- A **piecewise function** is a function in which more than one formula is used to define the output.
- Each formula has its own domain, and the domain of the function is the union of all these smaller domains.
- The general notation looks like:

$$f(x) = \begin{cases} \text{formula 1}, & \text{if } x \text{ is in domain 1} \\ \text{formula 2}, & \text{if } x \text{ is in domain 2} \\ \text{formula 3}, & \text{if } x \text{ is in domain 3} \end{cases}$$

Devise Piecewise-Defined Functions

- Consider a simple tax system in which incomes up to \$10,000 are taxed at 10%, and any additional income is taxed at 20%.

  Write a function for the income tax T in terms of the income x.

  The tax on a total income x would be
  - $0.1x$ if $x \leq 10,000$;
  - $1000 + 0.2(x - 10,000)$ if $x > 10,000$.

  That is

$$T(x) = \begin{cases} 0.1x, & \text{if } x \leq 10,000 \\ 1000 + 0.2(x - 10,000), & \text{if } x > 10,000 \end{cases}$$

Devise Piecewise-Defined Functions

- A museum charges \$5 per person for a guided tour with a group of 1 to 9 people or a fixed \$50 fee for a group of 10 or more people.

  Write a function relating the number of people, n, to the cost, C.

  The cost C in terms of the number n of people is calculated as follows:

  - if $n < 10$, $C = 5n$;
  - if $n \geq 10$, $C = 50$.

  That is,

$$C(n) = \begin{cases} 5n, & \text{if } n < 10 \\ 50, & \text{if } n \geq 10 \end{cases}$$

Working With Piecewise-Defined Functions

- A cell phone company uses the function below to determine the cost, C, in dollars, for g gigabytes of data transfer.

$$C(g) = \begin{cases} 25, & \text{if } 0 < g < 2 \\ 25 + 10(g-2), & \text{if } g \geq 2 \end{cases}$$

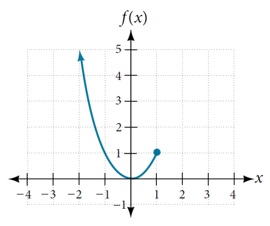
Find the cost of using 1.5 gigabytes of data and the cost of using 4 gigabytes of data.

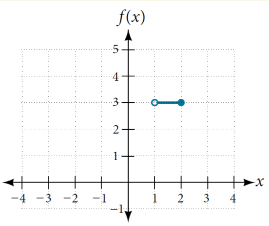
$C(1.5) = \$25$

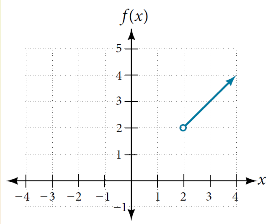
$C(4) = 25 + 10(4-2) = \$45.$

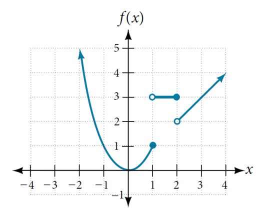
Graph Piecewise-Defined Functions

- Sketch a graph of the function $f(x) = \begin{cases} x^2, & \text{if } x \leq 1 \\ 3, & \text{if } 1 < x \leq 2 \\ x, & \text{if } x > 2 \end{cases}$









Subsection 3

Rates of Change and Behavior of Graphs

We Will Learn How To:

- Find the average rate of change of a function;
- Use a graph to determine where a function is increasing, decreasing or constant;
- Use a graph to locate local maxima and local minima;
- Use a graph to locate the absolute maximum and the absolute minimum.

Average Rate of Change of a Function

- The **average rate of change** of a function $y = f(x)$ between x_1 and x_2 is defined by

$$\begin{aligned} \text{Average rate of change} &= \frac{\text{Change in output}}{\text{Change in input}} \\ &= \frac{\Delta y}{\Delta x} \\ &= \frac{y_2 - y_1}{x_2 - x_1} \\ &= \frac{f(x_2) - f(x_1)}{x_2 - x_1}. \end{aligned}$$

Finding the Average Rate of Change (Table)

- The average cost, in dollars, of a gallon of gasoline for the years 2005-2012 is given by

x	2005	2006	2007	2008	2009	2010	2011	2012
$C(x)$	2.31	2.62	2.84	3.30	2.41	2.84	3.58	3.68

Find the average rate of change of the price of gasoline between 2007 and 2009.

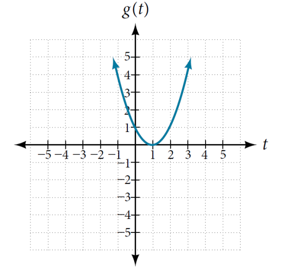
We get

$$\frac{\Delta y}{\Delta x} = \frac{C(2009) - C(2007)}{2009 - 2007} = \frac{2.41 - 2.84}{2} = \frac{-0.43}{2} = -0.22.$$

Thus, gasoline priced changed by an average of $-\$0.22$ per year between 2007 and 2009.

Finding the Average Rate of Change (Graph)

- Given the function $g(t)$ shown, find the average rate of change on the interval $[-1, 2]$.



We have

$$\frac{\Delta y}{\Delta x} = \frac{g(2) - g(-1)}{2 - (-1)} = = \frac{1 - 4}{2 + 1} = \frac{-3}{3} = -1.$$

Finding the Average Rate of Change (Formula)

- Compute the average rate of change of $f(x) = x^2 - \frac{1}{x}$ on the interval $[2, 4]$.

$$\begin{aligned}\frac{\Delta y}{\Delta x} &= \frac{f(4) - f(2)}{4 - 2} \\ &= \frac{(16 - \frac{1}{4}) - (4 - \frac{1}{2})}{2} \\ &= \frac{\frac{63}{4} - \frac{7}{2}}{2} \\ &= \frac{\frac{49}{4}}{2} = \frac{49}{8}.\end{aligned}$$

Finding the Average Rate of Change (Expression)

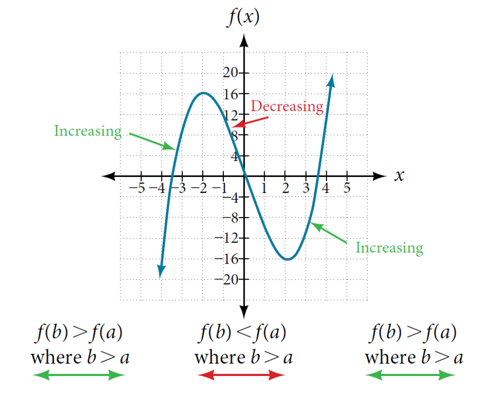
- Find the average rate of change of $g(t) = t^2 + 3t + 1$ on $[0, a]$. The answer will be an expression involving a in simplest form.

$$\frac{\Delta y}{\Delta x} = \frac{g(a) - g(0)}{a - 0} = \frac{(a^2 + 3a + 1) - (0^2 + 3 \cdot 0 + 1)}{a}$$

$$= \frac{a^2 + 3a + 1 - 1}{a} = \frac{a^2 + 3a}{a} = \frac{a(a+3)}{a} = a + 3.$$

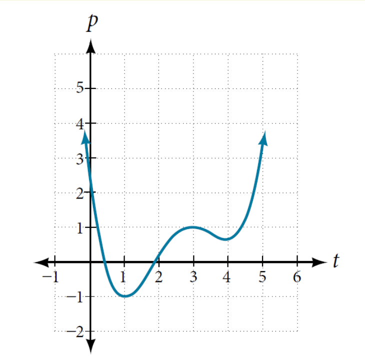
Increasing and Decreasing Functions

- A function f is an **increasing function** on an open interval if, for any two input values $a < b$ in the given interval, $f(a) < f(b)$.
- A function f is a **decreasing function** on an open interval if, for any two input values $a < b$ in the given interval, $f(a) > f(b)$.



Determine Where a Function is Increasing/Decreasing

- Given the function $p(t)$, identify the intervals on which the function appears to be increasing.

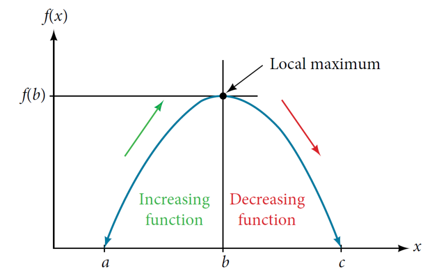


The function is increasing from $t = 1$ to $t = 3$ and from $t = 4$ on.

In interval notation, the function is increasing on $(1, 3)$ and on $(4, \infty)$.

Local Maxima and Local Minima

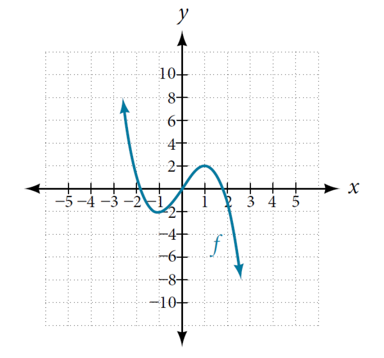
- A function f has a **local maximum** at $x = b$ if there exists an interval (a, c) with $a < b < c$ such that, for any x in the interval (a, c), $f(x) \leq f(b)$.



- Similarly, f has a **local minimum** at $x = b$ if there exists an interval (a, c) with $a < b < c$ such that, for any x in the interval (a, c), $f(x) \geq f(b)$.

Locate the Local Maxima and Local Minima

- For the function f whose graph is shown, find all local maxima and minima.

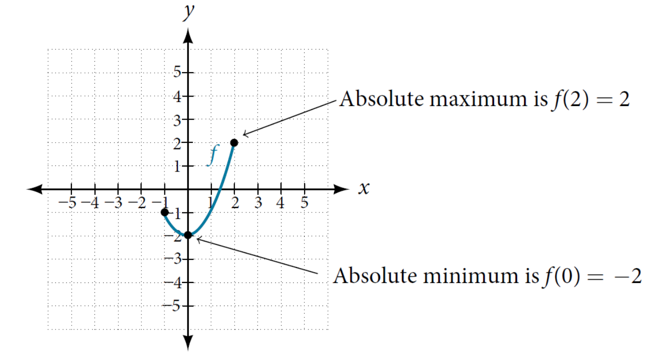


The graph attains a local maximum $y = 2$ at $x = 1$.

The graph attains a local minimum $y = -2$ at $x = -1$.

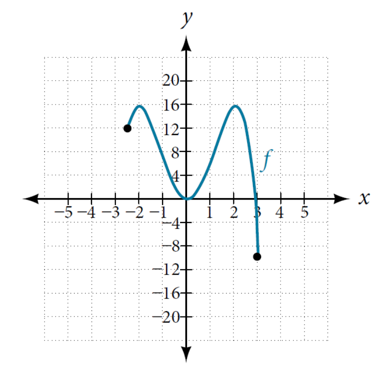
Absolute Maximum and Absolute Minimum

- The **absolute maximum** of f at $x = c$ is $f(c)$ where $f(c) \geq f(x)$ for all x in the domain of f.
- The **absolute minimum** of f at $x = d$ is $f(d)$ where $f(d) \leq f(x)$ for all x in the domain of f.



Locate the Absolute Maximum and Absolute Minimum

- For the function f shown, find all absolute maxima and minima.



The graph attains an absolute maximum $y = 16$ at $x = -2$ and at $x = 2$.

The graph attains an absolute minimum $y = -10$ at $x = 3$.

Subsection 4

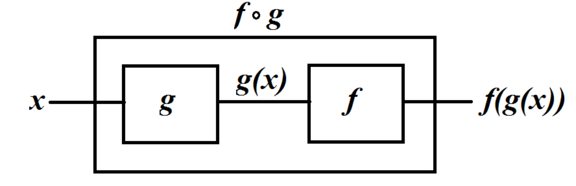
Composition of Functions

We Will Learn How To:

- Create a new function by composition of functions;
- Evaluate composite functions;
- Find the domain of a composite function;
- Decompose a composite function into its component functions.

Composition of Functions

- When the output of one function is used as the input of another, we call the entire operation a *composition of functions*.



- For any input x and functions f and g, this action defines a **composite function**, which we write as $f \circ g$ such that

$$(f \circ g)(x) = f(g(x)).$$

- The domain of the composite function $f \circ g$ is all x such that:
  - x is in the domain of g and
  - $g(x)$ is in the domain of f.

Create a Function by Composition of Functions

- Using the functions $f(x) = 2x + 1$, $g(x) = 3 - x$, find $f(g(x))$ and $g(f(x))$.

$$f(g(x)) = f(3 - x) = 2(3 - x) + 1 = 6 - 2x + 1 = 7 - 2x.$$

$$g(f(x)) = g(2x + 1) = 3 - (2x + 1) = 3 - 2x - 1 = 2 - 2x.$$

Are $f(g(x))$ and $g(f(x))$ the same functions?

- No! E.g., $f(g(0)) = 7 \neq 2 = g(f(0))$

Evaluating Composite Functions (Tables)

- Using the given table, evaluate $f(g(3))$ and $g(f(3))$.

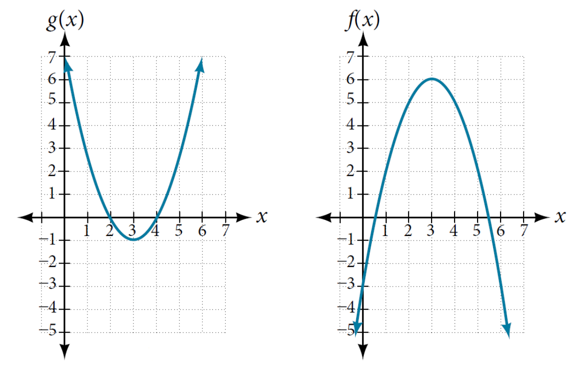
x	$f(x)$	$g(x)$
1	6	3
2	8	5
3	3	2
4	1	7

$$f(g(3)) = f(2) = 8;$$

$$g(f(3)) = g(3) = 2.$$

Evaluating Composite Functions (Graphs)

- Given f and g, as shown, evaluate $f(g(1))$.



We have

$$f(g(1)) = f(3) = 6.$$

Evaluating Composite Functions (Formulas)

- Given $f(t) = t^2 - t$ and $h(x) = 3x + 2$, evaluate $f(h(1))$.

  We get
  $$f(h(1)) = f(3 \cdot 1 + 2) = f(5) = 5^2 - 5 = 20.$$

Finding the Domain of a Composite Function

- Find the domain of $(f \circ g)(x)$ where $f(x) = \frac{5}{x-1}$ and $g(x) = \frac{4}{3x-2}$.

  We implement the steps outlined above:
  - $\mathrm{Dom}(g) = \mathbb{R} - \{\frac{2}{3}\}$;
  - $\mathrm{Dom}(f) = \mathbb{R} - \{1\}$;
  - We must exclude those values in $\mathrm{Dom}(g)$, such that $g(x)$ not in $\mathrm{Dom}(f)$.

    This means we should nor allow $\frac{4}{3x-2} = 1$.

$$\frac{4}{3x-2} = 1 \Rightarrow 4 = 3x - 2 \Rightarrow 3x = 6 \Rightarrow x = 2.$$

We conclude $\mathrm{Dom}(f \circ g) = \mathbb{R} - \{\frac{2}{3}, 2\}$.

Finding the Domain of a Composite Function

- Find the domain of $(f \circ g)(x)$ where $f(x) = \sqrt{x+2}$ and $g(x) = \sqrt{3-x}$.

  We implement the steps outlined above:
  - $\mathrm{Dom}(g) = (-\infty, 3]$;
  - $\mathrm{Dom}(f) = [-2, +\infty)$;
  - We must find those values in $\mathrm{Dom}(g)$, such that $g(x)$ is in $\mathrm{Dom}(f)$. This means we must have $\sqrt{3-x} \geq -2$.

    $$\sqrt{3-x} \geq -2 \;\Rightarrow\; \text{all } x \leq 3.$$

    We conclude $\mathrm{Dom}(f \circ g) = (-\infty, 3]$.

Decomposing a Function into its Components

- Write $f(x) = \sqrt{5 - x^2}$ as the composition of two functions.
- Think of f as transforming an input x to an output $f(x) = \sqrt{5 - x^2}$. Which steps does it apply?
  - It first computes $x \mapsto 5 - x^2$;
  - It then calculates the square root of the previous step $x \mapsto \sqrt{x}$.

  Therefore $f(x) = h(g(x))$, where
  - $g(x) = 5 - x^2$;
  - $h(x) = \sqrt{x}$.
- Are there any other ways?

Subsection 5

Transformation of Functions

We Will Learn How To:

- Graph functions using vertical and horizontal shifts;
- Graph functions using reflections about the x-axis and the y-axis;;
- Determine whether a function is even, odd or neither from its graph;
- Graph functions using compressions and stretches;
- Combine transformations.

Vertical Shift

- Given a function $f(x)$, a new function $g(x) = f(x) + k$, where k is a constant, is a vertical shift of the function $f(x)$.
- All the output values change by k units.
  - If k is positive, the graph will shift up.
  - If k is negative, the graph will shift own.
- A function $f(x)$ is given in the table

x	2	4	6	8
$f(x)$	1	3	7	11

Create a table for the function $g(x) = f(x) - 3$.

We have

x	2	4	6	8
$g(x)$	-2	0	4	8

Horizontal Shift

- Given a function f, a new function $g(x) = f(x - h)$, where h is a constant, is a horizontal shift of the function f.
  - If h is positive, the graph will shift right.
  - If h is negative, the graph will shift left.
- A function $f(x)$ is given by

x	2	4	6	8
$f(x)$	1	3	7	11

Create a table for the function $g(x) = f(x - 3)$.

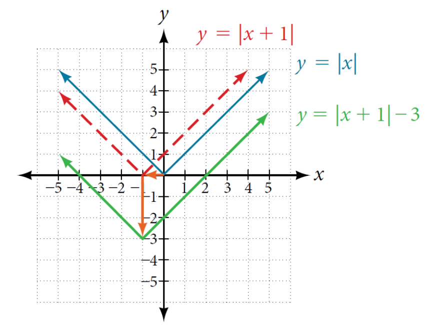
We have

x	5	7	9	11
$g(x)$	1	3	7	11

Graphing Functions Using Vertical and Horizontal Shifts

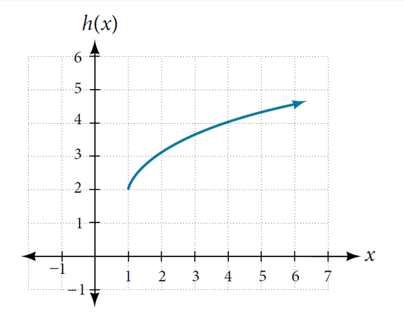
- Given $f(x) = |x|$, sketch a graph of $h(x) = f(x+1) - 3$.
  The key is to identify the sequence of transformations leading from input $f(x)$ to output $h(x) = f(x+1) - 3$:

$$\begin{array}{lcll} f(x) & \longrightarrow & f(x+1) & \text{(Shift Left by 1)} \\ & \longrightarrow & f(x+1) - 3 & \text{(Shift Down by 3)} \end{array}$$



Identifying Combined Vertical and Horizontal Shifts

- Write a formula for the graph shown in the figure, which is a transformation of $f(x) = \sqrt{x}$.

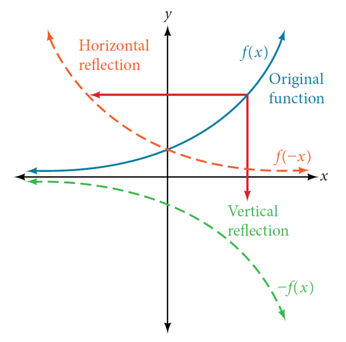


- This is clearly a shift by:
  - 1 unit to the right;
  - 2 units up.

  Therefore, $h(x) = f(x-1) + 2 = \sqrt{x-1} + 2$.

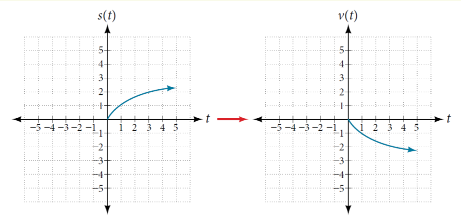
Graphing Functions Using Reflections about the Axes

- Given a function $f(x)$, a new function $g(x) = -f(x)$ is a **vertical reflection** of the function $f(x)$, sometimes called a **reflection about** (or **over**, or **through**) the x-axis.
- Given a function $f(x)$, a new function $g(x) = f(-x)$ is a **horizontal reflection** of the function $f(x)$, sometimes called a **reflection about** the y-axis.

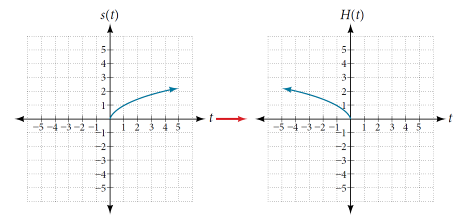


Graphing a Function

- Reflect the graph of $s(t) = \sqrt{t}$ (a) vertically and (b) horizontally.
  (a) The vertical reflection would give $V(x) = -\sqrt{t}$.



  (b) The horizontal reflection would give $H(t) = \sqrt{-t}$.



Graphing a Function

- A function $f(x)$ is given by the table

x	2	4	6	8
$f(x)$	1	3	7	11

Create a table for the functions a. $g(x) = -f(x)$ b. $h(x) = f(-x)$.

a. This is a vertical reflection of f.
So we have

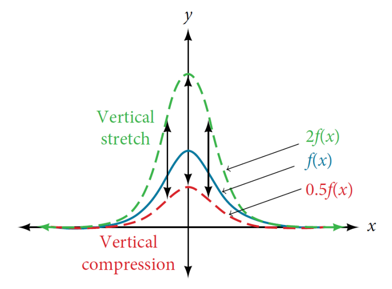
x	2	4	6	8
$g(x)$	-1	-3	-7	-11

b. This is a horizontal reflection of f.
So we have

x	-2	-4	-6	-8
$h(x)$	1	3	7	11

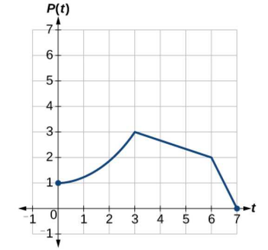
Vertical Stretches and Compressions

- Given a function $f(x)$, a new function $g(x) = af(x)$, where a is a constant, is a vertical stretch or vertical compression of the function $f(x)$.
  - If $a > 1$, then the graph will be stretched.
  - If $0 < a < 1$, then the graph will be compressed.
  - If $a < 0$, then there will be combination of a vertical stretch or compression with a vertical reflection.

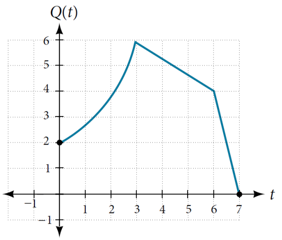


Graphing Functions Using Stretches

- A function $P(t)$, whose graph is shown, models the population of fruit flies.



$(0, 1) \longrightarrow (0, 2)$
$(3, 3) \longrightarrow (3, 6)$
$(6, 2) \longrightarrow (6, 4)$
$(7, 0) \longrightarrow (7, 0)$



A scientist is comparing this population to another population, Q, whose growth follows the same pattern, but is twice as large. Sketch a graph of this population.

We have $Q(t) = 2P(t)$.

The graph of $Q(t)$ is shown on the right above.

Graphing Functions Using Compressions

- A function f is given by the table

x	2	4	6	8
$f(x)$	1	3	7	11

Create a table for the function $g(x) = \frac{1}{2}f(x)$.

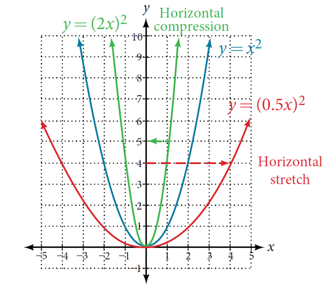
The function g is a vertical compression of f by a factor of $\frac{1}{2}$.

So we have

x	2	4	6	8
$g(x)$	$\frac{1}{2}$	$\frac{3}{2}$	$\frac{7}{2}$	$\frac{11}{2}$

Horizontal Stretches and Compressions

- Given a function $f(x)$, a new function $g(x) = f(bx)$, where b is a constant, is a horizontal stretch or horizontal compression of the function $f(x)$.
  - If $b > 1$, then the graph will be compressed by $\frac{1}{b}$.
  - If $0 < b < 1$, then the graph will be stretched by $\frac{1}{b}$.
  - If $b < 0$, then there will be combination of a horizontal stretch or compression with a horizontal reflection.



Horizontal Stretches and Compressions

- A function $f(x)$ is given as

x	2	4	6	8
$f(x)$	1	3	7	11

Create a table for the function $g(x) = f(\frac{1}{2}x)$.

The function g is a horizontal stretch of f by a factor of 2.

So we have

x	4	8	12	16
$g(x)$	1	3	7	11

Performing a Sequence of Transformations

- Combining vertical transformations $af(x) + k$:
  - first vertically stretch by a;
  - then vertically shift by k.
- Combining horizontal transformations $f(bx - h)$:
  - first horizontally shift by h;
  - then horizontally stretch by $\frac{1}{b}$.
- Combining horizontal transformations $f(b(x - h))$:
  - first horizontally stretch by $\frac{1}{b}$;
  - then horizontally shift by h.
- Horizontal and vertical transformations are independent.

  It does not matter whether horizontal or vertical transformations are performed first.

Performing a Sequence of Transformations

- Given the function $f(x)$

x	6	12	18	24
$f(x)$	10	14	15	17

Create a table of values for the function $g(x) = 2f(3x) + 1$.

We apply

$$\begin{array}{lll} f(x) & \longrightarrow & f(3x) \quad \text{(horizontal compression)} \\ & \longrightarrow & 2f(3x) \quad \text{(vertical stretch)} \\ & \longrightarrow & 2f(3x) + 1 \quad \text{(vertical shift)} \end{array}$$

These result in

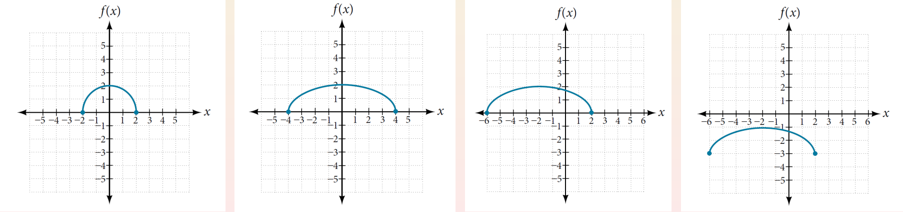
x	2	4	6	8
$g(x)$	21	29	31	35

Performing a Sequence of Transformations

- Use the graph of $f(x)$ in the figure to sketch a graph of $k(x) = f(\frac{1}{2}x + 1) - 3$.

  We apply

$$
\begin{array}{rcll}
f(x) & \longrightarrow & f(\frac{1}{2}x) & \text{(horizontal stretch)} \\
 & \longrightarrow & f(\frac{1}{2}(x+2)) & \text{(horizontal shift)} \\
 & \longrightarrow & f(\frac{1}{2}x+1)-3 & \text{(vertical shift)}
\end{array}
$$



Subsection 6

Absolute Value Functions

We Will Learn How To:

- Graph an absolute value function;
- Solve an absolute value equation.

Absolute Value

- The **absolute value** function can be defined as a piecewise function

$$|x| = \begin{cases} x, & \text{if } x \geq 0 \\ -x, & \text{if } x < 0 \end{cases}$$

- E.g., $|5.4| = 5.4$, whereas $|-3.7| = -(-3.7) = 3.7$.
- The absolute value has a significant geometric interpretation:
  - $|x - y|$ is the distance between x and y on the real line;
  - in particular, $|x| = |x - 0|$ is the distance of x from 0 on the real line.

Understanding Absolute Value

- Describe all numbers x that are at a distance of $\frac{1}{2}$ from the number -4.

  Express this using absolute value notation.

  Verbal description: The distance between x and -4 equals $\frac{1}{2}$.

  Translation into an equation:

$$|x-(-4)|=\frac{1}{2} \;\Rightarrow\; |x+4|=\frac{1}{2}.$$

Graphing a Function

- Describe all function values $f(x)$ such that the distance from $f(x)$ to the value 8 is less than 0.03 units.

  Express this using absolute value notation.

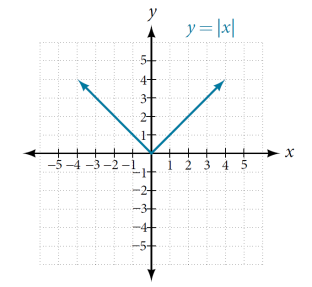
  Verbal description: The distance between $f(x)$ and 8 is less than 0.03.

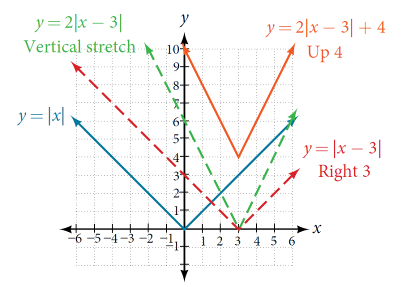
  Translation into an equation:

$$|f(x) - 8| < 0.03.$$

Graphing an Absolute Value Function

- The graph of $f(x) = |x|$ is shown on the left below.





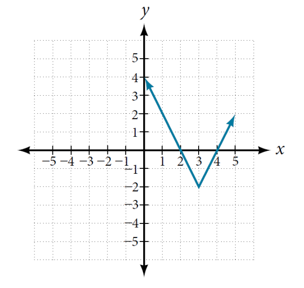
Use it to obtain the graph of $g(x) = 2|x-3| + 4$.

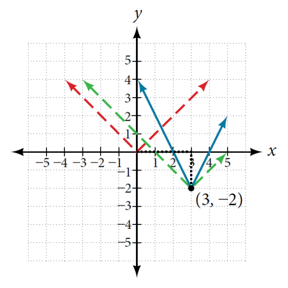
$$\begin{array}{rcll} |x| & \longrightarrow & |x-3| & \text{(horizontal shift 3 to the right)} \\ & \longrightarrow & 2|x-3| & \text{(vertical stretch by 2)} \\ & \longrightarrow & 2|x-3|+4 & \text{(vertical shift up by 4)} \end{array}$$

Following one by one this transformations we get the graph.

Graphing a Function

- Write an equation for the function whose graph is shown on the left.





Use the graph to obtain a formula.

We have

$$\begin{aligned} |x| &\longrightarrow |x-3| \quad \text{(horizontal shift 3 to the right)} \\ &\longrightarrow 2|x-3| \quad \text{(vertical stretch by 2)} \\ &\longrightarrow 2|x-3|-2 \quad \text{(vertical shift 2 down)} \end{aligned}$$

Solving an Absolute Value Equation

- For real numbers A and B, consider the equation $|A| = B$:
  - If $B \geq 0$, it has solutions
    $$A = B \quad \text{or} \quad A = -B;$$
  - If $B < 0$, it has no solution.
- E.g., solve $|5x + 2| - 4 = 9$.
$$\begin{gathered} |5x+2| - 4 = 9 \\ |5x+2| = 13 \\ 5x + 2 = -13 \quad \text{or} \quad 5x + 2 = 13 \\ 5x = -15 \quad \text{or} \quad 5x = 11 \\ x = -3 \quad \text{or} \quad x = \tfrac{11}{5}. \end{gathered}$$

Finding the Zeros of an Absolute Value Function

- For the function $f(x) = |4x + 1| - 7$, find the values of x such that $f(x) = 0$.

  We have

$$\begin{gathered} f(x) = 0 \\ |4x + 1| - 7 = 0 \\ |4x + 1| = 7 \\ 4x + 1 = -7 \quad \text{or} \quad 4x + 1 = 7 \\ 4x = -8 \quad \text{or} \quad 4x = 6 \\ x = -2 \quad \text{or} \quad x = \frac{3}{2}. \end{gathered}$$

Subsection 7

Inverse Functions

We Will Learn How To:

- Verify inverse functions;
- Determine the domain and range of an inverse function;
- Restrict the domain of a function to make it one-to-one;
- Find or evaluate the inverse of a function;
- Graph the inverse function, given the graph of the original.

Inverse Function

- A function f must be one-to-one (i.e., must pass the horizontal line test) to have an inverse.
- If that is the case, its **inverse function**, f^{-1}, is related to f by

  $$f(x) = y \quad \text{if and only if} \quad f^{-1}(y) = x.$$

  That is f and f^{-1} "exchange" inputs and outputs.
- We then have:

  $$\begin{aligned} f^{-1}(f(x)) &= f^{-1}(y) \\ &= x \end{aligned}$$

  and

  $$\begin{aligned} f(f^{-1}(y)) &= f(x) \\ &= y. \end{aligned}$$

  That is if one composes the two (in any order), the output of the composition is identical to the original input.

Identifying an Inverse Function for an Input-Output Pair

- If for a particular one-to-one function $f(2) = 4$ and $f(5) = 12$, what are the corresponding input and output values for the inverse function?

  We have

$$\begin{aligned} f^{-1}(4) &= 2 \\ f^{-1}(12) &= 5. \end{aligned}$$

Testing Inverse Relationships Algebraically

- If $f(x) = \frac{1}{x+2}$ and $g(x) = \frac{1}{x} - 2$, is $g = f^{-1}$?
  One needs to check whether $f(g(x)) = x$ and $g(f(x)) = x$.
  - For the first, we have
    $$f(g(x)) = f\left(\frac{1}{x} - 2\right) = \frac{1}{\frac{1}{x} - 2 + 2} = \frac{1}{\frac{1}{x}} = x.$$
  - For the second
    $$g(f(x)) = g\left(\frac{1}{x+2}\right) = \frac{1}{\frac{1}{x+2}} - 2 = x + 2 - 2 = x.$$

  Since both hold, $g = f^{-1}$.

Finding Domain and Range of Inverse Functions

- Since f and f^{-1} exchange inputs and outputs:
  - The range of $f(x)$ is the domain of $f^{-1}(x)$;
  - The domain of $f(x)$ is the range of $f^{-1}(x)$.
- Suppose that f has an inverse and that

$$\mathrm{Dom}(f) = [3, +\infty), \quad \mathrm{Ran}(f) = [0, +\infty).$$

What are the domain and range of f^{-1}?

We have

$$\begin{aligned} \mathrm{Dom}(f^{-1}) &= \mathrm{Ran}(f) = [0, +\infty); \\ \mathrm{Ran}(f^{-1}) &= \mathrm{Dom}(f) = [3, +\infty). \end{aligned}$$

Finding and Evaluating Inverse Functions (Table)

- A function $f(t)$ is given by the table

t (minutes)	30	50	70	90
$f(t)$ (miles)	20	40	60	70

showing distance in miles that a car has traveled in t minutes.

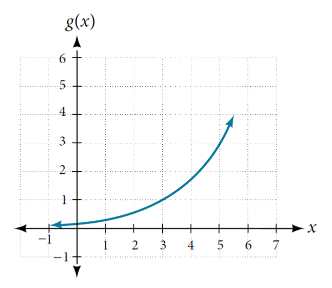
Find $f^{-1}(70)$ and $f^{-1}(40)$.

We have

$$\begin{aligned} f^{-1}(70) &= 90; \\ f^{-1}(40) &= 50. \end{aligned}$$

Finding and Evaluating Inverse Functions (Graph)

- A function $g(x)$ is given in the figure.



Find $g(3)$ and $g^{-1}(3)$.

We have

$$\begin{aligned} g(3) &= 1; \\ g^{-1}(3) &= 5. \end{aligned}$$

Finding and Evaluating Inverse Functions (Formula)

- Find the inverse of the function $f(x) = \frac{2}{x-3} + 4$.

  We must find a formula for $f^{-1}(x)$.

  Since f^{-1} reverses the roles of x (input) and y (output), we solve the equation $x = \frac{2}{y-3} + 4$ for y.

$$x = \frac{2}{y-3} + 4$$

$$x - 4 = \frac{2}{y-3}$$

$$\frac{1}{x-4} = \frac{y-3}{2}$$

$$\frac{2}{x-4} = y - 3$$

$$\frac{2}{x-4} + 3 = y.$$

  We conclude that $f^{-1}(x) = \frac{2}{x-4} + 3$.

Finding and Evaluating Inverse Functions (Formula)

- Find the inverse of the function $f(x) = 2 + \sqrt{x-4}$.

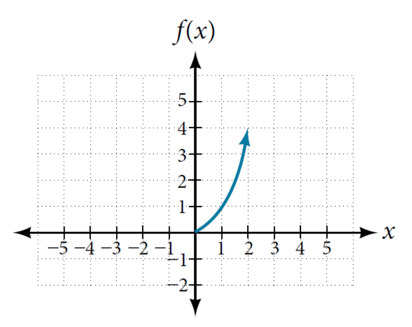
  Again, we write $x = 2 + \sqrt{y-4}$ and try to solve for y.

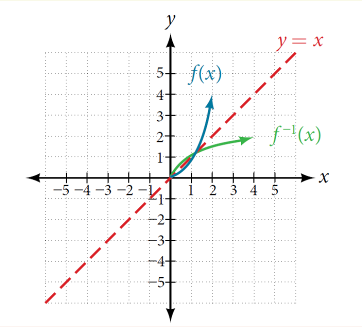
$$x = 2 + \sqrt{y-4}$$
$$x - 2 = \sqrt{y-4}$$
$$(x-2)^2 = y - 4$$
$$(x-2)^2 + 4 = y.$$

  Thus, $f^{-1}(x) = (x-2)^2 + 4$.

Finding Inverse Functions and Their Graphs

- The graph of $f^{-1}(x)$ is the graph of $f(x)$ reflected about the diagonal line $y = x$, which we will call the *identity line*.

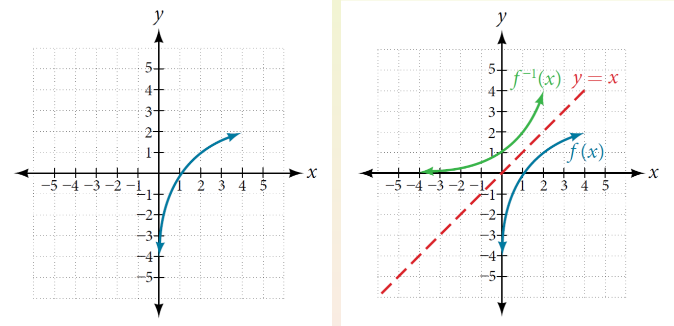




- This holds for all one-to-one functions, because the inverse swaps inputs and outputs.

Finding Inverse Functions and Their Graphs

- Given the graph of $f(x)$ in the figure, sketch a graph of $f^{-1}(x)$.



If we reflect this graph over the line $y = x$:

- the point $(1, 0)$ reflects to $(0, 1)$;
- the point $(4, 2)$ reflects to $(2, 4)$.