College Algebra

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LSSU Math 111

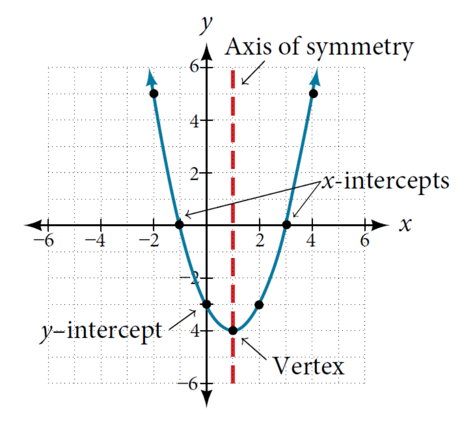
Subsection 1

Quadratic Functions

We Will Learn How To:

- Recognize characteristics of parabolas;
- Graphing a parabola given a quadratic function;
- Writing a quadratic function given the graph of a parabola;
- Determine a quadratic function's minimum or maximum value;
- Solve problems involving the minimum or maximum value.

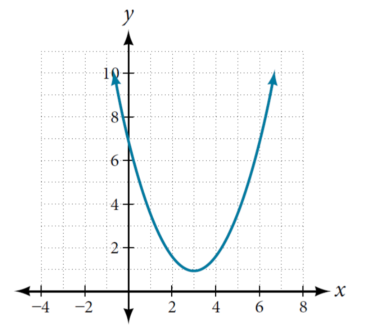
Characteristics of Parabolas



- General Form: $f(x) = ax^2 + bx + c$;
- Vertex: $(h,k) = \left(-\frac{b}{2a}, f(-\frac{b}{2a})\right)$;
- Axis of Symmetry: $x = -\frac{b}{2a}$;
- y-intercept: $(0,c)$;
- x-intercepts:
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$
if $b^2 - 4ac \geq 0$.

Identifying the Characteristics of a Parabola

- Determine the vertex, axis of symmetry, zeros, and y-intercept of the parabola shown in the figure.



Vertex is at $(3, 1)$.
Axis of symmetry is the line $x = 3$.
It has no zeros (no x-intercepts).
The y-intercept is $(0, 7)$.

Forms of Quadratic Functions

- A **quadratic function** is a polynomial function of degree two.
- The graph of a quadratic function is a **parabola**.
- The **general form** of a quadratic function is

$$f(x) = ax^2 + bx + c,$$

  where a, b, and c are real numbers and $a \neq 0$.
- The **standard form** of a quadratic function is

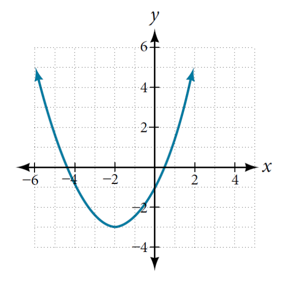
$$f(x) = a(x-h)^2 + k, \quad \text{where } a \neq 0.$$

- The vertex (h, k) is located at

$$h = -\frac{b}{2a}, \quad k = f(h) = f\left(-\frac{b}{2a}\right).$$

Writing the Equation from the Graph

- Write an equation in the standard form for the quadratic function g shown and then convert the equation to general form.



Since the vertex is at $(h,k) = (-2,-3)$, we get:

$$g(x) = a(x+2)^2 - 3.$$

Note that it passes through $(0,-1)$. So we must have $-1 = a(0+2)^2 - 3$ implies $-1 = 4a - 3$ implies $a = \frac{1}{2}$.
So it has equation $g(x) = \frac{1}{2}(x+2)^2 - 3$.

- To write in general form, we expand the square and distribute:

$$g(x) = \frac{1}{2}(x^2 + 4x + 4) - 3 \text{ or } g(x) = \frac{1}{2}x^2 + 2x - 1.$$

Finding the Vertex of a Quadratic Function

- Find the vertex of the quadratic function $f(x) = 2x^2 - 6x + 7$.

  Rewrite the quadratic in standard form.

  We work as follows:
  - First, find $h = -\frac{b}{2a} = -\frac{-6}{2 \cdot 2} = \frac{3}{2}$.
  - Next, find $k = f(\frac{3}{2}) = 2(\frac{3}{2})^2 - 6 \cdot \frac{3}{2} + 7 = 2 \cdot \frac{9}{4} - 9 + 7 = \frac{9}{2} - 2 = \frac{5}{2}$.
  - Finally we write

$$f(x) = a(x-h)^2 + k \quad \text{or} \quad f(x) = 2\left(x - \frac{3}{2}\right)^2 + \frac{5}{2}.$$

Finding the Domain and Range of a Quadratic Function

- Find the domain and range of $f(x) = -5x^2 + 9x - 1$.
- The domain is (for any quadratic function) $\mathbb{R}$.
- Suppose the vertex is at (h, k).
  - If the parabola open up ($a > 0$), the range is $[k, +\infty)$;
  - If the parabola open down ($a < 0$), the range is $(-\infty, k]$.

  So compute the vertex:
  - $h = -\frac{9}{2(-5)} = \frac{9}{10}$;
  - $k = f(\frac{9}{10}) = -5(\frac{9}{10})^2 + 9 \cdot \frac{9}{10} - 1 = -5 \cdot \frac{81}{100} + \frac{81}{10} - 1 = -\frac{81}{20} + \frac{81}{10} - 1 = -\frac{81}{20} + \frac{162}{20} - \frac{20}{20} = \frac{61}{20}$;

  Thus $\mathrm{Ran}(f) = (-\infty, \frac{61}{20}]$.

Finding the Maximum Value of a Quadratic Function

- A backyard farmer wants to enclose a rectangular space for a new garden within her fenced backyard. She has purchased 80 feet of wire fencing to enclose three sides, and she will use a section of the backyard fence as the fourth side.
  a. Find a formula for the area enclosed by the fence if the sides of fencing perpendicular to the existing fence have length L.
  b. What dimensions should she make her garden to maximize the enclosed area?

a. First, note that $2L + W = 80$, which implies $W = 80 - 2L$. So we get:

$$A = LW = L(80 - 2L) = -2L^2 + 80L.$$

b. A is maximum at the vertex:

$$L = -\frac{b}{2a} = -\frac{80}{2(-2)} = 20.$$

Thus the garden must be 40×20 to maximize the area.

Finding Maximum Revenue

- A newspaper has 84,000 subscribers at a quarterly charge of \$30. Suppose if the price rises to \$32, 5,000 subscribers will be lost. Assuming that subscriptions are linearly related to the price, what price should the newspaper charge for a quarterly subscription to maximize their revenue?
  Set x the price charged and y the number of subscribers.
  Data points $(30, 84000)$ and $(32, 79000)$.
  Slope $m = \frac{79000-84000}{32-30} = -\frac{5000}{2} = -2500$.
  Equation of line $y - 84000 = -2500(x - 30)$ or
  $y = -2500x + 159000$.
  Thus, the revenue of the paper is:
  $$R = xy = x(-2500x + 159000) = -2500x^2 + 159000x.$$
  To maximize this we find the vertex:
  $x = -\frac{b}{2a} = -\frac{159000}{2(-2500)} = \frac{159000}{5000} = 31.8$.

Finding the y- and x-Intercepts of a Parabola

- Find the y- and x-intercepts of the quadratic $f(x) = 3x^2 + 5x - 2$.
- We work as follows:
  - Since a y-intercept is on the y-axis, we set $x = 0$.
  - Since an x-intercept is on the x-axis, we set $y = 0$.
- For the y-intercept:

$$f(0) = -2.$$

  Thus y-intercept is at $(0, -2)$.
- For x-intercepts:

$$0 = 3x^2 + 5x - 2 \Rightarrow 0 = (3x - 1)(x + 2) \Rightarrow x = \frac{1}{3} \text{ or } x = -2.$$

  Thus x-intercepts occur at $(-2, 0)$ and $(\frac{1}{3}, 0)$.

Applying the Vertex and x-Intercepts of a Parabola

- A ball is thrown upward from the top of a 40 foot high building at a speed of 80 feet per second. The ball's height above ground can be modeled by the equation $H(t) = -16t^2 + 80t + 40$.
  a. When does the ball reach the maximum height?
  b. What is the maximum height of the ball?
  c. When does the ball hit the ground?

a. $t = -\frac{b}{2a} = -\frac{80}{2(-16)} = \frac{80}{32} = \frac{5}{2}$.

b. $H_{\text{max}} = H(-\frac{5}{2}) = -16(\frac{5}{2})^2 + 80 \cdot \frac{5}{2} + 40 = -16 \cdot \frac{25}{4} + 200 + 40 = -100 + 200 + 40 = 140$.

c. $H(t) = 0$ implies $-16t^2 + 80t + 40 = 0$.

$$t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-80 \pm \sqrt{8960}}{-32} \approx 5.458.$$

Subsection 2

Power Functions and Polynomial Functions

We Will Learn How To:

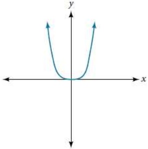
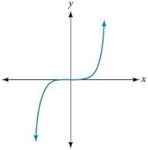
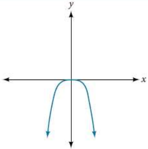
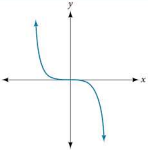
- Identify power functions;
- Identify end behavior of power functions;
- Identify polynomial functions;
- Identify the degree and leading coefficient of polynomial functions.

Identifying Power Functions

- A **power function** is a function that can be represented in the form
$$f(x) = kx^p,$$
where k and p are real numbers, and k is known as the **coefficient**.
- E.g., we have
$$\begin{array}{rclcl} f(x) & = & 1 & = & 1 \cdot x^0; \\ f(x) & = & x & = & 1 \cdot x^1; \\ f(x) & = & x^2 & = & 1 \cdot x^2; \\ f(x) & = & \frac{1}{x} & = & 1 \cdot x^{-1}; \\ f(x) & = & \frac{1}{x^2} & = & 1 \cdot x^{-2}; \\ f(x) & = & \sqrt{x} & = & 1 \cdot x^{1/2}; \\ f(x) & = & \sqrt[3]{x} & = & 1 \cdot x^{1/3}. \end{array}$$
- On the other hand $f(x) = 2^x$ is not a power function.

End Behavior of Power Functions

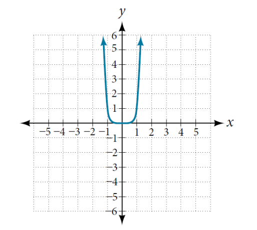
- The behavior of the graph of a function as the input values get very small ($x \to -\infty$) and and as they get very large ($x \to \infty$) is referred to as the **end behavior** of the function.

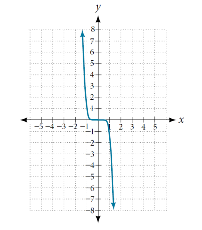
	Even power	Odd power
Positive constant $k > 0$	 $x \to -\infty, f(x) \to \infty$ and $x \to \infty, f(x) \to \infty$	 $x \to -\infty, f(x) \to \infty$ and $x \to \infty, f(x) \to -\infty$
Negative constant $k < 0$	 $x \to -\infty, f(x) \to \infty$ and $x \to \infty, f(x) \to -\infty$	 $x \to -\infty, f(x) \to -\infty$ and $x \to \infty, f(x) \to -\infty$

Identifying End Behavior of Power Functions

- Describe the end behavior of the graphs of a. $f(x) = x^8$ and b. $g(x) = -x^9$.

a. We have $k = 1$ positive and $p = 8$ even.
  - As $x \to -\infty$, $f(x) \to +\infty$;
  - As $x \to +\infty$, $f(x) \to +\infty$.





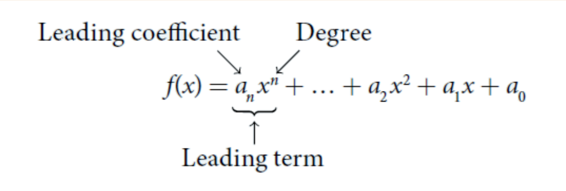
b. We have $k = -1$ negative and $p = 9$ odd.
  - As $x \to -\infty$, $g(x) \to +\infty$;
  - As $x \to +\infty$, $g(x) \to -\infty$.

Polynomial Functions

- Let n be a non-negative integer. A **polynomial function** is a function that can be written in the form

$$f(x) = a_n x^n + \cdots + a_2 x^2 + a_1 x + a_0.$$

- This is called the **general form** of a polynomial function.
- Each a_i is a **coefficient** and can be any real.
- Each expression $a_i x^i$ is a **term** of a polynomial function.



Identifying the Degree and Leading Coefficient

- Identify the degree, leading term, and leading coefficient of the following polynomial functions.
  a. $f(x) = 3 + 2x^2 - 4x^3$
  b. $g(t) = 5t^5 - 2t^3 + 7t$
  c. $h(p) = 6p - p^3 - 2$.

a. For f, we have:
   - Degree: 3;
   - Leading Term: $-4x^3$;
   - Leading Coefficient: -4.

b. For g, we have:
   - Degree: 5;
   - Leading Term: $5t^5$;
   - Leading Coefficient: 5.

c. For h, we have:
   - Degree: 3;
   - Leading Term: $-p^3$;
   - Leading Coefficient: -1.

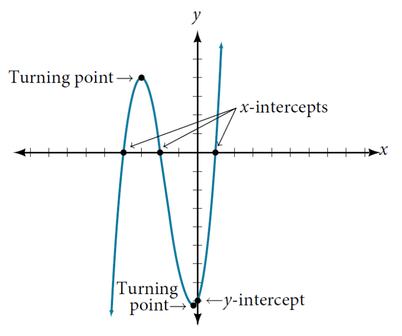
Identifying End Behavior of Polynomial Functions

- For any polynomial, the end behavior of the polynomial will match the end behavior of the power function consisting of the leading term.
- E.g., $f(x) = -x^2 + 7x - 12$ has the same end behavior as $-x^2$.
- E.g., $g(x) = 3x^5 - x^4 + 7x^3 + 11x^2 - x + 2$ has the same end behavior as $3x^5$.

Identifying End Behavior of a Polynomial Function

- Given the function
$$f(x) = -3x^2(x-1)(x+4),$$
express the function as a polynomial in general form, and determine the leading term, degree, and end behavior of the function.
Write in general form:
$$f(x) = -3x^2(x-1)(x+4)$$
$$f(x) = -3x^2(x^2+3x-4)$$
$$f(x) = -3x^4 - 9x^3 + 12x^2.$$
The degree is 4 and the leading term is $-3x^4$.
Since the coefficient is negative and the power even:
  - As $x \to -\infty$, $f(x) \to -\infty$;
  - As $x \to +\infty$, $f(x) \to -\infty$.

Intercepts and Turning Points of Polynomial Functions



- A **turning point** of a graph is a point at which the graph changes direction from increasing to decreasing or decreasing to increasing.
- The y-**intercept** is the point at which the function has an input value of zero.
- The x-**intercepts** are the points at which the output value is zero.

Determining the Intercepts

- Given the polynomial function $f(x) = (x-2)(x+1)(x-4)$, written in factored form, determine the y- and x-intercepts.

  For the y-intercept, set $x = 0$:

  $$f(0) = (-2)(1)(-4) = 8.$$

  So the y-intercept is $(0, 8)$.

  For the x-intercepts, we must solve

  $$\begin{gathered}(x-2)(x+1)(x-4) = 0 \\ x-2=0 \text{ or } x+1=0 \text{ or } x-4=0 \\ x=2 \text{ or } x=-1 \text{ or } x=4.\end{gathered}$$

  Thus the x-intercepts are $(-1, 0)$, $(2, 0)$ and $(4, 0)$.

Determining the Intercepts

- Given the polynomial function $f(x) = x^4 - 4x^2 - 45$, determine the y- and x-intercepts.

  For the y-intercept, set $x = 0$:

  $$f(0) = -45.$$

  So the y-intercept is $(0, -45)$.

  For the x-intercepts, we must solve

  $$\begin{gathered} x^4 - 4x^2 - 45 = 0 \\ (x^2 - 9)(x^2 + 5) = 0 \\ (x+3)(x-3)(x^2+5) = 0 \\ x+3 = 0 \text{ or } x-3 = 0 \text{ or } x^2+5 = 0 \\ x = -3 \text{ or } x = 3 \quad (x^2 \neq -5). \end{gathered}$$

  Thus the x-intercepts are $(-3, 0)$, $(3, 0)$.

Number of Intercepts and Turning Points

- A polynomial of degree n will have:
  - at most n x-intercepts;
  - at most $n-1$ turning points.
- Without graphing the function, determine the local behavior of the function by finding the maximum number of x-intercepts and turning points for $f(x) = -3x^{10} + 4x^7 - x^4 + 2x^3$.

  The graph will have
  - At most 10 x-intercepts;
  - At most 9 turning points.
- Given the function $f(x) = -4x(x+3)(x-4)$, determine the local behavior.

  Since $f(x) = -4x^3 + 4x^2 + 48x$, it will have:
  - At most 3 x-intercepts;
  - At most 2 turning points.

Subsection 3

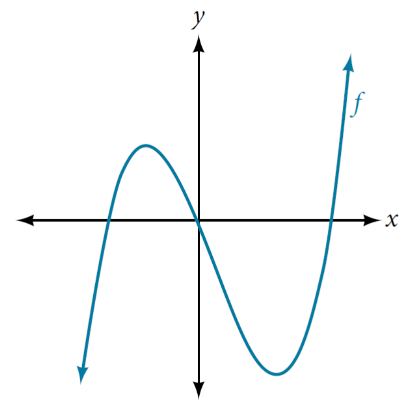
Graphs of Polynomial Functions

We Will Learn How To:

- Recognize characteristics of graphs of polynomial functions;
- Use factoring to find zeros of polynomial functions;
- Identify zeros and their multiplicities;
- Determine end behavior;
- Understand the relationship between degree and turning points;
- Graph polynomial functions;

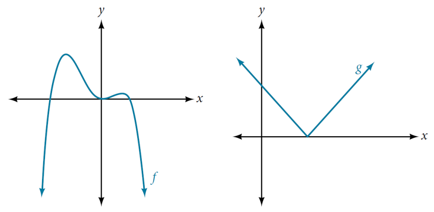
Characteristics of Graphs of Polynomial Functions

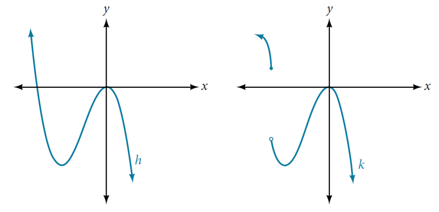
- Polynomial functions have graphs that:
  - do not have sharp corners (are **smooth**);
  - do not have breaks (are **continuous**).



Recognizing Polynomial Functions

- Which of the graphs represents a polynomial function?





- The graphs of f and h are both continuous and smooth.
  So they represent graphs of polynomial functions.
- On the other hand, g is not smooth and k is not continuous.
  So these do not qualify as polynomial functions.

Finding the x-Intercepts by Factoring

- Find the x-intercepts of $f(x) = x^6 - 3x^4 + 2x^2$.

  Factor and use the zero-factor property:

$$x^6 - 3x^4 + 2x^2 = 0$$
$$x^2(x^4 - 3x^2 + 2) = 0$$
$$x^2(x^2 - 1)(x^2 - 2) = 0$$
$$x^2 = 0 \text{ or } x^2 - 1 = 0 \text{ or } x^2 - 2 = 0$$
$$x = 0 \text{ or } x^2 = 1 \text{ or } x^2 = 2$$
$$x = 0 \text{ or } x = \pm 1 \text{ or } x = \pm\sqrt{2}$$

  So f has x-intercepts $(0,0)$, $(-1,0)$, $(1,0)$, $(-\sqrt{2},0)$ and $(\sqrt{2},0)$.

Finding the x-Intercepts by Factoring

- Find the x-intercepts of $f(x) = x^3 - 5x^2 - x + 5$.

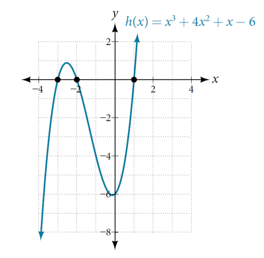
  We work in the same way:

$$x^3 - 5x^2 - x + 5 = 0$$
$$x^2(x-5) - (x-5) = 0$$
$$(x^2-1)(x-5) = 0$$
$$(x+1)(x-1)(x-5) = 0$$
$$x+1 = 0 \text{ or } x-1 = 0 \text{ or } x-5 = 0$$
$$x = -1 \text{ or } x = 1 \text{ or } x = 5$$

  We conclude that the x-intercepts are $(-1,0)$, $(1,0)$ and $(5,0)$.

Finding the x-Intercepts Using a Graph

- Find the x-intercepts of $h(x) = x^3 + 4x^2 + x - 6$ whose graph is shown below

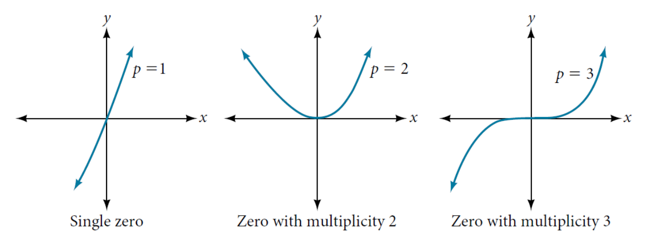


We can see that the x-intercepts are $(-3, 0)$, $(-2, 0)$ and $(1, 0)$.
We can use one to factor and find the others.

$$x^3 + 4x^2 + x - 6 = 0$$
$$x^3 + 3x^2 + x^2 + 3x - 2x - 6 = 0$$
$$x^2(x + 3) + x(x + 3) - 2(x + 3) = 0$$
$$(x^2 + x - 2)(x + 3) = 0$$
$$(x + 2)(x - 1)(x + 3) = 0$$
$$x = -2 \text{ or } x = 1 \text{ or } x = -3.$$

Graphical Behavior of Polynomials at x-Intercepts

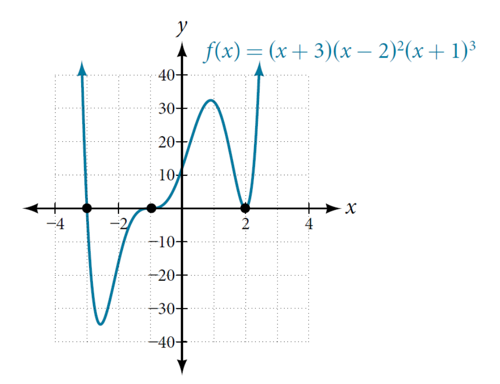
- If a polynomial contains a factor of the form $(x-h)^p$, the behavior near the x-intercept h is determined by the power p.
- We say that $x = h$ is a **zero of multiplicity** p.
  - The graph of a polynomial function will touch the x-axis at zeros with even multiplicities.
  - The graph will cross the x-axis at zeros with odd multiplicities.



Graphing a Function

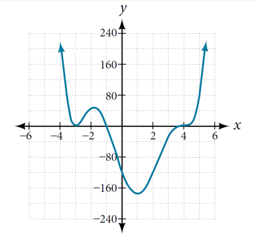
- For example, we graph the function shown.

$$f(x) = (x+3)(x-2)^2(x+1)^3.$$



Identifying Zeros and Their Multiplicities

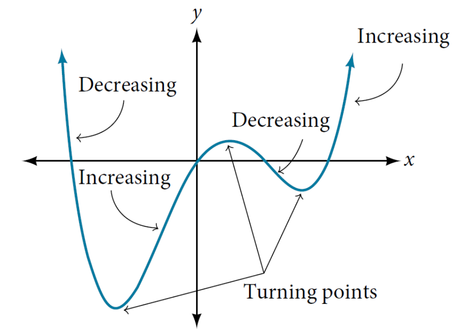
- Use the graph of the function of degree 6 to identify the zeros of the function and their possible multiplicities.



  - The first zero occurs at $x = -3$.
    The graph touches the x-axis, so the multiplicity must be even.
  - The next zero occurs at $x = -1$.
    This is a single zero of multiplicity 1.
  - The last zero occurs at $x = 4$.
    The graph crosses the x-axis, so the multiplicity must be odd.
    We know that the multiplicity is likely 3.

Turning Points

- A **turning point** is a point of the graph where the graph changes from increasing to decreasing (rising to falling) or decreasing to increasing (falling to rising).
- A polynomial of degree n will have at most $n-1$ turning points.



Finding the Maximum Number of Turning Points

- Find the maximum number of turning points of each polynomial function.
  a. $f(x) = -x^3 + 4x^5 - 3x^2 + 1$
  b. $f(x) = -(x-1)^2(1+2x^2)$

a. The degree is 5.

Thus, f has at most 4 turning points.

b. The degree is 4.

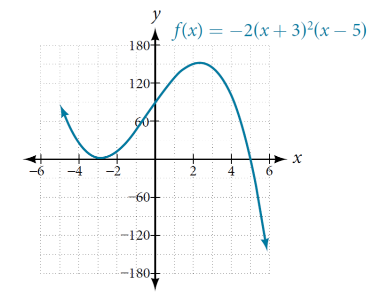
So this polynomial has at most 3 turning points.

Sketching the Graph of a Polynomial Function

- Sketch a graph of $f(x) = -2(x+3)^2(x-5)$.

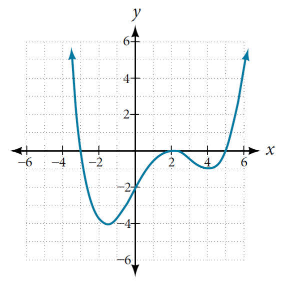
  Use:

  - End behavior:
    - if $x \to -\infty$, $f(x) \to +\infty$;
    - if $x \to +\infty$, $f(x) \to -\infty$.
  - The roots and their multiplicities:
    - $x = -3$ of multiplicity 2;
    - $x = 5$ of multiplicity 1.



Writing Formulas for Polynomial Functions

- Write a formula for the polynomial function shown



Taking into account the zeros and their multiplicities we come up with a candidate formula:

$$f(x) = a(x+3)(x-2)^2(x-5).$$

Then we find a using a point on the graph.

$$f(0) = -2 \Rightarrow a \cdot 3 \cdot (-2)^2 \cdot (-5) = -2 \Rightarrow -60a = -2 \Rightarrow a = \frac{1}{30}.$$

So we have $f(x) = \frac{1}{30}(x+3)(x-2)^2(x-5)$.

Subsection 4

Dividing Polynomials

We Will Learn How To:

- Use long division to divide polynomials;
- Use synthetic division to divide polynomials.

The Division Algorithm

- Recall the **division of numbers**.
  - **Dividend** 17
  - **Divisor** 3

    Divide $17 \div 3$ to get:
  - **Quotient** 5
  - **Remainder** 2

    and write: $17 = 3 \cdot 5 + 2$ or $\frac{17}{3} = 5 + \frac{2}{3}$.
- Similarly for polynomials:
  - **Dividend** $f(x)$
  - **Divisor** $d(x)$, with $\deg(d(x)) \leq \deg(f(x))$

    Divide $f(x) \div d(x)$ to get:
  - **Quotient** $q(x)$
  - **Remainder** $r(x)$, with $0 \leq \deg(r(x)) < \deg(d(x))$

    and write $f(x) = d(x) \cdot q(x) + r(x)$ or $\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$.

Using Long Division to Divide a Polynomial

- Divide $5x^2 + 3x - 2$ by $x + 1$ and write your answer in an appropriate form.

$$\begin{array}{rr|rrr} & & 5x & -2 & \\ \cline{3-5} x & +1 & 5x^2 & +3x & -2 \\ & & 5x^2 & +5x & \\ \cline{3-4} & & & -2x & -2 \\ & & & -2x & -2 \\ \cline{4-5} & & & & 0 \end{array}$$

So we have

$$5x^2 + 3x - 2 = (x + 1)(5x - 2).$$

Using Long Division to Divide a Polynomial

- Divide $6x^3 + 11x^2 - 31x + 15$ by $3x - 2$ and write your answer in an appropriate form.

$$
\begin{array}{rrrrrr}
 & & 2x^2 & +5x & -7 & \\ \cline{3-6}
3x & -2 \,| & 6x^3 & +11x^2 & -31x & +15 \\
 & & 6x^3 & -4x^2 & & \\ \cline{3-4}
 & & & 15x^2 & -31x & \\
 & & & 15x^2 & -10x & \\ \cline{4-5}
 & & & & -21x & +15 \\
 & & & & -21x & +14 \\ \cline{5-6}
 & & & & & 1
\end{array}
$$

So we have

$$6x^3 + 11x^2 - 31x + 15 = (3x - 2)(2x^2 + 5x - 7) + 1.$$

Synthetic Division

- **Synthetic division** is a shortcut that can be used when the divisor is a binomial in the form $x - k$ where k is a real number.
- Only the coefficients are used, omitting the powers of x.
- Use synthetic division to divide $5x^2 - 3x - 36$ by $x - 3$.

$$\begin{array}{r|rrr} 3 & 5 & -3 & -36 \\ & & 15 & 36 \\ \hline & 5 & 12 & 0 \end{array}$$

So, $5x^2 - 3x - 36 = (x - 3)(5x + 12)$.

Using Synthetic Division

- Use synthetic division to divide $4x^3 + 10x^2 - 6x - 20$ by $x + 2$.

$$\begin{array}{r|rrrr} -2 & 4 & 10 & -6 & -20 \\ & & -8 & -4 & 20 \\ \hline & 4 & 2 & -10 & 0 \end{array}$$

So, $4x^3 + 10x^2 - 6x - 20 = (x + 2)(4x^2 + 2x - 10)$.

- Use synthetic division to divide $-9x^4 + 10x^3 + 7x^2 - 6$ by $x - 1$.

$$\begin{array}{r|rrrrr} 1 & -9 & 10 & 7 & 0 & -6 \\ & & -9 & 1 & 8 & 8 \\ \hline & -9 & 1 & 8 & 8 & 2 \end{array}$$

Therefore $-9x^4 + 10x^3 + 7x^2 - 6 = (x - 1)(-9x^3 + x^2 + 8x + 8) + 2$.

Using Polynomial Division in an Application Problem

- The volume of a rectangular solid is given by the polynomial $3x^4 - 3x^3 - 33x^2 + 54x$.
  The length of the solid is given by $3x$ and the width is given by $x - 2$. Find the height, t, of the solid.
  We know that the volume equals length times width times height.
  So, according to the data, we have:

$$\begin{aligned} 3x(x-2)t &= 3x^4 - 3x^3 - 33x^2 + 54x \\ (x-2)t &= \frac{3x^4 - 3x^3 - 33x^2 + 54x}{3x} \\ (x-2)t &= x^3 - x^2 - 11x + 18 \\ t &= \frac{x^3 - x^2 - 11x + 18}{x-2}. \end{aligned}$$

  To find t, we must perform the division:

$$\begin{array}{r|rrrr} 2 & 1 & -1 & -11 & 18 \\ & & 2 & 2 & -18 \\ \hline & 1 & 1 & -9 & 0 \end{array}$$

  So $t = \frac{x^3 - x^2 - 11x + 18}{x-2} = x^2 + x - 9$.

Subsection 5

Zeros of Polynomial Functions

We Will Learn How To:

- Evaluate a polynomial using the Remainder Theorem;
- Use the Factor Theorem to solve a polynomial equation;
- Use the Rational Zero Theorem to find rational zeros;
- Find zeros of a polynomial function;
- Find polynomials with given zeros.

The Remainder Theorem

- If a polynomial $f(x)$ is divided by $x - k$, we get

$$f(x) = (x - k)q(x) + r, \text{ where } r \text{ is a constant.}$$

- Note that

$$f(k) = (k - k)q(k) + r = r.$$

- That is, the remainder of the division $f(x) \div (x - k)$ equals $f(k)$!

Using the Remainder Theorem

- Use the **Remainder Theorem** to compute $f(2)$ if

$$f(x) = 6x^4 - x^3 - 15x^2 + 2x - 7.$$

We must divide $f(x)$ by $x - 2$ and find the remainder.

$$\begin{array}{r|rrrrr} 2 & 6 & -1 & -15 & 2 & -7 \\ & & 12 & 22 & 14 & 32 \\ \hline & 6 & 11 & 7 & 16 & 25 \end{array}$$

Therefore, $f(2) = 25$.

The Factor Theorem

- We saw that $f(x) \div (x-k)$ has remainder $r = f(k)$.
- It follows that

$$\begin{aligned} k \text{ is a zero of } f(x) \quad & \text{if and only if} \quad \text{the remainder } r = 0 \\ & \text{if and only if} \quad (x-k) \text{ is a factor of } f(x). \end{aligned}$$

- Thus $(x-k)$ is a factor of $f(x)$ if and only if $f(k) = 0$.

Using the Factor Theorem

- Let $f(x) = x^3 - 6x^2 - x + 30$.
  (a) Use the **Factor Theorem** to show that $(x+2)$ is a factor of $f(x)$.
  (b) Find the remaining factors.
  (c) Use the factors to determine the zeros of the polynomial.

(a) Show $f(-2) = 0$:

$$f(-2) = (-2)^3 - 6(-2)^2 - (-2) + 30 = -8 - 24 + 2 + 30 = 0.$$

So $x+2$ is a factor of $f(x)$.

(b) We divide $f(x)$ by $x+2$:

$$\begin{array}{r|rrrr} -2 & 1 & -6 & -1 & 30 \\ & & -2 & 16 & -30 \\ \hline & 1 & -8 & 15 & 0 \end{array}$$

So $x^3 - 6x^2 - x + 30 = (x+2)(x^2 - 8x + 15) = (x+2)(x-3)(x-5)$.

(c) The zeros are $x = -2$, $x = 3$ and $x = 5$.

The Rational Zero Theorem

- The **Rational Zero Theorem** states that, if the polynomial

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

has integer coefficients, then every rational zero of $f(x)$ has the form $\frac{p}{q}$ where
  - p is a factor of the constant term a_0;
  - q is a factor of the leading coefficient a_n.
- When the leading coefficient is 1, the possible rational zeros are the factors of the constant term.

Listing All Possible Rational Zeros

- List all possible rational zeros of $f(x) = 2x^4 - 5x^3 + x^2 - 4$.

  Relying on the Rational Zero Theorem,
  - we first list the factors of $a_0 = -4$: $\pm 1, \pm 2, \pm 4$;
  - we then list the factors of $a_4 = 2$: $\pm 1, \pm 2$.

  Finally, we form all possible ratios:

$$\pm 1, \pm \frac{1}{2}, \pm 2, \pm 4.$$

Finding the Zeros of Polynomial Functions

- Find the zeros of $f(x) = 4x^3 - 3x - 1$.
- We follow the strategy:
  - Quickly identify a zero, possibly by using the Rational Zero Theorem.
  - Use Synthetic Division to find the quotient.
  - Repeat these steps until obtaining a quadratic.

Observe that $f(1) = 0$.

Divide f by $(x - 1)$:

$$\begin{array}{r|rrrr} 1 & 4 & 0 & -3 & -1 \\ & & 4 & 4 & 1 \\ \hline & 4 & 4 & 1 & 0 \end{array}$$

So we get

$$f(x) = 4x^3 - 3x - 1 = (x - 1)(4x^2 + 4x + 1) = (x - 1)(2x + 1)^2.$$

Thus, the zeros are $x = 1$ and $x = -\frac{1}{2}$.

Find a Polynomial with Given Zeros

- Find a third degree polynomial that has zeros of -3, 1 and 2, such that $f(-2) = 60$.

  We must have
  $$f(x) = a(x+3)(x-1)(x-2).$$
  Now use $f(-2) = 60$ to compute a:
  $$\begin{aligned} a(-2+3)(-2-1)(-2-2) &= 60 \\ a \cdot 1 \cdot (-3) \cdot (-4) &= 60 \\ 12a &= 60 \\ a &= 5. \end{aligned}$$
  So we get
  $$\begin{aligned} f(x) &= 5(x^2+2x-3)(x-2) = 5(x^3-2x^2+2x^2-4x-3x+6) \\ &= 5(x^3-7x+6) = 5x^3-35x+30. \end{aligned}$$

Subsection 6

Rational Functions

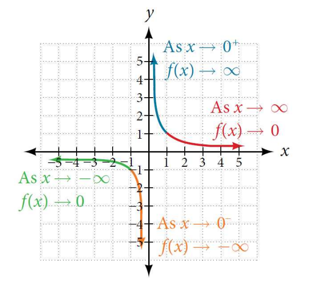
We Will Learn How To:

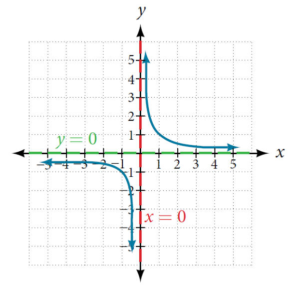
- Find the domain of a rational function;
- Identify vertical and horizontal asymptotes;
- Find x- and y-intercepts;
- Sketch the graph of a rational function, given a formula;
- Obtain a formula for a rational function, given a graph.

Vertical and Horizontal Asymptotes

- Follow the trends in the graph:
  - As $x \to 0^-$, $f(x) \to -\infty$
  - As $x \to 0^+$, $f(x) \to +\infty$

  We say that the line $x = 0$ is a **vertical asymptote** of the graph.



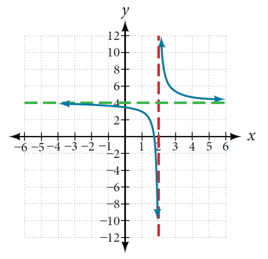


- Similarly:
  - As $x \to -\infty$, $f(x) \to 0$
  - As $x \to +\infty$, $f(x) \to 0$

  We say that the line $y = 0$ is a **horizontal asymptote** of the graph.

Using Arrow Notation

- Use arrow notation to describe the end behavior and local behavior of the function shown.



  - As $x \to 2^-$, $y \to -\infty$
  - As $x \to 2^+$, $y \to +\infty$

So $x = 2$ is a vertical asymptote.

  - As $x \to -\infty$, $y \to 4$
  - As $x \to +\infty$, $y \to 4$

So $y = 4$ is a horizontal asymptote.

Domain of a Rational Function

- The domain of a rational function includes all real numbers except those that cause the denominator to equal zero.
- Find the domain of $f(x) = \frac{x+3}{x^2-9}$.

  Begin by setting the denominator equal to zero and solving.

$$\begin{aligned} x^2 - 9 &= 0 \\ x^2 &= 9 \\ x &= \pm 3. \end{aligned}$$

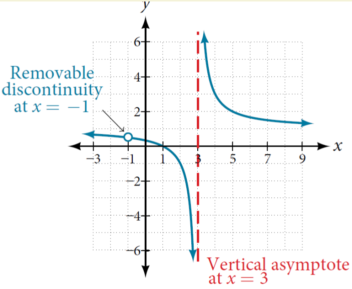
  The denominator is equal to zero when $x = \pm 3$.

  The domain of the function is all real numbers except $x = \pm 3$.

  Formally and succinctly, we write $\mathrm{Dom}(f) = \mathbb{R} - \{-3, 3\}$.

Identifying Vertical Asymptotes

- In general the vertical asymptotes occur at those values that zero the denominator, i.e., those values we exclude from the domain.
- Exceptions may occur if those same values zero also the numerator.
- If the multiplicity of the zero in the numerator is greater than or equal to that in the denominator, we have a hole in the graph.
- If the multiplicity is greater in the denominator, then we have a vertical asymptote at that value.



- The hole is called a **removable discontinuity**.

Vertical Asymptotes and Removable Discontinuities

- Find the vertical asymptotes of $k(x) = \frac{5+2x^2}{(2+x)(1-x)}$.

  We have
  $$k(x) = \frac{5+2x^2}{(2+x)(1-x)}.$$
  So $x = -2$ and $x = 1$ are vertical asymptotes.

- Find the vertical asymptotes and removable discontinuities of the graph of $k(x) = \frac{x-2}{x^2-4}$.

  Factor the numerator and the denominator, $k(x) = \frac{x-2}{(x-2)(x+2)}$.
  - There is a common factor in the numerator and the denominator, $x-2$. The multiplicities are both equal to 1.
    So, at $x = 2$, k has a removable discontinuity.
  - There is a factor in the denominator that is not in the numerator, $x+2$.
    So $x = -2$ is a vertical asymptote.

Horizontal Asymptotes of Rational Functions

- The horizontal asymptote of a rational function can be determined by looking at the degrees of the numerator and denominator.
  - Degree of numerator is less than degree of denominator: horizontal asymptote at $y = 0$.
  - Degree of numerator is greater than degree of denominator: no horizontal asymptote.
  - Degree of numerator is equal to degree of denominator: horizontal asymptote at ratio of leading coefficients.

Identifying Horizontal Asymptotes

- For the functions listed, identify the horizontal asymptote.
  a. $g(x) = \frac{6x^3-10x}{2x^3+5x^2}$
  b. $h(x) = \frac{x^2-4x+1}{x+2}$
  c. $k(x) = \frac{x^2+4x}{x^3-8}$.

a. Numerator and denominator have the same degree 3.
   So g has a horizontal asymptote $y = \frac{6}{2}$ or $y = 3$.

b. The degree of the numerator exceeds that of the denominator.
   So there is no horizontal asymptote.

c. The degree of the denominator exceeds that of the numerator.
   So $y = 0$ is the horizontal asymptote.

Identifying Horizontal and Vertical Asymptotes

- Find the horizontal and vertical asymptotes of the function

$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}.$$

Start by finding the domain $\mathrm{Dom}(f) = \mathbb{R} - \{-2, 1, 5\}$.

Then find the x-intercepts (these are the numbers that zero the numerator): $x = -3$, $x = 2$.

Note that numerator and denominator do not share any zeros.

So we get the following:

  - The vertical asymptotes are the lines

$$x = -2, \quad x = 1, \quad x = 5.$$

  - The horizontal asymptote (since the degree of the denominator is bigger than that of the numerator) is $y = 0$.

Finding the Intercepts of a Rational Function

- Find the intercepts of
$$f(x) = \frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)}.$$
For the y-intercept, we set $x = 0$:
$$f(0) = \frac{-2 \cdot 3}{-1 \cdot 2 \cdot (-5)} = \frac{-6}{10} = -\frac{3}{5}.$$
So the y-intercept is $(0, -\frac{3}{5})$.
For the x-intercepts, set $y = 0$.
$$\frac{(x-2)(x+3)}{(x-1)(x+2)(x-5)} = 0 \quad \Rightarrow \quad (x-2)(x+3) = 0$$
$$\Rightarrow \quad x = -3 \text{ or } x = 2.$$
Thus, the x-intercepts are $(-3, 0)$ and $(2, 0)$.

Graphing and Writing a Rational Function

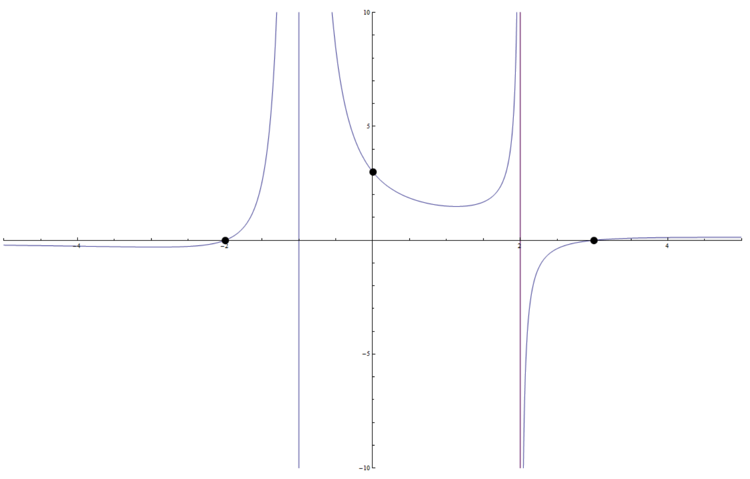
- Sketch a graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.

  We follow a series of steps to facilitate graphing:
- The domain is $\mathrm{Dom}(f) = \mathbb{R} - \{-1, 2\}$;
- The vertical asymptotes are: $x = -1$ and $x = 2$.
- The horizontal asymptote is: $y = 0$.
- The x-intercepts are: $(-2, 0)$ and $(3, 0)$.
- The y-intercept is: $(0, 3)$.
- Finally, set up the sign table for $f(x)$:

interval	$(-\infty, -2)$	$(-2, -1)$	$(-1, 2)$	$(2, 3)$	$(3, +\infty)$
sign of $f(x)$	$-$	$+$	$+$	$-$	$+$

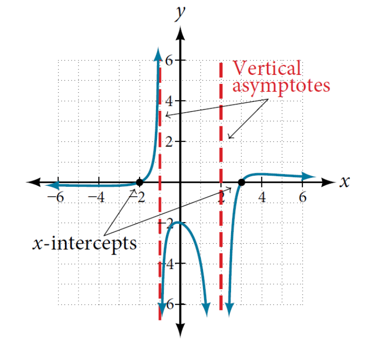
Graphing and Writing a Rational Function

- Sketch a graph of $f(x) = \frac{(x+2)(x-3)}{(x+1)^2(x-2)}$.



Graphing and Writing a Rational Function

- Write an equation for the rational function shown



- x-intercepts are $(-2, 0)$ and $(3, 0)$. Numerator factors: $x + 2$ and $x - 3$.
- Vertical asymptotes are $x = -1$ and $x = 2$. Denominator factors: $x + 1$ and $x - 2$.
- Horizontal asymptote $y = 0$. So denominator is of higher degree.

  At $x = 2$, $f(x)$ does not switch signs.

A guess for the formula is $f(x) = \frac{a(x+2)(x-3)}{(x+1)(x-2)^2}$.

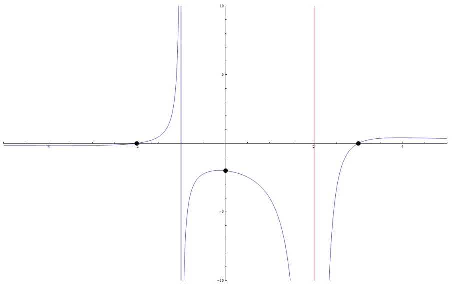
Graphing and Writing a Rational Function (Cont'd)

- We obtained $f(x) = \frac{a(x+2)(x-3)}{(x+1)(x-2)^2}$.

  y-intercept is $(0, -2)$.

$$f(0) = -2 \Rightarrow \frac{a \cdot 2 \cdot (-3)}{1 \cdot (-2)^2} = -2 \Rightarrow \frac{-6a}{4} = -2 \Rightarrow a = \frac{4}{3}.$$

  So we have the formula $f(x) = \frac{4(x+2)(x-3)}{3(x+1)(x-2)^2}$.



Subsection 7

Inverses and Radical Functions

We Will Learn How To:

- Find the inverse of an invertible polynomial function;
- Restrict the domain to find the inverse of a polynomial function.

Verifying two Functions are Inverses

- Two functions, f and g, are inverses of one another if
  - for all x in the domain of f, $g(f(x)) = x$;
  - for all x in the domain of g, $f(g(x)) = x$.
- Show that $f(x) = \frac{1}{x+1}$ and $g(x) = \frac{1}{x} - 1$ are inverses.

  We verify that both composition operations yield the identity:

$$g(f(x)) = g\left(\frac{1}{x+1}\right) = \frac{1}{\frac{1}{x+1}} - 1 = x + 1 - 1 = x.$$

$$f(g(x)) = f\left(\frac{1}{x} - 1\right) = \frac{1}{\frac{1}{x} - 1 + 1} = \frac{1}{\frac{1}{x}} = x.$$

Finding the Inverse of a Cubic Function

- Find the inverse of the function $f(x) = 5x^3 + 1$.
  Rewrite $y = 5x^3 + 1$.
  Interchange $x \leftrightarrow y$:
  $$x = 5y^3 + 1.$$
  Solve for y:
  $$x = 5y^3 + 1$$
  $$x - 1 = 5y^3$$
  $$\frac{x-1}{5} = y^3$$
  $$\sqrt[3]{\frac{x-1}{5}} = y.$$
  So $f^{-1}(x) = \sqrt[3]{\frac{x-1}{5}}$.

Restricting the Domain to Find the Inverse

- If a function is not one-to-one, it cannot have an inverse.
- If we restrict the domain of the function so that it becomes one-to-one, thus creating a new function, this new function will have an inverse.
- Find the inverse function of f:
  a. $f(x) = (x-4)^2$, $x \geq 4$;
  b. $f(x) = (x-4)^2$, $x \leq 4$.

a. $y = (x-4)^2 \Rightarrow x = (y-4)^2 \overset{y \geq 4}{\Rightarrow} \sqrt{x} = y - 4 \Rightarrow \sqrt{x} + 4 = y$.
So, in this case, $f^{-1}(x) = \sqrt{x} + 4$.

a. $y = (x-4)^2 \Rightarrow x = (y-4)^2 \overset{y \leq 4}{\Rightarrow} -\sqrt{x} = y - 4 \Rightarrow -\sqrt{x} + 4 = y$.
So, in this case, $f^{-1}(x) = -\sqrt{x} + 4$.

Finding the Inverse When the Restriction Is Not Specified

- Restrict the domain and then find the inverse of

$$f(x) = (x-2)^2 - 3.$$

The graph is that of $y = x^2$ shifted 2 units right and 3 units down.

To pass the horizontal line test, we must restrict its domain to $[2, \infty)$.

Now we work to find the inverse:

$$\begin{aligned} y &= (x-2)^2 - 3 \\ x &= (y-2)^2 - 3 \\ x+3 &= (y-2)^2 \\ \sqrt{x+3} &= y-2 \\ \sqrt{x+3}+2 &= y. \end{aligned}$$

So $f^{-1}(x) = \sqrt{x+3} + 2$.

Finding the Inverse of a Radical Function

- Restrict the domain of the function $f(x) = \sqrt{x-4}$ and then find the inverse.

  The graph is that of $y = \sqrt{x}$ shifted 4 units right.

  The graph passes the horizontal line test on $[4, \infty)$.

  Now we work to find the inverse:

$$\begin{aligned} y &= \sqrt{x-4} \\ x &= \sqrt{y-4} \\ x^2 &= y-4 \\ x^2+4 &= y. \end{aligned}$$

  So $f^{-1}(x) = x^2 + 4$, but defined only on $[0, \infty)$.

Solving Applications of Radical Functions

- A mound of gravel is in the shape of a cone with the height equal to twice the radius, whose volume in terms of the radius is $V = \frac{2}{3}\pi r^3$.
  a. Find the inverse of the function $V = \frac{2}{3}\pi r^3$ that determines the volume V of a cone and is a function of the radius r.
  b. Then use the inverse function to calculate the radius of such a mound of gravel measuring 100 cubic feet. Use $\pi = 3.14$.

a. We need to solve for r:

$$V = \frac{2}{3}\pi r^3 \Rightarrow \frac{3V}{2\pi} = r^3 \Rightarrow \sqrt[3]{\frac{3V}{2\pi}} = r.$$

Thus, $r = \sqrt[3]{\frac{3V}{2\pi}}$.

b. We have $r = \sqrt[3]{\frac{3 \cdot 100}{2 \cdot 3.14}} \approx 3.63$ feet.

Determining the Domain of a Radical Function

- Find the domain of the function $f(x) = \sqrt{\frac{(x+2)(x-3)}{x-1}}$.

  One has to impose two restrictions:

$$x - 1 \neq 0 \quad \text{and} \quad \frac{(x-2)(x-3)}{x-1} \geq 0.$$

  We use the sign table method:

	$(-\infty, -2]$	$[-2, 1)$	$(1, 3]$	$[3, \infty)$
$\frac{(x-2)(x-3)}{x-1}$	$-$	$+$	$-$	$+$

  Hence, we must have x in $[-2, 1) \cup [3, \infty)$.

Finding the Inverse of a Rational Function

- The function $C = \frac{20+0.4n}{100+n}$ represents the concentration C of an acid solution after n mL of 40% solution has been added to 100 mL of a 20% solution.
  a. Find the inverse of the function; that is, find an expression for n in terms of C.
  b. Use your result to determine how much of the 40% solution should be added so that the final mixture is a 35% solution.

a. We have

$$C = \frac{20+0.4n}{100+n} \Rightarrow C(100+n) = 20+0.4n$$

$$\Rightarrow 100C + Cn = 20 + 0.4n \Rightarrow 100C - 20 = 0.4n - Cn$$

$$\Rightarrow 100C - 20 = (0.4 - C)n \Rightarrow \frac{100C-20}{0.4-C} = n.$$

So $n = \frac{100C-20}{0.35-C}$.

(b) Now we get $n = \frac{100 \cdot 0.35 - 20}{0.4-0.35} = \frac{15}{0.05} = 300.$

Subsection 8

Modeling Using Variation

We Will Learn How To:

- Solve direct variation problems;
- Solve inverse variation problems;
- Solve problems involving joint variation.

Direct Variation

- If x and y are related by an equation of the form

$$y = kx^n,$$

  then we say that the relationship is **direct variation** and y **varies directly with**, or **is proportional to**, the nth power of x.
- In direct variation relationships, there is a nonzero constant ratio $k = \frac{y}{x^n}$, where k is called the **constant of variation**, which helps define the relationship between the variables.

Solving a Direct Variation Problem

- The quantity y varies directly with the cube of x.
  If $y = 25$ when $x = 2$, find y when x is 6.
  The hypothesis implies that there exists a constant k, such that
  $$y = kx^3.$$
  Since when $x = 2$, $y = 25$, we get
  $$25 = k \cdot 2^3 \Rightarrow k = \frac{25}{8}.$$
  Thus, the relation of direct variation is
  $$y = \frac{25}{8}x^3.$$
  Therefore, for $x = 6$,
  $$y = \frac{25}{8} \cdot 6^3 = 675.$$

Inverse Variation

- If x and y are related by an equation of the form

$$y = \frac{k}{x^n},$$

  where k is a nonzero constant, then we say that y **varies inversely with** the nth power of x.
- In **inversely proportional** relationships, or **inverse variations**, there is a constant multiple $k = x^n y$.

Solving an Inverse Variation Problem

- A quantity y varies inversely with the cube of x.
  If $y = 25$ when $x = 2$, find y when x is 6.
  The hypothesis implies that there exists a constant k, such that
  $$y = \frac{k}{x^3}.$$
  Since when $x = 2$, $y = 25$, we get
  $$25 = \frac{k}{2^3} \Rightarrow k = 25 \cdot 8 \Rightarrow k = 200.$$
  Thus, the relation of inverse variation is
  $$y = \frac{200}{x^3}.$$
  Therefore, for $x = 6$,
  $$y = \frac{200}{6^3} = \frac{25}{27}.$$

Joint Variation

- **Joint variation** occurs when a variable varies directly or inversely with multiple variables.
- For instance, if x varies directly with both y and z, we have

$$x = kyz.$$

- If x varies directly with y and inversely with z, we have

$$x = \frac{ky}{z}.$$

- Notice that we only use one constant in a joint variation equation.

Solving Problems Involving Joint Variation

- A quantity x varies
  - directly with the square of y and
  - inversely with the cube root of z.

  If $x = 6$ when $y = 2$ and $z = 8$, find x when $y = 1$ and $z = 27$.
  The hypothesis implies that there exists a constant k, such that

$$x = \frac{ky^2}{\sqrt[3]{z}}.$$

Since when $y = 2$ and $z = 8$, $x = 6$, we get

$$6 = \frac{k \cdot 2^2}{\sqrt[3]{8}} \Rightarrow k = \frac{6 \cdot 2}{4} \Rightarrow k = 3.$$

Thus, the relation of inverse variation is $x = \frac{3y^2}{\sqrt[3]{z}}$.
Therefore, for $y = 1$ and $z = 27$,

$$x = \frac{3 \cdot 1^2}{\sqrt[3]{27}} = \frac{3 \cdot 1}{3} = 1.$$