College Algebra

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LSSU Math 111

Subsection 1

Exponential Functions

We Will Learn How To:

- Undrstand exponential functions;
- Find the equation of an exponential function;
- Use compound interest formulas;
- Evaluate exponential functions with base e.

Exponential Growth/Decay

- **Percent change** refers to a change based on a percent of the original amount.
- **Exponential growth** refers to an increase based on multiplication by a constant over equal increments of time.

x	-3	-2	-1	0	1	2	3
y	$\frac{1}{8}$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

- **Exponential decay** refers to a decrease based on multiplication by a constant over equal increments of time.

x	-3	-2	-1	0	1	2	3
y	64	16	4	1	$\frac{1}{4}$	$\frac{1}{16}$	$\frac{1}{64}$

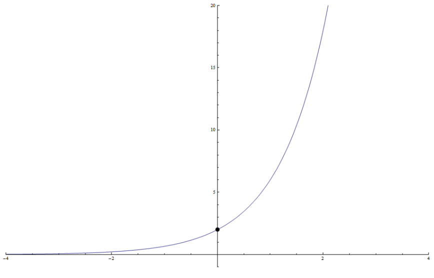
Exponential versus Linear Growth

- **Exponential growth** refers to the original value from the range increased by the same percentage over equal increments found in the domain.
- **Linear growth** refers to the original value from the range increased by the same amount over equal increments found in the domain.

x	$f(x) = 2^x$	$g(x) = 2x$
0	1	0
1	2	2
2	4	4
3	8	6
4	16	8
5	32	10
6	64	12

Exponential Function

- For any real number x, an exponential function is a function with the form
$$f(x) = a \cdot b^x,$$
where
    - a is the a non-zero real number called the **initial value**;
    - b is any positive real number such that $b \neq 1$.
- The domain of f is all real numbers.
- The range of f is
    - $(0, \infty)$ if $a > 0$;
    - $(-\infty, 0)$ if $a < 0$.
- The y-intercept is $(0, a)$.
- The horizontal asymptote is $y = 0$.



Exponential Growth

- A function that models **exponential growth** grows by a rate proportional to the amount present.
- For any real number x and any positive real numbers a and b such that $b \neq 1$, an exponential growth function has the form

$$f(x) = a \cdot b^x,$$

  where
  - a is the **initial** or **starting value** of the function;
  - b is the **growth factor** or **growth multiplier** per unit x.

Evaluating a Real-World Exponential Model

- The population of India was about 1.25 billion in the year 2013, with an annual growth rate of about 1.2%.
  (a) Find an exponential function modeling the population growth.
  (b) To the nearest thousandth, what is the population predicted to be in 2031?

(a) Let $P(t)$ be the population, where t is the number of years since 2013.
Then, we get
$$P(t) = 1.25 \cdot (1.012)^t.$$

(b) Year 2031 corresponds to $t = 18$.
So we get
$$P(18) = 1.25 \cdot (1.012)^{18} \approx 1.549 \text{ billion people.}$$

Finding Equations of Exponential Functions

- In 2006, 80 deer were introduced into a wildlife refuge.
  By 2012, the population had grown to 180 deer.
  Assuming exponential growth, find a function $N(t)$ representing the population N of deer as a function of time t.
  We have
  $$N(t) = a \cdot b^t, \quad t \text{ in years since 2006.}$$
  Since, at $t = 0$ (2006), $N(0) = 80$, we get
  $$80 = a \cdot b^0 \Rightarrow 80 = a.$$
  Since at $t = 6$ (2012), $N(6) = 180$, we get
  $$180 = 80 \cdot b^6 \Rightarrow b^6 = \frac{9}{4} \Rightarrow b = \sqrt[6]{\frac{9}{4}} \Rightarrow b \approx 1.1447.$$
  So the model we obtain is $N(t) = 80 \cdot 1.1447^t$.

Writing a Model When the Initial Value is Not Known

- Find an exponential function that passes through the points $(-2,6)$ and $(2,1)$.

  Assume a function $y = a \cdot b^x$.

  The first data point gives

  $$6 = a \cdot b^{-2} \Rightarrow 6 = \frac{a}{b^2} \Rightarrow a = 6b^2.$$

  The second data point yields

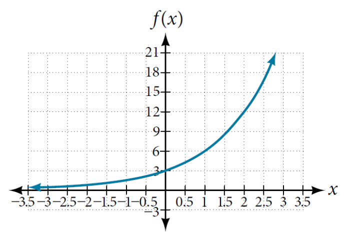
  $$\begin{aligned} 1 = a \cdot b^2 &\Rightarrow 1 = 6b^2 b^2 \Rightarrow 1 = 6b^4 \\ &\Rightarrow b^4 = \tfrac{1}{6} \Rightarrow b = \sqrt[4]{\tfrac{1}{6}} \Rightarrow b \approx 0.6389. \end{aligned}$$

  Thus, $a = 6b^2 = 6 \cdot 0.6389^2 \approx 2.4492$.

  So the function is $y = 2.4492 \cdot 0.6389^x$.

Writing an Exponential Function Given Its Graph

- Find an equation for the exponential function whose graph is shown



Assume the model $y = a \cdot b^x$.
Since $(0, 3)$ is on graph,

$$3 = a \cdot b^0 \;\Rightarrow\; 3 = a.$$

Since $(1, 6)$ is on graph

$$6 = 3 \cdot b^1 \;\Rightarrow\; b = 2.$$

Hence we get the model $y = 3 \cdot 2^x$.

The Compound Interest Formula

- **Compound interest** can be calculated using the formula

$$A(t) = P\left(1 + \frac{r}{n}\right)^{nt},$$

where

  - $A(t)$ is the account value;
  - t is measured in years;
  - P is the starting amount of the account, often called the **principal**, or more generally **present value**;
  - r is the annual percentage rate (APR) expressed as a decimal;
  - n is the number of compounding periods in one year.

Calculating Compound Interest

- If we invest \$3,000 in an investment account paying 3% interest compounded quarterly, how much will the account be worth in 10 years?

  We use $A = P(1+\frac{r}{n})^{nt}$, where

  $$P = 3000, \quad r = 0.03, \quad n = 4, \quad t = 10.$$

  So we get

  $$A = 3000\left(1+\frac{0.03}{4}\right)^{4\cdot 10} = 3000\cdot 1.0075^{40} \approx 4,045.05.$$

Using the Formula to Solve for the Principal

- A 529 Plan is a savings plan that allows relatives to invest money to pay for a child's future college tuition with tax-free growth.
- Lily wants to set up a 529 account for her new granddaughter and wants the account to grow to $40,000 over 18 years.

  She believes the account will earn 6% compounded semi-annually.

  To the nearest dollar, how much needs to be invested now?

  We use $A = P(1 + \frac{r}{n})^{nt}$, where

  $$A = 40000, \quad r = 0.06, \quad n = 2, \quad t = 18.$$

  Hence,

  $$40000 = P(1 + \frac{0.06}{2})^{2 \cdot 18} \Rightarrow 40000 = P \cdot 1.03^{36}$$
  $$\Rightarrow P = \frac{40000}{1.03^{36}} \Rightarrow P \approx 13,801.$$

The Continuous Growth/Decay Formula

- For all real numbers t, and all positive numbers a and r, continuous growth or decay is represented by the formula

$$A(t) = a \cdot e^{rt},$$

  where
  - a is the initial value;
  - r is the continuous growth rate per unit time;
    - If $r > 0$, then the formula represents continuous growth;
    - If $r < 0$, then the formula represents continuous decay.
  - t is the elapsed time.
- For business applications, the continuous growth formula is called the **continuous compounding formula** and takes the form

$$A(t) = P \cdot e^{rt},$$

  where
  - P is the principal or the initial amount invested;
  - r is the growth or interest rate per unit time;
  - t is the period or term of the investment.

Calculating Continuous Growth

- A person invested \$1,000 in an account earning a nominal 10% per year compounded continuously.

  How much was in the account at the end of one year?

  We use $A = Pe^{rt}$, where

  $$P = 1000, \quad r = 0.1, \quad t = 1.$$

  So we get

  $$A = 1000 \cdot e^{0.1 \cdot 1} \Rightarrow A \approx 1105.17.$$

Calculating Continuous Decay

- Radon-222 decays at a continuous rate of 17.3% per day. How much will 100 mg of Radon-222 decay to in 3 days?
  We use $A = ae^{rt}$, where
  $$a = 100, \quad r = -0.173, \quad t = 3.$$
  So we get
  $$A = 100 \cdot e^{-0.173 \cdot 3} \Rightarrow A \approx 59.5115.$$

Subsection 2

Graphs of Exponential Functions

We Will Learn How To:

- Graph exponential functions;
- Graph exponential functions using transformations.

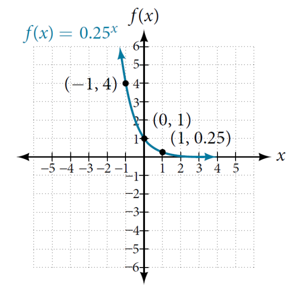
Graphing Exponential Functions

- Sketch a graph of $f(x) = (\frac{1}{4})^x$.

  State the domain, range and asymptote.

  Create a small table of values and plot the corresponding points.

x	$f(x)$
-2	16
-1	4
0	1
1	$\frac{1}{4}$
2	$\frac{1}{16}$



  The domain is $\mathbb{R}$, the range is $(0, \infty)$ and the asymptote is $y = 0$ (the x-axis).

Translations of the Parent Function $f(x) = b^x$

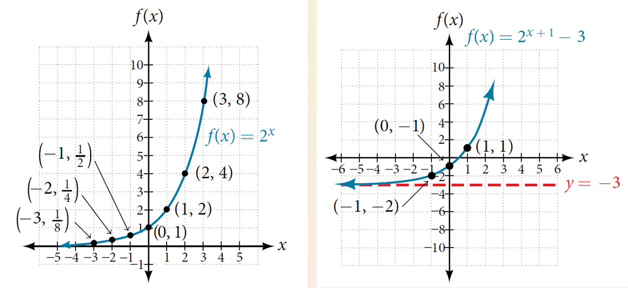
Transformation	Form
Shift c Units Right and d Units Up	$f(x) = b^{x-c} + d$
Stretch/Compress by $a > 0$	$f(x) = a \cdot b^x$
Reflect WRT x-axis	$f(x) = -b^x$
Reflect WRT y-axis	$f(x) = b^{-x} = \left(\frac{1}{b}\right)^x$
General Equation	$f(x) = a \cdot b^{x-c} + d$

Graphing Transformations of Exponential Functions

- Graph $f(x) = 2^{x+1} - 3$.

  State the domain, range and asymptote.

  We start by graphing $g(x) = 2^x$.



  Then, we move it left 1 point and down 3 points.

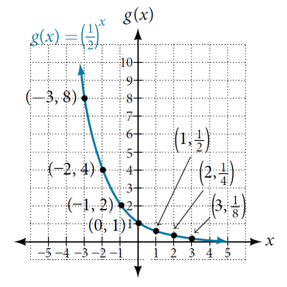
  The domain is $\mathbb{R}$, the range is $(-3, \infty)$ and the asymptote is $y = -3$.

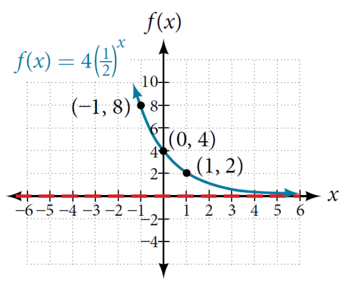
Graphing the Stretch of an Exponential Function

- Sketch a graph of $f(x) = 4\left(\frac{1}{2}\right)^x$.

  State the domain, range and asymptote.

  We start by graphing $g(x) = (\frac{1}{2})^x$.





  Then, we stretch it by a factor of 4.

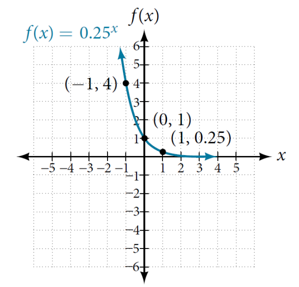
  The domain is $\mathbb{R}$, the range is $(0, \infty)$ and the asymptote is $y = 0$.

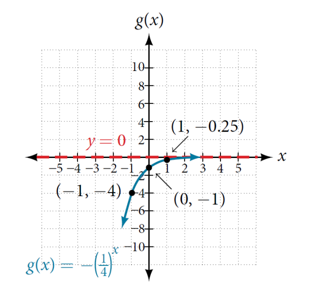
Writing and Graphing a Reflection

- Find and graph the equation for a function, $g(x)$, that reflects $f(x) = (\frac{1}{4})^x$ about the x-axis.

  State its domain, range and asymptote.

  We start by graphing $f(x) = (\frac{1}{4})^x$.





  Then, reflect it about the x-axis to get $g(x) = -(\frac{1}{4})^x$.

  The domain is $\mathbb{R}$, the range is $(-\infty, 0)$ and the asymptote is $y = 0$.

Writing a Function from a Description

- Write the equation for the function described by

  $f(x) = e^x$ is vertically stretched by a factor of 2, reflected across the y-axis, and then shifted up 4 units.

  Give the horizontal asymptote, the domain and the range.

  We start from $f(x) = e^x$ and perform the transformations one-by-one as described:

$$\begin{aligned} f(x) = e^x \quad & \xrightarrow{\updownarrow 2} \quad g(x) = 2e^x \\ & \xrightarrow{\leftrightarrow} \quad h(x) = 2e^{-x} \\ & \xrightarrow{\uparrow 4} \quad k(x) = 2e^{-x} + 4. \end{aligned}$$

  The domain is $\mathbb{R}$, the range is $(4, \infty)$ and the asymptote is $y = 4$.

Subsection 3

Logarithmic Functions

We Will Learn How To:

- Convert from logarithmic to exponential form;
- Convert from exponential to logarithmic form;
- Evaluate logarithms;
- Use common and natural logarithms.

Definition of the Logarithmic Function

- A **logarithm base** b **of a positive number** x satisfies the following definition.

  For $x > 0$, $0 < b \neq 1$,

$$y = \log_b(x) \quad \text{is equivalent to} \quad b^y = x.$$

- We read $\log_b(x)$ as, "the logarithm with base b of x" or the "log base b of x".
- The logarithm y is the exponent to which b must be raised to get x.
- The logarithmic and exponential functions switch input-output values.
- The domain and range of the exponential function are interchanged for the logarithmic function.
  - The domain of the logarithm function with base b is $(0, \infty)$.
  - The range of the logarithm function with base b is $(-\infty, \infty)$.

Converting from Logarithmic Form to Exponential Form

- Write the following logarithmic equations in exponential form.
  a. $\log_6(\sqrt{6}) = \frac{1}{2}$
  b. $\log_3(9) = 2$

a.
$$\log_6(\sqrt{6}) = \frac{1}{2} \quad \Leftrightarrow \quad 6^{1/2} = \sqrt{6}.$$

b.
$$\log_3(9) = 2 \quad \Leftrightarrow \quad 3^2 = 9.$$

Converting from Exponential to Logarithmic Form

- Write the following exponential equations in logarithmic form.
  a. $2^3 = 8$
  b. $5^2 = 25$
  c. $10^{-4} = \frac{1}{10000}$

a.

$$2^3 = 8 \quad \Leftrightarrow \quad \log_2(8) = 3.$$

b.

$$5^2 = 25 \quad \Leftrightarrow \quad \log_5(25) = 2.$$

b.

$$10^{-4} = \frac{1}{10000} \quad \Leftrightarrow \quad \log_{10}\left(\frac{1}{10000}\right) = -4.$$

Solving Logarithms Mentally

- Solve $y = \log_4(64)$ without using a calculator.
  We convert to an exponential:

$$\begin{aligned} y = \log_4(64) \quad &\Leftrightarrow \quad 4^y = 64 \\ &\Leftrightarrow \quad y = 3. \end{aligned}$$

- Evaluate $y = \log_3 \frac{1}{27}$ without using a calculator.

$$\begin{aligned} y = \log_3 \frac{1}{27} \quad &\Leftrightarrow \quad 3^y = \frac{1}{27} \\ &\Leftrightarrow \quad y = -3. \end{aligned}$$

Definition of the Common Logarithm

- A **common logarithm** is a logarithm with base 10.
- We write $\log_{10}(x)$ simply as $\log(x)$.
- The common logarithm of a positive number x satisfies,

$$y = \log(x) \quad \text{is equivalent to} \quad 10^y = x.$$

- We read $\log(x)$ as, "the logarithm with base 10 of x" or "log base 10 of x".
- The logarithm y is the exponent to which 10 must be raised to get x.

Finding the Value of a Common Logarithm Mentally

- Evaluate $y = \log(1000)$ without using a calculator.

  We convert to an exponential:

$$y = \log(1000) \quad \Leftrightarrow \quad 10^y = 1000$$
$$\Leftrightarrow \quad y = 3.$$

Definition of the Natural Logarithm

- A **natural logarithm** is a logarithm with base e.
- We write $\log_e(x)$ simply as $\ln(x)$.
- The natural logarithm of a positive number x satisfies

$$y = \ln(x) \quad \text{is equivalent to} \quad e^y = x.$$

- We read $\ln(x)$ as, "the logarithm with base e of x" or "the natural logarithm of x".
- The logarithm y is the exponent to which e must be raised to get x.
- Since the functions $y = e$ and $y = \ln(x)$ are inverse functions,

$$\ln(e^x) = x, \text{ for all } x, \text{ and } e^{\ln x} = x, \text{ for all } x > 0.$$

Using Natural Logarithms

- Evaluate or solve for x, as appropriate.

(a) $\ln e = 1$;

(b) $\ln 1 = 0$;

(c) $e^{3x} = 5$

$$\begin{aligned} e^{3x} = 5 \quad &\Leftrightarrow \quad 3x = \ln 5 \\ &\Leftrightarrow \quad x = \frac{\ln 5}{3}. \end{aligned}$$

(d) $\ln x = 7$.

$$\ln x = 7 \quad \Leftrightarrow \quad x = e^7.$$

Subsection 4

Graphs of Logarithmic Functions

We Will Learn How To:

- Identify the domain of a logarithmic function;
- Graph logarithmic functions.

Identifying the Domain of a Logarithmic Shift

- What is the domain of $f(x) = \log_2(x+3)$?

  The argument of a logarithmic function must be strictly positive:

$$x + 3 > 0 \;\Rightarrow\; x > -3.$$

  Therefore, the domain of f is $(-3, \infty)$.

Domain of a Logarithmic Shift and Reflection

- What is the domain of $f(x) = \log(5 - 2x)$?

  The argument of a logarithmic function must be strictly positive:

$$5 - 2x > 0 \Rightarrow -2x > -5 \Rightarrow x < \frac{5}{2}.$$

  Therefore, the domain of f is $(-\infty, \frac{5}{2})$.

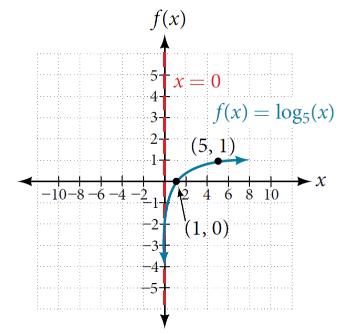
Graphing $f(x) = \log_b(x)$.

- Graph $f(x) = \log_5(x)$.

  State the domain, range and asymptote.

  Create a small table of values and plot the corresponding points.

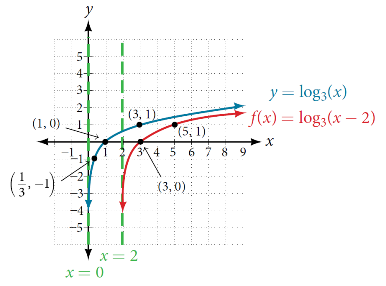
x	$f(x)$
$\frac{1}{25}$	-2
$\frac{1}{5}$	-1
1	0
5	1
25	2



The domain is $(0, \infty)$, the range is $\mathbb{R}$ and the asymptote is $x = 0$ (the y-axis).

Graphing a Horizontal Shift of $y = \log_b(x)$

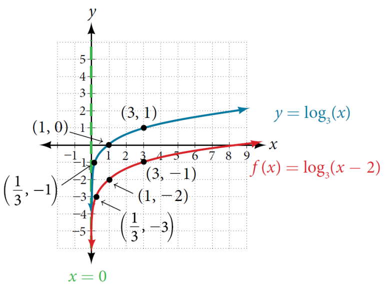
- Sketch the shift $f(x) = \log_3(x-2)$ alongside its parent function.
  Include the key points and asymptotes on the graph.
  State the domain, range and asymptote.
  First, sketch the graph of $g(x) = \log_3(x)$, which passes through $(1,0)$ and $(3,1)$.
  Then shift it two points to the right to obtain the graph of $f(x)$.



The domain is $(2, \infty)$, the range is $\mathbb{R}$ and the asymptote is $x = 2$.

Graphing a Vertical Shift of $y = log_b(x)$

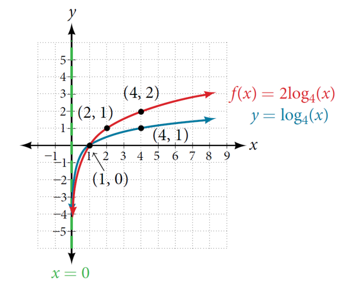
- Sketch a graph of $f(x) = \log_3(x) - 2$ alongside its parent function.
  Include the key points and asymptote on the graph.
  State the domain, range and asymptote.
  First, sketch the graph of $g(x) = \log_3(x)$, which passes through $(1, 0)$ and $(3, 1)$.
  Then shift it two points down to obtain the graph of $f(x)$.



The domain is $(0, \infty)$, the range is $\mathbb{R}$ and the asymptote is $x = 0$.

Graphing a Stretch or Compression of $y = \log_b(x)$

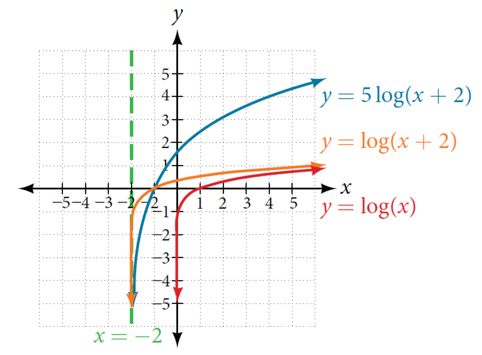
- Sketch a graph of $f(x) = 2\log_4(x)$ alongside its parent function. Include the key points and asymptote on the graph.
  State the domain, range and asymptote.
  First, sketch the graph of $g(x) = \log_4(x)$, which passes through $(1, 0)$ and $(4, 1)$.
  Then stretch it by a factor of 2 to obtain the graph of $f(x)$.



The domain is $(0, \infty)$, the range is $\mathbb{R}$ and the asymptote is $x = 0$.

Combining a Shift and a Stretch

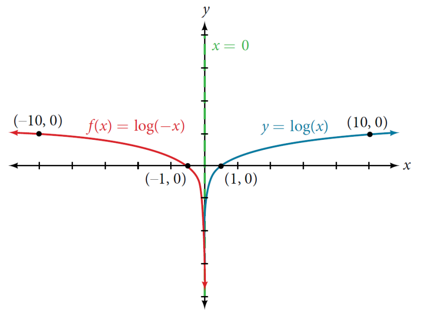
- Sketch a graph of $f(x) = 5\log(x+2)$.
  State the domain, range and asymptote.
  First, sketch the graph of $g(x) = \log(x)$, through $(1,0)$ and $(10,1)$.
  Then move it left 2 points and stretch it by a factor of 5.



The domain is $(-2,\infty)$, the range is $\mathbb{R}$ and the asymptote is $x=-2$.

Graphing a Reflection of a Logarithmic Function

- Sketch a graph of $f(x) = \log(-x)$ alongside its parent function.
  Include the key points and asymptote on the graph.
  State the domain, range and asymptote.
  First, sketch the graph of $g(x) = \log(x)$, through $(1, 0)$ and $(10, 1)$.
  Then reflect it about the y-axis.



The domain is $(-\infty, 0)$, the range is $\mathbb{R}$ and the asymptote is $x = 0$.

Transformations of the Parent Function $y = \log_b(x)$

Translation	Form
Shift c Units Right and d Units Up	$y = \log_b(x - c) + d$
Stretch/Compress	$y = a \log_b(x)$
Reflect WRT x-axis	$y = -\log_b(x)$
Reflect WRT y-axis	$y = \log_b(-x)$
General Equation	$y = a \log_b(x - c) + d$

Finding the Vertical Asymptote of a Logarithm Graph

- What is the vertical asymptote of $f(x) = -2\log_3(x+4)+5$?

  The function $g(x) = \log_3(x)$ has vertical asymptote $x = 0$ (the y-axis).

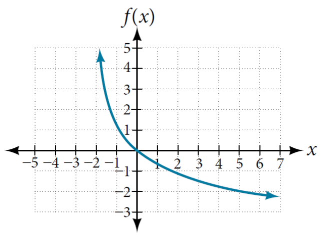
  f is obtained from g by the following moves:

$$\begin{array}{rcl} y = \log_3(x) & \xrightarrow{\leftarrow 4} & y = \log_3(x+4) \\ & \xrightarrow{\updownarrow 2} & y = 2\log_3(x+4) \\ & \xrightarrow{\updownarrow} & y = -2\log_3(x+4) \\ & \xrightarrow{\uparrow 5} & y = -2\log_3(x+4)+5. \end{array}$$

  Thus, f has vertical asymptote $x = -4$.

Finding the Equation from a Graph

- Find a possible equation for the common logarithmic function shown



Adopt the most general form $f(x) = a\log(x-c)+d$.
Then try to determine a, c and d.

- The vertical asymptote is at $x=-2$. So $c=-2$.
- Graph passes through $(-1,1)$. So $1=a\log(-1+2)+d$, i.e., $d=1$.
- Finally, the graph passes through $(5,-2)$. So we get
  $-2=a\log(5+2)+1 \Rightarrow -3=a\log(7) \Rightarrow a=\frac{-3}{\log(7)}$.

So $f(x)=\frac{-3}{\log(7)}\log(x+2)+1$.

Subsection 5

Logarithmic Properties

We Will Learn How To:

- Use the product rule for logarithms;
- Use the quotient rule for logarithms;
- Use the power rule for logarithms;
- Expand logarithmic expressions;
- Condense logarithmic expressions;
- Use the change-of-base formula for logarithms.

The Product Rule for Logarithms

- The **product rule for logarithms** can be used to simplify a logarithm of a product by rewriting it as a sum of individual logarithms.

$$\log_b(MN) = \log_b(M) + \log_b(N), \text{ for } b > 0.$$

- Expand $\log_3(5x(3x+4))$.

  We have:

$$\log_3(5x(3x+4)) = \log_3(5) + \log_3(x) + \log_3(3x+4).$$

The Quotient Rule for Logarithms

- The **quotient rule for logarithms** can be used to simplify a logarithm or a quotient by rewriting it as the difference of individual logarithms.

$$\log_b\left(\frac{M}{N}\right) = \log_b(M) - \log_b(N).$$

- Expand $\log_2\left(\frac{15x(x-1)}{(3x+4)(2-x)}\right)$.

  We have

$$\begin{aligned}
\log_2\left(\tfrac{15x(x-1)}{(3x+4)(2-x)}\right) &= \log_2(15\cdot x(x-1)) \\
&\quad - \log_2((3x+4)(2-x)) \\
&= \log_2(15) + \log_2(x) + \log_2(x-1) \\
&\quad - (\log_2(3x+4) + \log_2(2-x)) \\
&= \log_2(15) + \log_2(x) + \log_2(x-1) \\
&\quad - \log_2(3x+4) - \log_2(2-x).
\end{aligned}$$

The Power Rule for Logarithms

- The **power rule for logarithms** can be used to simplify the logarithm of a power by rewriting it as the product of the exponent times the logarithm of the base.

$$\log_b(M^n) = n\log_b(M).$$

- Expand $\log_2(x^5)$.
  $\log_2(x^5) = 5\log_2(x)$.
- Expand $\log_3(25)$ using the power rule for logs.
  $\log_3(25) = \log_3(5^2) = 2\log_3(5)$.
- Rewrite $4\ln(x)$ using the power rule for logs to a single logarithm with a leading coefficient of 1.
  $4\ln(x) = \ln(x^4)$.

Expanding Using the Product, Quotient and Power Rules

- Rewrite $\ln\left(\frac{x^4 y}{7}\right)$ as a sum or difference of logs.

  We have
  $$\begin{aligned}\ln\left(\frac{x^4 y}{7}\right) &= \ln(x^4 y) - \ln(7)\\ &= \ln(x^4) + \ln(y) - \ln(7)\\ &= 4\ln(x) + \ln(y) - \ln(7).\end{aligned}$$

- Expand $\log(\sqrt{x})$.

  We have
  $$\log(\sqrt{x}) = \log(x^{1/2}) = \frac{1}{2}\log(x).$$

Expanding Using the Product, Quotient and Power Rules

- Expand $\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right)$.

  We have

$$\begin{array}{rcl}\log_6\left(\frac{64x^3(4x+1)}{(2x-1)}\right) & = & \log_6(2^6x^3(4x+1)) - \log_6(2x-1) \\ & = & \log_6(2^6) + \log_6(x^3) \\ & & \quad + \log_6(4x+1) - \log_6(2x-1) \\ & = & 6\log_6(2) + 3\log_6(x) \\ & & \quad + \log_6(4x+1) - \log_6(2x-1).\end{array}$$

Using the Rules to Combine Logarithms

- Write $\log_3(5) + \log_3(8) - \log_3(2)$ as a single logarithm.
  We have

$$\begin{aligned} & \log_3(5) + \log_3(8) - \log_3(2) \\ = & \log_3(5 \cdot 8) - \log_3(2) \\ = & \log_3\left(\tfrac{5 \cdot 8}{2}\right) \\ = & \log_3(20). \end{aligned}$$

Using the Rules to Combine Logarithms

- Condense $\log_2(x^2) + \frac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2)$.

  We have

$$\begin{aligned} & \log_2(x^2) + \tfrac{1}{2}\log_2(x-1) - 3\log_2((x+3)^2) \\ = & \log_2(x^2) + \log_2((x-1)^{1/2}) - \log_2(((x+3)^2)^3) \\ = & \log_2(x^2(x-1)^{1/2}) - \log_2((x+3)^6) \\ = & \log_2\left(\tfrac{x^2(x-1)^{1/2}}{(x+3)^6}\right). \end{aligned}$$

Using the Rules to Combine Logarithms

- Rewrite $2\log(x) - 4\log(x+5) + \frac{1}{x}\log(3x+5)$ as a single logarithm.

  We have

$$\begin{aligned} & 2\log(x) - 4\log(x+5) + \tfrac{1}{x}\log(3x+5) \\ = & \log(x^2) - \log((x+5)^4) + \log((3x+5)^{1/x}) \\ = & \log\left(\frac{x^2}{(x+5)^4}\right) + \log((3x+5)^{1/x}) \\ = & \log\left(\frac{x^2(3x+5)^{1/x}}{(x+5)^4}\right). \end{aligned}$$

The Change-of-Base Formula

- The **change-of-base formula** can be used to evaluate a logarithm with any base.
- For any positive real numbers M, b, and n, where $n \neq 1$ and $b \neq 1$,

$$\log_b(M) = \frac{\log_n(M)}{\log_n(b)}.$$

- It follows that the change-of-base formula can be used to rewrite a logarithm with any base as the quotient of common or natural logs.

$$\log_b(M) = \frac{\ln(M)}{\ln(b)} \quad \text{and} \quad \log_b(M) = \frac{\log(M)}{\log(b)}.$$

Applying the Change-of-Base Formula

- Change $\log_5(3)$ to a quotient of natural logarithms.

$$\log_5(3) = \frac{\ln(3)}{\ln(5)}.$$

- Evaluate $\log_2(10)$ using the change-of-base formula to convert to common logarithms and then using a calculator.

$$\log_2(10) = \frac{\log(10)}{\log(2)} = \frac{1}{\log(2)} \approx 3.3219.$$

Subsection 6

Exponential and Logarithmic Equations

We Will Learn How To:

- Use like bases to solve exponential equations;
- Use logarithms to solve exponential equations;
- Use the definition of a logarithm to solve logarithmic equations;
- Use the one-to-one property of logarithms to solve logarithmic equations;
- Solve applied problems involving exponential and logarithmic equations.

One-to-One Property of Exponential Functions

- For any algebraic expressions S and T, and any positive real number $b \neq 1$,
$$b^S = b^T \quad \text{if and only if} \quad S = T.$$
- Solve $2^{x-1} = 2^{2x-4}$.
  We get
$$2^{x-1} = 2^{2x-4} \Rightarrow x - 1 = 2x - 4 \Rightarrow x = 3.$$
- Solve the exponential equation $3^{4x-7} = \frac{3^{2x}}{3}$.
  Similarly, we get
$$3^{4x-7} = \frac{3^{2x}}{3} \Rightarrow 3^{4x-7} = 3^{2x-1} \Rightarrow 4x - 7 = 2x - 1$$
$$\Rightarrow 2x = 6 \Rightarrow x = 3.$$

Rewriting Equations So All Powers Have the Same Base

- Solve $256 = 4^{x-5}$.

  Rewrite both sides over same base:

  $$256 = 4^{x-5} \Rightarrow 4^4 = 4^{x-5} \Rightarrow 4 = x - 5 \Rightarrow x = 9.$$

- Solve $8^{x+2} = 16^{x+1}$.

  Rewrite both sides over same base:

  $$8^{x+2} = 16^{x+1} \Rightarrow (2^3)^{x+2} = (2^4)^{x+1} \Rightarrow 2^{3(x+2)} = 2^{4(x+1)}$$
  $$\Rightarrow 2^{3x+6} = 2^{4x+4} \Rightarrow 3x + 6 = 4x + 4 \Rightarrow x = 2.$$

- Solve $2^{5x} = \sqrt{2}$.

  Rewrite both sides over same base:

  $$2^{5x} = \sqrt{2} \Rightarrow 2^{5x} = 2^{1/2} \Rightarrow 5x = \frac{1}{2} \Rightarrow x = \frac{1}{10}.$$

Solving an Equation Containing Powers of Different Bases

- Solve $5^{x+2} = 4^x$.

  If we cannot match bases, the technique calls for taking logarithms of both sides:

$$5^{x+2} = 4^x \Rightarrow \ln(5^{x+2}) = \ln(4^x) \Rightarrow (x+2)\ln(5) = x\ln(4)$$
$$\Rightarrow x\ln(5) + 2\ln(5) = x\ln(4) \Rightarrow x\ln(5) - x\ln(4) = -2\ln(5)$$
$$\Rightarrow x(\ln(5) - \ln(4)) = -2\ln(5) \Rightarrow x = \frac{-2\ln(5)}{\ln(5)-\ln(4)}.$$

Solve an Equation of the Form $y = Ae^{kt}$

- Solve $100 = 20e^{2t}$.

  Isolate the exponential and convert into a logarithm:

$$100 = 20e^{2t} \Rightarrow 5 = e^{2t} \Rightarrow 2t = \ln(5) \Rightarrow t = \frac{1}{2}\ln(5).$$

- Solve $4e^{2x} + 5 = 12$.

  Isolate the exponential and convert into a logarithm:

$$4e^{2x} + 5 = 12 \Rightarrow 4e^{2x} = 7 \Rightarrow e^{2x} = \tfrac{7}{4}$$
$$\Rightarrow 2x = \ln\left(\tfrac{7}{4}\right) \Rightarrow x = \tfrac{1}{2}\ln\left(\tfrac{7}{4}\right).$$

Solving Exponential Functions in Quadratic Form

- Solve $e^{2x} - e^x = 56$.

  Set $y = e^x$.

  Then $y^2 = (e^x)^2 = e^{2x}$.

  So we obtain

$$e^{2x} - e^x = 56 \Rightarrow y^2 - y = 56 \Rightarrow y^2 - y - 56 = 0$$

$$\Rightarrow (y+7)(y-8) = 0 \Rightarrow y+7=0 \text{ or } y-8=0$$

$$\Rightarrow y = -7 \text{ or } y = 8.$$

  Finally,

$$e^x = -7 \text{ or } e^x = 8 \Rightarrow x = \ln(8).$$

  (Note that e^x cannot be negative.)

Using Algebra to Solve a Logarithmic Equation

- Solve $\log_2(2) + \log_2(3x-5) = 3$.
  Combine on the left:
  $$\log_2(2) + \log_2(3x-5) = 3 \Rightarrow \log_2(2(3x-5)) = 3$$
  $$\Rightarrow 2(3x-5) = 2^3 \Rightarrow 6x - 10 = 8$$
  $$\Rightarrow 6x = 18 \Rightarrow x = 3 \checkmark$$
- Solve $2\ln(x) + 3 = 7$.
  $$2\ln(x) + 3 = 7 \Rightarrow 2\ln(x) = 4 \Rightarrow \ln(x) = 2 \Rightarrow x = e^2 \checkmark$$
- Solve $2\ln(6x) = 7$.
  $$2\ln(6x) = 7 \Rightarrow \ln(6x) = \frac{7}{2} \Rightarrow 6x = e^{7/2} \Rightarrow x = \frac{e^{7/2}}{6} \checkmark$$

One-to-One Property of Logarithms

- For any algebraic expressions S and T and any positive real number b, where $b \neq 1$,

$$\log_b(S) = \log_b(T) \quad \text{if and only if} \quad S = T.$$

- When solving an equation involving logarithms, always check to see if the answer is correct or if it is an extraneous solution.

Using the One-to-One Property of Logarithms

- Solve $\log(3x-2) - \log(2) = \log(x+4)$.

  We have

$$\begin{aligned} &\log(3x-2) - \log(2) = \log(x+4) \\ \Rightarrow\ &\log(3x-2) = \log(x+4) + \log(2) \\ \Rightarrow\ &\log(3x-2) = \log(2(x+4)) \\ \Rightarrow\ &3x-2 = 2x+8 \\ \Rightarrow\ &x = 10\ \checkmark \end{aligned}$$

Checking for Extraneous Solutions

- Solve $\ln(x^2) = \ln(2x+3)$.

  We work similarly

$$\begin{aligned} & \ln(x^2) = \ln(2x+3) \\ \Rightarrow\ & x^2 = 2x+3 \\ \Rightarrow\ & x^2 - 2x - 3 = 0 \\ \Rightarrow\ & (x+1)(x-3) = 0 \\ \Rightarrow\ & x+1 = 0 \text{ or } x-3 = 0 \\ \Rightarrow\ & x = -1 \text{ or } x = 3. \end{aligned}$$

  Both are admissible solutions.

Subsection 7

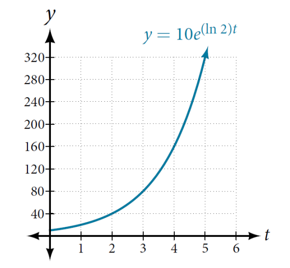
Exponential and Logarithmic Models

We Will Learn How To:

- Model exponential growth and decay;
- Use Newton's Law of Cooling;
- Use logistic-growth models;
- Choose an appropriate model for data;
- Express an exponential model in base e.

Graphing Exponential Growth $y = A_0 e^{kt}$

- A population of bacteria doubles every hour.
  If the culture started with 10 bacteria, graph the population as a function of time.
  We have $A_0 = 10$.
  Moreover, when $t = 1$, $A = 20$. Therefore,
  $$20 = 10e^{k \cdot 1} \Rightarrow 2 = e^k \Rightarrow k = \ln(2).$$
  Hence, the model is $A = 10e^{(\ln(2))t}$.



Finding the Function that Describes Radioactive Decay

- The half-life of carbon-14 is 5,730 years.

  Express the amount of carbon-14 remaining as a function of time, t.

  If initially the quantity is A_0, then the model is $A = A_0 e^{kt}$.

  For $t = 5730$, we have $A = \frac{1}{2} A_0$.

  So we get

$$\frac{1}{2} A_0 = A_0 e^{k \cdot 5730} \Rightarrow \frac{1}{2} = e^{k \cdot 5730}$$

$$\Rightarrow k \cdot 5730 = \ln\left(\frac{1}{2}\right) \Rightarrow k = \frac{\ln(1/2)}{5730}.$$

  Therefore the model is

$$A = A_0 e^{\left(\frac{\ln(1/2)}{5730}\right) t}.$$

Finding the Age of a Bone

- A bone fragment is found that contains 20% of its original carbon-14. To the nearest year, how old is the bone?

  Here we use the formula we obtained in the preceding slide, keeping in mind the interpretations of the parameters and the variables.

$$A = A_0 e^{\left(\frac{\ln(1/2)}{5730}\right)t}.$$

  We have $A = 0.2A_0$.

  Therefore,

$$0.2A_0 = A_0 e^{\left(\frac{\ln(1/2)}{5730}\right)t} \Rightarrow 0.2 = e^{\left(\frac{\ln(1/2)}{5730}\right)t}$$

$$\frac{\ln(1/2)}{5730} t = \ln(0.2) \Rightarrow t = 5730 \frac{\ln(0.2)}{\ln(0.5)}.$$

  The bone fragment is about 13,305 years old.

Finding a Function That Describes Exponential Growth

- According to Moore's Law, the doubling time for the number of transistors that can be put on a chip is approximately two years.

  Give a function that describes this behavior.

  If initially the quantity is A_0, then the model is $A = A_0 e^{kt}$.

  For $t = 2$, we have $A = 2A_0$.

  So we get

$$2A_0 = A_0 e^{2k} \Rightarrow 2 = e^{2k}$$
$$\Rightarrow 2k = \ln(2) \Rightarrow k = \frac{\ln(2)}{2}.$$

  Therefore the model is

$$A = A_0 e^{(\frac{\ln(2)}{2})t}.$$

Newton's Law of Cooling

- The temperature of an object, T, in surrounding air with temperature T_s will behave according to the formula

$$T(t) = Ae^{kt} + T_s,$$

  where
  - t is time;
  - $A = T(0) - T_s$ is the difference between the initial temperature of the object and the surroundings;
  - k is a constant, the continuous rate of cooling of the object.

Applying Newton's Law of Cooling $T(t) = Ae^{kt} + T_s$

- A cheesecake is taken out of the oven with an ideal internal temperature of 165°F, and is placed into a 35°F refrigerator. After 10 minutes, the cheesecake has cooled to 150°F. If we must wait until the cheesecake has cooled to 70°F before we eat it, how long will we have to wait?
  The temperature where cooling takes place is $T_s = 35$.
  The difference between initial temperature and surroundings is $A = 165 - 35 = 130$.
  Thus, we get $T(t) = 130e^{kt} + 35$.
  At $t = 10$, we get $T = 150$.
  So we get
  $$150 = 130e^{10k} + 35 \Rightarrow 115 = 130e^{10k} \Rightarrow e^{10k} = \frac{115}{130}$$
  $$\Rightarrow 10k = \ln\left(\frac{115}{130}\right) \Rightarrow k = \frac{1}{10}\ln\left(\frac{115}{130}\right).$$
  Therefore, $T(t) = 130e^{\frac{1}{10}\ln\left(\frac{115}{130}\right)t} + 35$.

Applying Newton's Law (Cont'd)

- We found
$$T(t) = 130e^{\frac{1}{10}\ln(\frac{115}{130})t} + 35.$$
To find how long we have to wait until the cheesecake has cooled to 70°F, we set $T = 70$ and solve for t:
$$70 = 130e^{\frac{1}{10}\ln(\frac{115}{130})t} + 35 \Rightarrow 35 = 130e^{\frac{1}{10}\ln(\frac{115}{130})t}$$
$$\Rightarrow e^{\frac{1}{10}\ln(\frac{115}{130})t} = \frac{35}{130} \Rightarrow \frac{1}{10}\ln\left(\frac{115}{130}\right)t = \ln\left(\frac{35}{130}\right)$$
$$\Rightarrow t = 10\frac{\ln(\frac{35}{130})}{\ln(\frac{115}{130})}.$$
So, we' ll have to wait for approximately 107 minutes.

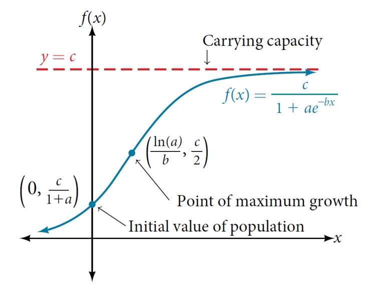
Logistic Growth

- The logistic growth model is

$$f(x) = \frac{c}{1 + ae^{-bx}},$$

where

  - $\frac{c}{1+a}$ is the initial value;
  - c is the **carrying capacity**, or **limiting value**;
  - b is a constant determined by the rate of growth.



Using the Logistic-Growth Model $f(x) = \frac{c}{1+ae^{-bx}}$

- An influenza epidemic spreads according to the logistic model.

  At time $t = 0$, one person in a community of 1,000 has the flu.

  For this strain of the flu, the growth constant is $b = 0.6030$.

  (a) Estimate the number of people in this community who will have had this flu after ten days.

  (b) Predict how many people in this community will have had this flu after a long period of time has passed.

  First we work to establish the model.

  When $x = 0$, we have $f(x) = 1$. So we get $1 = \frac{c}{1+a}$.

  The carrying capacity $c = 1000$.

  So we get $1 = \frac{1000}{1+a} \Rightarrow 1 + a = 1000 \Rightarrow a = 999$.

  So the model is $f(x) = \frac{1000}{1+999e^{-0.6030x}}$.

Using the Logistic-Growth Model (Cont'd)

- We came up with

$$f(x) = \frac{1000}{1 + 999e^{-0.6030x}}.$$

(a) To find the the number of people in this community who will have had this flu after ten days, we set $x = 10$:

$$f(10) = \frac{1000}{1 + 999e^{-0.6030 \cdot 10}} \approx 294.$$

(b) The number of people in this community will have had this flu after a long period of time is approximated by the carrying capacity

$$c = 1000.$$

Changing to base e

- Change the function $y = 2.5 \cdot (3.1)^x$ so that this same function is written in the form $y = A_0 e^{kx}$.

  The trick is to take advantage of

  $$e^{\ln x} = x.$$

  So we do the rewriting as follows:

  $$\begin{aligned} y &= 2.5 \cdot (3.1)^x \\ &= 2.5 \cdot (e^{\ln (3.1)})^x \\ &= 2.5 \cdot e^{(\ln (3.1))x}. \end{aligned}$$