Introduction to Convexity

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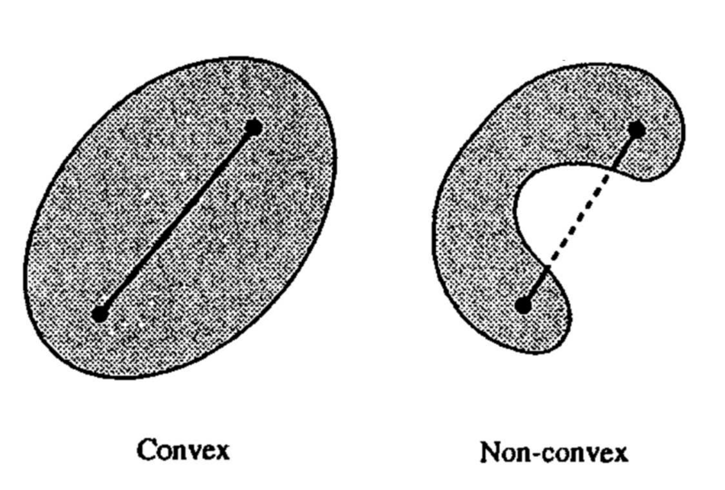
Subsection 1

Basic Properties of Convex Sets

Convex Sets in Space

- A set in space is **convex** if whenever it contains two points, it also contains the line segment joining them.
- Elementary geometry abounds in convex sets:
  - ellipses;
  - triangles;
  - parallelograms;
  - balls;
  - halfspaces;
  - cubes.
- Examples of non-convex sets are:
  - an annulus;
  - a crescent;
  - the vertex set of a cube.

Convex Sets in Space: Illustration



Convex Sets in $\mathbb{R}^n$

- Let $\boldsymbol{x}$ and $\boldsymbol{y}$ be distinct points of $\mathbb{R}^n$.
- Then the subset

$$\{\lambda \boldsymbol{x} + \mu \boldsymbol{y} : \lambda, \mu \geq 0, \lambda + \mu = 1\}$$

  of the line through $\boldsymbol{x}$ and $\boldsymbol{y}$ is called the **line segment** joining $\boldsymbol{x}$ and $\boldsymbol{y}$.
- The set A in $\mathbb{R}^n$ is said to be **convex** if whenever it contains two points, it also contains the line segment joining them.
- Expressed algebraically, A is convex if $\lambda \boldsymbol{x} + \mu \boldsymbol{y} \in A$ whenever $\boldsymbol{x}, \boldsymbol{y} \in A$ and $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$.
- Equivalently, A is convex if $\lambda A + \mu A \subseteq A$ whenever $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$.

First Examples

- The condition for a set to be convex is less restrictive than for it to be a flat.
- So every flat is a convex set.
- In particular, the following are convex:
  - the empty set;
  - singletons;
  - lines;
  - hyperplanes;
  - $\mathbb{R}^n$ itself.

Balls are Convex

- We show that the closed ball $B[\boldsymbol{a};r]$ in $\mathbb{R}^n$ is convex.
- Let $\boldsymbol{x},\boldsymbol{y} \in B[\boldsymbol{a};r]$.
- Let $\lambda,\mu \geq 0$ with $\lambda+\mu=1$.
- Then $\|\boldsymbol{x}-\boldsymbol{a}\| < r$, $\|\boldsymbol{y}-\boldsymbol{a}\| < r$.
- So
$$\begin{aligned} \|\lambda\boldsymbol{x}+\mu\boldsymbol{y}-\boldsymbol{a}\| &= \|\lambda(\boldsymbol{x}-\boldsymbol{a})+\mu(\boldsymbol{y}-\boldsymbol{a})\| \\ &\leq \lambda\|\boldsymbol{x}-\boldsymbol{a}\|+\mu\|\boldsymbol{y}-\boldsymbol{a}\| \\ &\leq \lambda r+\mu r = r. \end{aligned}$$
- Thus $\lambda\boldsymbol{x}+\mu\boldsymbol{y} \in B[\boldsymbol{a};r]$.
- This proves that $B[\boldsymbol{a};r]$ is convex.
- A similar argument shows that the open ball $B(\boldsymbol{a};r)$ is convex.

Halfspaces are Convex

- We show that the closed halfspace A in $\mathbb{R}^n$ defined by the inequality $\boldsymbol{u}\cdot\boldsymbol{x}\le u_0$ is convex.
- Let $\boldsymbol{x},\boldsymbol{y}\in A$ and let $\lambda,\mu\ge 0$ with $\lambda+\mu=1$.
- Then $\boldsymbol{u}\cdot\boldsymbol{x}\le u_0$, $\boldsymbol{u}\cdot\boldsymbol{y}\le u_0$.
- So
$$\boldsymbol{u}\cdot(\lambda\boldsymbol{x}+\mu\boldsymbol{y})=\lambda\boldsymbol{u}\cdot\boldsymbol{x}+\mu\boldsymbol{u}\cdot\boldsymbol{y}\le\lambda u_0+\mu u_0=u_0.$$
- Thus $\lambda\boldsymbol{x}+\mu\boldsymbol{y}\in A$.
- This proves that A is convex.
- A similar argument shows that open halfspaces are convex.

Closure Under Intersections

Theorem

The intersection of an arbitrary family of convex sets in $\mathbb{R}^n$ is convex.

- Let $(A_i : i \in I)$ be a family of convex sets in $\mathbb{R}^n$.

  If $\boldsymbol{a}, \boldsymbol{b} \in \bigcap(A_i : i \in I)$ and $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$, then $\boldsymbol{a}, \boldsymbol{b} \in A_i$.

  As A_i is convex, $\lambda\boldsymbol{a} + \mu\boldsymbol{b} \in A$, for each $i \in I$.

  Thus $\lambda\boldsymbol{a} + \mu\boldsymbol{b} \in \bigcap(A_i : i \in I)$.

  This shows that the intersection is convex.

Closure Under Restricted Linear Combinations

Theorem

Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ be points of a convex set A in $\mathbb{R}^n$. Let $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$. Then $\lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m \in A$.

- We argue by induction on m.
- When $m = 1$ the assertion is trivial.
- Suppose that the assertion holds when m is some positive integer k.
- Let
$$\boldsymbol{x} = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_{k+1} \boldsymbol{a}_{k+1},$$
where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_{k+1} \in A$ and $\lambda_1, \ldots, \lambda_{k+1} \geq 0$ with $\lambda_1 + \cdots + \lambda_{k+1} = 1$. At least one λ_i must be less than 1, say $\lambda_{k+1} < 1$. Write $\boldsymbol{y} = \frac{\lambda_1}{\lambda} \boldsymbol{a}_1 + \cdots + \frac{\lambda_k}{\lambda} \boldsymbol{a}_k$, where $\lambda = \lambda_1 + \cdots + \lambda_k = 1 - \lambda_{k+1} > 0$. By the induction hypothesis, $\boldsymbol{y} \in A$. Since A is convex and contains both $\boldsymbol{y}$ and $\boldsymbol{a}_{k+1}$, the equation $\boldsymbol{x} = \lambda \boldsymbol{y} + \lambda_{k+1} \boldsymbol{a}_{k+1}$ shows that $\boldsymbol{x} \in A$. This completes the proof by induction.

Convex Combinations

- A point $\boldsymbol{x}$ is said to be a **convex combination** of points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ in $\mathbb{R}^n$ if there exist scalars $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$ such that

$$\boldsymbol{x} = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m.$$

- The preceding theorem can thus be expressed as:

  Every convex combination of points of a convex set in $\mathbb{R}^n$ belongs to that set.

Convexity, Vector Addition and Scalar Multiplication

Theorem

Let A, B be convex sets in $\mathbb{R}^n$ and let α be a scalar. Then $A+B$ and αA are convex.

- Let $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$. Since A, B are convex,

$$\lambda(A+B)+\mu(A+B)=(\lambda A+\mu A)+(\lambda B+\mu B)\subseteq A+B;$$
$$\lambda(\alpha A)+\mu(\alpha A)=\alpha(\lambda A+\mu A)\subseteq \alpha A.$$

  This shows that $A+B$ and αA are convex.

Corollary

Let $A_1,\ldots,A_m$ be convex sets in $\mathbb{R}^n$ and let $\lambda_1,\ldots,\lambda_m$ be scalars. Then $\lambda_1 A_1+\cdots+\lambda_m A_m$ is convex.

A Distributivity Property

Theorem

Let A be a convex set in $\mathbb{R}^n$ and let $\lambda_1, \ldots, \lambda_m \geq 0$. Then

$$(\lambda_1 + \cdots + \lambda_m)A = \lambda_1 A + \cdots + \lambda_m A.$$

- The result is trivial when each λ_i is zero.

  Suppose that $\lambda = \lambda_1 + \cdots + \lambda_m > 0$.

  With the help of a previous theorem, we can deduce that

$$\begin{array}{rcl}(\lambda_1 + \cdots + \lambda_m)A & \subseteq & \lambda_1 A + \cdots + \lambda_m A \\ & = & \lambda(\frac{\lambda_1}{\lambda} A + \cdots + \frac{\lambda_m}{\lambda} A) \\ & \subseteq & \lambda A \\ & = & (\lambda_1 + \cdots + \lambda_m)A.\end{array}$$

  Thus $(\lambda_1 + \cdots + \lambda_m)A = \lambda_1 A + \cdots + \lambda_m A$.

A Cancelation Property

Theorem

Let A, B, C be sets in $\mathbb{R}^n$. Suppose that A is non-empty and bounded, that C is closed and convex, and that $A + B \subseteq A + C$. Then $B \subseteq C$.

- Let $\boldsymbol{a}_0 \in A$. If $\boldsymbol{b} \in B$, then $\boldsymbol{a}_0 + \boldsymbol{b} \in A + B \subseteq A + C$. So there exist $\boldsymbol{a}_1 \in A$, $\boldsymbol{c}_1 \in C$, such that $\boldsymbol{a}_0 + \boldsymbol{b} = \boldsymbol{a}_1 + \boldsymbol{c}_1$. Similarly, there exist $\boldsymbol{a}_2, \ldots, \boldsymbol{a}_i \in A$ and $\boldsymbol{c}_2, \ldots, \boldsymbol{c}_i \in C$ with $\boldsymbol{a}_1 + \boldsymbol{b} = \boldsymbol{a}_2 + \boldsymbol{c}_2$, ..., $\boldsymbol{a}_{i-1} + \boldsymbol{b} = \boldsymbol{a}_i + \boldsymbol{c}_i$. We add the i equations above together to deduce that

$$\boldsymbol{a}_0 + i\boldsymbol{b} = \boldsymbol{a}_i + \boldsymbol{c}_1 + \cdots + \boldsymbol{c}_i.$$

Since C is convex, the point $\boldsymbol{x}_i = \frac{1}{i}(\boldsymbol{c}_1 + \cdots + \boldsymbol{c}_i)$ lies in C. Since A is bounded,

$$\begin{aligned} \|\boldsymbol{b} - \boldsymbol{x}_i\| &= \|\tfrac{1}{i}(\boldsymbol{a}_i + \boldsymbol{c}_1 + \cdots + \boldsymbol{c}_i - \boldsymbol{a}_0) - \tfrac{1}{i}(\boldsymbol{c}_1 + \cdots + \boldsymbol{c}_i)\| \\ &= \tfrac{1}{i}\|\boldsymbol{a}_i - \boldsymbol{a}_0\| \to 0 \text{ as } i \to \infty. \end{aligned}$$

Thus $\boldsymbol{x}_i \to \boldsymbol{b}$ as $i \to \infty$. But C is closed. So $\boldsymbol{b} \in C$. Hence, $B \subseteq C$.

A Cancelation Property (Cont'd)

Corollary

Let A, B, C be sets in $\mathbb{R}^n$. Suppose that A is non-empty and bounded, that B and C are closed and convex, and that $A+B=A+C$. Then $B=C$.

Affine Transformations and Convex Sets

Theorem

Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be an affine transformation. Then $f(A)$ is convex for each convex set A in $\mathbb{R}^n$, and $f^{-1}(B)$ is convex for each convex set B in $\mathbb{R}^m$.

- Let A be a convex set in $\mathbb{R}^n$. Let $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$. If $\boldsymbol{x}, \boldsymbol{y} \in f(A)$, then $\boldsymbol{x} = f(\boldsymbol{a})$, $\boldsymbol{y} = f(\boldsymbol{b})$ for some $\boldsymbol{a}, \boldsymbol{b} \in A$. Since A is convex, $\lambda \boldsymbol{a} + \mu \boldsymbol{b} \in A$. Since f is affine,

$$\lambda \boldsymbol{x} + \mu \boldsymbol{y} = \lambda f(\boldsymbol{a}) + \mu f(\boldsymbol{b}) = f(\lambda \boldsymbol{a} + \mu \boldsymbol{b}).$$

  Thus $\lambda \boldsymbol{x} + \mu \boldsymbol{y} \in f(A)$. This shows that $f(A)$ is convex.

  Let B be a convex set in $\mathbb{R}^m$. Let $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$. If $\boldsymbol{x}, \boldsymbol{y} \in f^{-1}(B)$, then $f(\boldsymbol{x}), f(\boldsymbol{y}) \in B$. Since B is convex, $\lambda f(\boldsymbol{x}) + \mu f(\boldsymbol{y}) \in B$. Since f is affine, $f(\lambda \boldsymbol{x} + \mu \boldsymbol{y}) = \lambda f(\boldsymbol{x}) + \mu f(\boldsymbol{y}) \in B$. Thus $\lambda \boldsymbol{x} + \mu \boldsymbol{y} \in f^{-1}(B)$. This shows that $f^{-1}(B)$ is convex.

Subsection 2

The Convex Hull

The Convex Hull

- The **convex hull** conv$A$ of a set A in $\mathbb{R}^n$ is the intersection of all convex sets in $\mathbb{R}^n$ containing A.
- The definition of conv$A$, together with a previous theorem, shows that conv$A$ is a convex set containing A.
- Moreover, if C is any convex set in $\mathbb{R}^n$ containing A, then $\text{conv}A \subseteq C$.
- Thus we may refer to conv$A$ as the smallest convex set in $\mathbb{R}^n$ containing A.
- Clearly, A is convex if and only if $A = \text{conv}A$.
- Moreover $\text{conv}(\text{conv}A) = \text{conv}A$.
- Also $\text{conv}A \subseteq \text{conv}B$ whenever $A \subseteq B$.

Examples

- In space:
  - The convex hull of two distinct points is the line segment joining them;
  - The convex hull of three non-collinear points is the triangle which they determine;
  - The convex hull of four non-coplanar points is the tetrahedron which they determine.
- In $\mathbb{R}^2$ the convex hull of m points symmetrically placed on the circumference of a circle, where $m \geq 3$, is a regular m-sided polygon.

Example

- The convex hull of the set $A = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| = 1\}$ is the closed unit ball $U = \{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \leq 1\}$.
- The ball U is convex and contains A, so $\text{conv}A \subseteq U$.
- We now show that $U \subseteq \text{conv}A$.

  Let $\boldsymbol{x} \in U$. If $\boldsymbol{x} = 0$ and $\boldsymbol{y} \in A$, then $\boldsymbol{x} = \frac{1}{2}\boldsymbol{y} + \frac{1}{2}(-\boldsymbol{y})$. Since $\text{conv}A$ is convex and contains $\boldsymbol{y}$ and $-\boldsymbol{y}$, this shows that $\boldsymbol{x} \in \text{conv}A$. If $\boldsymbol{x} \neq \boldsymbol{0}$, then $0 < \|\boldsymbol{x}\| \leq 1$. The equation

$$\boldsymbol{x} = \left(\frac{1 + \|\boldsymbol{x}\|}{2}\right)\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} + \left(\frac{1 - \|\boldsymbol{x}\|}{2}\right)\frac{-\boldsymbol{x}}{\|\boldsymbol{x}\|}$$

  shows that $\boldsymbol{x} \in \text{conv}A$, since $\text{conv}A$ is convex and contains $\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$ and $-\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|}$. Thus $U \subseteq \text{conv}A$.
- We now have $U = \text{conv}A$.

Description of Convex Hull in Terms of Points

Theorem

Let A be a set in $\mathbb{R}^n$. Then conv$A$ is the set of all convex combinations of points of A.

- Denote by B the set of all convex combinations of points of A.

  That $B \subseteq \text{conv}A$ follows from a previous theorem and the inclusion $A \subseteq \text{conv}A$.

  We next show that B is convex. If $\boldsymbol{x}, \boldsymbol{y} \in B$, then

$$\boldsymbol{x} = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m, \quad \boldsymbol{y} = \mu_1 \boldsymbol{b}_1 + \cdots + \mu_p \boldsymbol{b}_p,$$

  for some $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$, $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_p \in A$ and $\lambda_1, \ldots, \lambda_m$, $\mu_1, \ldots, \mu_p \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$ and $\mu_1 + \cdots + \mu_p = 1$.

Description of Convex Hull in Terms of Points (Cont'd)

- Let $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$. Then

$$\lambda \boldsymbol{x} + \mu \boldsymbol{y} = \lambda\lambda_1 \boldsymbol{a}_1 + \cdots + \lambda\lambda_m \boldsymbol{a}_m + \mu\mu_1 \boldsymbol{b}_1 + \cdots + \mu\mu_p \boldsymbol{b}_p$$

and

$$\begin{aligned} \lambda\lambda_1 + \cdots + \lambda\lambda_m + \mu\mu_1 + \cdots + \mu\mu_p \\ &= \lambda(\lambda_1 + \cdots + \lambda_m) + \mu(\mu_1 + \cdots + \mu_p) \\ &= \lambda + \mu = 1. \end{aligned}$$

Thus $\lambda \boldsymbol{x} + \mu \boldsymbol{y} \in B$, so B is convex. Since B is convex and $B \supseteq A$, it follows that $B \supseteq \mathrm{conv} A$. Hence $B = \mathrm{conv} A$.

Corollary

Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbb{R}^n$. Then

$$\begin{aligned} \mathrm{conv}\{\boldsymbol{a}_1, \ldots \boldsymbol{a}_m\} &= \{\lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m : \lambda_1, \ldots, \lambda_m \geq 0, \\ &\qquad \lambda_1 + \cdots + \lambda_m = 1\}. \end{aligned}$$

On the Number of Points

- The preceding theorem shows that each point of the convex hull of a set in $\mathbb{R}^n$ is a convex combination of points of that set.
- The theorem makes no reference to the number of points in the combination.
- Carathéodory's Theorem, which is proved next, states that each point of the convex hull of an r-dimensional set can be expressed as a convex combination of $r+1$ or fewer points of the set.
- Thus a point in the convex hull of a set in $\mathbb{R}^3$ is either a point of the set or belongs to a line segment, a triangle, or a tetrahedron with vertices in the set.

Carathéodory's Theorem

Theorem (Carathéodory's Theorem)

Let $\boldsymbol{a} \in \text{conv}A$, where A is an r-dimensional set in $\mathbb{R}^n$. Then $\boldsymbol{a}$ can be expressed as a convex combination of $r+1$ or fewer points of A.

- The preceding theorem shows the existence of points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ of A and scalars $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$ such that

$$\boldsymbol{a} = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m.$$

We assume that this representation of $\boldsymbol{a}$ is so chosen that $\boldsymbol{a}$ cannot be expressed as a convex combination of fewer than m points of A.

It follows that no two of the points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ are equal and that $\lambda_1, \ldots, \lambda_m > 0$. We prove the theorem by showing that $m \leq r+1$.

We use a contradiction argument.

Carathéodory's Theorem (Cont'd)

- Suppose that $m > r+1$. Then, since A is r-dimensional, the set $\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m\}$ must be affinely dependent. So there exist scalars $\mu_1,\ldots,\mu_m$, not all zero, such that

$$\mathbf{0} = \mu_1\boldsymbol{a}_1 + \cdots + \mu_m\boldsymbol{a}_m, \quad \mu_1 + \cdots + \mu_m = 0.$$

Let $t > 0$ be such that the scalars $\lambda_1 + \mu_1 t,\ldots,\lambda_m + \mu_m t$ are nonnegative with at least one of them zero. Such a t exists since the λ's are all positive and at least one of the μ's is negative. The equation

$$\boldsymbol{a} = (\lambda_1 + \mu_1 t)\boldsymbol{a}_1 + \cdots + (\lambda_m + \mu_m t)\boldsymbol{a}_m,$$

when its terms with zero coefficients are omitted, exhibits $\boldsymbol{a}$ as a convex combination of fewer than m points of A. This contradiction to the minimality of m shows that $m \le r+1$.

Radon's Theorem

Theorem (Radon's Theorem)

Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbb{R}^n$ $(m \geq n+2)$. Then the set $\{1, \ldots, m\}$ can be partitioned into two subsets I and J such that $\text{conv}\{\boldsymbol{a}_i : i \in I\}$ meets $\text{conv}\{\boldsymbol{a}_j : j \in J\}$.

- We consider the non-trivial case when the $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ are distinct.

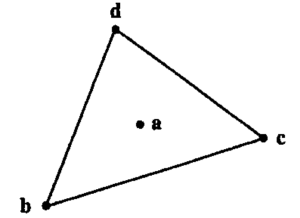
  It follows from a previous corollary that there exist scalars $\lambda_1, \ldots, \lambda_m$, not all zero, such that $\lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m = \boldsymbol{0}$ and $\lambda_1 + \cdots + \lambda_m = 0$. Some of the λ's will be positive, others negative. Let $I = \{i : \lambda_i \geq 0\}$ and $J = \{j : \lambda_j < 0\}$. Then

$$\frac{\sum_{i \in I} \lambda_i \boldsymbol{a}_i}{\sum_{i \in I} \lambda_i} = \frac{\sum_{j \in J} (-\lambda_j) \boldsymbol{a}_j}{\sum_{j \in J} (-\lambda_j)} = \boldsymbol{x} \text{ say.}$$

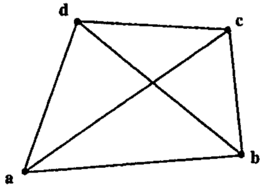
  Thus $\boldsymbol{x}$ is a convex combination of points of both $\{\boldsymbol{a}_i : i \in I\}$ and $\{\boldsymbol{a}_j : j \in J\}$. Hence $\boldsymbol{x} \in \text{conv}\{\boldsymbol{a}_i : i \in I\} \cap \text{conv}\{\boldsymbol{a}_j : j \in J\}$.

Four Points in $\mathbb{R}^2$

- Radon's theorem yields information about the possible configurations of four points in $\mathbb{R}^2$.
- It shows that:
  - Either one of the four points belongs to the (possibly degenerate) triangle determined by the remaining three;
  - Or the four points are the vertices of a convex quadrilateral.



conv{a} meets conv{b,c,d}



conv{a,c} meets conv{b,d}

Convex Hull of Open or Compact Sets

Theorem

In $\mathbb{R}^n$ the convex hull of an open set is open and the convex hull of a compact set is compact.

- Let A be an open set in $\mathbb{R}^n$. If $\boldsymbol{a} \in \operatorname{conv} A$, then $\boldsymbol{a} = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m$ for some $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in A$ and $\lambda_1, \ldots, \lambda_m \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$. Since A is open, there exist $r_1, \ldots, r_m > 0$ such that $B(\boldsymbol{a}_1; r_1) \subseteq A, \ldots,$ $B(\boldsymbol{a}_m; r_m) \subseteq A$. Let $r = \min\{r_1, \ldots, r_m\}$, so $r > 0$. We show that $B(\boldsymbol{a}; r) \subseteq \operatorname{conv} A$. Let $\boldsymbol{x} \in B(\boldsymbol{a}; r)$. Then $\|\boldsymbol{x} - \boldsymbol{a}\| < r$. For $i = 1, \ldots, m$, the point $\boldsymbol{x}_i = \boldsymbol{a}_i + \boldsymbol{x} - \boldsymbol{a}$ lies in $B(\boldsymbol{a}_i; r)$. Hence also in $B(\boldsymbol{a}_i; r_i)$ and A. Now we get

$$\begin{aligned} \boldsymbol{x} &= \boldsymbol{a} + \boldsymbol{x} - \boldsymbol{a} = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m + (\lambda_1 + \cdots + \lambda_m)(\boldsymbol{x} - \boldsymbol{a}) \\ &= \lambda_1(\boldsymbol{a}_1 + \boldsymbol{x} - \boldsymbol{a}) + \cdots + \lambda_m(\boldsymbol{a}_m + \boldsymbol{x} - \boldsymbol{a}) = \lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_m \boldsymbol{x}_m. \end{aligned}$$

  So $\boldsymbol{x} \in \operatorname{conv} A$. Thus $B(\boldsymbol{a}; r) \subseteq \operatorname{conv} A$ and each point of $\operatorname{conv} A$ is an interior point of $\operatorname{conv} A$. I.e., $\operatorname{conv} A$ is open.

Convex Hull of Open or Compact Sets (Cont'd)

- Now let A be a compact set in $\mathbb{R}^n$. If $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k,\ldots$ is a sequence in conv$A$, then, by Carathéodory's Theorem, $\boldsymbol{x}_k$ can be expressed in the form $\boldsymbol{x}_k = \lambda_{k0}\boldsymbol{a}_{k0} + \cdots + \lambda_{kn}\boldsymbol{a}_{kn}$, for some $\boldsymbol{a}_{k0},\ldots,\boldsymbol{a}_{kn} \in A$ and $\lambda_{k0},\ldots,\lambda_{kn} \geq 0$ with $\lambda_{k0} + \cdots + \lambda_{kn} = 1$. It may be necessary to include some extra $\boldsymbol{a}$'s with zero coefficients to bring the number of $\boldsymbol{a}$'s in the expression for $\boldsymbol{x}_k$ up to $n+1$. Each sequence $\boldsymbol{a}_{1j},\ldots,\boldsymbol{a}_{kj},\ldots$ $(j=0,\ldots,n)$ belongs to the compact set A. Each real sequence $\lambda_{1j},\ldots,\lambda_{kj},\ldots$ $(j=0,\ldots,n)$ belongs to the compact interval $[0,1]$. Since there is only a finite number, namely $2n+2$, of these sequences, we can, by repeatedly forming convergent subsequences of sequences whose members lie in a compact set, find a subsequence $i_1,\ldots,i_k,\ldots$ of $1,\ldots,k,\ldots$, points $\boldsymbol{a}_0,\ldots,\boldsymbol{a}_n$ of A and scalars $\lambda_0,\ldots,\lambda_n \geq 0$ with $\lambda_0 + \cdots + \lambda_n = 1$, such that $\boldsymbol{a}_{kj} \to \boldsymbol{a}_j$ and $\lambda_{kj} \to \lambda_j$ $(k \to \infty,\ j=0,\ldots,n)$. Thus the subsequence $\boldsymbol{x}_{i_1},\ldots,\boldsymbol{x}_{i_k},\ldots$ converges to the point $\lambda_0\boldsymbol{a}_0 + \cdots + \lambda_n\boldsymbol{a}_n$ of conv$A$. This shows that conv$A$ is compact.

Finite Sets and Closed Sets

Corollary

The convex hull of a finite set in $\mathbb{R}^n$ is compact.

- The theorem makes no reference to the convex hull of a closed set.
- Except in $\mathbb{R}^1$, the convex hull of a closed set need not be closed.

  Example: In $\mathbb{R}^n$ the union of a line and a point not on it is a closed set.

  But its convex hull is not closed.

Diameter of Bounded Sets

- Since a set in $\mathbb{R}^n$ is bounded if and only if it lies in some ball, and balls are convex, it follows that the convex hull of a bounded set in $\mathbb{R}^n$ is bounded.
- The **diameter** of a nonempty bounded set A in $\mathbb{R}^n$ is the nonnegative real number

$$\sup\{\|\boldsymbol{a}-\boldsymbol{b}\| : \boldsymbol{a},\boldsymbol{b} \in A\}.$$

- In $\mathbb{R}^2$ the diameter of a triangle is the length of a longest side.

  The diameter of a rectangle is the length of a diagonal.
- In $\mathbb{R}^n$ the balls $B(a;r)$ and $B[a;r]$ both have diameter $2r$.
- The theorem below relates the diameters of a bounded set and its convex hull.

Diameter of a Bounded Set and its Convex Hull

Theorem

Let A be a nonempty bounded set in $\mathbb{R}^n$. Then A and $\text{conv}A$ have the same diameter.

- Suppose that A has diameter s. Let $\boldsymbol{x}, \boldsymbol{y} \in \text{conv}A$. Then

$$\boldsymbol{x} = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m, \quad \boldsymbol{y} = \mu_1 \boldsymbol{b}_1 + \cdots + \mu_p \boldsymbol{b}_p,$$

for some $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$, $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_p \in A$ and $\lambda_1, \ldots, \lambda_m$, $\mu_1, \ldots, \mu_p \geq 0$ with $\lambda_1 + \cdots + \lambda_m = 1$, $\mu_1 + \cdots + \mu_p = 1$. Thus $\boldsymbol{x} = \sum_{i=1}^m \sum_{j=1}^p \lambda_i \mu_j \boldsymbol{a}_i$ and $\boldsymbol{y} = \sum_{i=1}^m \sum_{j=1}^p \lambda_i \mu_j \boldsymbol{b}_j$. Hence, using the Triangle Inequality,

$$\begin{array}{rcl} \|\boldsymbol{x} - \boldsymbol{y}\| & = & \|\sum_{i=1}^m \sum_{j=1}^p \lambda_i \mu_j (\boldsymbol{a}_i - \boldsymbol{b}_j)\| \\ & \leq & \sum_{i=1}^m \sum_{j=1}^p \lambda_i \mu_j \|\boldsymbol{a}_i - \boldsymbol{b}_j\| \\ & \leq & \sum_{i=1}^m \sum_{j=1}^p \lambda_i \mu_j s = s. \end{array}$$

Hence the diameter of $\text{conv}A$ does not exceed s. Since $A \subseteq \text{conv}A$, the diameter of $\text{conv}A$ is at least s. Thus $\text{conv}A$ has diameter s.

Convex Hull of Set of Complex Numbers

- We now prove a result concerning the location of the roots of the derivative of a complex polynomial.
- In a natural way we can identify the Euclidean space $\mathbb{R}^2$ with the complex plane by identifying each point (x,y) of $\mathbb{R}^2$ with the complex number $x+iy$ and vice versa.
- This identification allows us to refer to the convex hull of a set of complex numbers or to a convex combination of complex numbers.

  Example: Consider a complex polynomial $P(z)=az^2+bz+c$.

  Then P has roots $\frac{-b\pm\sqrt{b^2-4ac}}{2a}$, and its derivative P' has root $-\frac{b}{2a}$.

  Hence the root of P' lies midway between the roots of P.

  So the root of P' is in the convex hull of the roots of P.

The Gauss-Lucas Theorem

Theorem (Gauss-Lucas Theorem)

The roots of the derivative of a non-constant complex polynomial belong to the convex hull of the set of roots of the polynomial itself.

- Let P be the complex polynomial defined for complex z by the equation

$$P(z) = a_n z^n + \cdots + a_1 z + a_0,$$

where $n \geq 1$ and $a_0, a_1, \ldots, a_n$ are complex numbers with $a_n \neq 0$. Then

$$P(z) = a_n (z - z_1) \cdots (z - z_n),$$

where $z_1, \ldots, z_n$ are the roots of P, each being repeated according to its multiplicity. A routine verification shows that, for $z \neq z_1, \ldots, z_n$,

$$\frac{P'(z)}{P(z)} = \frac{1}{z - z_1} + \cdots + \frac{1}{z - z_n} = \frac{\overline{z} - \overline{z}_1}{|z - z_1|^2} + \cdots + \frac{\overline{z} - \overline{z}_n}{|z - z_n|^2}.$$

The Gauss-Lucas Theorem (Cont'd)

- Suppose now that z is a root of P'. We establish the theorem by exhibiting z as a convex combination of $z_1, \ldots, z_n$. This can be done trivially if z is one of $z_1, \ldots, z_n$. So assume that this is not the case. Putting $P'(z) = 0$ in the preceding equation, we find easily that

$$z = \frac{\frac{1}{|z-z_1|^2} z_1 + \cdots + \frac{1}{|z-z_n|^2} z_n}{\frac{1}{|z-z_1|^2} + \cdots + \frac{1}{|z-z_n|^2}}.$$

This expresses z as a convex combination of $z_1, \ldots, z_n$.

Corollary

Suppose that the roots of a non-constant complex polynomial lie in some given convex set. Then the roots of its derivative lie in the same convex set.

- A simple application of the corollary:

  If all the roots of a non-constant complex polynomial have positive imaginary parts, then the same is also true of the roots of its derivative.

Subsection 3

Interiors and Closures

Relative Interior of a Nonempty Convex Set

Theorem

The relative interior of a non-empty convex set in $\mathbb{R}^n$ is non-empty.

- Let A be a non-empty r-dimensional convex set in $\mathbb{R}^n$. Then A contains points $\boldsymbol{a}_0,\ldots,\boldsymbol{a}_r$ which form an affine basis for the r-flat $\operatorname{aff} A$, and the barycentric coordinates $\lambda_0,\ldots,\lambda_r$ of a point $\boldsymbol{x}$ of $\operatorname{aff} A$ relative to $\boldsymbol{a}_0,\ldots,\boldsymbol{a}_r$ are continuous functions of $\boldsymbol{x}$, a fact which follows easily from a previous theorem. Let $\boldsymbol{a}=\frac{1}{r+1}(\boldsymbol{a}_0+\cdots+\boldsymbol{a}_r)$. Then $\boldsymbol{a}$ lies in A and its barycentric coordinates $\lambda_0,\ldots,\lambda_r$ are positive, each being $\frac{1}{r+1}$. By the continuity of the barycentric coordinates, for each $i=0,\ldots,r$, there exists $s_i>0$, such that $\lambda_i>0$ whenever $\boldsymbol{x}$ lies in $B(\boldsymbol{a};s_i)\cap\operatorname{aff} A$. Let s be the minimum of $s_0,\ldots,s_r$. So $s>0$. Then if $\boldsymbol{x}$ lies in $B(\boldsymbol{a};s)\cap\operatorname{aff} A$, all its barycentric coordinates $\lambda_0,\ldots,\lambda_r$ are positive. So, since A is convex, $\boldsymbol{x}$ lies in A. Thus, the relative interior of A contains the point $\boldsymbol{a}$.

Convex Sets with Empty Interior

Corollary

A convex set in $\mathbb{R}^n$ has an empty interior if and only if it lies in some hyper plane of $\mathbb{R}^n$.

- Since a hyperplane in $\mathbb{R}^n$ has an empty interior, so does each of its subsets.

  A convex set in $\mathbb{R}^n$ which does not lie in any hyperplane of $\mathbb{R}^n$ must be n-dimensional. Therefore its interior coincides with its relative interior. This relative interior is non-empty by the theorem.

Line Segment Between Relative Interior and Set

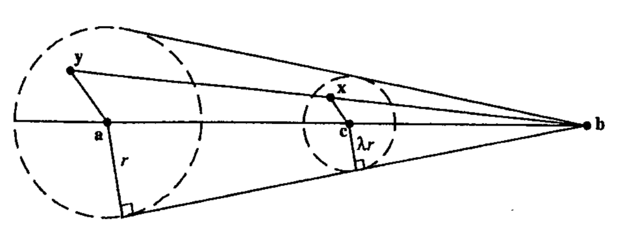
Lemma

Let A be a convex set in $\mathbb{R}^n$. Let $\boldsymbol{a} \in \operatorname{ri} A$ and $\boldsymbol{b} \in A$. Then, for $0 < \lambda \leq 1$,

$$\lambda \boldsymbol{a} + (1 - \lambda)\boldsymbol{b} \in \operatorname{ri} A.$$

- Since $\boldsymbol{a} \in \operatorname{ri} A$, there is an $r > 0$ such that $B(\boldsymbol{a}; r) \cap \operatorname{aff} A \subseteq A$.
  Let $\boldsymbol{c} = \lambda \boldsymbol{a} + (1 - \lambda)\boldsymbol{b}$, where $0 < \lambda \leq 1$.
  We show that $B(\boldsymbol{c}; \lambda r) \cap \operatorname{aff} A \subseteq A$.

Line Segment Between Relative Interior and Set (Cont'd)



- Let $\boldsymbol{x} \in B(\boldsymbol{c};\lambda r) \cap \operatorname{aff} A$. Let

$$\boldsymbol{y} = \boldsymbol{a} + \frac{1}{\lambda}(\boldsymbol{x} - \boldsymbol{c}) = \boldsymbol{a} + \frac{1}{\lambda}(\boldsymbol{x} - \lambda \boldsymbol{a} - (1-\lambda)\boldsymbol{b}) = \frac{1}{\lambda}\boldsymbol{x} + \left(1 - \frac{1}{\lambda}\right)\boldsymbol{b}.$$

Then $\boldsymbol{y} \in \operatorname{aff} A$ and $\|\boldsymbol{y} - \boldsymbol{a}\| = \frac{1}{\lambda}\|\boldsymbol{x} - \boldsymbol{c}\| < r$. Thus, $\boldsymbol{y} \in B(\boldsymbol{a};r) \cap \operatorname{aff} A \subseteq A$. The equation $\boldsymbol{x} = \lambda \boldsymbol{y} + (1-\lambda)\boldsymbol{b}$, together with the convexity of A, shows that $\boldsymbol{x} \in A$. Hence, $B(\boldsymbol{c};\lambda r) \cap \operatorname{aff} A \subseteq A$. So $\boldsymbol{c} \in \operatorname{ri} A$.

A Closure and a Relative Interior Point of a Convex Set

Theorem

Let A be a convex set in $\mathbb{R}^n$. Let $\boldsymbol{a} \in \text{ri}A$ and $\boldsymbol{b} \in \text{cl}A$. Then $\lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b} \in \text{ri}A$ for $0 < \lambda \le 1$.

- Since $\boldsymbol{a} \in \text{ri}A$, there is an $r > 0$ such that $B(\boldsymbol{a}; r) \cap \text{aff}A \subseteq A$. Let $\boldsymbol{c} = \lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b}$, where $0 < \lambda \le 1$. Since $\boldsymbol{b} \in \text{cl}A$, there exists $\boldsymbol{d} \in A$ satisfying $(1-\lambda)\|\boldsymbol{d} - \boldsymbol{b}\| < \lambda r$. Let

$$\begin{aligned} \boldsymbol{e} &= \boldsymbol{a} + \tfrac{1-\lambda}{\lambda}(\boldsymbol{b} - \boldsymbol{d}) = \tfrac{1}{\lambda}(\lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b} - (1-\lambda)\boldsymbol{d}) \\ &= \tfrac{1}{\lambda}(\boldsymbol{c} + (\lambda - 1)\boldsymbol{d}) = \tfrac{1}{\lambda}\boldsymbol{c} + \left(1 - \tfrac{1}{\lambda}\right)\boldsymbol{d}. \end{aligned}$$

  Then $\boldsymbol{e} \in \text{aff}A$ and $\|\boldsymbol{e} - \boldsymbol{a}\| = \frac{1-\lambda}{\lambda}\|\boldsymbol{b} - \boldsymbol{d}\| < r$. Thus $\boldsymbol{e}$ lies in $B(\boldsymbol{a}; r) \cap \text{aff}A$. Hence, it lies in $\text{ri}A$. The equation $\boldsymbol{c} = \lambda\boldsymbol{e} + (1-\lambda)\boldsymbol{d}$, together with the lemma, shows that $\boldsymbol{c} \in \text{ri}A$.

Relative Interior, Interior and Closure of Convex Sets

Theorem

Let A be a convex set in $\mathbb{R}^n$. Then ri$A$, int$A$ and cl$A$ are convex.

- If $\boldsymbol{a}, \boldsymbol{b} \in \text{ri}A$ and $0 \le \lambda \le 1$, then $\lambda \boldsymbol{a} + (1-\lambda)\boldsymbol{b} \in \text{ri}A$, either trivially, if $\lambda = 0$, or by the preceding theorem, otherwise. Thus ri$A$ is convex.

  That int$A$ is convex follows from the fact that either int$A$ is empty or coincides with ri$A$.

  If $\boldsymbol{a}, \boldsymbol{b} \in \text{cl}A$, then there are sequences $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k, \ldots$ and $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_k, \ldots$ of points of A such that $\boldsymbol{a}_k \to \boldsymbol{a}$, $\boldsymbol{b}_k \to \boldsymbol{b}$ as $k \to \infty$. Let $0 \le \lambda \le 1$. Then $\lambda \boldsymbol{a}_k + (1-\lambda)\boldsymbol{b}_k \in A$ for each k, since A is convex. Now

$$\lambda \boldsymbol{a}_k + (1-\lambda)\boldsymbol{b}_k \to \lambda \boldsymbol{a} + (1-\lambda)\boldsymbol{b} \text{ as } k \to \infty.$$

  This shows that $\lambda \boldsymbol{a} + (1-\lambda)\boldsymbol{b} \in \text{cl}A$. Thus cl$A$ is convex.

The Closed Convex Hull of a Set

- The preceding theorem shows that, for any set A in $\mathbb{R}^n$, the set cl(conv$A$) is a closed convex set containing A.
- If B is any closed convex set in $\mathbb{R}^n$ containing A, then

$$B = \text{cl}(\text{conv}B) \supseteq \text{cl}(\text{conv}A).$$

- So cl(conv$A$) is the smallest closed convex set containing A.
- It is called the **closed convex hull** of A.

Characterization of Relative Interior Points

- The next result asserts that a point $\boldsymbol{a}_0$ of a convex set A is a relative interior point of A if and only if every line segment lying in A, and having $\boldsymbol{a}_0$ as an endpoint, can be extended some distance beyond $\boldsymbol{a}_0$ without leaving A.

Theorem

Let $\boldsymbol{a}_0$ be a point of a convex set A in $\mathbb{R}^n$. Then $\boldsymbol{a}_0 \in \text{ri}A$ if and only if, for each $\boldsymbol{a} \in A$, there exists $\mu > 1$ such that $(1-\mu)\boldsymbol{a} + \mu\boldsymbol{a}_0 \in A$.

- Clearly, if $\boldsymbol{a}_0 \in \text{ri}A$, then the condition of the theorem is satisfied.

  Conversely, suppose that $\boldsymbol{a}_0$ satisfies this condition. Let $\boldsymbol{a} \in \text{ri}A$. Then there is $\mu > 1$ such that the point $\boldsymbol{x} = (1-\mu)\boldsymbol{a} + \mu\boldsymbol{a}_0$ lies in A. Hence $\boldsymbol{a}_0 = \lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{x}$, where $0 < \lambda = 1 - \frac{1}{\mu} < 1$. But $\boldsymbol{a} \in \text{ri}A$ and $\boldsymbol{x} \in A$, so $\boldsymbol{a}_0 \in \text{ri}A$ by a previous lemma.

Relative Interior of the Convex Hull of a Finite Set

Theorem

Let $A = \text{conv}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$, where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbb{R}^n$. Then

$$\text{ri}A = \{\lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m : \lambda_1, \ldots, \lambda_m > 0, \lambda_1 + \cdots + \lambda_m = 1\}.$$

- Suppose first that $\boldsymbol{a}_0 = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m$, where $\lambda_1, \ldots, \lambda_m > 0$ and $\lambda_1 + \cdots + \lambda_m = 1$, and that $\boldsymbol{a} \in A$. Then $\boldsymbol{a}_0 \in A$, and $\boldsymbol{a} = \mu_1 \boldsymbol{a}_1 + \cdots + \mu_m \boldsymbol{a}_m$ for some $\mu_1, \ldots, \mu_m \geq 0$ with $\mu_1 + \cdots + \mu_m = 1$. Choose $\mu > 1$ such that

$$\mu\lambda_1 + (1 - \mu)\mu_1 \geq 0, \ldots, \mu\lambda_m + (1 - \mu)\mu_m \geq 0.$$

  Then $(1 - \mu)\boldsymbol{a} + \mu \boldsymbol{a}_0 \in A$. So $\boldsymbol{a}_0 \in \text{ri}A$ by the preceding theorem.

Relative Interior of the Convex Hull of a Finite Set (Cont'd)

- Suppose next that $\boldsymbol{a}_0 \in \operatorname{ri} A$, and that $\boldsymbol{a}^* = \frac{1}{m}(\boldsymbol{a}_1 + \cdots + \boldsymbol{a}_m)$. Then $\boldsymbol{a}^* \in A$. By the preceding theorem, there exist $\mu > 1$ and $\boldsymbol{a} \in A$ such that $\boldsymbol{a} = (1-\mu)\boldsymbol{a}^* + \mu \boldsymbol{a}_0$, say $\boldsymbol{a} = \mu_1 \boldsymbol{a}_1 + \cdots + \mu_m \boldsymbol{a}_m$, where $\mu_1, \ldots, \mu_m \geq 0$ with $\mu_1 + \cdots + \mu_m = 1$. The equation

$$\boldsymbol{a}_0 = \frac{\mu_1 + \frac{\mu-1}{m}}{\mu} \boldsymbol{a}_1 + \cdots + \frac{\mu_m + \frac{\mu-1}{m}}{\mu} \boldsymbol{a}_m$$

now expresses $\boldsymbol{a}_0$ in the form $\lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m$, where $\lambda_1, \ldots, \lambda_m > 0$ with $\lambda_1 + \cdots + \lambda_m = 1$.

Closure and Relative Interior

Theorem

Let A be a convex set in $\mathbb{R}^n$. Then $\mathrm{ri}A = \mathrm{ri}(\mathrm{cl}A)$ and $\mathrm{cl}A = \mathrm{cl}(\mathrm{ri}A)$.

- We assume, throughout, that A is non-empty with $\boldsymbol{a} \in \mathrm{ri}A$.

  The inclusion $\mathrm{ri}A \subseteq \mathrm{ri}(\mathrm{cl}A)$ follows from the inclusion $A \subseteq \mathrm{cl}A$ and the fact that the affine hulls of A and $\mathrm{cl}A$ coincide. To establish the inclusion $\mathrm{ri}(\mathrm{cl}A) \subseteq \mathrm{ri}A$, suppose that $\boldsymbol{b} \in \mathrm{ri}(\mathrm{cl}A)$. By a previous theorem, there exist $\mu > 1$ and $\boldsymbol{c} \in \mathrm{cl}A$ such that $\boldsymbol{c} = (1-\mu)\boldsymbol{a} + \mu\boldsymbol{b}$. Hence $\boldsymbol{b} = (1-\lambda)\boldsymbol{a} + \lambda\boldsymbol{c}$, where $0 < \lambda = \frac{1}{\mu} < 1$. That $\boldsymbol{b} \in \mathrm{ri}A$ follows from a previous theorem. Thus, $\mathrm{ri}(\mathrm{cl}A) \subseteq \mathrm{ri}A$.

  The inclusion $\mathrm{cl}(\mathrm{ri}A) \subseteq \mathrm{cl}A$ is clear. To establish the inclusion $\mathrm{cl}A \subseteq \mathrm{cl}(\mathrm{ri}A)$, suppose that $\boldsymbol{b} \in \mathrm{cl}A$. A previous theorem shows that $\lambda\boldsymbol{a} + (1-\lambda)\boldsymbol{b} \in \mathrm{ri}A$ for $0 < \lambda \leq 1$. Hence, $\boldsymbol{b} \in \mathrm{cl}(\mathrm{ri}A)$. Thus, $\mathrm{cl}A \subseteq \mathrm{cl}(\mathrm{ri}A)$.

Interior, Closure and Boundaries

Corollary

Let A be a convex set in $\mathbb{R}^n$. Then $\text{int}A = \text{int}(\text{cl}A)$ and, when $\text{int}A$ is nonempty, $\text{cl}A = \text{cl}(\text{int}A)$.

- If $\text{int}A$ is non-empty, then $\text{ri}A = \text{int}A$ and $\text{ri}(\text{cl}A) = \text{int}(\text{cl}A)$, and the corollary follows from the theorem.
  If $\text{int}A$ is empty, then A, and hence $\text{cl}A$, lie in a hyperplane of $\mathbb{R}^n$.
  Hence, both $\text{int}A$ and $\text{int}(\text{cl}A)$ are empty.

Corollary

Let A be a convex set in $\mathbb{R}^n$. Then $\text{rebd}A = \text{rebd}(\text{cl}A)$ and $\text{bd}A = \text{bd}(\text{cl}A)$.

- By the theorem and its first corollary,

$$\begin{aligned} \text{rebd}(\text{cl}A) &= \text{cl}(\text{cl}A) \setminus \text{ri}(\text{cl}A) = \text{cl}A \setminus \text{ri}A = \text{rebd}A; \\ \text{bd}(\text{cl}A) &= \text{cl}(\text{cl}A) \setminus \text{int}(\text{cl}A) = \text{cl}A \setminus \text{int}A = \text{bd}A. \end{aligned}$$

Subsection 4

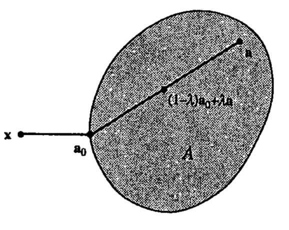
Separation and Support

Uniqueness of Closest Point

Theorem

In $\mathbb{R}^n$ let A be a nonempty closed convex set and let $\boldsymbol{x}$ be a point. Then there exists a unique point $\boldsymbol{a}_0$ of A such that $\|\boldsymbol{x}-\boldsymbol{a}_0\| = \inf\{\|\boldsymbol{x}-\boldsymbol{z}\| : \boldsymbol{z} \in A\}$. Moreover, $(\boldsymbol{x}-\boldsymbol{a}_0)\cdot(\boldsymbol{a}-\boldsymbol{a}_0) \le 0$, for each $\boldsymbol{a}$ in A.

- By a previous theorem, there exists $\boldsymbol{a}_0 \in A$ such that $\|\boldsymbol{x}-\boldsymbol{a}_0\| = \inf\{\|\boldsymbol{x}-\boldsymbol{z}\| : \boldsymbol{z} \in A\}$. Let $\boldsymbol{a} \in A$ and $0 < \lambda \le 1$. The convexity of A shows $(1-\lambda)\boldsymbol{a}_0 + \lambda \boldsymbol{a} \in A$.



The choice of $\boldsymbol{a}_0$ shows that

$$\|\boldsymbol{x}-((1-\lambda)\boldsymbol{a}_0+\lambda\boldsymbol{a})\| = \|(\boldsymbol{x}-\boldsymbol{a}_0)+\lambda(\boldsymbol{a}_0-\boldsymbol{a})\| \ge \|\boldsymbol{x}-\boldsymbol{a}_0\|.$$

We deduce, using a previous theorem that $(\boldsymbol{x}-\boldsymbol{a}_0)\cdot(\boldsymbol{a}-\boldsymbol{a}_0) \le 0$.

Uniqueness of Closest Point (Cont'd)

- Suppose that $\boldsymbol{a}_1 \in A$ also satisfies the equation

$$\|\boldsymbol{x}-\boldsymbol{a}_1\| = \inf\{\|\boldsymbol{x}-\boldsymbol{z}\| : \boldsymbol{z} \in A\}.$$

  Then, by what we have just proved, $(\boldsymbol{x}-\boldsymbol{a}_0)\cdot(\boldsymbol{a}_1-\boldsymbol{a}_0) \le 0$. Because of the symmetry between $\boldsymbol{a}_0$ and $\boldsymbol{a}_1$, we have $(\boldsymbol{x}-\boldsymbol{a}_1)\cdot(\boldsymbol{a}_0-\boldsymbol{a}_1) \le 0$. Adding these last two inequalities together, we deduce that

$$\|\boldsymbol{a}_1-\boldsymbol{a}_0\|^2 = (\boldsymbol{a}_1-\boldsymbol{a}_0)\cdot(\boldsymbol{a}_1-\boldsymbol{a}_0) \le 0.$$

  This proves that $\boldsymbol{a}_0 = \boldsymbol{a}_1$.
- The theorem shows how each non-empty closed convex set A in $\mathbb{R}^n$ gives rise to a mapping $f : \mathbb{R}^n \to A$ defined by $f(\boldsymbol{x}) = \boldsymbol{a}_0$, where $\boldsymbol{a}_0$ is the nearest point of A to a point $\boldsymbol{x}$ of $\mathbb{R}^n$.

  This mapping is called the **projection operator** of A.

Lipschitz Property of Projection

Corollary

Let A be a non-empty closed convex set in $\mathbb{R}^n$. Then the projection operator $f : \mathbb{R}^n \to A$ of A satisfies the Lipschitz condition $\|f(\boldsymbol{x}) - f(\boldsymbol{y})\| \le \|\boldsymbol{x} - \boldsymbol{y}\|$ for all $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. So it is continuous.

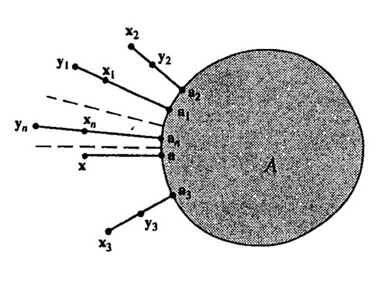
- Let $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$. Write $\boldsymbol{u} = \boldsymbol{x} - f(\boldsymbol{x})$, $\boldsymbol{v} = \boldsymbol{y} - f(\boldsymbol{y})$. Then, by the theorem, $\boldsymbol{u} \cdot (f(\boldsymbol{y}) - f(\boldsymbol{x})) \le 0$ and $\boldsymbol{v} \cdot (f(\boldsymbol{x}) - f(\boldsymbol{y})) \le 0$. So, we get $(\boldsymbol{u} - \boldsymbol{v}) \cdot (f(\boldsymbol{x}) - f(\boldsymbol{y})) \ge 0$. Thus,

$$\begin{array}{rcl}\|\boldsymbol{x} - \boldsymbol{y}\|^2 & = & \|(\boldsymbol{u} - \boldsymbol{v}) + (f(\boldsymbol{x}) - f(\boldsymbol{y}))\|^2 \\ & = & \|\boldsymbol{u} - \boldsymbol{v}\|^2 + 2(\boldsymbol{u} - \boldsymbol{v}) \cdot (f(\boldsymbol{x}) - f(\boldsymbol{y})) + \|f(\boldsymbol{x}) - f(\boldsymbol{y})\|^2 \\ & \ge & \|f(\boldsymbol{x}) - f(\boldsymbol{y})\|^2.\end{array}$$

  So $\|f(\boldsymbol{x}) - f(\boldsymbol{y})\| \le \|\boldsymbol{x} - \boldsymbol{y}\|$.

Geometry of Nearest Points

- Geometrically, the following corollary states that, if $f(\boldsymbol{x})$ is the nearest point of a non-empty closed convex set A in $\mathbb{R}^n$ to a point $\boldsymbol{x}$ of $\mathbb{R}^n$ not belonging to A,



then it is also the nearest point of A to any point on the halfline starting at $f(\boldsymbol{x})$ and passing through $\boldsymbol{x}$.

Nearest Points and Line Segments

Corollary

Let A be a non-empty closed convex set in $\mathbb{R}^n$ with projection operator f. Then, for all $\boldsymbol{x} \in \mathbb{R}^n$ and $\lambda \geq 0$,

$$f(f(\boldsymbol{x}) + \lambda(\boldsymbol{x} - f(\boldsymbol{x}))) = f(\boldsymbol{x}).$$

- Write $\boldsymbol{y} = f(\boldsymbol{x}) + \lambda(\boldsymbol{x} - f(\boldsymbol{x}))$, where $\boldsymbol{x} \in \mathbb{R}^n$, $\lambda \geq 0$. By the theorem, $(\boldsymbol{x} - f(\boldsymbol{x})) \cdot (f(\boldsymbol{y}) - f(\boldsymbol{x})) \leq 0$ and $(\boldsymbol{y} - f(\boldsymbol{y})) \cdot (f(\boldsymbol{x}) - f(\boldsymbol{y})) \leq 0$. From these inequalities, we deduce that

$$\begin{array}{rcl} 0 & \leq & (f(\boldsymbol{y}) - f(\boldsymbol{x})) \cdot (f(\boldsymbol{y}) - f(\boldsymbol{x})) \\ & = & (f(\boldsymbol{y}) + \lambda(\boldsymbol{x} - f(\boldsymbol{x})) - \boldsymbol{y}) \cdot (f(\boldsymbol{y}) - f(\boldsymbol{x})) \\ & = & (f(\boldsymbol{y}) - \boldsymbol{y}) \cdot (f(\boldsymbol{y}) - f(\boldsymbol{x})) + \lambda(\boldsymbol{x} - f(\boldsymbol{x})) \cdot (f(\boldsymbol{y}) - f(\boldsymbol{x})) \\ & \leq & 0. \end{array}$$

Hence, $\|f(\boldsymbol{y}) - f(\boldsymbol{x})\|^2 = 0$. Therefore, $f(\boldsymbol{y}) = f(\boldsymbol{x})$.

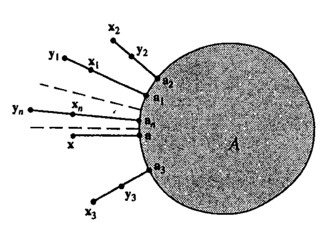
Relative Boundaries and Closest Points

Theorem

Let $\boldsymbol{a}$ be a relative boundary point of a non-empty closed convex set A in $\mathbb{R}^n$ with projection operator f. Then there exists $\boldsymbol{x} \in (\text{aff} A) \backslash A$ such that $f(\boldsymbol{x}) = \boldsymbol{a}$.

- Since $\boldsymbol{a} \in \text{rebd} A$, there exists, for each positive integer m, a point $\boldsymbol{y}_m$ of $(\text{aff} A) \backslash A$ satisfying $\|\boldsymbol{y}_m - \boldsymbol{a}\| \leq \frac{1}{m}$. Write $\boldsymbol{a}_m = f(\boldsymbol{y}_m)$ and

$$\boldsymbol{x}_m = \boldsymbol{a}_m + \frac{\boldsymbol{y}_m - \boldsymbol{a}_m}{\|\boldsymbol{y}_m - \boldsymbol{a}_m\|}.$$



Relative Boundaries and Closest Points (Cont'd)

- Then $\|\boldsymbol{x}_m - \boldsymbol{a}_m\| = 1$ and, by the preceding corollary, $f(\boldsymbol{x}_m) = \boldsymbol{a}_m$. A previous corollary shows that

$$\|\boldsymbol{a}_m - \boldsymbol{a}\| = \|f(\boldsymbol{y}_m) - f(\boldsymbol{a})\| \le \|\boldsymbol{y}_m - \boldsymbol{a}\| \le \frac{1}{m}.$$

So $\boldsymbol{a}_m \to \boldsymbol{a}$ as $m \to \infty$. We have

$$\|\boldsymbol{x}_m\| \le \|\boldsymbol{x}_m - \boldsymbol{a}_m\| + \|\boldsymbol{a}_m - \boldsymbol{a}\| + \|\boldsymbol{a}\| \le 1 + \frac{1}{m} + \|\boldsymbol{a}\|.$$

So the sequence $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ is bounded. Thus $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ contains a convergent subsequence. Assume, without loss of generality, that $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ itself converges to some point $\boldsymbol{x}$ of $\mathbb{R}^n$. Since $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots$ belong to $\operatorname{aff} A$, we can deduce that $\boldsymbol{x} \in \operatorname{aff} A$. The continuity of f shows that $f(\boldsymbol{x}_m) \to f(\boldsymbol{x})$ as $m \to \infty$, i.e., $\boldsymbol{a}_m \to f(\boldsymbol{x})$ as $m \to \infty$. But $\boldsymbol{a}_m \to \boldsymbol{a}$ as $m \to \infty$. Hence, $f(\boldsymbol{x}) = \boldsymbol{a}$. Clearly $\|\boldsymbol{x} - \boldsymbol{a}\| = 1$. So $\boldsymbol{x} \notin A$.

Bounded Sets and Boundaries

Corollary

Let A be a non-empty closed convex set in $\mathbb{R}^n$ with projection operator f and let B be a bounded set in $\mathbb{R}^n$ such that $A \subseteq B$. Then $f(\text{bd}B) = \text{bd}A$.

- We show that $\text{bd}A \subseteq f(\text{bd}B)$, the opposite inclusion being obvious.

  Let $\boldsymbol{a} \in \text{bd}A$. Then there exists $\boldsymbol{x} \in \mathbb{R}^n \backslash A$ such that $f(\boldsymbol{x}) = \boldsymbol{a}$. This can be proved by substituting $\text{bd}A$ for $\text{rebd}A$, and $\mathbb{R}^n$ for $\text{aff}A$, in the proof of the theorem. As B is bounded, there is some $\mu > 0$ such that, for $\lambda \geq \mu$, $\boldsymbol{a} + \lambda(\boldsymbol{x} - \boldsymbol{a}) \notin B$. But $\boldsymbol{a} \in B$. So $\boldsymbol{a} + \lambda_0(\boldsymbol{x} - \boldsymbol{a}) \in \text{bd}B$ for some $\lambda_0 \geq 0$. The preceding corollary shows that $f(\boldsymbol{a} + \lambda_0(\boldsymbol{x} - \boldsymbol{a})) = \boldsymbol{a}$. Hence, $\text{bd}A \subseteq f(\text{bd}B)$.

Separation

- Let A and B be sets, and let H be a hyperplane in $\mathbb{R}^n$.
- Then H is said to **separate** A and B if A lies in one of the closed halfspaces determined by H and B lies in the other.
- H is said to **separate** A and B **properly** if it separates them, but not both A and B lie in H.
- If A and B lie in opposite open halfspaces determined by H, then H is said to **separate** A and B **strictly**.
- It follows from the convexity of halfspaces that, if a hyperplane separates two sets, then it also separates their convex hulls.

  For this reason, we consider only the separation of convex sets.

Examples

- It is not always possible to separate two convex sets by a hyperplane.

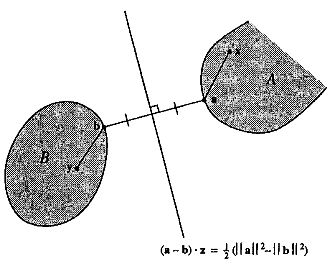
  For example, there is no line separating the set $\{0\}$ and the closed unit disc $\{(x,y) : x^2 + y^2 \leq 1\}$ in $\mathbb{R}^2$.
- Any two sets that can be strictly separated can be properly separated, unless they are both empty.
- The convex sets $\{(x,y) : x \leq 0\}$ and $\{(x,y) : x > 0, y \geq \frac{1}{x}\}$ in $\mathbb{R}^2$ cannot be strictly separated, but they are properly separated by the y-axis.
- A hyperplane in $\mathbb{R}^n$ separates any two of its subsets, but does not separate them properly.

Strict Separation of a Closed and a Compact Set

Theorem

Let A and B be disjoint non-empty convex sets in $\mathbb{R}^n$ with A closed and B compact. Then A and B can be strictly separated by a hyperplane in $\mathbb{R}^n$.

- The geometry of the proof is as follows. Let $\boldsymbol{a}$ and $\boldsymbol{b}$ be nearest points of A and B. Then the hyperplane through the midpoint of the line segment joining $\boldsymbol{a}$ and $\boldsymbol{b}$ with normal vector $\boldsymbol{a}-\boldsymbol{b}$ strictly separates A and B.



Strict Separation of a Closed and a Compact Set (Cont'd)

- Let $\boldsymbol{a} \in A$, $\boldsymbol{b} \in B$ be such that $\boldsymbol{a}$ is the nearest point of A to $\boldsymbol{b}$, and $\boldsymbol{b}$ is the nearest point of $\boldsymbol{B}$ to $\boldsymbol{a}$. This is possible by a previous theorem. Since A and B are disjoint, $\boldsymbol{a} \neq \boldsymbol{b}$. Let $\boldsymbol{x} \in A$, $\boldsymbol{y} \in B$. Then, by a previous theorem, $(\boldsymbol{b}-\boldsymbol{a})\cdot(\boldsymbol{x}-\boldsymbol{a}) \leq 0$ and $(\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{y}-\boldsymbol{b}) \leq 0$. Thus,

$$\begin{array}{rcl}(\boldsymbol{a}-\boldsymbol{b})\cdot\boldsymbol{x} & \geq & (\boldsymbol{a}-\boldsymbol{b})\cdot\boldsymbol{a} \\ & = & \frac{1}{2}(\|\boldsymbol{a}\|^2-\|\boldsymbol{b}\|^2+\|\boldsymbol{a}-\boldsymbol{b}\|^2) \\ & > & \frac{1}{2}(\|\boldsymbol{a}\|^2-\|\boldsymbol{b}\|^2) \\ & > & \frac{1}{2}(\|\boldsymbol{a}\|^2-\|\boldsymbol{b}\|^2-\|\boldsymbol{a}-\boldsymbol{b}\|^2) \\ & = & (\boldsymbol{a}-\boldsymbol{b})\cdot\boldsymbol{b} \\ & \geq & (\boldsymbol{a}-\boldsymbol{b})\cdot\boldsymbol{y}.\end{array}$$

  Write $\boldsymbol{c}=\boldsymbol{a}-\boldsymbol{b}$ and $c_0=\frac{1}{2}(\|\boldsymbol{a}\|^2-\|\boldsymbol{b}\|^2)$. Then we have shown that the hyperplane $\boldsymbol{c}\cdot\boldsymbol{z}=c_0$ strictly separates A and B.

Consequences

Corollary

In $\mathbb{R}^n$ let A be a closed convex set and let $\boldsymbol{b}$ be a point not lying in A. Then A and $\{\boldsymbol{b}\}$ can be strictly separated by a hyperplane in $\mathbb{R}^n$.

Corollary

Each closed convex set A in $\mathbb{R}^n$ is the intersection of all the closed halfspaces in $\mathbb{R}^n$ containing A.

- Denote by B the intersection of all the closed haifspaces in $\mathbb{R}^n$ containing A. Then B is a closed convex set containing A. If $\boldsymbol{b} \notin A$, then the corollary above shows that there exists some closed halfspace in $\mathbb{R}^n$ which contains A but not $\boldsymbol{b}$. Hence $\boldsymbol{b} \notin B$. Thus $B \subseteq A$, and $A = B$.

Convex Sets Not Containing the Origin

Lemma

In $\mathbb{R}^n$ let A be a non-empty convex set not containing the origin. Then there exists a hyperplane in $\mathbb{R}^n$ which separates the origin and A, and does not contain A.

- Suppose first that $\mathbf{0} \notin \text{cl}A$. Then the lemma follows from a previous corollary applied to the closed convex set $\text{cl}A$. Suppose next that $\mathbf{0} \in \text{cl}A$. Then $\mathbf{0} \in \text{rebd}A$. By a previous corollary, $\text{rebd}A = \text{rebd}(\text{cl}A)$. So $\mathbf{0} \in \text{rebd}(\text{cl}A)$. A previous theorem asserts the existence of a point $\boldsymbol{x}$ of $(\text{aff}(\text{cl}A)) \setminus \text{cl}A$ whose nearest point in $\text{cl}A$ is $\mathbf{0}$. By a previous theorem, $\text{cl}A \subseteq \{\boldsymbol{z} \in \mathbb{R}^n : \boldsymbol{x} \cdot \boldsymbol{z} \leq 0\}$. This shows that the hyperplane $H = \{\boldsymbol{z} \in \mathbb{R}^n : \boldsymbol{x} \cdot \boldsymbol{z} = 0\}$ separates $\mathbf{0}$ and A.

  We cannot have $A \subseteq H$: This would imply that $\boldsymbol{x} \in \text{aff}(\text{cl}A) \subseteq H$. But, this is impossible since $\boldsymbol{x} \cdot \boldsymbol{x} > 0$. Thus, H separates $\{\mathbf{0}\}$ and A, and does not contain A.

Disjoint Nonempty Convex Sets

Theorem

Each pair of disjoint non-empty convex sets A and B in $\mathbb{R}^n$ can be properly separated by a hyperplane in $\mathbb{R}^n$.

- The non-empty convex set $A-B$ does not contain the origin.
  By the lemma, there exists a hyperplane in $\mathbb{R}^n$ which separates $\{\mathbf{0}\}$ and $A-B$, and that does not contain $A-B$. Thus, there exist $\boldsymbol{c}\in\mathbb{R}^n$ with $\boldsymbol{c}\neq\mathbf{0}$ and $c_0\in\mathbb{R}$ such that

  $$0=\boldsymbol{c}\cdot\mathbf{0}\leq c_0 \quad \text{and} \quad \boldsymbol{c}\cdot(\boldsymbol{a}-\boldsymbol{b})\geq c_0, \quad \text{for } \boldsymbol{a}\in A,\ \boldsymbol{b}\in B.$$

  Also, for some $\boldsymbol{a}_0\in A$, $\boldsymbol{b}_0\in B$, we have $\boldsymbol{c}\cdot(\boldsymbol{a}_0-\boldsymbol{b}_0)>c_0\geq 0$.
  For every $\boldsymbol{a}\in A$, $\boldsymbol{b}\in B$,

  $$\boldsymbol{c}\cdot\boldsymbol{a}\geq\boldsymbol{c}\cdot\boldsymbol{b}+c_0\geq\boldsymbol{c}\cdot\boldsymbol{b}.$$

Disjoint Nonempty Convex Sets (Cont'd)

- Thus there is a scalar d satisfying the inequalities

$$\inf\{\boldsymbol{c}\cdot\boldsymbol{a}:\boldsymbol{a}\in A\}\geq d\geq\sup\{\boldsymbol{c}\cdot\boldsymbol{b}:\boldsymbol{b}\in B\}.$$

For any $\boldsymbol{a}'\in A$, $\boldsymbol{b}'\in B$, we have $\boldsymbol{c}\cdot\boldsymbol{a}'\geq d\geq\boldsymbol{c}\cdot\boldsymbol{b}'$. So the hyperplane H with equation $\boldsymbol{c}\cdot\boldsymbol{z}=d$ separates A and B.

H cannot contain both A and B, for this would imply that $\boldsymbol{c}\cdot(\boldsymbol{a}_0-\boldsymbol{b}_0)=0$, which contradicts the inequality above. Thus H separates A and B properly.

Convex Sets With Disjoint Relative Interiors

Corollary

Each pair of non-empty convex sets A and B in $\mathbb{R}^n$ whose relative interiors are disjoint can be properly separated by a hyperplane in $\mathbb{R}^n$.

- The non-empty convex sets $\text{ri}A$ and $\text{ri}B$ are disjoint. So, by the theorem, there exists a hyperplane H in $\mathbb{R}^n$ which properly separates them. Since closed halfspaces are closed, H also properly separates $\text{cl}(\text{ri}A) = \text{cl}A$ and $\text{cl}(\text{ri}B) = \text{cl}B$. Hence, it also properly separates A and B.

Support Hyperplanes

- In $\mathbb{R}^n$ a hyperplane H is called a **support hyperplane** to a set A if H meets cl$A$ and A lies in one of the closed halfspaces determined by H.
- Such a hyperplane H is said to **support** A at those points where H meets cl$A$.
- A hyperplane H cannot support a set A at an interior point of A, because every ball with center in H meets both the open halfspaces determined by H.
- A hyperplane in $\mathbb{R}^n$ is a **trivial support hyperplane** to each of its non-empty subsets.
- A support hyperplane to a set in $\mathbb{R}^n$ is said to be a **non-trivial support hyperplane** to the set if it does not contain the set itself.

Boundary Points and Support Hyperplanes

Theorem

Through each boundary point of a convex set A in $\mathbb{R}^n$ there passes a support hyperplane to A, and through each relative boundary point of A there passes a non-trivial support hyperplane to A.

- Suppose first that $\boldsymbol{a}$ is a boundary point of A, but not a relative boundary point of A. Then A cannot be n-dimensional. So it lies in some hyperplane H of $\mathbb{R}^n$. Clearly H is a support hyperplane to A passing through $\boldsymbol{a}$.

Boundary Points and Support Hyperplanes (Cont'd)

- Suppose next that $\boldsymbol{a}$ is a relative boundary point of A.

  The preceding theorem shows the existence of a hyperplane H which properly separates $\{\boldsymbol{a}\}$ and $\mathrm{ri}A$.

  H cannot contain $\mathrm{ri}A$, for this would imply that H also contains $\mathrm{cl}(\mathrm{ri}A) = \mathrm{cl}A$, and hence $\boldsymbol{a}$.

  By the definition of separation, $\{\boldsymbol{a}\}$ and $\mathrm{ri}A$, and thus $\{\boldsymbol{a}\}$ and $\mathrm{cl}(\mathrm{ri}A) = \mathrm{cl}A$, belong to opposite closed halfspaces determined by H.

  Since $\boldsymbol{a} \in \mathrm{cl}A$, we must have $\boldsymbol{a} \in H$. Thus H is a non-trivial support hyperplane to A passing through $\boldsymbol{a}$.

Example

- Let H be a support hyperplane to a closed ball $B[\boldsymbol{a};r]$ in $\mathbb{R}^n$ at some point $\boldsymbol{c}$.
- Since H does not meet the interior of the ball, every point of H must be a distance of at least r from $\boldsymbol{a}$.
- Hence $\boldsymbol{c}$ must be a nearest point of H to $\boldsymbol{a}$.
- By the uniqueness of nearest points of convex sets, H can meet the ball only in the point $\boldsymbol{c}$.
- A previous theorem shows that $(\boldsymbol{h}-\boldsymbol{c})\cdot(\boldsymbol{a}-\boldsymbol{c})\leq 0$ for all $\boldsymbol{h}$ in H.
- We cannot have $(\boldsymbol{h}-\boldsymbol{c})\cdot(\boldsymbol{a}-\boldsymbol{c})<0$ for some $\boldsymbol{h}$ in H.
  This would imply that the point $\boldsymbol{h}'=2\boldsymbol{c}-\boldsymbol{h}$ of H satisfies
  $$(\boldsymbol{h}'-\boldsymbol{c})\cdot(\boldsymbol{a}-\boldsymbol{c})=(2\boldsymbol{c}-\boldsymbol{h}-\boldsymbol{c})\cdot(\boldsymbol{a}-\boldsymbol{c})=(\boldsymbol{c}-\boldsymbol{h})\cdot(\boldsymbol{a}-\boldsymbol{c})>0.$$
  But this is impossible.
- Thus H must be the hyperplane with equation $(\boldsymbol{x}-\boldsymbol{c})\cdot(\boldsymbol{a}-\boldsymbol{c})=0$.
- So H is the unique support hyperplane to $B[\boldsymbol{a};r]$ at $\boldsymbol{c}$.

Distance Between a Point and a Hyperplane

- We conclude this section by establishing a formula for the distance between a point $\boldsymbol{a}$ and a hyperplane H with equation $\boldsymbol{c} \cdot \boldsymbol{x} = c_0$ in $\mathbb{R}^n$.
- Denote by $\boldsymbol{a}_0$ the point defined by the equation

$$\boldsymbol{a}_0 = \boldsymbol{a} + \frac{c_0 - \boldsymbol{c} \cdot \boldsymbol{a}}{\|\boldsymbol{c}\|^2} \boldsymbol{c}.$$

- Then $\boldsymbol{a}_0$ lies in H, and for any $\boldsymbol{x}$ in H, we have

$$\|\boldsymbol{a} - \boldsymbol{x}\|^2 = \|(\boldsymbol{a} - \boldsymbol{a}_0) + (\boldsymbol{a}_0 - \boldsymbol{x})\|^2 = \|\boldsymbol{a} - \boldsymbol{a}_0\|^2 + \|\boldsymbol{a}_0 - \boldsymbol{x}\|^2.$$

  This shows that $\boldsymbol{a}_0$ is the unique nearest point of H to $\boldsymbol{a}$, and that the (shortest) distance between $\boldsymbol{a}$ and H is $\|\boldsymbol{a} - \boldsymbol{a}_0\| = \frac{|\boldsymbol{c} \cdot \boldsymbol{a} - c_0|}{\|\boldsymbol{c}\|}$.
- When $\boldsymbol{c}$ is a unit vector and $\boldsymbol{a}$ is the origin, this distance becomes $|c_0|$.
- The (shortest) distance between parallel hyperplanes $\boldsymbol{c} \cdot \boldsymbol{x} = c_0$ and $\boldsymbol{c} \cdot \boldsymbol{x} = d_0$ is $\frac{|d_0 - c_0|}{\|\boldsymbol{c}\|}$ which becomes $|d_0 - c_0|$ when $\boldsymbol{c}$ is a unit vector.

Subsection 5

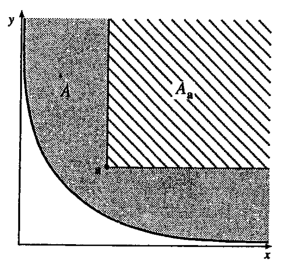
Unbounded Convex Sets

Example: Recession Cone

- Let A be the closed unbounded convex set in $\mathbb{R}^2$ that is defined by the equation

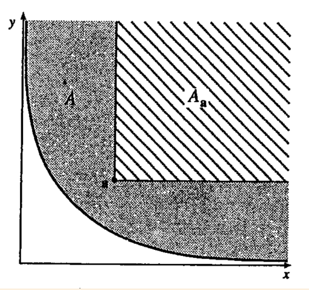
$$A = \{(x,y) : y \geq \frac{1}{x}, x > 0\}$$

and let $\boldsymbol{a} \in A$.



- Then there are halflines starting at $\boldsymbol{a}$ which are contained in A.
- If we denote by $A_{\boldsymbol{a}}$ the union of all these halflines, then $A_{\boldsymbol{a}}$ is a closed convex set that is a union of halflines starting at $\boldsymbol{a}$.
- Such a set $A_{\boldsymbol{a}}$ is called a **closed convex cone with apex $\boldsymbol{a}$**.
- In fact, $A_{\boldsymbol{a}} = \boldsymbol{a} + P$, where P is the non-negative quadrant $\{(x,y) : x \geq 0, y \geq 0\}$ of $\mathbb{R}^2$.

Recession Cone



- $A_{\boldsymbol{a}} = \boldsymbol{a} + P$, where $P = \{(x, y) : x \geq 0, y \geq 0\}$.
- The important observation here is that P is determined by the set A alone, being independent of the initial choice of the point $\boldsymbol{a}$ in A.
- We refer to P as the **recession cone** of A.
- Roughly speaking, the recession cone of a convex set indicates in which directions the set recedes to infinity.

Halflines

- A **halfline** L^+ in $\mathbb{R}^n$ is a set of the form $\{\boldsymbol{x}_0 + \lambda \boldsymbol{y} : \lambda \geq 0\}$, where $\boldsymbol{x}_0, \boldsymbol{y} \in \mathbb{R}^n$ and $\boldsymbol{y} \neq \boldsymbol{0}$.
- The reason for this is that the line joining the points $\boldsymbol{x}_0$ and $\boldsymbol{x}_0 + \boldsymbol{y}$ is the set

$$\{(1-\lambda)\boldsymbol{x}_0 + \lambda(\boldsymbol{x}_0 + \boldsymbol{y}) : \lambda \in \mathbb{R}\} = \{\boldsymbol{x}_0 + \lambda \boldsymbol{y} : \lambda \in \mathbb{R}\}.$$

- A halfline L_0^+ of the form $\{\lambda \boldsymbol{y} : \lambda \geq 0\}$, where $\boldsymbol{y} \in \mathbb{R}^n$ and $\boldsymbol{y} \neq \boldsymbol{0}$, is called a **ray**.
- The equation $L^+ = \boldsymbol{x}_0 + L_0^+$ expresses L^+ as a translate of the ray L_0^+.
- Since $\boldsymbol{x}_0$ is the only point of L^+ whose removal from L^+ leaves a convex set, and $L_0^+ = L^+ - \boldsymbol{x}_0$, it follows that $\boldsymbol{x}_0$ and L_0^+ are uniquely determined by L^+.
- L^+ is the **halfline with direction L_0^+ and initial point $\boldsymbol{x}_0$**.
- The word **direction** will be used as a synonym for ray.

Closed Unbounded Convex Sets and Halflines

Theorem

Let A be a closed unbounded convex set in $\mathbb{R}^n$. Then A contains a halfline. Moreover, if A contains some halfline with direction L_0^+, then it contains every halfline with direction L_0^+ whose initial point is in A.

- Since A is unbounded, it contains a sequence $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k, \ldots$ of non-zero vectors such that $\|\boldsymbol{a}_k\| \to \infty$ as $k \to \infty$. Let $\lambda_k = \frac{1}{\|\boldsymbol{a}_k\|}$. Then the sequence $\lambda_1 \boldsymbol{a}_1, \ldots, \lambda_k \boldsymbol{a}_k, \ldots$ lies in the compact set $\{\boldsymbol{x} : \|\boldsymbol{x}\| = 1\}$. So it contains some subsequence converging to a point $\boldsymbol{a}$ with $\|\boldsymbol{a}\| = 1$. We may suppose that the sequence itself converges to $\boldsymbol{a}$. Let L^+ be the direction $\{\lambda \boldsymbol{a} : \lambda \geq 0\}$. Then we show that $\boldsymbol{a}_0 + L^+ \subseteq A$ for every $\boldsymbol{a}_0$ in A.

Closed Unbounded Convex Sets and Halflines (Cont'd)

- Let $\boldsymbol{a}_0 \in A$ and let $\lambda \geq 0$. Since $\lambda_k \to 0$ as $k \to \infty$, we must have, for all but a finite number of k's, $0 \leq \lambda\lambda_k \leq 1$ and $(1-\lambda\lambda_k)\boldsymbol{a}_0 + \lambda\lambda_k \boldsymbol{a}_k \in A$. Clearly,

$$(1-\lambda\lambda_k)\boldsymbol{a}_0 + \lambda\lambda_k \boldsymbol{a}_k \to \boldsymbol{a}_0 + \lambda\boldsymbol{a}, \text{ as } k \to \infty.$$

So $\boldsymbol{a}_0 + \lambda\boldsymbol{a} \in A$, since A is closed. Thus the halfline $\boldsymbol{a}_0 + L^+$ is contained in A.

Suppose next that A contains the halfline $\boldsymbol{b}_0 + L_0^+$, where L_0^+ is the direction $\{\lambda\boldsymbol{b} : \lambda \geq 0\}$ for some $\boldsymbol{b} \neq \mathbf{0}$. We show that every halfline $\boldsymbol{c}_0 + L_0^+$, where $\boldsymbol{c}_0 \in A$, is contained in A. Let $\mu \geq 0$. Then, for all $\lambda > \mu$,

$$\left(1 - \frac{\mu}{\lambda}\right)\boldsymbol{c}_0 + \frac{\mu}{\lambda}(\boldsymbol{b}_0 + \lambda\boldsymbol{b}) \in A.$$

Letting $\lambda \to \infty$ in this last relation and using the fact that A is closed, we deduce that $\boldsymbol{c}_0 + \mu\boldsymbol{b} \in A$. Thus $\boldsymbol{c}_0 + L_0^+ \subseteq A$.

Relative Interior, Closure and Halflines

Corollary

Let A be an unbounded convex set in $\mathbb{R}^n$. Then ri$A$ contains a halfline. Moreover, if cl$A$ contains some halfline with direction L_0^+, then ri$A$ contains every halfline with direction L_0^+ whose initial point is in ri$A$.

- We apply the theorem to the closed unbounded convex set cl$A$.

  Suppose that cl$A$ contains the halfline $\boldsymbol{b}_0 + L_0^+$, where L_0^+ is the direction $\{\lambda \boldsymbol{b} : \lambda \geq 0\}$. Let $\boldsymbol{a}_0 \in \text{ri}A$ and let $\mu \geq 0$. Then $\boldsymbol{a}_0 + 2\mu\boldsymbol{b} \in \text{cl}A$. So $\boldsymbol{a}_0 + \mu\boldsymbol{b} = \frac{1}{2}\boldsymbol{a}_0 + \frac{1}{2}(\boldsymbol{a}_0 + 2\mu\boldsymbol{b}) \in \text{ri}A$, by a previous theorem. Thus $\boldsymbol{a}_0 + L_0^+ \subseteq \text{ri}A$.

Cones

- A nonempty set A in $\mathbb{R}^n$ is called a **cone** if $\lambda \boldsymbol{a} \in A$ whenever $\boldsymbol{a} \in A$ and $\lambda \geq 0$.
- Examples of cones are:
  - subspaces;
  - rays;
  - the **nonnegative orthant**

$$\{(x_1, \ldots, x_n) : x_1 \geq 0, \ldots, x_n \geq 0\}$$

    of $\mathbb{R}^n$.
- All cones contain the origin and are, with the exception of the trivial cone $\{\mathbf{0}\}$, unbounded.
- Cones need not be convex.

  The set $\{(x, y) : xy \geq 0\}$ is a non-convex cone in $\mathbb{R}^2$.

Characterization of Convex Cones

Theorem

Let A be a non-empty set in $\mathbb{R}^n$. Then A is a convex cone if and only if $\lambda \boldsymbol{a} + \mu \boldsymbol{b} \in A$ whenever $\boldsymbol{a}, \boldsymbol{b} \in A$ and $\lambda, \mu \geq 0$.

- Let A be a convex cone. Suppose that $\boldsymbol{a}, \boldsymbol{b} \in A$ and $\lambda, \mu \geq 0$. If $\lambda + \mu = 0$, then $\lambda = \mu = 0$ and trivially $\lambda \boldsymbol{a} + \mu \boldsymbol{b} \in A$. If $\lambda + \mu > 0$, then

$$\frac{\lambda}{\lambda + \mu} \boldsymbol{a} + \frac{\mu}{\lambda + \mu} \boldsymbol{b} \in A,$$

since A is convex. Hence

$$\lambda \boldsymbol{a} + \mu \boldsymbol{b} = (\lambda + \mu) \left(\frac{\lambda}{\lambda + \mu} \boldsymbol{a} + \frac{\mu}{\lambda + \mu} \boldsymbol{b} \right) \in A,$$

since A is a cone. Thus, in all cases, $\lambda \boldsymbol{a} + \mu \boldsymbol{b} \in A$.

Characterization of Convex Cones (Cont'd)

- Suppose next that $\lambda \boldsymbol{a} + \mu \boldsymbol{b} \in A$ whenever $\boldsymbol{a}, \boldsymbol{b} \in A$ and $\lambda, \mu \geq 0$.

  Clearly A is convex.

  To show that A is a cone, let $\boldsymbol{a} \in A$ and $\lambda \geq 0$. Then, by our hypothesis, $\lambda \boldsymbol{a} = \lambda \boldsymbol{a} + \boldsymbol{0} \in A$.

Corollary

Let A be a non-empty set in $\mathbb{R}^n$. Then A is a convex cone if and only if $\boldsymbol{a} + \boldsymbol{b} \in A$ and $\lambda \boldsymbol{a} \in A$ whenever $\boldsymbol{a}, \boldsymbol{b} \in A$ and $\lambda \geq 0$.

Convex Cone Generated by a Set

- It is a routine matter to show that the intersection of any family of convex cones in $\mathbb{R}^n$ is a convex cone.
- Hence cone$A$, defined as the intersection of all convex cones containing a set A in $\mathbb{R}^n$, is a convex cone.
- It is called the **convex cone generated by** A.
- Clearly cone$A$ is the smallest convex cone containing A.
- We note that cone$\emptyset = \{\mathbf{0}\}$.
- We now characterize cone$A$, in the case when A is non-empty, as the set of all **nonnegative linear combinations** of points of A,

  i.e., points of the form

$$\lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m,$$

  where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in A$ and $\lambda_1, \ldots, \lambda_m \geq 0$.

Characterization of Generated Convex Cones

Theorem

Let A be a nonempty set in $\mathbb{R}^n$. Then cone$A$ is the set of all non-negative linear combinations of points of A.

- Denote by B the set of all nonnegative linear combinations of points of A. Let $\boldsymbol{x} \in B$. Then

$$\boldsymbol{x} = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m, \text{ for some } \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in A \text{ and } \lambda_1, \ldots, \lambda_m \geq 0.$$

Then $\boldsymbol{x} \in \text{cone}A$ by repeated use of the corollary to cone$A$. Hence $B \subseteq \text{cone}A$. The corollary shows that B is a convex cone. Clearly $A \subseteq B$. So $\text{cone}A \subseteq B$. Thus, $B = \text{cone}A$.

Recession Cone

- A nonempty convex set A in $\mathbb{R}^n$ is said to **recede in a direction** L_0^+, or to have a **direction of recession** L_0^+, if every halfline with initial point in A and direction L_0^+ lies in A, i.e., if $A + L_0^+ \subseteq A$.
- The union of all directions of recession of A, together with the zero vector, is called the **recession cone** of A.
- A previous theorem shows that a nonempty closed convex set in $\mathbb{R}^n$ is bounded if and only if its recession cone consists of the zero vector alone.
- The recession cone of a non-empty flat is the unique subspace which is parallel to it.
- The set $\{(x,y) : x > 0, y > 0\} \cup \{(0,0)\}$ is its own recession cone.

  It is an example of a set whose recession cone is not closed.

Characterization of Recession Cone

Theorem

Let A be a non-empty convex set in $\mathbb{R}^n$. Then the recession cone of A consists of all those points $\boldsymbol{x}$ such that $A+\boldsymbol{x}\subseteq A$. Moreover, the recession cone of A is a convex cone, which is closed when A is closed.

- If $\boldsymbol{x}$ belongs to the recession cone of A, then trivially $A+\boldsymbol{x}\subseteq A$. Conversely, if $A+\boldsymbol{x}\subseteq A$, then

$$A+2\boldsymbol{x}=(A+\boldsymbol{x})+\boldsymbol{x}\subseteq A+\boldsymbol{x}\subseteq A.$$

  By repeated application of this argument, $A+m\boldsymbol{x}\subseteq A$ for each positive integer m. But A is convex. So $A+\lambda\boldsymbol{x}\subseteq A$, for all $\lambda\geq 0$. Hence $\boldsymbol{x}$ lies in the recession cone of A.

Characterization of Recession Cone (Cont'd)

- Denote by C the recession cone of A. Let $\boldsymbol{x}, \boldsymbol{y} \in C$ and $\lambda, \mu \geq 0$. Then

$$A + \lambda \boldsymbol{x} + \mu \boldsymbol{y} = (A + \lambda \boldsymbol{x}) + \mu \boldsymbol{y} \subseteq A + \mu \boldsymbol{y} \subseteq A.$$

  So $\lambda \boldsymbol{x} + \mu \boldsymbol{y} \in C$. Hence C is a convex cone by a previous theorem.

  Suppose now that A is closed. Let $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_k, \ldots$ be a sequence of points of the recession cone C that converges to some point $\boldsymbol{x}$ of $\mathbb{R}^n$. Then $\boldsymbol{a} + \boldsymbol{x}_k \in A$ for each k and for each point $\boldsymbol{a}$ of A. But A is closed, so $\boldsymbol{a} + \boldsymbol{x} \in A$ for each point $\boldsymbol{a}$ of A. I.e., $A + \boldsymbol{x} \subseteq A$. Thus, $\boldsymbol{x} \in C$. This shows that C is closed.

Direction of a Line

- Let L be a line in $\mathbb{R}^n$.
- Then by the **direction** of L is meant the *unique* line L_0 in $\mathbb{R}^n$, which is parallel to L and passes through the origin.
- A line is uniquely determined by specifying one of its points and giving its direction.

  Indeed, if $\boldsymbol{x}$ lies on a line L in $\mathbb{R}^n$, then:
  - The direction L_0 of L is simply the line $L - \boldsymbol{x}$;
  - $L = \boldsymbol{x} + L_0$.

Lineality Space

- Let L_0 be a line in $\mathbb{R}^n$ that passes through the origin.
- Then a nonempty convex set A in $\mathbb{R}^n$ is said to be **linear in the direction** L_0, or to have a **direction of linearity** L_0, if every line meeting A which has direction L_0 lies in A, i.e., if $A + L_0 \subseteq A$.
- The union of all the directions of linearity of A, together with the zero vector, is called the **lineality space** of A.
- A previous theorem shows that, if a closed convex set A contains a line with direction L_0, then it contains every line with direction L_0 which meets A, i.e., L_0 is a direction of linearity of A.

Lineality Space: Examples

- A non-empty closed convex set contains a line if and only if its lineality space does not consist of the zero vector alone.
- The lineality space of a non-empty flat is the unique subspace which is parallel to it.
- The lineality space of the unbounded circular cylinder

$$\{(x,y,z) : x^2 + y^2 \leq 1\}$$

is the subspace $\{(0,0,z) : z \in \mathbb{R}\}$, i.e., the z-axis.

Characterization of Lineality Space

Theorem

Let A be a non-empty convex set in $\mathbb{R}^n$. Then the lineality space of A consists of all those points $\boldsymbol{x}$ of $\mathbb{R}^n$ such that $A+\boldsymbol{x}=A$, and is a subspace of $\mathbb{R}^n$.

- If $\boldsymbol{x}$ belongs to the lineality space of A, then trivially $A+\boldsymbol{x}\subseteq A$ and $A-\boldsymbol{x}\subseteq A$. Hence, $A+\boldsymbol{x}=A$.

  Conversely, if $A+\boldsymbol{x}=A$, then, as in the proof of the preceding theorem, $A+m\boldsymbol{x}=A$, for each positive integer m. If m is a negative integer, then $A+m\boldsymbol{x}=(A+(-m)\boldsymbol{x})+m\boldsymbol{x}=A$. Hence $A+m\boldsymbol{x}=A$, for all integers m. But A is convex, so $A+\lambda\boldsymbol{x}=A$ for all real λ. Thus $\boldsymbol{x}$ belongs to the lineality space of A.

  Let S be the lineality space of A. Let $\boldsymbol{x},\boldsymbol{y}\in S$ and $\lambda,\mu\in\mathbb{R}$. Then trivially $A+\lambda\boldsymbol{x}=A$ and $A+\mu\boldsymbol{y}=A$. Thus

  $$A+\lambda\boldsymbol{x}+\mu\boldsymbol{y}=(A+\lambda\boldsymbol{x})+\mu\boldsymbol{y}=A+\mu\boldsymbol{y}=A.$$

  So $\lambda\boldsymbol{x}+\mu\boldsymbol{y}\in S$. This shows that S is a subspace of $\mathbb{R}^n$.

Decomposition of a Closed Convex Set

Theorem

Let A be a non-empty closed convex set in $\mathbb{R}^n$ with lineality space S. Then

$$A = S + (A \cap S^\perp),$$

and the convex set $A \cap S^\perp$ contains no lines.

- Let $\boldsymbol{a} \in A$. Then $\boldsymbol{a}$ can be expressed uniquely in the form $\boldsymbol{a} = \boldsymbol{b} + \boldsymbol{c}$, where $\boldsymbol{b} \in S$ and $\boldsymbol{c} \in S^\perp$. Since $\boldsymbol{b} \in S$, $-\boldsymbol{b} \in S$. Hence, by the preceding theorem, $\boldsymbol{c} = \boldsymbol{a} - \boldsymbol{b} \in A$. Thus $\boldsymbol{c} \in A \cap S^\perp$. So $A \subseteq S + (A \cap S^\perp)$. The opposite inclusion follows immediately from the preceding theorem. Thus $A = S + (A \cap S^\perp)$.

  Suppose that $A \cap S^\perp$ does contain a line. Then there exist $\boldsymbol{x}, \boldsymbol{y}$ in $\mathbb{R}^n$ with $\boldsymbol{y} \neq \boldsymbol{0}$ such that $\boldsymbol{x} + \lambda \boldsymbol{y} \in A \cap S^\perp$ for all real λ. A previous theorem shows that $\boldsymbol{y} \in S$. Hence, for all real λ, $(\boldsymbol{x} + \lambda \boldsymbol{y}) \cdot \boldsymbol{y} = \boldsymbol{x} \cdot \boldsymbol{y} + \lambda \|\boldsymbol{y}\|^2 = 0$, which is clearly impossible. Thus $A \cap S^\perp$ contains no lines.

Subsection 6

Facial Structure

Faces of a Convex Set

- Each face F of a three-dimensional convex polyhedron P is a convex subset of P with the property that whenever the relative interior of a line segment L lying in P meets F, then the endpoints of L lie in F.
- This observation motivates the definition of a face of a general convex set.
- A **face** of a convex set A in $\mathbb{R}^n$ is a convex subset B of A such that whenever $\lambda \boldsymbol{x} + \mu \boldsymbol{y} \in B$, where $\boldsymbol{x}, \boldsymbol{y} \in A$ and $\lambda, \mu > 0$ with $\lambda + \mu = 1$, then $\boldsymbol{x}, \boldsymbol{y} \in B$.
- Every convex set A in $\mathbb{R}^n$ has the faces $\emptyset$ and A, called **improper faces** of A.
- Faces of A other than $\emptyset$ and A are called **proper faces** of A.

Example and k-Faces

- The above definition of a face is more comprehensive than the one usually understood in elementary geometry.

  Example: A cube has:
  - six two-dimensional faces;
  - one three-dimensional face (itself);
  - twelve one-dimensional faces (its edges);
  - eight zero-dimensional faces (its vertices);
  - one face of dimension -1 (the empty set).
- In general, we refer to a k-dimensional face of a convex set as a k-**face**.

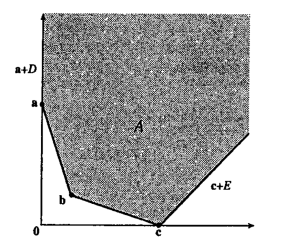
Special Names for Particular Faces

- Certain faces of a convex set are of particular importance and are given special names.
- The 0-faces of a convex set are called its **extreme points**.
- The faces that are halflines are called its **extreme half lines**.
- The directions of the extreme halflines of a convex set are called its **extreme directions**.
- Clearly, a point $\boldsymbol{a}$ of a convex set A in $\mathbb{R}^n$ is an extreme point of A if and only if whenever $\boldsymbol{a} = \lambda \boldsymbol{x} + \mu \boldsymbol{y}$, where $\boldsymbol{x}, \boldsymbol{y} \in A$ and $\lambda, \mu > 0$ with $\lambda + \mu = 1$, then $\boldsymbol{x} = \boldsymbol{y} = \boldsymbol{a}$.

Example

- Denote by A the set of those points (x,y) in $\mathbb{R}^2$ which satisfy the four inequalities

$$\begin{array}{rcl} x & \geq & 0, \\ -x+y+1 & \geq & 0, \\ x+3y-1 & \geq & 0, \\ 3x+y-1 & \geq & 0. \end{array}$$



- The extreme points of A are the points $\boldsymbol{a}=(0,1)$, $\boldsymbol{b}=(\frac{1}{4},\frac{1}{4})$ and $\boldsymbol{c}=(1,0)$.
- The extreme directions of A are the directions $D=\{(0,\lambda):\lambda\geq 0\}$ and $E=\{(\lambda,\lambda):\lambda\geq 0\}$.
- The extreme halflines of A are the halflines $\boldsymbol{a}+D$ and $\boldsymbol{c}+E$.

Characterization of the Faces

- Consider the case of a 2-face B of a cube A in $\mathbb{R}^3$.
  - The set $A \backslash B$, i.e., the cube A with its face B removed, is convex.
  - Also B is the intersection of its affine hull $\operatorname{aff} B$, i.e., the plane containing B, with the cube A itself.

Theorem

Let B be a convex subset of a convex set A in $\mathbb{R}^n$. Then B is a face of A if and only if $A \backslash B$ is convex and $B = (\operatorname{aff} B) \cap A$. In particular, a point $\boldsymbol{a}$ of A is an extreme point of A if and only if $A \backslash \{\boldsymbol{a}\}$ is convex.

- Suppose that B is a face of A. If $\boldsymbol{x}, \boldsymbol{y} \in A \backslash B$ and $\lambda, \mu > 0$ with $\lambda + \mu = 1$, then $\lambda \boldsymbol{x} + \mu \boldsymbol{y} \in A$, since $\boldsymbol{x}, \boldsymbol{y} \in A$ and A is convex. We cannot have $\lambda \boldsymbol{x} + \mu \boldsymbol{y} \in B$, for this would imply that $\boldsymbol{x}, \boldsymbol{y} \in B$. Thus $\lambda \boldsymbol{x} + \mu \boldsymbol{y} \in A \backslash B$ and $A \backslash B$ is convex.

Characterization of the Faces (Cont'd)

- Trivially, $B \subseteq (\text{aff}\, B) \cap A$.

  We now establish the opposite inclusion. Suppose that $\boldsymbol{u} \in (\text{aff}\, B) \cap A$. Let $\boldsymbol{b} \in \text{ri}B$. Then there exist $\boldsymbol{v} \in B$ and $\alpha, \beta > 0$ with $\alpha + \beta = 1$ such that $\boldsymbol{b} = \alpha \boldsymbol{u} + \beta \boldsymbol{v}$. Since $\boldsymbol{u}, \boldsymbol{v} \in A$ and B is a face of A, $\boldsymbol{u} \in B$. Hence, $(\text{aff}\, B) \cap A \subseteq B$. So $B = (\text{aff}\, B) \cap A$.

  Suppose next that $A \backslash B$ is convex and $B = (\text{aff}\, B) \cap A$. If $\lambda \boldsymbol{x} + \mu \boldsymbol{y} \in B$, where $\boldsymbol{x}, \boldsymbol{y} \in A$ and $\lambda, \mu > 0$ with $\lambda + \mu = 1$, then not both $\boldsymbol{x}$ and $\boldsymbol{y}$ can lie in $A \backslash B$, for the convexity of $A \backslash B$ would imply that $\lambda \boldsymbol{x} + \mu \boldsymbol{y} \in A \backslash B$. Suppose that $\boldsymbol{x} \notin A \backslash B$. Then $\boldsymbol{x} \in B$. So

$$\boldsymbol{y} = \frac{1}{\mu}(\lambda \boldsymbol{x} + \mu \boldsymbol{y}) - \frac{\lambda}{\mu} \boldsymbol{x} = \left(1 + \frac{\lambda}{\mu}\right)(\lambda \boldsymbol{x} + \mu \boldsymbol{y}) - \frac{\lambda}{\mu} \boldsymbol{x} \in \text{aff}\, B.$$

  Thus, $\boldsymbol{y} \in (\text{aff}\, B) \cap A$. Hence, $\boldsymbol{y} \in B$. So $\boldsymbol{x}, \boldsymbol{y} \in B$ and B is a face of A.

  The final assertion of the theorem follows from what we have just proved and the fact that a singleton set is its own affine hull.

Consequences

Corollary

Each face of a closed convex set in $\mathbb{R}^n$ is closed.

- Let B be a face of a closed convex set A in $\mathbb{R}^n$. Then B, being the intersection of the closed sets $\operatorname{aff} B$ and A, is itself closed.

Corollary

Let $A = \operatorname{conv} C$, where C is a set in $\mathbb{R}^n$. Then each extreme point of A lies in C.

- Suppose that there is an extreme point $\boldsymbol{a}$ of A which does not lie in C. Then $A \backslash \{\boldsymbol{a}\}$ is a proper convex subset of A containing C. Hence A properly contains $\operatorname{conv} C$, i.e. A. This contradiction shows that each extreme point of A lies in C.

Properties of Faces

Theorem

Let A be a convex set in $\mathbb{R}^n$. Then:

(i) The intersection of any non-empty family of faces of A is a face of A;
(ii) If B is a face of A, and C is a face of B, then C is a face of A;
(iii) The intersection of A with each of its support hyperplanes is a face of A.

(i) Let $(A_i : i \in I)$ be a non-empty family of faces of A. Then $\bigcap(A_i : i \in I)$ is a convex subset of A. If $\lambda \boldsymbol{x} + \mu \boldsymbol{y} \in \bigcap(A_i : i \in I)$, where $\boldsymbol{x}, \boldsymbol{y} \in A$ and $\lambda, \mu > 0$ with $\lambda + \mu = 1$, then $\lambda \boldsymbol{x} + \mu \boldsymbol{y} \in A_i$ for all $i \in I$. So, since each A_i is a face of A, we have $\boldsymbol{x}, \boldsymbol{y} \in A_i$, for all $i \in I$. Hence, $\boldsymbol{x}, \boldsymbol{y} \in \bigcap(A_i : i \in I)$. Therefore, $\bigcap(A_i : i \in I)$ is a face of A.

Properties of Faces (Cont'd)

(ii) Let B be a face of A and let C be a face of B. If $\lambda\boldsymbol{x}+\mu\boldsymbol{y}\in C$, where $\boldsymbol{x},\boldsymbol{y}\in A$ and $\lambda,\mu>0$ with $\lambda+\mu=1$, then $\lambda\boldsymbol{x}+\mu\boldsymbol{y}\in B$, since $C\subseteq B$. Since B is a face of A, $\boldsymbol{x},\boldsymbol{y}\in B$. But C is a face of B, so $\boldsymbol{x},\boldsymbol{y}\in C$. This proves that C is a face of A.

(iii) Let $H=\{\boldsymbol{z}\in\mathbb{R}^n:\boldsymbol{c}\cdot\boldsymbol{z}=c_0\}$, where $c_0\in\mathbb{R}$, $\boldsymbol{c}\in\mathbb{R}^n$ and $\boldsymbol{c}\neq\boldsymbol{0}$, be a support hyperplane to A. Suppose that $\boldsymbol{c}\cdot\boldsymbol{a}\leq c_0$ whenever $\boldsymbol{a}\in A$. If $\lambda\boldsymbol{x}+\mu\boldsymbol{y}\in A\cap H$, where $\boldsymbol{x},\boldsymbol{y}\in A$ and $\lambda,\mu>0$ with $\lambda+\mu=1$, then

$$c_0=\boldsymbol{c}\cdot(\lambda\boldsymbol{x}+\mu\boldsymbol{y})=\lambda\boldsymbol{c}\cdot\boldsymbol{x}+\mu\boldsymbol{c}\cdot\boldsymbol{y}\leq\lambda c_0+\mu c_0=c_0.$$

Since $\lambda,\mu>0$, we must have $\boldsymbol{c}\cdot\boldsymbol{x}=\boldsymbol{c}\cdot\boldsymbol{y}=c_0$. Hence, $\boldsymbol{x},\boldsymbol{y}\in A\cap H$. Thus, $A\cap H$ is a face of A, for clearly it is a convex subset of A.

Strengthening the Defining Property

Theorem

Let B be a face of a convex set A in $\mathbb{R}^n$. Suppose that C is a subset of A such that $\mathrm{ri} C$ meets B. Then $C \subseteq B$.

- Let $\boldsymbol{c} \in C$ and $\boldsymbol{b} \in B \cap \mathrm{ri} C$. Then there exist $\boldsymbol{d} \in C$ and $\lambda, \mu > 0$ with $\lambda + \mu = 1$ such that $\boldsymbol{b} = \lambda \boldsymbol{c} + \mu \boldsymbol{d}$. Since $\boldsymbol{c}, \boldsymbol{d} \in A$ and B is a face of A, we see that $\boldsymbol{c} \in B$. Thus, $C \subseteq B$.

Corollary

Let B and C be faces of a convex set A in $\mathbb{R}^n$ such that $\mathrm{ri} B$ and $\mathrm{ri} C$ meet. Then $B = C$.

- Since $\mathrm{ri} C$ meets B, we have $C \subseteq B$. Similarly, $B \subseteq C$. Thus, $B = C$.

On Dimensional Properties of Faces

Corollary

Let B be a face of a convex set A in $\mathbb{R}^n$, other than A itself. Then $\dim B < \dim A$.

- We suppose that B is non-empty. Clearly, $\operatorname{aff} B \subseteq \operatorname{aff} A$ and $\dim B \leq \dim A$. If $\dim B = \dim A$, then $\operatorname{aff} B = \operatorname{aff} A$ and $\emptyset \subset \operatorname{ri} B \subseteq \operatorname{ri} A$. Hence $A = B$ by the preceding corollary.

Corollary

The intersection of any family of faces of a convex set in $\mathbb{R}^n$ can be expressed as an intersection of $n+1$ or fewer members of the family.

- Suppose that the result is false. Then there exist faces $A_1, \ldots, A_{n+2}$ of some convex set A in $\mathbb{R}^n$ such that $A_1 \subset A_2 \subset \cdots \subset A_{n+2} \subset A$. Since A_i is a face of A_{i+1} for $i = 1, \ldots, n+1$, the preceding corollary shows that

$$-1 < \dim A_1 < \dim A_2 < \cdots < \dim A_{n+2} \leq n-1,$$

which is impossible.

Smallest Face Containing a Point

- Each point of a convex set belongs to at least one face of the set, namely the set itself, and in general belongs to several different faces.

  Example: A vertex of a three-dimensional cube belongs to one 0-face, three 1-faces, three 2-faces, and one 3-face of the cube.
- Suppose that $\boldsymbol{a}$ is a point of a convex set A in $\mathbb{R}^n$ and that $F_{\boldsymbol{a}}$, is the intersection of all faces of A containing $\boldsymbol{a}$.
- Then it follows from the preceding theorem that $F_{\boldsymbol{a}}$ is the **smallest face of A containing $\boldsymbol{a}$**.

Characterization of Smallest Face Containing a Point

Theorem

Let $\boldsymbol{a}$ be a point of a convex set A in $\mathbb{R}^n$ and let $F_{\boldsymbol{a}}$ be the intersection of all faces of A containing $\boldsymbol{a}$. Then $\boldsymbol{a} \in \text{ri}F_{\boldsymbol{a}}$ and the relative interiors of the faces of A form a partition of A.

- If $\boldsymbol{a} \notin \text{ri}F_{\boldsymbol{a}}$, then $\boldsymbol{a} \in \text{rebd}F_{\boldsymbol{a}}$. So, by a previous theorem, there exists a support hyperplane H of $F_{\boldsymbol{a}}$ passing through $\boldsymbol{a}$ but not containing $F_{\boldsymbol{a}}$. Hence, by the preceding theorem, Part (iii), $H \cap F_{\boldsymbol{a}}$ is a face of A containing $\boldsymbol{a}$ which is strictly contained in $F_{\boldsymbol{a}}$. Since this is impossible, $\boldsymbol{a} \in \text{ri}F_{\boldsymbol{a}}$. Thus each point $\boldsymbol{a}$ of A belongs to the relative interior of the face $F_{\boldsymbol{a}}$ of A. The relative interiors of two different faces of A are disjoint by a previous corollary. Hence the relative interiors of the faces of A form a partition of A.

Smallest Face of a Relative Boundary Point

Corollary

Let $\boldsymbol{a}$ be a relative boundary point of a convex set A in $\mathbb{R}^n$. Then $\dim F_{\boldsymbol{a}} < \dim A$.

- The faces A and $F_{\boldsymbol{a}}$ of A cannot be equal because $\boldsymbol{a} \in \operatorname{ri} F_{\boldsymbol{a}}$. Thus $F_{\boldsymbol{a}} \subset A$. Hence $\dim F_{\boldsymbol{a}} < \dim A$ by a previous corollary.

Primitive Convex Sets and Halfflats

- A closed convex set in $\mathbb{R}^n$ which is not the convex hull of its relative boundary is said be **primitive**.
- The reader should have little difficulty in discovering that the only primitive sets in $\mathbb{R}^2$ are:
  - points;
  - lines;
  - halflines;
  - closed halfplanes;
  - $\mathbb{R}^2$ itself.
- Before we can extend this last result to $\mathbb{R}^n$, we need to generalize the concepts of halflines and halfplanes from $\mathbb{R}^2$ to $\mathbb{R}^n$.
- In $\mathbb{R}^n$ a **closed halfflat** is the intersection of a flat with a closed halfspace which meets it, but does not contain it.

Characterization of Primitive Sets

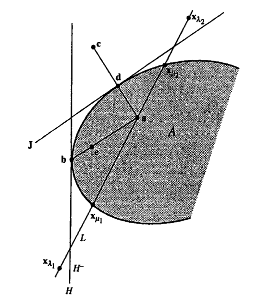
Theorem

A closed convex set in $\mathbb{R}^n$ is primitive if and only if it is either a nonempty flat or a closed halfflat.

- We establish only the non-trivial part of the theorem, i.e., that, for each closed convex set A in $\mathbb{R}^n$ other than a flat or a closed halfflat, $A = \text{conv}(\text{rebd}A)$.

  We know $\text{conv}(\text{rebd}A) \subseteq A$. Moreover, $A = (\text{rebd}A) \cup (\text{ri}A)$ and $\text{rebd}A \subseteq \text{conv}(\text{rebd}A)$. So, we must show $\text{ri}A \subseteq \text{conv}(\text{rebd}A)$.

  Let $\boldsymbol{a} \in \text{ri}A$. Since A is not a flat, its relative boundary is not empty, say $\boldsymbol{b} \in \text{rebd}A$. A previous theorem shows that there is a non-trivial support hyperplane H to A at $\boldsymbol{b}$.



Characterization of Primitive Sets (Cont'd)

- Since $A \cap H$ is a proper face of A, a previous theorem shows that $\boldsymbol{a} \notin H$. Thus there exist $u_0 \in \mathbb{R}$, $\boldsymbol{u} \in \mathbb{R}^n$ such that H has equation $\boldsymbol{n} \cdot \boldsymbol{x} = u_0$ and $\boldsymbol{u} \cdot \boldsymbol{a} < u_0$, $\boldsymbol{u} \cdot \boldsymbol{b} = u_0$ with A lying in the closed halfspace

  $$H^- = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{u} \cdot \boldsymbol{x} \le u_0\}.$$

  By the hypothesis, A is not a closed halfflat, and so cannot be $(\operatorname{aff} A) \cap H^-$. Thus there exists a point $\boldsymbol{c}$ of $(\operatorname{aff} A) \cap H^-$ that does not lie in A. Denote by $\boldsymbol{d}$ the point where the line segment joining $\boldsymbol{a}$ and $\boldsymbol{c}$ meets $\operatorname{rebd} A$. Since $\boldsymbol{u} \cdot \boldsymbol{a} < u_0$ and $\boldsymbol{u} \cdot \boldsymbol{c} \le u_0$, we get $\boldsymbol{u} \cdot \boldsymbol{d} < u_0$.

  The existence of a non-trivial support hyperplane to A at $\boldsymbol{d}$, J say, shows that there exist $v_0 \in \mathbb{R}$, $\boldsymbol{v} \in \mathbb{R}^n$ such that J has equation $\boldsymbol{v} \cdot \boldsymbol{x} = v_0$ and $\boldsymbol{v} \cdot \boldsymbol{a} < v_0$, $\boldsymbol{v} \cdot \boldsymbol{d} = v_0$, with A lying in the closed halfspace $\{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{v} \cdot \boldsymbol{x} \le v_0\}$.

Characterization of Primitive Sets (Cont'd)

- Since $\boldsymbol{u}\cdot\boldsymbol{a}<u_0$, $\boldsymbol{u}\cdot\boldsymbol{b}=u_0$ and $\boldsymbol{u}\cdot\boldsymbol{d}<u_0$, there is a point $\boldsymbol{e}$ of $\mathrm{ri}A$ lying on the line segment joining $\boldsymbol{a}$ and $\boldsymbol{b}$ such that $\boldsymbol{u}\cdot\boldsymbol{e}>\boldsymbol{u}\cdot\boldsymbol{d}$. Because $\boldsymbol{e}$ lies in $\mathrm{ri}A$, we must have $\boldsymbol{v}\cdot\boldsymbol{e}<v_0$. For each scalar λ, denote by $\boldsymbol{x}_\lambda$ the point $\boldsymbol{a}+\lambda(\boldsymbol{d}-\boldsymbol{e})$ on the line L joining the points $\boldsymbol{a}$ and $\boldsymbol{a}+\boldsymbol{d}-\boldsymbol{e}$ of $\mathrm{aff}A$. Choose scalars λ_1,λ_2 such that

$$\lambda_1<\frac{u_0-\boldsymbol{u}\cdot\boldsymbol{a}}{\boldsymbol{u}\cdot\boldsymbol{d}-\boldsymbol{u}\cdot\boldsymbol{e}}<0<\frac{v_0-\boldsymbol{v}\cdot\boldsymbol{a}}{\boldsymbol{v}\cdot\boldsymbol{d}-\boldsymbol{v}\cdot\boldsymbol{e}}<\lambda_2.$$

  Then

$$\begin{array}{rcl}\boldsymbol{u}\cdot\boldsymbol{x}_{\lambda_1} &=& \boldsymbol{u}\cdot(\boldsymbol{a}+\lambda_1(\boldsymbol{d}-\boldsymbol{e}))=\boldsymbol{u}\cdot\boldsymbol{a}+\lambda_1\boldsymbol{u}\cdot(\boldsymbol{d}-\boldsymbol{e})\\ &>& \boldsymbol{u}\cdot\boldsymbol{a}+u_0-\boldsymbol{u}\cdot\boldsymbol{a}=u_0;\\ \boldsymbol{v}\cdot\boldsymbol{x}_{\lambda_2} &=& \boldsymbol{v}\cdot(\boldsymbol{a}+\lambda_2(\boldsymbol{d}-\boldsymbol{e}))=\boldsymbol{v}\cdot\boldsymbol{a}+\lambda_2\boldsymbol{v}\cdot(\boldsymbol{d}-\boldsymbol{e})\\ &>& \boldsymbol{v}\cdot\boldsymbol{a}+v_0-\boldsymbol{v}\cdot\boldsymbol{a}=v_0.\end{array}$$

  Hence neither $\boldsymbol{x}_{\lambda_1}$ nor $\boldsymbol{x}_{\lambda_2}$ lies in A. Thus, there are scalars μ_1,μ_2 with $\lambda_1<\mu_1<0<\mu_2<\lambda_2$ such that $\boldsymbol{x}_{\mu_1},\boldsymbol{x}_{\mu_2}\in\mathrm{rebd}A$. Hence, $\boldsymbol{a}=\frac{\mu_2}{\mu_2-\mu_1}\boldsymbol{x}_{\mu_1}-\frac{\mu_1}{\mu_2-\mu_1}\boldsymbol{x}_{\mu_2}\in\mathrm{conv}\{\boldsymbol{x}_{\mu_1},\boldsymbol{x}_{\mu_2}\}$. So $\mathrm{ri}A\subseteq\mathrm{conv}(\mathrm{rebd}A)$.

Facial Structure of Closed Convex Sets

- By the convex hull of a family of sets in $\mathbb{R}^n$ is meant the convex hull of its union.

Theorem

Every closed convex set in $\mathbb{R}^n$ is the convex hull of its primitive faces.

- Let A be a closed convex set in $\mathbb{R}^n$. We argue by induction on the dimension of A.

  The case $\dim A = -1$ is trivial.

  Suppose that $\dim A = m$, where $m > -1$, and that the assertion is true for all closed convex sets in $\mathbb{R}^n$ with dimension less than m. The theorem is trivial when A is primitive. Suppose, then, that A is not primitive. Denote by B the convex hull of the primitive faces of A. Then $B \subseteq A$. So we need only show that $A \subseteq B$.

Facial Structure of Closed Convex Sets (Cont'd)

- Since A is not primitive, we have $A = \text{conv}(\text{rebd}A)$. Let $\boldsymbol{a} \in \text{rebd}A$. Then $\boldsymbol{a}$ lies in $F_{\boldsymbol{a}}$, the smallest face of A containing $\boldsymbol{a}$. By a previous corollary, $\dim F_{\boldsymbol{a}} < \dim A$. The induction hypothesis shows that $F_{\boldsymbol{a}}$ is the convex hull of its primitive faces. Since each primitive face of $F_{\boldsymbol{a}}$ is a primitive face of A, $F_{\boldsymbol{a}} \subseteq B$. Hence, $\boldsymbol{a} \in F_{\boldsymbol{a}} \subseteq B$ and $\text{rebd}A \subseteq B$. So $A = \text{conv}(\text{rebd}A) \subseteq B$. Thus, $A = B$, i.e., A is the convex hull of its primitive faces.

Consequences

Corollary

Every closed convex set in $\mathbb{R}^n$ is the convex hull of those of its faces which are flats or closed halfflats.

- The result follows from the theorem and a previous theorem.

Corollary

Every closed convex set in $\mathbb{R}^n$ that contains no lines is the convex hull of its extreme points and extreme halflines.

- The corollary follows from the theorem and the fact that points and halflines are the only primitive sets which contain no lines.

Theorem (Krein-Milman)

Every compact convex set in $\mathbb{R}^n$ is the convex hull of its extreme points.

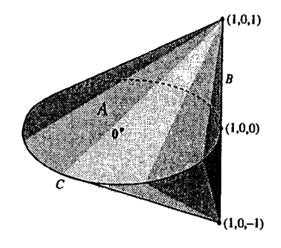
- The theorem follows from the preceding theorem and the fact that points are the only compact primitive sets.

Comment on Set of Extreme Points

- The Krein-Milman theorem shows that the convex hull of the extreme points of a compact convex set in $\mathbb{R}^n$ is closed.
- It is not true, however, that the set of extreme points itself is necessarily closed.
- To see this, let A and B denote the circular disc and the line segment in $\mathbb{R}^3$ given by the equations

$$A = \{(x,y,0) : x^2 + y^2 \leq 1\} \quad \text{and} \quad B = \{(1,0,z) : -1 \leq z \leq 1\}.$$

Let $C = \text{conv}(A \cup B)$. Then C is a compact convex set. Its set of extreme points consists of $(1,0,1)$ and $(1,0,-1)$ together with the points on the relative boundary of A with the exception of $(1,0,0)$. This set is not closed.

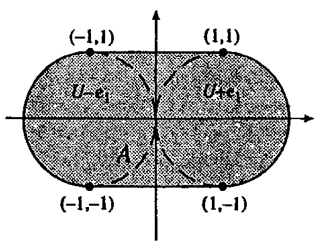


Exposed Faces

- By a previous theorem, the intersection of a convex set in $\mathbb{R}^n$ with one of its support hyperplanes is a face of the set.
- A face which arises in this way is called an **exposed face** of the set.
- It is technically convenient to allow the empty set and the set itself as exposed faces of any convex set in $\mathbb{R}^n$.
- Thus, an **exposed face** of a convex set in $\mathbb{R}^n$ is either the empty set, the set itself, or the intersection of the set with one of its support hyperplanes.

Example: Exposed Faces

- Faces of a convex set are not always exposed.
  Example: Let A be the convex hull of the union of the circular discs $U+\boldsymbol{e}_1$, and $U-\boldsymbol{e}_1$, in $\mathbb{R}^2$, where
  $$U=\{(x,y):x^2+y^2\leq 1\} \quad \text{and} \quad \boldsymbol{e}_1=(1,0).$$
  Then the points $(1,1)$, $(1,-1)$, $(-1,1)$, $(-1,-1)$ are faces of A that are not exposed.



- This example also serves to show that an exposed face of an exposed face of a convex set need not be an exposed face of that convex set.
  The line segment joining $(-1,1)$ and $(1,1)$ is an exposed face of A and $(1,1)$ is an exposed face of this line segment, but $(1,1)$ is not an exposed face of A.

Closure of Exposed Faces Under Intersections

Theorem

The intersection of any non-empty family of exposed faces of a convex set in $\mathbb{R}^n$ is an exposed face of the set.

- In view of a previous corollary, we may assume that the family of exposed faces is finite.

  Let $A_1,\ldots,A_m$ be exposed faces of a convex set A in $\mathbb{R}^n$. We show that $A_1 \cap \cdots \cap A_m$ is an exposed face of A, considering only the non-trivial case when the $A_1,\ldots,A_m$ are proper exposed faces of A, whose intersection is non-empty, containing some point $\boldsymbol{a}_0$, say. For each $i=1,\ldots,m$, there exists $\boldsymbol{u}_i \in \mathbb{R}^n$ such that

$$A_i = \{\boldsymbol{a} \in A : \boldsymbol{u}_i \cdot \boldsymbol{a} = \boldsymbol{u}_i \cdot \boldsymbol{a}_0\} \quad \text{and} \quad A \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{u}_i \cdot \boldsymbol{x} \leq \boldsymbol{u}_i \cdot \boldsymbol{a}_0\}.$$

Closure of Exposed Faces Under Intersections (Cont'd)

- It follows easily that

$$A_1 \cap \cdots \cap A_m = \{\boldsymbol{a} \in A : (\boldsymbol{u}_1 + \cdots + \boldsymbol{u}_m) \cdot \boldsymbol{a} = (\boldsymbol{u}_1 + \cdots + \boldsymbol{u}_m) \cdot \boldsymbol{a}_0\},$$

and that

$$A \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : (\boldsymbol{u}_1 + \cdots \boldsymbol{u}_m) \cdot \boldsymbol{x} \leq (\boldsymbol{u}_1 + \cdots + \boldsymbol{u}_m) \cdot \boldsymbol{a}_0\}.$$

This shows that $A_1 \cap \cdots \cap A_m$ is an exposed face of A.

Exposed Faces of Sums

Theorem

Let A and B be convex sets in $\mathbb{R}^n$. Then each exposed face of $A+B$ has the form $C+D$, where C is an exposed face of A and D is an exposed face of B.

- Suppose that F is a *proper* exposed face of $A+B$. Then there exist $\boldsymbol{a}_0 \in A$, $\boldsymbol{b}_0 \in B$, and a non-zero $\boldsymbol{u}$ in $\mathbb{R}^n$ such that

$$\begin{aligned} A+B &\subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{u} \cdot \boldsymbol{x} \le \boldsymbol{u} \cdot (\boldsymbol{a}_0 + \boldsymbol{b}_0)\}, \\ F &= \{\boldsymbol{x} \in A+B : \boldsymbol{u} \cdot \boldsymbol{x} = \boldsymbol{u} \cdot (\boldsymbol{a}_0 + \boldsymbol{b}_0)\}. \end{aligned}$$

If $\boldsymbol{a} \in A$, then $\boldsymbol{a} + \boldsymbol{b}_0 \in A+B$. Hence $\boldsymbol{u} \cdot (\boldsymbol{a} + \boldsymbol{b}_0) \le \boldsymbol{u} \cdot (\boldsymbol{a}_0 + \boldsymbol{b}_0)$ and $\boldsymbol{u} \cdot \boldsymbol{a} \le \boldsymbol{u} \cdot \boldsymbol{a}_0$. Similarly, if $\boldsymbol{b} \in B$, then $\boldsymbol{u} \cdot \boldsymbol{b} \le \boldsymbol{u} \cdot \boldsymbol{b}_0$. Thus

$$C = \{\boldsymbol{x} \in A : \boldsymbol{u} \cdot \boldsymbol{x} = \boldsymbol{u} \cdot \boldsymbol{a}_0\} \quad \text{and} \quad D = \{\boldsymbol{x} \in B : \boldsymbol{u} \cdot \boldsymbol{x} = \boldsymbol{u} \cdot \boldsymbol{b}_0\}$$

are, respectively, exposed faces of A and B.

Exposed Faces of Sums (Cont'd)

- We derived that

$$C = \{\boldsymbol{x} \in A : \boldsymbol{u} \cdot \boldsymbol{x} = \boldsymbol{u} \cdot \boldsymbol{a}_0\} \quad \text{and} \quad D = \{\boldsymbol{x} \in B : \boldsymbol{u} \cdot \boldsymbol{x} = \boldsymbol{u} \cdot \boldsymbol{b}_0\}$$

are, respectively, exposed faces of A and B. Clearly $C + D \subseteq F$.

If $\boldsymbol{f} \in F$, then $\boldsymbol{f} = \boldsymbol{a} + \boldsymbol{b}$ for some $\boldsymbol{a} \in A$, $\boldsymbol{b} \in B$.

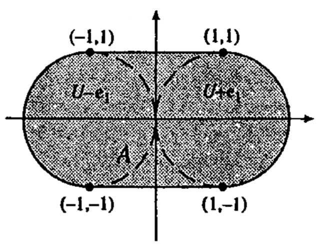
Now $\boldsymbol{u} \cdot (\boldsymbol{a} + \boldsymbol{b}) = \boldsymbol{u} \cdot (\boldsymbol{a}_0 + \boldsymbol{b}_0)$, $\boldsymbol{u} \cdot \boldsymbol{a} \leq \boldsymbol{u} \cdot \boldsymbol{a}_0$ and $\boldsymbol{u} \cdot \boldsymbol{b} \leq \boldsymbol{u} \cdot \boldsymbol{b}_0$.

Hence, $\boldsymbol{u} \cdot \boldsymbol{a} = \boldsymbol{u} \cdot \boldsymbol{a}_0$ and $\boldsymbol{u} \cdot \boldsymbol{b} = \boldsymbol{u} \cdot \boldsymbol{b}_0$.

Thus $\boldsymbol{a} \in C$, $\boldsymbol{b} \in D$, and $F = C + D$.

Exposed Points

- The zero-dimensional exposed faces of a convex set are called its **exposed points**.
- Thus a point $\boldsymbol{a}$ of a convex set A in $\mathbb{R}^n$ is an exposed point of A if and only if there is some support hyperplane to A meeting it in the single point $\boldsymbol{a}$.
- Every exposed point of a convex set in $\mathbb{R}^n$ is one of its extreme points, but not necessarily conversely.
- The point $(1,1)$ of the set A of the preceding example is an extreme, but not an exposed, point of A.



Farthest Points

- Let A be a non-empty compact set in $\mathbb{R}^n$ and let $\boldsymbol{b}$ be a point of $\mathbb{R}^n$.
- For each point $\boldsymbol{x}$ of A, denote by $f(\boldsymbol{x})$ the distance $\|\boldsymbol{x}-\boldsymbol{b}\|$ of $\boldsymbol{x}$ from $\boldsymbol{b}$.
- Then f is a continuous real-valued function defined on a non-empty compact set A.
- So it is bounded and attains its bounds.
- In particular, f attains its upper bound.
- So there is a point $\boldsymbol{a}$ of A such that $\|\boldsymbol{x}-\boldsymbol{b}\| \leq \|\boldsymbol{a}-\boldsymbol{b}\|$ for all $\boldsymbol{x}$ in A.
- Each such point $\boldsymbol{a}$ of A is called a **farthest point** of A from $\boldsymbol{b}$.

Farthest Points and Exposed Points

Theorem

Let $\boldsymbol{a}$ be a farthest point of a compact convex set A in $\mathbb{R}^n$ from some point $\boldsymbol{b}$ of $\mathbb{R}^n$. Then $\boldsymbol{a}$ is an exposed point of A.

- We consider the non-trivial case when $\boldsymbol{a} \neq \boldsymbol{b}$. Since $\boldsymbol{a}$ is a farthest point of A from $\boldsymbol{b}$, we have, for each point $\boldsymbol{x}$ of A,

$$\begin{array}{rcl} \|\boldsymbol{a}-\boldsymbol{b}\|^2 & \geq & \|\boldsymbol{x}-\boldsymbol{b}\|^2 \\ & = & \|(\boldsymbol{x}-\boldsymbol{a})+(\boldsymbol{a}-\boldsymbol{b})\|^2 \\ & = & \|\boldsymbol{x}-\boldsymbol{a}\|^2+2(\boldsymbol{x}-\boldsymbol{a})\cdot(\boldsymbol{a}-\boldsymbol{b})+\|\boldsymbol{a}-\boldsymbol{b}\|^2. \end{array}$$

  Hence $0 \geq (\boldsymbol{x}-\boldsymbol{a})\cdot(\boldsymbol{a}-\boldsymbol{b})$. Equality occurs in the last inequality if and only if $\boldsymbol{x}=\boldsymbol{a}$. So the hyperplane $H=\{\boldsymbol{z}\in\mathbb{R}^n : (\boldsymbol{a}-\boldsymbol{b})\cdot(\boldsymbol{z}-\boldsymbol{a})=0\}$ supports A at $\boldsymbol{a}$ and $H\cap A=\{\boldsymbol{a}\}$. Thus $\boldsymbol{a}$ is an exposed point of A.

Compact Convex Sets and Exposed Points

Lemma

A compact convex set in $\mathbb{R}^n$ has an exposed point in every open halfspace which meets it.

- Suppose that the compact convex set A in $\mathbb{R}^n$ meets the open halfspace $J = \{\boldsymbol{z} \in \mathbb{R}^n : \boldsymbol{u} \cdot \boldsymbol{z} + u_0 < 0\}$, where $u_0 \in \mathbb{R}$, $\boldsymbol{u} \in \mathbb{R}^n$, and $\boldsymbol{u} \neq \boldsymbol{0}$, say $\boldsymbol{a} \in A \cap J$. Let $\lambda > 0$ satisfy $s^2 + 2\lambda(\boldsymbol{u} \cdot \boldsymbol{a} + u_0) < 0$, where s is the diameter of A. Let $\boldsymbol{c} = \boldsymbol{a} + \lambda \boldsymbol{u}$. For each point $\boldsymbol{x}$ of $A \backslash J$, $\boldsymbol{u} \cdot \boldsymbol{x} + u_0 \geq 0$ and $\boldsymbol{u} \cdot (\boldsymbol{a} - \boldsymbol{x}) \leq \boldsymbol{u} \cdot \boldsymbol{a} + u_0$. Hence,

$$\begin{array}{rcl} \|\boldsymbol{c} - \boldsymbol{x}\|^2 & = & \|\boldsymbol{a} - \boldsymbol{x} + \lambda \boldsymbol{u}\|^2 \\ & = & \|\boldsymbol{a} - \boldsymbol{x}\|^2 + 2\lambda \boldsymbol{u} \cdot (\boldsymbol{a} - \boldsymbol{x}) + \lambda^2 \|\boldsymbol{u}\|^2 \\ & \leq & s^2 + 2\lambda(\boldsymbol{u} \cdot \boldsymbol{a} + u_0) + \lambda^2 \|\boldsymbol{u}\|^2 \\ & < & \lambda^2 \|\boldsymbol{u}\|^2 = \|\boldsymbol{c} - \boldsymbol{a}\|^2. \end{array}$$

  Thus no point $\boldsymbol{x}$ of $A \backslash J$ is a farthest point of A from $\boldsymbol{c}$. So every farthest point of A from $\boldsymbol{c}$ is an exposed point of A lying in J.

Compact Convex Sets as Hulls of Exposed Points

Theorem (Straszewicz)

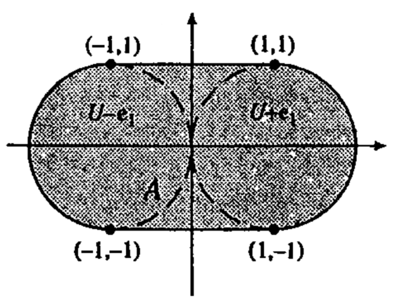
Every compact convex set in $\mathbb{R}^n$ is the closure of the convex hull of its exposed points.

- Let B be the set of the exposed points of a compact convex set A in $\mathbb{R}^n$. Trivially $\mathrm{cl}(\mathrm{conv} B) \subseteq A$. So we must show that $A \subseteq \mathrm{cl}(\mathrm{conv} B)$.

  Suppose that this is not so. Then there is a point $\boldsymbol{a}$ of A which does not belong to the closed convex set $\mathrm{cl}(\mathrm{conv} B)$. It follows immediately from a previous corollary that there is an open halfspace J in $\mathbb{R}^n$ which contains $\boldsymbol{a}$ but is disjoint from $\mathrm{cl}(\mathrm{conv} B)$. By the preceding lemma, there is a point of B lying in J, which is impossible. Thus $A \subseteq \mathrm{cl}(\mathrm{conv} B)$ as desired.

Necessity of the Closure Requirement

- The two-dimensional set illustrated in the figure



shows that the closure requirement in Straszewicz's theorem cannot be omitted.

Subsection 7

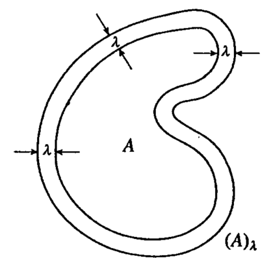
The Blaschke Selection Principle

Distance Between Two Sets

- Sets A and B in $\mathbb{R}^n$ are said to be a **finite distance apart** if there exists $\lambda \geq 0$ such that, for each point $\boldsymbol{a}$ of A, there is a point $\boldsymbol{b}$ of B whose distance $\|\boldsymbol{a}-\boldsymbol{b}\|$ from $\boldsymbol{a}$ does not exceed λ, and vice versa.
- In this situation, we say that the **distance between** A **and** B **does not exceed** λ.
- The **distance between** sets A and B in $\mathbb{R}^n$ that are a finite distance apart is defined to be the infimum of the set of all those $\lambda \geq 0$ for which the distance between A and B does not exceed λ.
- This definition does not assign a distance between the empty set and a non-empty set or between a bounded set and an unbounded one.
- On the other hand, the definition always assigns a distance between two non-empty bounded sets.
- For our purposes here, it will be sufficient to restrict attention to non-empty compact sets.

λ-Neighborhood of a Set

- Let A be a set in $\mathbb{R}^n$ and let $\lambda \geq 0$.
- Then the λ-**neighborhood** $(A)_\lambda$ of A is the set $A + \lambda U$, where U denotes the closed unit ball $\{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\| \leq 1\}$.
  The figure makes it clear why the set $(A)_\lambda$ is often referred to as the **outer parallel set of** A **at distance** λ.



- Clearly, if $\boldsymbol{a} \in \mathbb{R}^n$, $r > 0$, and $\lambda \geq 0$, then:
  - The r-neighborhood of $\{\boldsymbol{a}\}$ is the closed ball $B[\boldsymbol{a}; r]$;
  - The λ-neighborhood of $B[\boldsymbol{a}; r]$ is the closed ball $B[\boldsymbol{a}; r + \lambda]$.

Properties of Neighborhoods

Theorem

Let A, B be sets in $\mathbb{R}^n$ and let $\lambda, \mu \geq 0$. Then:

(i) $(A)_0 = A$ and $A \subseteq (A)_\lambda$;

(ii) $(A)_\lambda \subseteq (B)_\lambda$ when $A \subseteq B$;

(iii) $(A)_\lambda$ is convex when A is;

(iv) $((A)_\lambda)_\mu = (A)_{\lambda+\mu}$.

- Parts (i) and (ii) are easy consequences of the definition of a λ-neighborhood.

  Part (iii) follows from a previous example and theorem.

  To prove Part (iv) we note, using a previous theorem, that

$$((A)_\lambda)_\mu = (A)_\lambda + \mu U = (A + \lambda U) + \mu U = A + (\lambda + \mu) U = (A)_{\lambda+\mu}.$$

The Hausdorff Distance

- The assertions that, for each point $\boldsymbol{a}$ of a set A in $\mathbb{R}^n$, there is a point $\boldsymbol{b}$ of a set B in $\mathbb{R}^n$ such that $\|\boldsymbol{a}-\boldsymbol{b}\| \leq \lambda$, and that $A \subseteq (B)_\lambda$, where $\lambda \geq 0$, are equivalent.
- Thus the definition we now give of the distance between non-empty compact sets in $\mathbb{R}^n$ coincides with the one given earlier.
- The **distance** $\rho(A,B)$ between non-empty compact sets A,B in $\mathbb{R}^n$ is defined by the equation

$$\rho(A,B) = \inf\{\lambda \geq 0 : A \subseteq (B)_\lambda \text{ and } B \subseteq (A)_\lambda\}.$$

- The assumptions that A and B are non-empty and compact ensure that $\rho(A,B)$ is well-defined.
- The function ρ is known as the **Hausdorff metric** or **Hausdorff distance**.

Properties of the Distance

- It is easily seen that the Hausdorff distance $\rho(\{\boldsymbol{a}\},\{\boldsymbol{b}\})$ between the singleton sets $\{\boldsymbol{a}\}$ and $\{\boldsymbol{b}\}$ in $\mathbb{R}^n$ is $\|\boldsymbol{a}-\boldsymbol{b}\|$, i.e., the distance between the points $\boldsymbol{a}$ and $\boldsymbol{b}$ themselves.
- Another readily verified fact is that the Hausdorff distance is invariant under translation in the sense that, if A and B are non-empty compact sets in $\mathbb{R}^n$ and $\boldsymbol{x}$ is a point of $\mathbb{R}^n$, then $\rho(A,B)=\rho(A+\boldsymbol{x},B+\boldsymbol{x})$.

Example

- Let A and B be, respectively, the closed balls $B[\boldsymbol{a};r]$ and $B[\boldsymbol{b};s]$ in $\mathbb{R}^n$. Then $\rho(A,B)=\|\boldsymbol{b}-\boldsymbol{a}\|+|s-r|$.
- Suppose first that $r\leq s$. We have

$$\begin{aligned} A &\subseteq B-(\boldsymbol{b}-\boldsymbol{a})\subseteq(B)_{\|\boldsymbol{b}-\boldsymbol{a}\|};\\ B &= A+\boldsymbol{b}-\boldsymbol{a}+(s-r)U\subseteq(A)_{\|\boldsymbol{b}-\boldsymbol{a}\|+s-r}.\end{aligned}$$

Hence, $\rho(A,B)\leq\|\boldsymbol{b}-\boldsymbol{a}\|+s-r$.

Now B contains a point whose distance from $\boldsymbol{a}$ is $\|\boldsymbol{b}-\boldsymbol{a}\|+s$.

Thus, if $\lambda\geq 0$ and $B\subseteq(A)_\lambda=B[\boldsymbol{a};\lambda+r]$, then $\|\boldsymbol{b}-\boldsymbol{a}\|+s\leq\lambda+r$.

Hence, $\rho(A,B)\geq\|\boldsymbol{b}-\boldsymbol{a}\|+s-r$. Thus, $\rho(A,B)=\|\boldsymbol{b}-\boldsymbol{a}\|+s-r$.

I.e., $\rho(A,B)=\|\boldsymbol{b}-\boldsymbol{a}\|+|s-r|$.

The case $s\leq r$ is similar.

Necessary Condition

Theorem

Let A and B be nonempty compact sets in $\mathbb{R}^n$ with $\rho(A,B)=\lambda$. Then $A\subseteq(B)_\lambda$ and $B\subseteq(A)_\lambda$.

- Let $\boldsymbol{a}\in A$. For each $\varepsilon>0$, $A\subseteq(B)_{\lambda+\varepsilon}$. Hence there is a point $\boldsymbol{b}_\varepsilon$ of B for which $\|\boldsymbol{a}-\boldsymbol{b}_\varepsilon\|\leq\lambda+\varepsilon$. So $\inf\{\|\boldsymbol{a}-\boldsymbol{b}\|:\boldsymbol{b}\in B\}\leq\lambda$.

  A previous theorem shows that there exists some point $\boldsymbol{b}_0$ of B such that $\|\boldsymbol{a}-\boldsymbol{b}_0\|\leq\lambda$. Thus, $\boldsymbol{a}\in(B)_\lambda$ and $A\subseteq(B)_\lambda$.

  Similarly, $B\subseteq(A)_\lambda$.

Metric Properties of the Distance

Theorem

Let A, B, C be non-empty compact sets in $\mathbb{R}^n$ and let $\theta \geq 0$. Then:

(i) $\rho(A,B) \geq 0$ and $\rho(A,B) = 0$ if and only if $A = B$;

(ii) $\rho(A,B) = \rho(B,A)$;

(iii) $\rho(A,C) \leq \rho(A,B) + \rho(B,C)$;

(iv) $\rho(\text{conv}A, \text{conv}B) \leq \rho(A,B)$;

(v) if A and B are convex, then $\rho(A,B) = \rho((A)_\theta, (B)_\theta)$.

(i) Trivially $\rho(A,B) \geq 0$. Also $\rho(A,A) = 0$. If $\rho(A,B) = 0$, then $A \subseteq (B)_0 = B$ and $B \subseteq (A)_0 = A$. Hence $A = B$.

(ii) This follows immediately from the definition of ρ.

(iii) Let $\rho(A,B) = \alpha$ and $\rho(B,C) = \beta$. Then $A \subseteq (B)_\alpha \subseteq ((C)_\beta)_\alpha = (C)_{\alpha+\beta}$ and $C \subseteq (B)_\beta \subseteq ((A)_\alpha)_\beta = (A)_{\alpha+\beta}$. Hence $\rho(A,C) \leq \alpha + \beta = \rho(A,B)$ $+\rho(B,C)$.

Metric Properties of the Distance (Cont'd)

(iv) Let $\rho(A,B)=\alpha$. Then $(\text{conv}A)_\alpha$ is convex and $B\subseteq(A)_\alpha\subseteq(\text{conv}A)_\alpha$. Hence $\text{conv}B\subseteq(\text{conv}A)_\alpha$. Similarly, $\text{conv}A\subseteq(\text{conv}B)_\alpha$. Thus, $\rho(\text{conv}A,\text{conv}B)\le\alpha=\rho(A,B)$. We note that $\text{conv}A$ and $\text{conv}B$ are compact by a previous theorem.

(v) Let A and B be convex. The sets $(A)_\theta$ and $(B)_\theta$ are compact by a previous theorem. Let $\rho(A,B)=\alpha$ and $\rho((A)_\theta,(B)_\theta)=\beta$. Then

$$(A)_\theta\subseteq((B)_\alpha)_\theta=((B)_\theta)_\alpha \quad\text{and}\quad (B)_\theta\subseteq((A)_\alpha)_\theta=((A)_\theta)_\alpha.$$

This shows that $\beta\le\alpha$. Also

$$A+\theta U\subseteq(B+\theta U)+\beta U \quad\text{and}\quad B+\theta U\subseteq(A+\theta U)+\beta U,$$

i.e.,

$$A+\theta U\subseteq(B+\beta U)+\theta U \quad\text{and}\quad B+\theta U\subseteq(A+\beta U)+\theta U.$$

Hence, $A\subseteq B+\beta U$ and $B\subseteq A+\beta U$ by a previous theorem. Thus, $\alpha\le\beta$. So $\alpha=\beta$.

Convergence of Compact Sets

- The sequence $A_1, \ldots, A_j, \ldots$ of non-empty compact sets in $\mathbb{R}^n$ is said to **converge** to the non-empty compact set A in $\mathbb{R}^n$, written $A_j \to A$ as $j \to \infty$, if $\rho(A_j, A) \to 0$ as $j \to \infty$.
- Such a sequence cannot converge to more than one nonempty compact set A.

  If it also converges to a nonempty compact set B in $\mathbb{R}^n$, then

  $$0 \le \rho(A, B) \le \rho(A, A_j) + \rho(A_j, B) \to 0,$$

  as $j \to \infty$. Hence $\rho(A, B) = 0$ and $A = B$.

Convergence and Convexity

- If the sequence $A_1, \ldots, A_j, \ldots$ converges to A and each A_j is convex, then so too is A.

  Suppose that a sequence $A_1, \ldots, A_j, \ldots$ of nonempty compact convex sets in $\mathbb{R}^n$ converges to a nonempty compact set A in $\mathbb{R}^n$. The preceding theorem shows that

  $$\rho(A_j, \text{conv}A) = \rho(\text{conv}A_j, \text{conv}A) \leq \rho(A_j, A) \to 0$$

  as $j \to \infty$. Thus $A_1, \ldots, A_j, \ldots$ also converges to conv$A$. Since a sequence cannot converge to two different limits, $A = \text{conv}A$ and A is convex.

Convergence and Linear Combinations

Theorem

Let sequences $A_1, \ldots, A_j, \ldots$ and $B_1, \ldots, B_j, \ldots$ converge, respectively, to A_0 and B_0, where all the A's and B's are nonempty compact sets in $\mathbb{R}^n$. Let real sequences $\alpha_1, \ldots, \alpha_j, \ldots$ and $\beta_1, \ldots, \beta_j, \ldots$ converge, respectively, to α and β. Then the sequence $\alpha_1 A_1 + \beta_1 B_1$, ..., $\alpha_j A_j + \beta_j B_j$, ... converges to $\alpha A_0 + \beta B_0$.

- For $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a}_j, \boldsymbol{b}_j \in \mathbb{R}^n$,

$$\begin{aligned}
&\|\alpha_j \boldsymbol{a}_j + \beta_j \boldsymbol{b}_j - \alpha \boldsymbol{a} - \beta \boldsymbol{b}\| \\
&= \|\alpha_j(\boldsymbol{a}_j - \boldsymbol{a}) + (\alpha_j - \alpha)\boldsymbol{a} + \beta_j(\boldsymbol{b}_j - \boldsymbol{b}) + (\beta_j - \beta)\boldsymbol{b}\| \\
&\leq |\alpha_j| \|\boldsymbol{a}_j - \boldsymbol{a}\| + |\alpha_j - \alpha| \|\boldsymbol{a}\| + |\beta_j| \|\boldsymbol{b}_j - \boldsymbol{b}\| + |\beta_j - \beta| \|\boldsymbol{b}\|.
\end{aligned}$$

  Write $\theta_j = \rho(A_j, A_0)$ and $\varphi_j = \rho(B_j, B_0)$. Let $r > 0$ be such that $\|\boldsymbol{a}\|, \|\boldsymbol{b}\| < r$ whenever $\boldsymbol{a} \in A_0$, $\boldsymbol{b} \in B_0$.

Convergence and Linear Combinations (Cont'd)

- It follows easily from the above inequality that

$$\rho(\alpha_j A_j + \beta_j B_j, \alpha A_0 + \beta B_0) \\ \leq |\alpha_j|\theta_j + |\alpha_j - \alpha| r + |\beta_j|\varphi_j + |\beta_j - \beta| r.$$

  Hence, $\alpha_j A_j + \beta_j B_j \to \alpha A_0 + \beta B_0$.

Corollary

For $i = 1, \ldots, m$, let the sequence $A_1^i, \ldots, A_j^i, \ldots$ converge to A_0^i, where all the A's are nonempty compact sets in $\mathbb{R}^n$. Let the real sequence $\alpha_1^i, \ldots, \alpha_j^i, \ldots$ converge to α_i. Then $\alpha_j^1 A_j^1 + \cdots + \alpha_j^m A_j^m \to \alpha_1 A_0^1 + \cdots + \alpha_m A_0^m$ as $j \to \infty$.

Convex Bodies, Inradius and Circumradius

- A **convex body** C is a compact convex set in $\mathbb{R}^n$ that has a nonempty interior.
- The **inradius** r of C is the supremum of the set of radii of closed balls lying in C.
- The **circumradius** R of C is the infimum of the set of radii of closed balls in $\mathbb{R}^n$ containing C.
- Clearly both r and R are positive real numbers satisfying $r \leq R$.

Convex Bodies and Balls

Theorem

Let C be a convex body in $\mathbb{R}^n$ with inradius r and circumradius R. Then C contains a closed ball of radius r and is contained in a unique closed ball of radius R.

- The definition of R implies that, for each, $j = 1, 2, \ldots$, there exist $\boldsymbol{a}_j \in \mathbb{R}^n$ and $R_j > 0$ such that $C \subseteq B[\boldsymbol{a}_j; R_j]$ and $R_j < R + \frac{1}{j}$. The sequence $R_1, \ldots, R_j, \ldots$ converges to R. The sequence $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_j, \ldots$ is bounded. Thus there is some subsequence $\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_j}, \ldots$ of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_j, \ldots$ that converges to some point $\boldsymbol{a}$ of $\mathbb{R}^n$. It follows from a previous example that $B[\boldsymbol{a}_{i_j}; R_{i_j}] \to B[\boldsymbol{a}; R]$ as $j \to \infty$.

  We show that $C \subseteq B[\boldsymbol{a}; R]$: Let $\boldsymbol{c} \in C$. Since $C \subseteq B[\boldsymbol{a}_{i_j}; R_{i_j}]$, $\|\boldsymbol{c} - \boldsymbol{a}_{i_j}\| \leq R_{i_j}$. Letting $j \to \infty$ in the last inequality, we find that $\|\boldsymbol{c} - \boldsymbol{a}\| \leq R$. Thus, $\boldsymbol{c} \in B[\boldsymbol{a}; R]$. So $C \subseteq B[\boldsymbol{a}; R]$.

Convex Bodies and Balls (Cont'd)

- The proof that C contains a closed ball of radius r is similar to the one which we have just given.

  Suppose that C lies in both of the closed balls $B[\boldsymbol{a};R]$ and $B[\boldsymbol{b};R]$ of radius R in $\mathbb{R}^n$. Then, for each $\boldsymbol{x}$ in C,

$$\begin{aligned}
\left\|\boldsymbol{x}-\tfrac{1}{2}(\boldsymbol{a}+\boldsymbol{b})\right\|^2 &= \|\boldsymbol{x}\|^2-\boldsymbol{x}\cdot\boldsymbol{a}-\boldsymbol{x}\cdot\boldsymbol{b}+\tfrac{1}{4}(\|\boldsymbol{a}\|^2+2\boldsymbol{a}\cdot\boldsymbol{b}+\|\boldsymbol{b}\|^2) \\
&= \tfrac{1}{2}(\|\boldsymbol{x}\|^2-2\boldsymbol{x}\cdot\boldsymbol{a}+\|\boldsymbol{a}\|^2)+\tfrac{1}{2}(\|\boldsymbol{x}\|^2-2\boldsymbol{x}\cdot\boldsymbol{b}+\|\boldsymbol{b}\|^2) \\
&\qquad -\tfrac{1}{4}(\|\boldsymbol{a}\|^2-2\boldsymbol{a}\cdot\boldsymbol{b}+\|\boldsymbol{b}\|^2) \\
&= \tfrac{1}{2}\|\boldsymbol{x}-\boldsymbol{a}\|^2+\tfrac{1}{2}\|\boldsymbol{x}-\boldsymbol{b}\|^2-\tfrac{1}{4}\|\boldsymbol{a}-\boldsymbol{b}\|^2 \\
&\leq R^2-\tfrac{1}{4}\|\boldsymbol{a}-\boldsymbol{b}\|^2.
\end{aligned}$$

  Hence, $C\subseteq B\left[\frac{1}{2}(\boldsymbol{a}+\boldsymbol{b});\sqrt{R^2-\frac{1}{4}\|\boldsymbol{a}-\boldsymbol{b}\|^2}\right]$. Since C cannot lie in a closed ball of radius less than R, we must have $\boldsymbol{a}=\boldsymbol{b}$. Thus there is precisely one closed ball of radius R in $\mathbb{R}^n$ which contains C.

Inballs, Incenters, Circumball, Circumcenter

- Let C be a convex body in $\mathbb{R}^n$ with inradius r and circumradius R.
- Then any closed ball of radius r lying in C is called an **inball** of C and its center an **incenter** of C.
- The unique closed ball of radius R which contains C is called the **circumball** of C and its center the **circumcenter** of C.
- A (non-square) rectangle in $\mathbb{R}^2$ is an example of a convex body that does not have a unique incentre.

Comparisons With Elementary Geometry

- Our definitions of the terms circumradius, circumcircle and circumcenter as applied to obtuse-angled triangles do not coincide with those used in elementary geometry.
- For example, consider an isosceles triangle with sides $2, 2, 2\sqrt{3}$.
  - In the parlance of elementary geometry, its circumradius is 2 and its circumcenter lies exterior to the triangle.
  - For us here its circumradius is $\sqrt{3}$ and its circumcenter is the midpoint of its longest side.

Extremal Problems and Uniform Boundedness

- The preceding theorem asserts the existence of solutions to two extremal problems in geometry.
  - The first to find a ball of minimal radius containing a given convex body;
  - The second to find a ball of maximal radius lying in the body.
- The key step in proving the theorem was the extraction of a convergent subsequence from a sequence of closed balls.
- It is a generalization of this idea that turns out to be useful in finding solutions to many extremal problems.
- What is needed is a criterion for a sequence of sets to contain a convergent subsequence.
- A sequence of sets in $\mathbb{R}^n$ is said to be **uniformly bounded** if there exists some ball in $\mathbb{R}^n$ that contains every member of the sequence.

Uniform Boundedness and Cauchy Sequences

Lemma

Let $A_1, \ldots, A_j, \ldots$ be a uniformly bounded sequence of nonempty compact sets in $\mathbb{R}^n$. Let $\varepsilon > 0$. Then there exists a subsequence $A_{i_1}, \ldots, A_{i_j}, \ldots$ of $A_1, \ldots, A_j, \ldots$ such that $\rho(A_{i_j}, A_{i_k}) \le \varepsilon$, for all $j, k = 1, 2, \ldots$.

- Since there is a ball in $\mathbb{R}^n$ which contains every member of the given sequence, there is a finite set E in $\mathbb{R}^n$ such that $A_j \subseteq (E)_{\frac{1}{2}\varepsilon}$, for $j = 1, 2, \ldots$. For each $j = 1, 2, \ldots$ denote by E_j the non-empty subset $E \cap (A_j)_{\frac{1}{2}\varepsilon}$ of E. It is easily verified that $\rho(E_j, A_j) \le \frac{1}{2}\varepsilon$. Because E is finite, there can only be a finite number of possible different sets E_j for $j = 1, 2, \ldots$. Hence the sequence $E_1, \ldots, E_j, \ldots$ must contain some constant subsequence, $E_{i_1}, \ldots, E_{i_j}, \ldots$ say. For $j, k = 1, 2, \ldots$, we have

$$\begin{aligned} \rho(A_{i_j}, A_{i_k}) &\le \rho(A_{i_j}, E_{i_j}) + \rho(E_{i_j}, A_{i_k}) \\ &= \rho(A_{i_j}, E_{i_j}) + \rho(E_{i_k}, A_{i_k}) \le \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

Uniform Boundedness and Convergence

Theorem

Every uniformly bounded sequence $A_1,\ldots,A_j,\ldots$ of nonempty compact sets in $\mathbb{R}^n$ contains a subsequence which converges to some nonempty compact set A in $\mathbb{R}^n$.

- It follows, by repeated applications of the lemma with $\varepsilon = 1, \frac{1}{2}, \ldots, \frac{1}{j}, \ldots$ that the sequence $A_1,\ldots,A_j,\ldots$ contains a sequence of subsequences

$$\begin{array}{l} A_{11}, A_{12}, \ldots, A_{1j}, \ldots \\ A_{21}, A_{22}, \ldots, A_{2j}, \ldots \\ \ldots \\ A_{r1}, A_{r2}, \ldots, A_{rj}, \ldots \\ \ldots \end{array}$$

  where each subsequence after the first in the list is a subsequence of the preceding one, and $\rho(A_{rj}, A_{rk}) \leq \frac{1}{r}$, for $j,k = 1,2,\ldots$.
  The diagonal sequence $A_{11}, A_{22}, \ldots, A_{jj}, \ldots$ is a subsequence of $A_1, A_2, \ldots, A_j, \ldots$ with the property that $\rho(A_{jj}, A_{kk}) \leq \frac{1}{j}$ whenever $j \leq k$.

Uniform Boundedness and Convergence (Cont'd)

- Write $B_j = A_{jj}$ for $j = 1,2,\ldots$. We complete the proof by showing that the subsequence $B_1,\ldots,B_j,\ldots$ of $A_1,\ldots,A_j,\ldots$ converges to the nonempty compact set B defined by $B = \bigcap((B_k)_{\frac{1}{k}} : k = 1,2,\ldots)$.

  Let j be a positive integer and let $\boldsymbol{b}_j \in B_j$. For $i = 1,2,\ldots$, choose $\boldsymbol{b}_{j+i} \in B_{j+i}$ such that $\|\boldsymbol{b}_{j+i} - \boldsymbol{b}_j\| \le \frac{1}{j}$; This is possible because $\rho(B_j, B_{j+i}) \le \frac{1}{j}$. The sequence $\boldsymbol{b}_{j+1}, \boldsymbol{b}_{j+2},\ldots$ lies in the compact set $(B_1)_1$. So it contains a subsequence converging to some point $\boldsymbol{b}$ of $\mathbb{R}^n$. Since $B_{j+i} \subseteq (B_k)_{\frac{1}{k}}$ whenever $j+i \ge k$, all but a finite number of terms of the sequence $\boldsymbol{b}_{j+1}, \boldsymbol{b}_{j+2},\ldots$ lie in the compact set $(B_k)_{\frac{1}{k}}$ for $k = 1,2,\ldots$. Hence $\boldsymbol{b} \in (B_k)_{\frac{1}{k}}$ and $\boldsymbol{b} \in B$. But $\boldsymbol{b}_j$ is an arbitrary point of B_j and clearly $\|\boldsymbol{b} - \boldsymbol{b}_j\| \le \frac{1}{j}$. So $B_j \subseteq (B)_{\frac{1}{j}}$. Trivially $B \subseteq (B_j)_{\frac{1}{j}}$. Thus, $\rho(B_j, B) \le \frac{1}{j}$ and $B_j \to B$ as $j \to \infty$.

Blaschke Selection Principle

Theorem (Blaschke Selection Principle)

Every uniformly bounded sequence of non-empty compact convex sets in $\mathbb{R}^n$ contains a subsequence which converges to some non-empty compact convex set in $\mathbb{R}^n$.

- The principle is a consequence of the theorem and the fact that a convergent sequence of convex sets must converge to a convex limit.

Continuity of Functions on Families of Compact Sets

- A typical extremal problem of elementary geometry is to maximize or minimize a real-valued function f defined on some family $\mathscr{F}$ of nonempty compact sets in $\mathbb{R}^n$.
- It is important to have a concept of continuity for such functions $f : \mathscr{F} \to \mathbb{R}$.
- The function f is said to be **continuous** on $\mathscr{F}$ if $f(A_j) \to f(A)$ as $j \to \infty$, whenever $A_j \to A$ as $j \to \infty$, where all the sets under consideration belong to $\mathscr{F}$.

Example

- We show that the diameter function D, which associates with each nonempty compact set A in $\mathbb{R}^n$ its diameter $D(A)$, is continuous on the family $\mathscr{F}$ of all nonempty compact sets in $\mathbb{R}^n$.
- It is easily verified that $D((A)_\lambda) = D(A) + 2\lambda$, where $A \in \mathscr{F}$ and $\lambda \geq 0$.
- Suppose now that $A, B \in \mathscr{F}$ and that $\rho(A, B) = \lambda$.
- Then

$$D(A) \leq D((B)_\lambda) = D(B) + 2\lambda \quad \text{and} \quad D(B) \leq D((A)_\lambda) = D(A) + 2\lambda.$$

- Hence $|D(A) - D(B)| \leq 2\lambda = 2\rho(A, B)$.
- The continuity of D is now clear.

An Isodiametric Problem

- We now indicate how the Blaschke selection principle can be used to show that in the family $\mathscr{F}$ of all compact convex sets in $\mathbb{R}^2$ with diameter 1, there exist sets having maximal area.
- For each set A in $\mathscr{F}$, denote by $f(A)$ the area of A.
- This area function will be defined formally later and it will be shown to be continuous on the family of all non-empty compact convex sets in $\mathbb{R}^2$.
- Let α be the supremum of the set of areas of members of $\mathscr{F}$.
- For each positive integer j, there is a member A_j of $\mathscr{F}$ such that $f(A_j) > \alpha - \frac{1}{j}$.
- We may suppose, by translating the A_j's if necessary, that they are uniformly bounded.

An Isodiametric Problem (Cont'd)

- The Blaschke selection principle guarantees the existence of a subsequence $A_{i_1},\ldots,A_{i_j},\ldots$ of $A_1,\ldots,A_j,\ldots$ which converges to some non-empty compact convex set A_0 in $\mathbb{R}^2$.
- The continuity of the diameter function shows that A_0 has diameter 1.
- So A_0 lies in $\mathscr{F}$.
- For each j,

$$\alpha \geq f(A_{i_j}) > \alpha - \frac{1}{i_j} \geq \alpha - \frac{1}{j}.$$

- Letting $j \to \infty$ in these inequalities, we deduce, using the continuity of f, that $f(A_0) = \alpha$.
- Thus A_0 is a member of $\mathscr{F}$ having maximal possible area.

Subsection 8

Duality

The Dual Operator

- For each nonzero vector $\boldsymbol{u}$ in $\mathbb{R}^n$, the set $H^- = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{u} \cdot \boldsymbol{x} \leq 1\}$ is a closed halfspace in $\mathbb{R}^n$ containing the origin as an interior point.
- Conversely, for each closed halfspace H^- in $\mathbb{R}^n$ containing the origin as an interior point, there is a unique non-zero vector $\boldsymbol{u}$ in $\mathbb{R}^n$ such that $H^- = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{u} \cdot \boldsymbol{x} \leq 1\}$.
- Thus there is a bijection between the set of nonzero vectors in $\mathbb{R}^n$ and the set of closed halfspaces in $\mathbb{R}^n$ containing the origin as an interior point.
- We define, for each point $\boldsymbol{u}$ of $\mathbb{R}^n$, a set $\boldsymbol{u}^*$ in $\mathbb{R}^n$ by the equation

$$\boldsymbol{u}^* = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{u} \cdot \boldsymbol{x} \leq 1\}.$$

- We note that $\mathbf{0}^* = \mathbb{R}^n$.

Polar Duals of Sets

- Define the **polar dual** A^* of a set A in $\mathbb{R}^n$ to be the intersection of all the sets $\boldsymbol{a}^*$, for $\boldsymbol{a} \in A$, i.e.,

$$A^* = \bigcap(\boldsymbol{a}^* : \boldsymbol{a} \in A) = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a} \cdot \boldsymbol{x} \leq 1, \text{ for all } \boldsymbol{a} \in A\}.$$

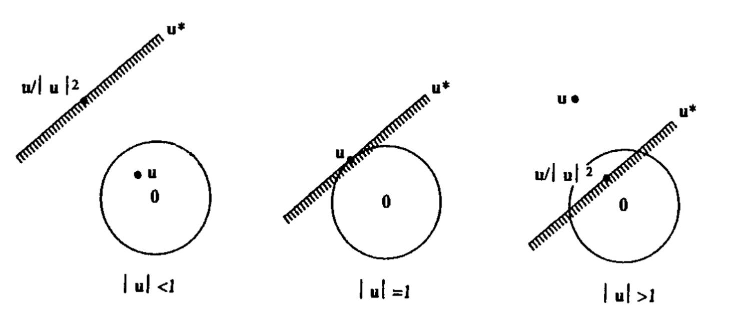
- For each set A in $\mathbb{R}^n$, its polar dual A^* is defined as an intersection of closed convex sets containing the origin.
- So A^* is itself a closed convex set containing the origin.
- The polar duals $\varnothing^*$ and $\{\mathbf{0}\}^*$ are both $\mathbb{R}^n$.
- The polar dual of $\mathbb{R}^n$ is $\{\mathbf{0}\}$.

Positions of Vector and its Dual

- It is instructive to examine the sets $\boldsymbol{u}^*$ for nonzero vectors $\boldsymbol{u}$ in $\mathbb{R}^n$.
- By definition, $\boldsymbol{u}^*$ is the closed halfspace which is bounded by the hyperplane $H = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{u} \cdot \boldsymbol{x} = 1\}$ and contains the origin.
- This hyperplane H has $\boldsymbol{u}$ as one of its normal vectors and passes through the point $\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|^2}$.
- The distance of $\frac{\boldsymbol{u}}{\|\boldsymbol{u}\|^2}$ from the origin is $\frac{1}{\|\boldsymbol{u}\|}$, which:
  - exceeds 1 if $\|\boldsymbol{u}\|$ is less than 1;
  - equals 1 if $\|\boldsymbol{u}\|$ equals 1;
  - is less than 1 if $\|\boldsymbol{u}\|$ exceeds 1.

Vector and Its Dual Illustrated

- The relative positions of $\boldsymbol{u}$ and $\boldsymbol{u}^*$ for the cases $\|\boldsymbol{u}\| < 1$, $\|\boldsymbol{u}\| = 1$ and $\|\boldsymbol{u}\| > 1$ are illustrated in the figure:



Properties of the Polar Dual

Theorem

Let A, B be sets in $\mathbb{R}^n$, U the closed unit ball centered on the origin of $\mathbb{R}^n$ and λ a nonzero scalar. Then:

(i) $A \subseteq B$ implies that $B^* \subseteq A^*$;

(ii) $(A \cup B)^* = A^* \cap B^*$;

(iii) $(\lambda A)^* = \frac{1}{\lambda} A^*$;

(iv) $U^* = U$.

(i) Suppose that $A \subseteq B$. Then

$$B^* = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{b} \cdot \boldsymbol{x} \leq 1, \boldsymbol{b} \in B\} \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a} \cdot \boldsymbol{x} \leq 1, \boldsymbol{a} \in A\} = A^*.$$

(ii) We have $\boldsymbol{x} \in (A \cup B)^*$ if and only if both $\boldsymbol{a} \cdot x \leq 1$, for $\boldsymbol{a} \in A$, and $\boldsymbol{b} \cdot \boldsymbol{x} \leq 1$, for $\boldsymbol{b} \in B$, if and only if $\boldsymbol{x} \in A^* \cap B^*$. So $(A \cup B)^* = A^* \cap B^*$.

Properties of the Polar Dual (Cont'd)

(iii) We have

$$\begin{aligned} \boldsymbol{x} \in (\lambda A)^* \quad & \text{iff} \quad \lambda \boldsymbol{a} \cdot \boldsymbol{x} \leq 1, \text{ for } \boldsymbol{a} \in A, \\ & \text{iff} \quad \boldsymbol{a} \cdot (\lambda \boldsymbol{x}) \leq 1, \text{ for } \boldsymbol{a} \in A, \\ & \text{iff} \quad \lambda \boldsymbol{x} \in A^* \\ & \text{iff} \quad \boldsymbol{x} \in \frac{1}{\lambda} A^*. \end{aligned}$$

Thus $(\lambda A)^* = \frac{1}{\lambda} A^*$.

(iv) Suppose, first, that $\boldsymbol{x} \in U$. Then, for all $\boldsymbol{u} \in U$, $\boldsymbol{u} \cdot \boldsymbol{x} \leq \|\boldsymbol{u}\| \|\boldsymbol{x}\| \leq 1$. Hence $\boldsymbol{x} \in U^*$ and $U \subseteq U^*$.

Conversely, let $\boldsymbol{x} \in U^*$ with $\boldsymbol{x} \neq \mathbf{0}$. Then $\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \in U$. So $\frac{\boldsymbol{x}}{\|\boldsymbol{x}\|} \cdot \boldsymbol{x} = \|\boldsymbol{x}\| \leq 1$. This shows that $\boldsymbol{x} \in U$. Thus $U^* \subseteq U$.

Example

- We find the polar dual A^* of the n-cube A defined by the equation
$$A = \{(a_1, \ldots, a_n) : |a_1| \le 1, \ldots, |a_n| \le 1\}.$$
- Suppose $(x_1, \ldots, x_n) \in A^*$. Define $(a_1, \ldots, a_n) \in A$ by $a_i = 1$, if $x_i \ge 0$, and $a_i = -1$, if $x_i < 0$. Then
$$|x_1| + \cdots + |x_n| = a_1 x_1 + \cdots + a_n x_n = (a_1, \ldots, a_n) \cdot (x_1, \ldots, x_n) \le 1.$$
- Conversely, suppose $(x_1, \ldots, x_n)$ satisfies $|x_1| + \cdots + |x_n| \le 1$. Then, for any point $(a_1, \ldots, a_n) \in A$,
$$\begin{array}{rcl} (a_1, \ldots, a_n) \cdot (x_1, \ldots, x_n) & = & a_1 x_1 + \cdots + a_n x_n \\ & \le & |a_1||x_1| + \cdots + |a_n||x_n| \\ & \le & |x_1| + \cdots + |x_n| \le 1. \end{array}$$
Thus, $(x_1, \ldots, x_n) \in A^*$.
- So A^* is the set, known as a **regular** ***n*****-crosspolytope**, defined by the equation $A^* = \{(x_1, \ldots, x_n) : |x_1| + \cdots + |x_n| \le 1\}$.

Example (Cont'd)

- We now find A^{**} the polar dual of the polar dual A^* of the n-cube A.
- For each point $(u_1, \ldots, u_n)$ of A^{**}, define a point $(x_1, 0, \ldots, 0)$ of A^* by the conditions that $x_1 = 1$ if $u_1 \geq 0$, and $x_1 = -1$ if $u_1 < 0$. Then

$$(x_1, 0, \ldots, 0) \cdot (u_1, u_2, \ldots, u_n) = |u_1| < 1.$$

  Similarly, $|u_2| \leq 1, \ldots, |u_n| \leq 1$. Hence, $(u_1, \ldots, u_n) \in A$ and $A^{**} \subseteq A$.
- The inclusion $A \subseteq A^{**}$, which holds for every set A in $\mathbb{R}^n$, follows immediately from the definition of the polar dual.
- Hence $A^{**} = A$.

The Double-Polar Dual of a Set

- This last example suggests that we examine the **double-polar dual** A^{**} of an arbitrary set A in $\mathbb{R}^n$ and see how it is related to A.
- The polar dual of any set in $\mathbb{R}^n$ is always a closed convex set containing the origin.
- So a necessary condition for the equality of the sets A and A^{**} is that A is a closed convex set containing the origin.
- We aim to show that this condition is also sufficient, and establish the exact relationship between A^{**} and A.

A Set and Its Double-Polar Dual

Theorem

Let A be a set in $\mathbb{R}^n$. Then $A^{**} = \text{cl}(\text{conv}(A \cup \{\mathbf{0}\}))$. In particular, if A is closed, convex and contains the origin, then $A^{**} = A$.

- For all $\boldsymbol{a} \in A$, $\boldsymbol{x} \in A^*$, we have $\boldsymbol{a} \cdot \boldsymbol{x} \leq 1$. Hence $\boldsymbol{a} \in A^{**}$. So $A \subseteq A^{**}$. But A^{**} is a closed convex set containing $\mathbf{0}$ and A. Therefore, $\text{cl}(\text{conv}(A \cup \{\mathbf{0}\})) \subseteq A^{**}$.

  For the reverse inclusion, suppose $\boldsymbol{z}$ is a point of $\mathbb{R}^n$ not lying in $\text{cl}(\text{conv}(A \cup \{\mathbf{0}\}))$. By a previous corollary, there exists a hyperplane strictly separating $\{\boldsymbol{z}\}$ and $\text{cl}(\text{conv}(A \cup \{\mathbf{0}\}))$. Thus, since such a hyperplane cannot pass through the origin, there exists $\boldsymbol{u}$ in $\mathbb{R}^n$ such that $\boldsymbol{u} \cdot \boldsymbol{z} > 1$ and $\boldsymbol{u} \cdot \boldsymbol{a} < 1$, for all $\boldsymbol{a}$ in A. This shows that $\boldsymbol{u} \in A^*$ and $\boldsymbol{z} \notin A^{**}$. Hence, $A^{**} \subseteq \text{cl}(\text{conv}(A \cup \{\mathbf{0}\}))$.

Boundedness and the Origin

Theorem

Let A be a closed convex set in $\mathbb{R}^n$ containing the origin. Then A is bounded if and only if the origin is an interior point of A^*, and A^* is bounded if and only if the origin is an interior point of A.

- We use two previous theorems.

  Suppose first that A is bounded. Then, for some $r > 0$, $A \subseteq rU$. Hence, $\frac{1}{r}U \subseteq A^*$. So the origin is an interior point of A^*.

  By applying the last result to the set A^*, we deduce that, if A^* is bounded, then the origin is an interior point of $A^{**} = A$.

  Suppose next that the origin is an interior point of A. Then, for some $s > 0$, $sU \subseteq A$. Hence, $A^* \subseteq \frac{1}{s}U$. So A^* is bounded.

  By applying the last result to A^*, we deduce that, if the origin is an interior point of A^*, then $A^{**} = A$ is bounded.

Polar Duals of Compacts With Origin an Interior Point

Corollary

Let $\mathscr{F}$ be the family of all compact convex sets in $\mathbb{R}^n$ which contain the origin as an interior point. Then the mapping $\theta : \mathscr{F} \to \mathscr{F}$ defined for $A \in \mathscr{F}$ by the equation $\theta(A) = A^*$ is a bijection.

- Let $A, B \in \mathscr{F}$. The theorem shows that $\theta(A) \in \mathscr{F}$.

  θ is injective, for $A^* = B^*$ implies $A^{**} = B^{**}$, i.e., $A = B$.

  It is surjective, since $\theta(A^*) = A$.

The Polar Face Mapping

- Suppose that A is a compact convex set in $\mathbb{R}^n$ that contains the origin as an interior point.
- Let B be an exposed face of A.
- Then, for each point $\boldsymbol{b}$ in B, the set $\{\boldsymbol{x} \in A^* : \boldsymbol{b} \cdot \boldsymbol{x} = 1\}$ is an exposed face (possibly empty) of A^*.
- Thus the set $\varphi(B)$, defined by the equation

$$\varphi(B) = \{\boldsymbol{x} \in A^* : \boldsymbol{b} \cdot \boldsymbol{x} = 1, \text{ for } \boldsymbol{b} \in B\},$$

  being an intersection of exposed faces of A^*, is itself an exposed face of A^*.
- In this way we have constructed a mapping φ from the set of exposed faces of A to the set of exposed faces of A^*.
- We call φ the **polar face mapping** of A.

Properties of The Polar Face Mapping

Theorem

Let A be a compact convex set in $\mathbb{R}^n$ which contains the origin as an interior point. Then the polar face mapping φ of A is an inclusion-reversing bijection.

- That φ is inclusion reversing follows immediately from its definition.

  Let ψ be the polar face mapping of A^*. We show that, for each exposed face B of A, $\psi(\varphi(B)) = B$. This is clear when B is either $\emptyset$ or A. We assume that B is a proper exposed face of A. Thus there is $\boldsymbol{u}$ in $\mathbb{R}^n$ such that

$$B = \{\boldsymbol{a} \in A : \boldsymbol{u} \cdot \boldsymbol{a} = 1\} \quad \text{and} \quad \boldsymbol{u} \cdot \boldsymbol{a} \le 1, \text{ for } \boldsymbol{a} \in A.$$

  This shows that $\boldsymbol{u} \in A^*$ and $\boldsymbol{u} \in \varphi(B)$.

Properties of The Polar Face Mapping (Cont'd)

- Let $\boldsymbol{v} \in \psi(\varphi(B))$. Then $\boldsymbol{v} \in A^{**} = A$ and $\boldsymbol{u} \cdot \boldsymbol{v} = 1$. Hence $\boldsymbol{v} \in B$ and $\psi(\varphi(B)) \subseteq B$.

  Conversely, let $\boldsymbol{b} \in B$. Then $\boldsymbol{b} \in A = (A^*)^*$ and $\boldsymbol{b} \cdot \boldsymbol{w} = 1$, for all $\boldsymbol{w} \in \varphi(B)$. Hence, $\boldsymbol{b} \in \psi(\varphi(B))$ and $B \subseteq \psi(\varphi(B))$.

  Thus, $\psi(\varphi(B)) = B$.

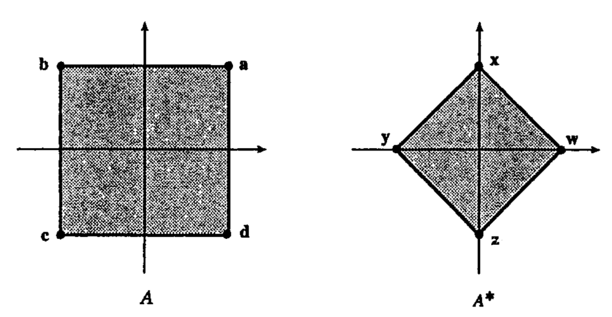
  We have just shown that the composite mapping $\psi \circ \varphi$ is the identity mapping on the set of exposed faces of A.

  By interchanging the roles of A and A^* in the discussion above, we can deduce that $\varphi \circ \psi$ is the identity mapping on the set of exposed faces of A^*.

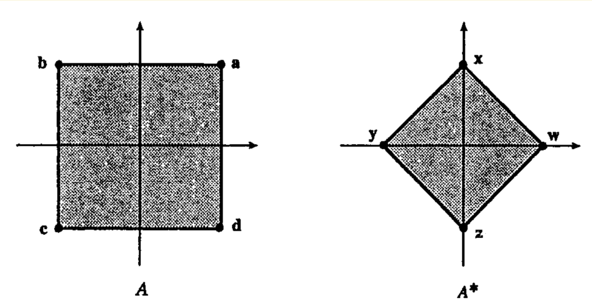
  It now follows easily that φ is a bijection.

Example

- Let A be the square conv$\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c},\boldsymbol{d}\}$ in $\mathbb{R}^2$, where $\boldsymbol{a}=(1,1)$, $\boldsymbol{b}=(-1,1)$, $\boldsymbol{c}=(-1,-1)$, $\boldsymbol{d}=(1,-1)$.
- We saw that the polar dual A^* of A is the square conv$\{\boldsymbol{w},\boldsymbol{x},\boldsymbol{y},\boldsymbol{z}\}$, where $\boldsymbol{w}=(1,0)$, $\boldsymbol{x}=(0,1)$, $\boldsymbol{y}=(-1,0)$, $\boldsymbol{w}=(0,-1)$.



Example (Cont'd)



- The polar face mapping φ of A is indicated below, where the faces of A and $A^*$are represented by their extreme points.

$$\begin{array}{ccccccccccc} & \emptyset & \boldsymbol{a} & \boldsymbol{b} & \boldsymbol{c} & \boldsymbol{d} & \boldsymbol{a},\boldsymbol{b} & \boldsymbol{b},\boldsymbol{c} & \boldsymbol{c},\boldsymbol{d} & \boldsymbol{d},\boldsymbol{a} & A \\ \varphi & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & A^* & \boldsymbol{w},\boldsymbol{x} & \boldsymbol{x},\boldsymbol{y} & \boldsymbol{y},\boldsymbol{z} & \boldsymbol{z},\boldsymbol{w} & \boldsymbol{x} & \boldsymbol{y} & \boldsymbol{z} & \boldsymbol{w} & \emptyset \end{array}$$