College Algebra

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 111

1 Systems of Equations and Inequalities
- Systems of Linear Equations: Two Variables
- Systems of Nonlinear Equations and Inequalities: Two Variables
- Partial Fractions
- Solving Systems with Gaussian Elimination
- Solving Systems with Cramer's Rule

Subsection 1

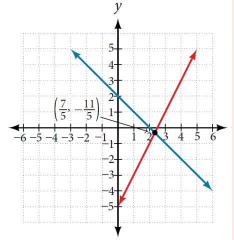
Systems of Linear Equations: Two Variables

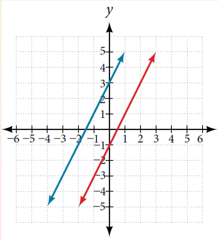
We Will Learn How To:

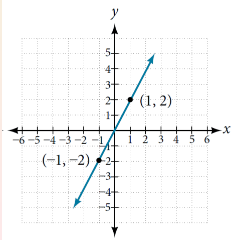
- Solve systems of equations by substitution;
- Solve systems of equations by addition;
- Identify inconsistent systems of equations containing two variables;
- Express the solution of a system of dependent equations containing two variables.

Types of Linear Systems

- There are three types of systems of linear equations in two variables, and three types of solutions.
  - An **independent system** has exactly one solution pair (x, y). The point where the two lines intersect is the only solution.
  - An **inconsistent system** has no solution. The two lines are parallel and will never intersect.
  - A **dependent system** has infinitely many solutions. The lines are coincident (the same line), so every coordinate pair on the line is a solution to both equations.







Solving by Substitution

- Solve the following system of equations by substitution.

$$\left\{\begin{array}{rcr} -x+y & = & -5 \\ 2x-5y & = & 1 \end{array}\right\}.$$

Solve the first equation for y:

$$y = x - 5.$$

Substitute into the second equation and solve for x:

$$2x - 5(x-5) = 1 \;\Rightarrow\; 2x - 5x + 25 = 1 \;\Rightarrow\; -3x = -24 \;\Rightarrow\; x = 8.$$

Find the value of y:

$$y = x - 5 \;\Rightarrow\; y = 8 - 5 \;\Rightarrow\; y = 3.$$

So the solution is $(x, y) = (8, 3)$.

Solving a System by the Addition Method

- Solve the given system of equations by addition.

$$\left\{\begin{array}{rcr} x+2y & = & -1 \\ -x+y & = & 3 \end{array}\right\}.$$

Add the two rows side-by-side (to cancel the x's):

$$3y = 2 \ \Rightarrow\ y = \frac{2}{3}.$$

Now use one of the two equations to find x:

$$x+2y=-1 \ \Rightarrow\ x=-2y-1 \ \Rightarrow\ x=-2\cdot\frac{2}{3}-1 \ \Rightarrow\ x=-\frac{7}{3}.$$

So the solution pair is $(x,y)=(-\frac{7}{3},\frac{2}{3})$.

Addition Method When Multiplication Is Required

- Solve the given system of equations by the addition method.

$$\left\{\begin{array}{rcr} 3x+5y & = & -11 \\ x-2y & = & 11 \end{array}\right\}.$$

To cancel the x's, we start by multiplying both sides of the second equation by -3:

$$\left\{\begin{array}{rcr} 3x+5y & = & -11 \\ -3x+6y & = & -33 \end{array}\right\}.$$

Now we add side-by-side:

$$11y=-44 \Rightarrow y=-4.$$

Finally we find x:

$$x-2y=11 \Rightarrow x=2y+11 \Rightarrow x=2(-4)+11 \Rightarrow x=3.$$

So the solution pair is $(x,y)=(3,-4)$.

Addition Method When Double Multiplication Is Required

- Solve the given system of equations in two variables by addition.

$$\left\{\begin{array}{rcr} 2x+3y & = & -16 \\ 5x-10y & = & 30 \end{array}\right\}.$$

To cancel the x's, we start by multiplying both sides of the first equation by 5 and both sides of the second equation by -2:

$$\left\{\begin{array}{rcr} 10x+15y & = & -80 \\ -10x+20y & = & -60 \end{array}\right\}.$$

Now we add side-by-side:

$$35y=-140 \Rightarrow y=-4.$$

Finally we find x:

$$2x+3y=-16 \Rightarrow 2x=-3y-16 \Rightarrow 2x=-3(-4)-16 \Rightarrow x=-2.$$

So the solution pair is $(x,y)=(-2,-4)$.

Addition Method in Systems Containing Fractions

- Solve the given system of equations in two variables by addition.

$$\left\{ \begin{array}{rcr} \frac{x}{3}+\frac{y}{6} & = & 3 \\ \frac{x}{2}-\frac{y}{4} & = & 1 \end{array} \right\}.$$

To get rid of fractions, we multiply both sides of the first equation by 6 and both sides of the second equation by 4:

$$\left\{ \begin{array}{rcr} 2x+y & = & 18 \\ 2x-y & = & 4 \end{array} \right\}.$$

Now we add side-by-side:

$$4x = 22 \;\Rightarrow\; x = \frac{11}{2}.$$

Finally we find y:

$$2x - y = 4 \;\Rightarrow\; y = 2x - 4 \;\Rightarrow\; y = 2 \cdot \frac{11}{2} - 4 \;\Rightarrow\; y = 7.$$

So the solution pair is $(x, y) = (\frac{11}{2}, 7)$.

Solving an Inconsistent System of Equations

- Solve the following system of equations.

$$\left\{\begin{array}{rcr} x & = & 9-2y \\ x+2y & = & 13 \end{array}\right\}.$$

Use substitution.

$$9-2y+2y=13 \;\Rightarrow\; 9=13.$$

Thus, the given system is inconsistent.

Finding a Solution to a Dependent System

- Find a solution to the system of equations using the addition method.

$$\left\{\begin{array}{rcr} x+3y & = & 2 \\ 3x+9y & = & 6 \end{array}\right\}.$$

Multiply the first equation by -3 and then add:

$$\left\{\begin{array}{rcr} -3x-9y & = & -6 \\ 3x+9y & = & 6 \end{array}\right\} \Rightarrow 0=0.$$

Therefore, the system is dependent.

We get

$$x+3y=2 \Rightarrow x=-3y+2.$$

So $(x,y)=(-3y+2,y)$, y any real number.

Finding the Break-Even Point and the Profit Function

- Given the cost function $C(x) = 0.85x + 35,000$ and the revenue function $R(x) = 1.55x$, find the break-even point and the profit function.

  The break-even point is the point where $R(x) = C(x)$.

  $$R(x) = C(x) \ \Rightarrow \ 1.55x = 0.85x + 35000$$
  $$\Rightarrow \ 0.70x = 35000 \ \Rightarrow \ x = 50000.$$

  The profit is

  $$\text{Profit} = \text{Revenue} - \text{Cost}.$$

  $$\begin{aligned} P(x) &= R(x) - C(x) \\ &= 1.55x - (0.85x + 35000) \\ &= 1.55x - 0.85x - 35000 \\ &= 0.70x - 35000. \end{aligned}$$

Writing and Solving a System of Equations

- The cost of a circus ticket is \$25.00 for children and \$50.00 for adults. On a certain day, attendance is 2,000 and total revenue \$70,000. How many children and how many adults bought tickets?

  Let x be number of children and y number of adults.

  Then we have
  $$\left\{ \begin{array}{rcr} x+y & = & 2000 \\ 25x+50y & = & 70000 \end{array} \right\}.$$
  Let us solve by substitution, starting by solving the first for y:
  $$y = 2000 - x.$$
  Substitute into the second equation:
  $$25x + 50(2000-x) = 70000 \;\Rightarrow\; 25x + 100000 - 50x = 70000$$
  $$-25x = -30000 \;\Rightarrow\; x = 1200.$$
  Thus, 1200 children and 800 adults attended.

Subsection 2

Systems of Nonlinear Equations and Inequalities: Two Variables

We Will Learn How To:

- Solve a system of nonlinear equations using substitution;
- Solve a system of nonlinear equations using elimination;
- Graph a nonlinear inequality;
- Graph a system of nonlinear inequalities.

Solving a System Representing a Parabola and a Line

- Solve the system of equations. $\left\{\begin{array}{rcr} x-y & = & -1 \\ y & = & x^2+1 \end{array}\right\}$.

  Substitute $y=x^2+1$ into the first equation and solve for x:

  $$x-y=-1 \;\Rightarrow\; x-(x^2+1)=-1 \;\Rightarrow\; x-x^2-1=-1$$

  $$\Rightarrow\; x-x^2=0 \;\Rightarrow\; x(1-x)=0 \;\Rightarrow\; x=0 \text{ or } 1-x=0$$

  $$\Rightarrow\; x=0 \text{ or } x=1.$$

  - If $x=0$, then $y=1$.
  - If $x=1$, then $y=2$.

  It follows that $(x,y)=(0,1)$ or $(x,y)=(1,2)$.

Finding the Intersection of a Circle and a Line

- Find the intersection of the given circle and the given line by substitution. $\left\{ \begin{array}{rcr} x^2 + y^2 & = & 5 \\ y & = & 3x - 5 \end{array} \right\}$.

  Substitute $y = 3x - 5$ into the first equation and solve for x:

  $$x^2 + y^2 = 5 \;\Rightarrow\; x^2 + (3x-5)^2 = 5$$
  $$\Rightarrow\; x^2 + 9x^2 - 30x + 25 = 5 \;\Rightarrow\; 10x^2 - 30x + 20 = 0$$
  $$\Rightarrow\; x^2 - 3x + 2 = 0 \;\Rightarrow\; (x-1)(x-2) = 0$$
  $$\Rightarrow\; x - 1 = 0 \text{ or } x - 2 = 0 \;\Rightarrow\; x = 1 \text{ or } x = 2.$$

  - If $x = 1$, then $y = -2$.
  - If $x = 2$, then $y = 1$.

  It follows that $(x, y) = (1, -2)$ or $(x, y) = (2, 1)$.

Solving a System Representing a Circle and an Ellipse

- Solve the system of nonlinear equations. $\left\{ \begin{array}{rcr} x^2+y^2 & = & 26 \\ 3x^2+25y^2 & = & 100 \end{array} \right\}$.

  We may use addition:

  $$\left\{ \begin{array}{rcr} x^2+y^2 & = & 26 \\ 3x^2+25y^2 & = & 100 \end{array} \right\} \Rightarrow \left\{ \begin{array}{rcr} -3x^2-3y^2 & = & -78 \\ 3x^2+25y^2 & = & 100 \end{array} \right\}$$

  $$\Rightarrow\ 22y^2=22\ \Rightarrow\ y^2=1\ \Rightarrow\ y=\pm 1.$$

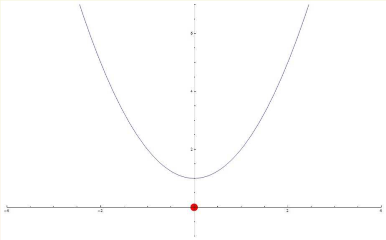
  - If $y=-1$, $x^2+1=26\ \Rightarrow\ x^2=25\ \Rightarrow\ x=\pm 5$.
  - If $y=1$, $x^2+1=26\ \Rightarrow\ x^2=25\ \Rightarrow\ x=\pm 5$.

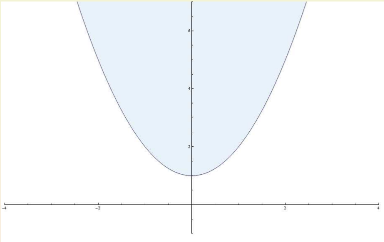
  Therefore, we have four solution pairs: $(x,y)=(-1,-5)$ or $(x,y)=(-1,5)$ or $(x,y)=(1,-5)$ or $(x,y)=(1,5)$.

Graphing an Inequality for a Parabola

- Graph the inequality $y > x^2 + 1$.

  First plot the graph $y = x^2 + 1$.





Pick a test point not on the graph, say $(0, 0)$.

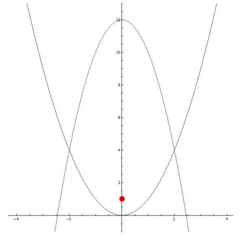
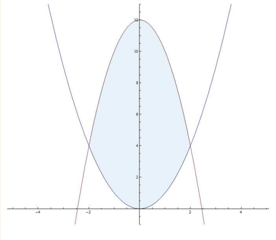
Test the inequality there: $0 > 0 + 1$ is FALSE.

So the region of solutions is the the opposite side.

Shade that region.

Graphing a System of Inequalities

- Graph the given system of inequalities. $\left\{ \begin{array}{rcr} x^2 - y & \leq & 0 \\ 2x^2 + y & \leq & 12 \end{array} \right\}.$
  First plot the graphs of
  - $x^2 - y = 0$ or $y = x^2$;
  - $2x^2 + y = 12$ or $y = 12 - 2x^2$.

Pick a test point not on the graphs, say $(0, 1)$.
Test the inequalities there:
  - $0^2 - 1 \leq 0$ is TRUE
  - $2 \cdot 0^2 + 1 \leq 12$ is TRUE

The solution region lies above the first and below the second parabola.

Subsection 3

Partial Fractions

We Will Learn How To:

- Decompose $\frac{P(x)}{Q(x)}$, where $Q(x)$ has only non-repeated linear factors;
- Decompose $\frac{P(x)}{Q(x)}$, where $Q(x)$ has repeated linear factors;
- Decompose $\frac{P(x)}{Q(x)}$, where $Q(x)$ has a non-repeated irreducible quadratic factor;
- Decompose $\frac{P(x)}{Q(x)}$, where $Q(x)$ has a repeated irreducible quadratic factor.

Denominator has Nonrepeated Linear Factors

- Suppose $\frac{P(x)}{Q(x)}$ is a rational expression, such that:
  - the degree of $P(x)$ is less than the degree of $Q(x)$;
  - $Q(x)$ has nonrepeated linear factors $a_1x+b_1, \ldots, a_nx+b_n$, i.e., we have

$$Q(x)=(a_1x+b_1)\cdots(a_nx+b_n).$$

Then the partial fraction decomposition of $\frac{P(x)}{Q(x)}$ is

$$\frac{P(x)}{Q(x)}=\frac{A_1}{a_1x+b_1}+\frac{A_2}{a_2x+b_2}+\frac{A_3}{a_3x+b_3}+\cdots+\frac{A_n}{a_nx+b_n},$$

where $A_1,\ldots,A_n$ are constants.

Decomposing Rationals with Distinct Linear Factors

- Decompose the rational expression $\frac{3x}{(x+2)(x-1)}$.

  We work as follows:

$$\frac{3x}{(x+2)(x-1)} = \frac{A}{x+2} + \frac{B}{x-1}$$

$$(x+2)(x-1)\frac{3x}{(x+2)(x-1)} = (x+2)(x-1)[\frac{A}{x+2} + \frac{B}{x-1}]$$

$$3x = A(x-1) + B(x+2)$$

$$3x = Ax - A + Bx + 2B$$

$$3x = (A+B)x + (-A+2B)$$

$$\left\{ \begin{array}{c} A+B=3 \\ -A+2B=0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} 2B+B=3 \\ A=2B \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} B=1 \\ A=2 \end{array} \right\}.$$

  So $\frac{3x}{(x+2)(x-1)} = \frac{1}{x+2} + \frac{2}{x-1}$.

Denominator has Repeated Linear Factors

- Suppose $\frac{P(x)}{Q(x)}$ is a rational expression, such that:
  - the degree of $P(x)$ is less than the degree of $Q(x)$;
  - $Q(x)$ has a repeated linear factor $ax+b$, occurring n times, i.e.,

$$Q(x) = (ax+b)^n.$$

  Then the partial fraction decomposition of $\frac{P(x)}{Q(x)}$ is

$$\frac{P(x)}{Q(x)} = \frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \frac{A_3}{(ax+b)^3} + \cdots + \frac{A_n}{(ax+b)^n},$$

  where $A_1, \ldots, A_n$ are constants.
- We write the denominator powers in increasing order.

Decomposing with Repeated Linear Factors

- Decompose the rational expression $\frac{-x^2+2x+4}{x^3-4x^2+4x}$.

  We have $x^3 - 4x^2 + 4x = x(x^2 - 4x + 4) = x(x-2)^2$.

$$\frac{-x^2+2x+4}{x(x-2)^2} = \frac{A}{x} + \frac{B}{x-2} + \frac{C}{(x-2)^2}$$

$$-x^2 + 2x + 4 = A(x^2 - 4x + 4) + Bx(x-2) + Cx$$

$$-x^2 + 2x + 4 = Ax^2 - 4Ax + 4A + Bx^2 - 2Bx + Cx$$

$$-x^2 + 2x + 4 = (A+B)x^2 + (-4A - 2B + C)x + 4A$$

$$\left\{ \begin{array}{c} A + B = -1 \\ -4A - 2B + C = 2 \\ 4A = 4 \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} A = 1 \\ B = -2 \\ C = 2 \end{array} \right\}$$

  Therefore, $\frac{-x^2+2x+4}{x(x-2)^2} = \frac{1}{x} + \frac{-2}{x-2} + \frac{2}{(x-2)^2}$.

Denominator has Nonrepeated Irreducible Quadratic Factor

- Suppose $\frac{P(x)}{Q(x)}$ is a rational expression, such that:
  - the degree of $P(x)$ is less than the degree of $Q(x)$;
  - $Q(x)$ has nonrepeated irreducible quadratic factors $a_1x^2 + b_1x + c_1$, $\ldots$,$a_nx^2 + b_nx + c_n$, i.e.,

$$Q(x) = (a_1x^2 + b_1x + c_1)\cdots(a_nx^2 + b_nx + c_n).$$

  Then the partial fraction decomposition of $\frac{P(x)}{Q(x)}$ is

$$\frac{P(x)}{Q(x)} = \frac{A_1x + B_1}{a_1x^2 + b_1x + c_1} + \frac{A_2x + B_2}{a_2x^2 + b_2x + c_2} + \cdots + \frac{A_nx + B_n}{a_nx^2 + b_nx + c_n},$$

  where $A_1, B_1, \ldots, A_n, B_n$ are constants.
- The decomposition may contain more rational expressions if there are linear factors.
- Each linear factor will have a different constant numerator: A, B, C, and so on.

Decomposing w/ Nonrepeated Irreducible Quadratic Factor

- Find a partial fraction decomposition of $\frac{8x^2+12x-20}{(x+3)(x^2+x+2)}$.

  We have

$$\frac{8x^2+12x-20}{(x+3)(x^2+x+2)} = \frac{A}{x+3} + \frac{Bx+C}{x^2+x+2}$$

$$8x^2+12x-20 = A(x^2+x+2)+(Bx+C)(x+3)$$

$$8x^2+12x-20 = Ax^2+Ax+2A+Bx^2+Cx+3Bx+3C$$

$$8x^2+12x-20 = (A+B)x^2+(A+3B+C)x+(2A+3C)$$

$$\left\{\begin{array}{c} A+B=8 \\ A+3B+C=12 \\ 2A+3C=-20 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} B=8-A \\ A+3(8-A)+C=12 \\ 2A+3C=-20 \end{array}\right\}$$

$$\Rightarrow \left\{\begin{array}{c} B=8-A \\ -2A+C=-12 \\ 2A+3C=-20 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} A=2 \\ B=6 \\ C=-8 \end{array}\right\}$$

  Thus, $\frac{8x^2+12x-20}{(x+3)(x^2+x+2)} = \frac{2}{x+3} + \frac{6x-8}{x^2+x+2}$.

Denominator Has a Repeated Irreducible Quadratic Factor

- Suppose $\frac{P(x)}{Q(x)}$ is a rational expression, such that:
  - the degree of $P(x)$ is less than the degree of $Q(x)$;
  - $Q(x)$ has a repeated irreducible quadratic factors $ax^2 + bx + c$, occurring n times, i.e.,

$$Q(x) = (ax^2 + bx + c)^n.$$

  Then the partial fraction decomposition of $\frac{P(x)}{Q(x)}$ is

$$\frac{P(x)}{Q(x)} = \frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_nx + B_n}{(ax^2 + bx + c)^n},$$

  where $A_1, B_1, \ldots, A_n, B_n$ are constants.
- We write the denominators in increasing powers.

Decomposing with a Repeated Irreducible Quadratic Factor

- Decompose the given expression that has a repeated irreducible factor in the denominator. $\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}$.

  We have

$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{x^2+1} + \frac{Dx+E}{(x^2+1)^2}$$

$$x^4+x^3+x^2-x+1 = A(x^2+1)^2 + (Bx+C)x(x^2+1) + (Dx+E)x$$

$$x^4+x^3+x^2-x+1 = A(x^4+2x^2+1) + (Bx+C)(x^3+x) + (Dx+E)x$$

$$x^4+x^3+x^2-x+1 = Ax^4+2Ax^2+A + Bx^4+Cx^3+Bx^2+Cx+Dx^2+Ex$$

$$x^4+x^3+x^2-x+1 = (A+B)x^4+Cx^3 + (2A+B+D)x^2+(C+E)x+A$$

Decomposing with a Repeated Irreducible Quadratic Factor

- We found

$$x^4+x^3+x^2-x+1=(A+B)x^4+Cx^3+(2A+B+D)x^2+(C+E)x+A.$$

Therefore, we get

$$\left\{\begin{array}{c} A+B=1 \\ C=1 \\ 2A+B+D=1 \\ C+E=-1 \\ A=1 \end{array}\right\} \Rightarrow \left\{\begin{array}{c} A=1 \\ B=0 \\ C=1 \\ D=-1 \\ E=-2 \end{array}\right\}$$

Therefore

$$\frac{x^4+x^3+x^2-x+1}{x(x^2+1)^2}=\frac{1}{x}+\frac{1}{x^2+1}+\frac{-x-2}{(x^2+1)^2}.$$

Subsection 4

Solving Systems with Gaussian Elimination

We Will Learn How To:

- Write the augmented matrix of a system of equations;
- Write the system of equations from an augmented matrix;
- Perform row operations on a matrix;
- Solve a system of linear equations using matrices.

Writing the Augmented Matrix for a System of Equations

- Write the augmented matrix for the given system of equations.

$$\left\{ \begin{array}{rcl} x+2y & = & 3 \\ 2x-y & = & 6 \end{array} \right\}$$

The left hand side consists of the **matrix of coefficients** and the right hand column of the right-hand side constants:

$$\left[\begin{array}{rr|r} 1 & 2 & 3 \\ 2 & -1 & 6 \end{array} \right]$$

Writing a System of Equations from Augmented Matrix

- Find the system of equations from the augmented matrix.

$$\left[\begin{array}{cc|c} 1 & -3 & -2 \\ 2 & -5 & 5 \end{array}\right].$$

We obtain the following system:

$$\left\{\begin{array}{rcr} x-3y & = & -2 \\ 2x-5y & = & 5 \end{array}\right\}$$

Elementary Row Operations

- To solve a system of equations we can perform the following **elementary row operations**:
  1. Interchange rows (Notation: $R_i \leftrightarrow R_j$)
  2. Multiply a row by a non-zero constant. (Notation: $R_i \leftarrow cR_i$)
  3. Add the product of a row multiplied by a constant to another row. (Notation: $R_i \leftarrow R_i + cR_j$)

Gaussian Elimination

- In **Gaussian elimination** the goal is to write matrix A with the number 1 as the entry down the main diagonal and have all zeros below.

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \xrightarrow[]{\text{Gaussian elimination}} A = \begin{bmatrix} 1 & b_{12} \\ 0 & 1 \end{bmatrix}.$$

- The first step of the Gaussian strategy includes obtaining a 1 as the first entry, so that row 1 may be used to alter the second row.

Solving a 2×2 System by Gaussian Elimination

- Solve by Gaussian elimination $\left\{\begin{array}{rcl} 2x+3y & = & 6 \\ x-y & = & \frac{1}{2} \end{array}\right\}$

  Write the augmented matrix and apply row operations to obtain the row echelon form:

$$\left[\begin{array}{rr|r} 2 & 3 & 6 \\ 1 & -1 & \frac{1}{2} \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rr|r} 1 & -1 & \frac{1}{2} \\ 2 & 3 & 6 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - 2R_1} \left[\begin{array}{rr|r} 1 & -1 & \frac{1}{2} \\ 0 & 5 & 5 \end{array}\right] \xrightarrow{R_2 \leftarrow \frac{1}{5}R_2} \left[\begin{array}{rr|r} 1 & -1 & \frac{1}{2} \\ 0 & 1 & 1 \end{array}\right].$$

  Now we have:

$$\left\{\begin{array}{rcl} x_1 - x_2 & = & \frac{1}{2} \\ x_2 & = & 1 \end{array}\right\} \Rightarrow \left\{\begin{array}{rcl} x_1 & = & \frac{3}{2} \\ x_2 & = & 1 \end{array}\right\}$$

Using Gaussian Elimination to Solve a System

- Use Gaussian elimination to solve $\left\{\begin{array}{rcl} 2x+y & = & 1 \\ 4x+2y & = & 6 \end{array}\right\}$.

  Form the augmented matrix and apply row operations:

$$\left[\begin{array}{cc|c} 2 & 1 & 1 \\ 4 & 2 & 6 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{2} R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 4 & 2 & 6 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - 4R_1} \left[\begin{array}{cc|c} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 4 \end{array}\right]$$

  The last row yields the equation $0 = 4$.

  So the given system is inconsistent.

Solving a Dependent System

- Solve the system of equations. $\left\{\begin{array}{rcl} 3x+4y & = & 12 \\ 6x+8y & = & 24 \end{array}\right\}$

  Form the augmented matrix and apply row operations:

$$\left[\begin{array}{cc|c} 3 & 4 & 12 \\ 6 & 8 & 24 \end{array}\right] \xrightarrow{R_1 \leftarrow \frac{1}{3}R_1} \left[\begin{array}{cc|c} 1 & \frac{4}{3} & 4 \\ 6 & 8 & 24 \end{array}\right]$$

$$\xrightarrow{R_2 \leftarrow R_2 - 6R_1} \left[\begin{array}{cc|c} 1 & \frac{4}{3} & 4 \\ 0 & 0 & 0 \end{array}\right]$$

  The last row yields the equation $0 = 0$.

  So the given system is dependent.

  From the first row, we get $x + \frac{4}{3}y = 4 \Rightarrow x = 4 - \frac{4}{3}y$.

  So $(x, y) = (4 - \frac{4}{3}y, y)$, y any real.

Applying 2×2 Matrices to Finance

- Carolyn invests a total of \$12,000 in two municipal bonds, one paying 10.5% interest and the other paying 12% interest.

  The annual interest earned on the two investments was \$1,335.

  How much was invested at each rate?

  Let x, y be the amounts invested in each kind.

  Then we must solve $\left\{ \begin{array}{rcr} x + y & = & 12000 \\ 0.105x + 0.12y & = & 1335 \end{array} \right\}$.

  We use the matrix method:

$$\left[\begin{array}{rr|r} 1 & 1 & 12000 \\ 105 & 120 & 1335000 \end{array} \right] \xrightarrow{R_2 \leftarrow R_2 - 105R_1} \left[\begin{array}{rr|r} 1 & 1 & 12000 \\ 0 & 15 & 75000 \end{array} \right]$$

$$\xrightarrow{R_2 \leftarrow \frac{1}{15}R_2} \left[\begin{array}{rr|r} 1 & 1 & 12000 \\ 0 & 1 & 5000 \end{array} \right]$$

  Therefore $(x, y) = (7000, 5000)$.

Subsection 5

Solving Systems with Cramer's Rule

We Will Learn How To:

- Evaluate 2×2 determinants;
- Use Cramer's Rule to solve a system of equations in two variables;

Find the Determinant of a 2×2 Matrix

- The **determinant** of a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is defined as

$$\det(A) = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - cb.$$

- Notice the change in notation.

  There are several ways to indicate the determinant, including $\det(A)$ and replacing the brackets in a matrix with straight lines, $|A|$.

Finding the Determinant of a 2×2 Matrix

- Find the determinant of the matrix $A = \begin{bmatrix} 5 & 2 \\ -6 & 3 \end{bmatrix}$.

  We compute

$$\det(A) = 5 \cdot 3 - 2 \cdot (-6) = 15 + 12 = 27.$$

Cramer's Rule for 2×2 systems

- **Cramer's Rule** is a method that uses determinants to solve systems of equations that have the same number of equations as variables.
- Consider a system of two linear equations in two variables.

$$\left\{ \begin{array}{rcl} a_1x + b_1y & = & c_1 \\ a_2x + b_2y & = & c_2 \end{array} \right\}$$

- The solution using Cramer's Rule is given as

$$x = \frac{D_x}{D} = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{D_y}{D} = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad D \neq 0.$$

  - If we are solving for x, the x column is replaced with the constant column.
  - If we are solving for y, the y column is replaced with the constant column.

Using Cramer's Rule to Solve a 2×2 System

- Solve the following 2×2 system using Cramer's Rule.

$$\left\{ \begin{array}{rcl} 12x + 3y & = & 15 \\ 2x - 3y & = & 13 \end{array} \right\}$$

Compute the three determinants D, D_x and D_y:

$$D = \left| \begin{array}{rr} 5 & 2 \\ -6 & 3 \end{array} \right| = 12 \cdot (-3) - 3 \cdot 2 = -42;$$

$$D_x = \left| \begin{array}{rr} 15 & 3 \\ 13 & -3 \end{array} \right| = 15 \cdot (-3) - 3 \cdot 13 = -84;$$

$$D_y = \left| \begin{array}{rr} 12 & 15 \\ 2 & 13 \end{array} \right| = 12 \cdot 13 - 15 \cdot 2 = 126.$$

Now we derive the solution:

$$(x, y) = \left(\frac{D_x}{D}, \frac{D_y}{D} \right) = \left(\frac{-84}{-42}, \frac{126}{-42} \right) = (2, -3).$$

Using Cramer's Rule to Solve an Inconsistent System

- Solve the system of equations using Cramer's Rule.
  $\left\{ \begin{array}{rcl} 3x - 2y & = & 4 \\ 6x - 4y & = & 0 \end{array} \right\}$.
  We have $D = \left| \begin{array}{cc} 3 & -2 \\ 6 & -4 \end{array} \right| = 0$.
  We work by addition
  $$\left\{ \begin{array}{c} 3x - 2y = 4 \\ 6x - 4y = 0 \end{array} \right\} \Rightarrow \left\{ \begin{array}{c} -6x + 4y = -8 \\ 6x - 4y = 0 \end{array} \right\} \Rightarrow 0 = -8.$$
  Therefore the system is inconsistent.