College Algebra

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LSSU Math 111

Subsection 1

Linear Functions

We Will Learn How To:

- Represent a linear function;
- Interpret slope as a rate of change;
- Write and interpret an equation for a linear function;
- Graph linear functions;
- Determine whether lines are parallel or perpendicular;
- Write the equation of a line parallel or perpendicular to a given line.

Linear Functions

- A **linear function** is a function whose graph is a line.
- Linear functions can be written in the slope-intercept form of a line

$$f(x) = mx + b,$$

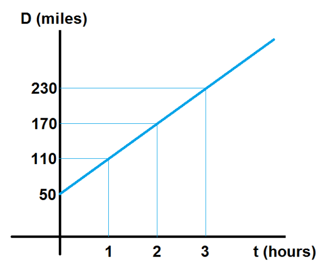
  where
  - b is the initial or starting value of the function (when $x = 0$),
  - m is the constant rate of change, or slope of the function.

Representing Linear Functions

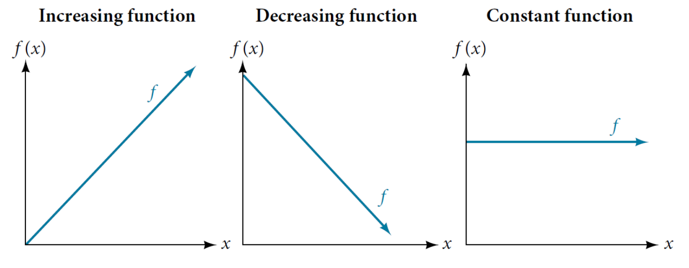
- Suppose the distance $D(t)$ of a train from the station at time $t = 0$ is 50 miles and is increasing at a constant rate of 60 miles per hour.
  - Function: $D(t) = 60t + 50$;
  - Table:

t	0	1	2	3
$D(t)$	50	110	170	230

  - Graph:



Linear Functions: Increasing, Decreasing, or Constant



- The slope determines if the function is an **increasing linear function**, a **decreasing linear function**, or a **constant function**:
  - $f(x) = mx + b$ is an increasing function if $m > 0$;
  - $f(x) = mx + b$ is an decreasing function if $m < 0$;
  - $f(x) = mx + b$ is a constant function if $m = 0$.

Slope as a Rate of Change

- The **slope**, or **rate of change**, m of a function can be calculated according to the following:

$$m = \frac{\text{change in output (rise)}}{\text{change in input (run)}} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1},$$

where

  - x_1 and x_2 are input values;
  - y_1 and y_2 are the corresponding output values.

Finding Slope

- If $f(x)$ is a linear function, and $(3,-2)$ and $(8,1)$ are points on the line, find the slope.

  Is this function increasing or decreasing?

  We have

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1-(-2)}{8-3} = \frac{3}{5}.$$

  This is an increasing line.

Finding Rate of Change

- The population of a city increased from 23,400 to 27,800 between 2008 and 2012.

  Find the change of population per year if we assume the change was constant from 2008 to 2012.

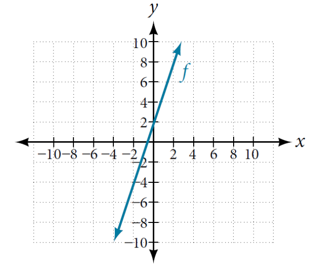
  We get

  $$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{27,800 - 23,400}{2012 - 2008} = \frac{4,400}{4} = 1,100.$$

  So the (constant) rate of change was 1,100 people per year.

Writing an Equation for a Linear Function

- Write an equation for a linear function given the graph of f



The y-intercept is $b = 2$.

The graph passes through $(0, 2)$ and $(2, 8)$.

So it has slope

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{8 - 2}{2 - 0} = 3.$$

Therefore, $f(x) = 3x + 2$.

Writing an Equation for a Linear Function

- Suppose Ben starts a company in which he incurs a fixed cost of \$1,250 per month for the overhead, which includes his office rent.

  His production costs are \$37.50 per item.

  Write a linear function C where $C(x)$ is the cost for x items produced in a given month.

  The y-intercept of C is \$1,250.

  The rate of change of the cost (average cost per item) is \$37.50.

  Therefore
$$C(x) = 37.50x + 1250.$$

Writing an Equation for a Linear Function

- If f is a linear function, with $f(3) = -2$, and $f(8) = 1$, find an equation for the function in slope-intercept form.

  Calculate the slope:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - (-2)}{8 - 3} = \frac{3}{5}.$$

  Now set up the point-slope form $y - y_0 = m(x - x_0)$.

$$y - (-2) = \frac{3}{5}(x - 3) \quad \Rightarrow \quad y + 2 = \frac{3}{5}(x - 3).$$

  Finally, solve for y to write in the slope-intercept form:

$$y + 2 = \frac{3}{5}x - \frac{9}{5} \quad \Rightarrow \quad y = \frac{3}{5}x - \frac{19}{5}.$$

Modeling Real-World Problems with Linear Functions

- Marcus currently has 200 songs in his music collection.
  Every month, he adds 15 new songs.
  (a) Write a formula for the number of songs, N, in his collection as a function of time, t, the number of months.
  (b) How many songs will he own in a year?

(a) We have

$$N(t) = 15t + 200.$$

(b) Therefore, in a year, he will own

$$N(12) = 15 \cdot 12 + 200 = 380 \text{ songs.}$$

Using a Table to Write an Equation for a Linear Function

- The table relates the number of rats in a population to time, in weeks

Number of weeks, w	0	2	4	6
Number of rats, $P(w)$	$1,000$	$1,080$	$1,160$	$1,240$

Use the table to write a linear equation.
First, note that the rat population increases by the same amount every two weeks.
This shows that the function $P(w)$ is linear and has slope

$$m = \frac{\Delta y}{\Delta x} = \frac{80}{2} = 40 \text{ rats per week.}$$

Therefore, taking into account that $P(0) = 1000$, we get

$$P(w) = 40w + 1000.$$

Slopes of Parallel or Perpendicular Lines

- Two lines are **parallel lines** if they do not intersect.

  The slopes of the lines are the same, $m_1 = m_2$.
- Two lines are **perpendicular lines** if they intersect at right angles.

  The slopes of the lines multiply to -1, $m_1 m_2 = -1$.

  Another way to say this is that the slope of one is the negative reciprocal of the slope of the other:

$$m_2 = -\frac{1}{m_1}.$$

Writing the Equation of a Line Parallel to a Given Line

- Find a line parallel to the graph of $f(x) = 3x + 6$ that passes through the point $(3, 0)$.

  The given line has slope $m_1 = 3$.

  Thus, the parallel line has slope $m_2 = 3$.

  Using the point-slope form $y - y_0 = m(x - x_0)$, we get

$$y - 0 = 3(x - 3) \quad \Rightarrow \quad y = 3x - 9.$$

Writing the Equation of a Perpendicular Line

- A line passes through the points $(-2, 6)$ and $(4, 5)$.

  Find the equation of a perpendicular line that passes through the point $(4, 5)$.

  The given line has slope

  $$m_1 = \frac{\Delta y}{\Delta x} = \frac{5-6}{4-(-2)} = -\frac{1}{6}.$$

  Thus, the perpendicular line has slope $m_2 = 6$.

  Using the point-slope form $y - y_0 = m(x - x_0)$, we get

  $$y - 5 = 6(x-4) \Rightarrow y - 5 = 6x - 24 \Rightarrow y = 6x - 19.$$

Subsection 2

Modeling with Linear Functions

We Will Learn How To:

- Build linear models from verbal descriptions;
- Model a set of data with a linear function.

Building Linear Models from Verbal Descriptions

- A town's population has been growing linearly.
  - In 2004 the population was 6,200.
  - By 2009 the population had grown to 8,100.
  - Assume this trend continues.

(a) Predict the population in 2013.

(b) Identify the year in which the population will reach 15,000.

We start counting from year $t = 0$, corresponding to 2004.
From the two given data points $(0, 6200)$ and $(5, 8100)$, we may construct a linear model for the population $P(t)$ in year t.

- The y intercept is 6200;
- The slope is $m = \frac{8100-6200}{5-0} = \frac{1900}{5} = 380$.

Thus, $P(t) = 380t + 6200$.

(a) $P(9) = 380 \cdot 9 + 6200 = 9620$.

(b) We must solve $P(t) = 15000$ implies $380t + 6200 = 15000$ implies $380t = 8800$ implies $t = 23.15$.

So the population will reach 15,000 in the year 2027-2028.

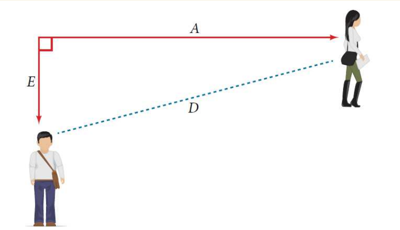
Using a Diagram to Model Distance Walked

- Anna and Emanuel start at the same intersection.
  - Anna walks east at 4 miles per hour;
  - Emanuel walks south at 3 miles per hour.

  They are communicating via a radio that has a range of 2 miles.

  How long after they start walking will they fall out of radio contact?

  Drawing makes the situation clearer.



We apply the Pythagorean Theorem:

$$\begin{aligned} D^2 &= A^2 + E^2 \\ D^2 &= (4t)^2 + (3t)^2 \\ D^2 &= 16t^2 + 9t^2 \\ D^2 &= 25t^2 \\ D &= 5t. \end{aligned}$$

We now solve $2 = 5t$ implying $t = \frac{2}{5}$ hours or $t = 24$ minutes.

Modeling a Set of Data with Linear Functions

- Jamal is choosing between two truck-rental companies.
  - The first, A, charges an up-front fee of \$20, then 59 cents a mile.
  - The second, B, charges an up-front fee of \$16, then 63 cents a mile.

When will A be the better choice for Jamal?

We model costs in terms of the moving distance x:

$$C_A(x) = 0.59x + 20, \quad C_B(x) = 0.63x + 16.$$

So the first company will be preferable if

$$\begin{aligned} C_A(x) &< C_B(x) \\ 0.59x + 20 &< 0.63x + 16 \\ 4 &< 0.04x \\ 100 &< x. \end{aligned}$$

Company A is the better choice if distance exceeds 100 miles.