Introduction to Convexity

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Subsection 1

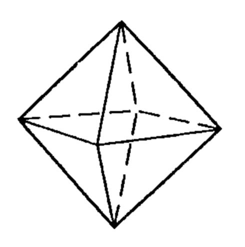
Polytopes

Polytopes and Simplexes

- A **convex polytope**, or simply a **polytope**, is the convex hull of a *finite* set of points in $\mathbb{R}^n$.
- Points, line segments, polygons, tetrahedra, cubes, octahedra, dodecahedra and icosahedra are all polytopes.
- Since the convex hull of a finite set in $\mathbb{R}^n$ is compact, polytopes are compact convex sets.
- A polytope of dimension r is called an r-**polytope**.
- The simplest example of an r-polytope is an r-**simplex** ($r = -1, \ldots, n$), which is defined to be the convex hull of an affinely independent set in $\mathbb{R}^n$ consisting of $r+1$ points.
- There is precisely one (-1)-simplex, namely the empty set.
- We refer to a 0-simplex as a **point**, a 1-simplex as a **line segment**, a 2-simplex as a **triangle**, and a 3-simplex as a **tetrahedron**.

Crosspolytopes

- An important example of an r-polytope is an r-**crosspolytope** ($r = 1, \ldots, n$), which is defined to be the convex hull of r linearly independent line segments in $\mathbb{R}^n$ whose midpoints coincide, i.e., a translate of a set of the form $\text{conv}\{\pm \boldsymbol{a}_1, \ldots, \pm \boldsymbol{a}_r\}$, where $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r\}_{\neq}$ is a linearly independent set of vectors in $\mathbb{R}^n$.
- Such a crosspolytope is called **regular** when the $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$ have equal lengths and are mutually orthogonal.
- Thus, $\text{conv}\{\pm \boldsymbol{e}_1, \ldots, \pm \boldsymbol{e}_r\}$, where $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_r$ are elementary vectors in $\mathbb{R}^n$, is a regular r-crosspolytope.
- In $\mathbb{R}^3$ a regular 2-crosspolytope is a square, and a regular 3-crosspolytope is a regular octahedron, which is a regular solid bounded by eight congruent equilateral triangles.



Addition and Scalar Multiplication

Theorem

Let A, B be polytopes in $\mathbb{R}^n$ and let $\alpha \in \mathbb{R}$. Then $A+B$ and αA are polytopes.

- We consider the non-trivial case when neither A nor B is empty.

  Let $A = \text{conv}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k\}$, $B = \text{conv}\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_m\}$, where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$, $\boldsymbol{b}_1, \ldots, \boldsymbol{b}_m \in \mathbb{R}^n$. Denote by C the finite set consisting of all those points of the form $\boldsymbol{a}_i + \boldsymbol{b}_j$, where $i = 1, \ldots, k$ and $j = 1, \ldots, m$, and by D the finite set whose points are $\alpha \boldsymbol{a}_1, \ldots, \alpha \boldsymbol{a}_k$. We prove the theorem by showing that $A + B = \text{conv}\, C$ and $\alpha A = \text{conv}\, D$.

Addition and Scalar Multiplication (Cont'd)

- Now $A+B$ is a convex set containing C. Hence, $\text{conv}\, C \subseteq A+B$. If $\boldsymbol{x} \in A+B$, then there exist scalars $\lambda_1,\ldots,\lambda_k$, $\mu_1,\ldots,\mu_m \geq 0$ with $\lambda_1+\cdots+\lambda_k=1$ and $\mu_1+\cdots+\mu_m=1$ such that

$$\begin{array}{rcl} \boldsymbol{x} & = & \lambda_1\boldsymbol{a}_1+\cdots+\lambda_k\boldsymbol{a}_k+\mu_1\boldsymbol{b}_1+\cdots+\mu_m\boldsymbol{b}_m \\ & = & \sum_{i=1}^k\sum_{j=1}^m \lambda_i\mu_j(\boldsymbol{a}_i+\boldsymbol{b}_j). \end{array}$$

  This shows that $\boldsymbol{x}$ is a convex combination of points of C. Hence, $\boldsymbol{x} \in \text{conv}\, C$ and $A+B \subseteq \text{conv}\, C$.

  Now αA is a convex set containing D. Hence, $\text{conv}\, D \subseteq \alpha A$. If $\boldsymbol{x} \in \alpha A$, then there exist $\lambda_1,\ldots,\lambda_k \geq 0$ with $\lambda_1+\cdots+\lambda_k=1$ such that

$$\boldsymbol{x} = \alpha(\lambda_1\boldsymbol{a}_1+\cdots+\lambda_k\boldsymbol{a}_k) = \lambda_1(\alpha\boldsymbol{a}_1)+\cdots+\lambda_k(\alpha\boldsymbol{a}_k).$$

  This shows that $\boldsymbol{x}$ is a convex combination of points of D. Hence, $\boldsymbol{x} \in \text{conv}\, D$ and $\alpha A \subseteq \text{conv}\, D$.

Zonotopes and r-Cubes

Corollary

Let $A_1,\ldots,A_m$ be polytopes in $\mathbb{R}^n$ and let $\alpha_1,\ldots,\alpha_m \in \mathbb{R}$. Then $\alpha_1 A_1 + \cdots + \alpha_m A_m$ is a polytope.

- Thus, the vector sum of a finite number of line segments in $\mathbb{R}^n$ is a polytope. Such a polytope is called a **zonotope**.
- An r-**cube** $(r = 1,\ldots,n)$ in $\mathbb{R}^n$ is the vector sum of r mutually orthogonal line segments in $\mathbb{R}^n$, all of equal length,
  i.e., a set of the form
  $$\text{conv}\{\boldsymbol{a}_1,\boldsymbol{b}_1\} + \cdots + \text{conv}\{\boldsymbol{a}_r,\boldsymbol{b}_r\},$$
  where $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_r$, $\boldsymbol{b}_1,\ldots,\boldsymbol{b}_r \in \mathbb{R}^n$, $(\boldsymbol{a}_i - \boldsymbol{b}_i)\cdot(\boldsymbol{a}_j - \boldsymbol{b}_j) = 0$ if and only if $i \neq j$, and $\|\boldsymbol{a}_1 - \boldsymbol{b}_i\| = \|\boldsymbol{a}_j - \boldsymbol{b}_j\|$ for all i,j.
- An example of an n-cube with **edge-length** 1 in $\mathbb{R}^n$ is the polytope
  $$\text{conv}\{\mathbf{0},\boldsymbol{e}_1\} + \cdots + \text{conv}\{\mathbf{0},\boldsymbol{e}_n\} = \{(x_1,\ldots,x_n) : 0 \le x_1,\ldots,x_n \le 1\}.$$

Vertices and Edges of a Polytope

- We now look at the facial structure of a polytope P in $\mathbb{R}^n$.
- It is customary to call the extreme points of P its **vertices** and its 1-faces its **edges**.
- The set of all P's vertices is called its **vertex set**.
- If $P = \text{conv}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$, for some $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbb{R}^n$, then a previous corollary shows that the vertex set of P is contained in $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$.

Property of Faces of a Polytope

Theorem

Every polytope in $\mathbb{R}^n$ has only a finite number of faces, and each of these is a polytope.

- Consider a non-empty polytope $A = \text{conv}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$, where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbb{R}^n$. By a previous theorem each face F of A is the convex hull of its extreme points. Another theorem shows that each extreme point of F is also an extreme point of A. Hence F is the convex hull of some subset of the vertex set of A. Since $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ contains the vertex set of A, it follows that F is the convex hull of some subset of $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$. The desired result is now immediate.

Subsets of Vertex Set Determining a Face

- Suppose that V is the vertex set of a polytope P in $\mathbb{R}^n$.
- Then the proof of the last theorem shows that each face of P has the form $\text{conv}W$, for some subset W of V.
- The question naturally arises as to which subsets W of V **determine** a face of P, i.e. are such that $\text{conv}W$ is a face of P.

Vertex Subsets That Determine Faces

Theorem

Let W be a subset of the vertex set V of a polytope P in $\mathbb{R}^n$. Then $\operatorname{conv} W$ is a face of P if and only if

$$(\operatorname{aff} W) \cap \operatorname{conv}(V \setminus W) = \emptyset.$$

- Suppose first that $\operatorname{conv} W$ is a face of P. If $\boldsymbol{v} \in V \setminus W$, then $P \setminus \{\boldsymbol{v}\}$ is convex, by a previous theorem, and contains W. Hence, $\operatorname{conv} W \subseteq P \setminus \{\boldsymbol{v}\}$. So $\boldsymbol{v} \notin \operatorname{conv} W$. Therefore, $V \setminus W \subseteq P \setminus \operatorname{conv} W$.

  By the same theorem, $P \setminus \operatorname{conv} W$ is convex. So $\operatorname{conv}(V \setminus W) \subseteq P \setminus \operatorname{conv} W$. Also by the same theorem, $(\operatorname{aff} W) \cap P = \operatorname{conv} W$. Hence,

$$\begin{aligned}(\operatorname{aff} W) \cap \operatorname{conv}(V \setminus W) &\subseteq (\operatorname{aff} W) \cap (P \setminus \operatorname{conv} W) \\ &\subseteq \operatorname{conv} W \cap (P \setminus \operatorname{conv} W) = \emptyset.\end{aligned}$$

Characterization of Face Determinators (Converse)

- Suppose $(\text{aff}\, W) \cap \text{conv}(V \setminus W) = \emptyset$ is satisfied.

  Clearly $\text{conv}\, W$ is a face of P if either W is empty or V.

  So we assume that this is not the case. Let $V = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_s\}_{\neq}$ and $W = \{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$, where $1 \le r < s$. Let $\boldsymbol{w} = \lambda \boldsymbol{x} + \mu \boldsymbol{y}$, where $\boldsymbol{w} \in \text{conv}\, W$, $\boldsymbol{x}, \boldsymbol{y} \in P$, and $\lambda, \mu > 0$ with $\lambda + \mu = 1$. Then $\boldsymbol{x} = \lambda_1 \boldsymbol{v}_1 + \cdots + \lambda_s \boldsymbol{v}_s$, $\boldsymbol{y} = \mu_1 \boldsymbol{v}_1 + \cdots + \mu_s \boldsymbol{v}_s$, for some $\lambda_1, \ldots, \lambda_s$, $\mu_1, \ldots, \mu_s \geq$ with $\lambda_1 + \cdots + \lambda_s = 1$ and $\mu_1 + \cdots + \mu_s = 1$. For $i = 1, \ldots, s$, write $\nu_i = \lambda \lambda_i + \mu \mu_i$. Then $\nu_1, \ldots, \nu_s \geq 0$, $\nu_1 + \cdots + \nu_s = 1$ and $\boldsymbol{w} = \nu_1 \boldsymbol{v}_1 + \cdots + \nu_s \boldsymbol{v}_s$. Write $\alpha = \nu_{r+1} + \cdots + \nu_s$. If $\alpha > 0$, then the point

$$\frac{1}{\alpha}(\boldsymbol{w} - \nu_1 \boldsymbol{v}_1 - \cdots - \nu_r \boldsymbol{v}_r) = \frac{1}{\alpha}(\nu_{r+1} \boldsymbol{v}_{r+1} + \cdots + \nu_s \boldsymbol{v}_s)$$

  lies both in $\text{aff}\, W$ and $\text{conv}(V \setminus W)$, which contradicts the hypothesis. Thus, $\alpha = 0$. This entails $\nu_{r+1}, \ldots \nu_s = 0$ and $\lambda_{r+1}, \ldots, \lambda_s, \mu_{r+1}, \ldots, \mu_s = 0$. Hence $\boldsymbol{x}, \boldsymbol{y} \in \text{conv}\, W$. So $\text{conv}\, W$ is a face of P.

Arbitrary Subsets and Faces

- In proving the "if" part of the last theorem, we used the fact that $\operatorname{conv} V = P$, but not the fact that each element of V was a vertex of P.
- We thus have the following:

Corollary

Let W be a subset of a finite set V in $\mathbb{R}^n$ such that

$$(\operatorname{aff} W) \cap \operatorname{conv}(V \setminus W) = \emptyset.$$

Then $\operatorname{conv} W$ is a face of the polytope $\operatorname{conv} V$.

Facial Structure of Simplexes

- Suppose that $S = \text{conv} K$, where V is an affinely independent set in $\mathbb{R}^n$.
- We have already seen that each face of S is the convex hull of some subset of V.
- Now we establish the converse:

  Let $W \subseteq V$. Since V is affinely independent,

  $$(\text{aff}\, W) \cap \text{conv}(V \setminus W) \subseteq (\text{aff}\, W) \cap \text{aff}(V \setminus W) = \emptyset.$$

  Therefore, $\text{conv}\, W$ is a face of S by the corollary.
- In particular, each point of V is a vertex of S.

Combinatorial Equivalence

- Let P, P' be polytopes, not necessarily lying in the same Euclidean space, with vertex sets V, V', respectively.
- Then P and P' are said to be **combinatorially equivalent** if there exists a bijection $\varphi : V \to V'$ such that a subset W of V determines a face of P if and only if $\varphi(W)$ determines a face of P'.
- Since 1-polytopes are simply line segments, they are all combinatorially equivalent to one another.
- Two 2-polytopes (polygons) are combinatorially equivalent if and only if they have the same number of vertices.

Combinatorial Equivalence and Number of Vertices

- Clearly, if two polytopes are combinatorially equivalent, then they must have the same number of vertices.
- The converse of this result is not true.

  In $\mathbb{R}^3$ consider:
  - A square pyramid P;
  - The polytope P' obtained by taking the union of a regular tetrahedron and its reflection in one of its triangular faces.

  Both P and P' have five vertices, but they are not combinatorially equivalent. P has a face with four vertices, but P' does not.
- We will show later that every 3-polytope with five vertices is combinatorially equivalent to either P or P'.

  So, there are just two **combinatorial types** for 3-polytopes having five vertices.

Approximation by Polytopes

Theorem

Let A be a non-empty compact convex set in $\mathbb{R}^n$ and let $\varepsilon > 0$. Then there exist polytopes P, Q in $\mathbb{R}^n$ such that $P \subseteq A \subseteq Q$, $\rho(A, P) \leq \varepsilon$, $\rho(A, Q) \leq \varepsilon$.

- By a previous theorem, there exists a finite set E in $\mathbb{R}^n$ such that $E \subseteq A \subseteq (E)_\varepsilon$. Let $P = \text{conv} E$. Then P is a polytope satisfying $P \subseteq A \subseteq (P)_\varepsilon$. Hence $\rho(A, P) \leq \varepsilon$. Replacing A by $(A)_\varepsilon$ in the last argument, we deduce the existence of a polytope Q satisfying $Q \subseteq (A)_\varepsilon \subseteq (Q)_\varepsilon$. The inclusion $(A)_\varepsilon \subseteq (Q)_\varepsilon$, i.e., $A + \varepsilon U \subseteq Q + \varepsilon U$, implies $A \subseteq Q$ by a previous theorem. The inequality $\rho(A, Q) \leq \varepsilon$ now follows.

Corollary

Let A be a non-empty compact convex set in $\mathbb{R}^n$. Then there exist sequences $P_1, \ldots, P_i, \ldots$ and $Q_1, \ldots, Q_i, \ldots$ of nonempty polytopes in $\mathbb{R}^n$ such that $P_i \subseteq A \subseteq Q_i$ for $i = 1, 2, \ldots$, and $P_i \to A$ and $Q_i \to A$ as $i \to \infty$.

Subsection 2

Polyhedral Sets

Polyhedral Sets

- A **polyhedral set** is the intersection of a finite family of closed halfspaces in $\mathbb{R}^n$.
- Equivalently, a polyhedral set is the set of all points $(x_1, \ldots, x_n)$ in $\mathbb{R}^n$ which satisfy a finite system of linear inequalities of the form
$$\begin{array}{rcl} a_{11}x_1 + \cdots + a_{1n}x_n & \leq & a_{10} \\ & \vdots & \\ a_{m1}x_1 + \cdots + a_{mn}x_n & \leq & a_{m0}. \end{array}$$
- Clearly, polyhedral sets are closed and convex.
- Moreover, the intersection of any finite family of polyhedral sets is a polyhedral set.
- Each hyperplane in $\mathbb{R}^n$ is an intersection of two closed halfspaces, and so is a polyhedral set.
- Since each flat in $\mathbb{R}^n$ is a finite intersection of hyperplanes, all flats are polyhedral sets.
- In particular, the empty set and $\mathbb{R}^n$ itself are polyhedral sets.

Facets of Polyhedral Sets

- A **facet** of an r-dimensional polyhedral set in $\mathbb{R}^n$ is a proper $(r-1)$-dimensional face of the set.
- In $\mathbb{R}^3$:
  - The non-negative orthant has three facets;
  - A tetrahedron has four facets;
  - A square pyramid has five facets;
  - A cube has six facets.
- Since flats have no proper faces, they have no facets.
- It will be shown in the following result that flats are the only polyhedral sets with this property.

Properties of Facets

Theorem

Suppose that the polyhedral set A in $\mathbb{R}^n$ is not a flat and that

$$A = (\operatorname{aff} A) \cap \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_1 \cdot \boldsymbol{x} \le \alpha_1\} \cap \cdots \cap \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_m \cdot \boldsymbol{x} \le \alpha_m\},$$

where $\alpha_1, \ldots, \alpha_m \in \mathbb{R}$, $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbb{R}^n \setminus \{\boldsymbol{0}\}$ and no one of the closed half spaces in the intersection can be omitted. For each $i = 1, \ldots m$, let

$$F_i = A \cap \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_i \cdot \boldsymbol{x} = \alpha_i\}.$$

Then:

(i) $\operatorname{ri} A = \{\boldsymbol{a} \in A : \boldsymbol{a}_1 \cdot \boldsymbol{a} < \alpha_1, \ldots, \boldsymbol{a}_m \cdot \boldsymbol{a} < \alpha_m\}$;

(ii) $\operatorname{rebd} A = F_1 \cup \cdots \cup F_m$;

(iii) The facets of A are precisely the sets $F_1, \ldots, F_m$;

Properties of Facets (Cont'd)

Theorem (Cont'd)

(iv) Each proper face of A is the intersection of those facets of A that contain it;

(v) A has a finite number of faces, each of which is exposed;

(vi) Each face of A is a polyhedral set;

(vii) Let B_j, B_k be j- and k-faces, respectively, of A $(0 \leq j \leq k-2)$ such that $B_j \subseteq B_k$. Then there are faces $B_{j+1}, \ldots, B_{k-1}$ of A such that, for each $i = j, \ldots, k-1$, the face B_i is a facet of B_{i+1}.

Proof of the Theorem (Parts (i) & (ii))

- Every polyhedral set A in $\mathbb{R}^n$ can be expressed in the form required by the theorem. The assumption that A is not a flat implies that $m \geq 1$.

(i) Suppose first that $\boldsymbol{a} \in A$ and that $\boldsymbol{a}_1 \cdot \boldsymbol{a} < \alpha_1$, ..., $\boldsymbol{a}_m \cdot \boldsymbol{a} < \alpha_m$. Then $\boldsymbol{a}$ belongs to the set $C = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_1 \cdot \boldsymbol{x} < \alpha_1, \ldots, \boldsymbol{a}_m \cdot \boldsymbol{x} < \alpha_m\}$, which is open, being a finite intersection of open halfspaces. Thus, there exists $r > 0$ such that $B(\boldsymbol{a}; r) \subseteq C$. Hence, $B(\boldsymbol{a}; r) \cap \operatorname{aff} A \subseteq C \cap \operatorname{aff} A \subseteq A$. Therefore, $\boldsymbol{a} \in \operatorname{ri} A$.

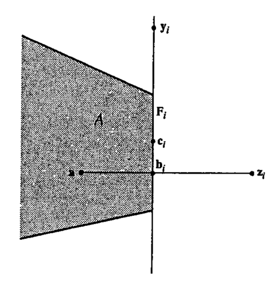
Suppose next that $\boldsymbol{a} \in \operatorname{ri} A$. Since no one of the closed halfspaces in the representation of A given in the statement of the theorem can be omitted, for each $i = 1, \ldots, m$, there exists $\boldsymbol{z}_i \in \operatorname{aff} A$ such that $\boldsymbol{a}_j \cdot \boldsymbol{z}_i \leq \alpha_j$, when $j \neq i$, and $\boldsymbol{a}_i \cdot \boldsymbol{z}_i > \alpha_i$. Hence, for each $i = 1, \ldots, m$, there exists $\lambda_i \in (0, 1)$ such that $\lambda_i \boldsymbol{z}_i + (1 - \lambda_i)\boldsymbol{a} \in A$. Therefore,

$$\begin{array}{rcl} \alpha_i & \geq & \boldsymbol{a}_i \cdot (\lambda \boldsymbol{z}_i + (1 - \lambda_i)\boldsymbol{a}) = \lambda_i \boldsymbol{a}_i \cdot \boldsymbol{z}_i + (1 - \lambda_i)\boldsymbol{a}_i \cdot \boldsymbol{a} \\ & > & \lambda_i \alpha_i + (1 - \lambda_i)\boldsymbol{a}_i \cdot \boldsymbol{a}. \qquad \text{So } \boldsymbol{a}_i \cdot \boldsymbol{a} < \alpha_i. \end{array}$$

(ii) This follows immediately from (i).

Proof of the Theorem (Part (iii))

(iii) We now show that, for each $i = 1, \ldots, m$, F_i is a facet of A.
Let $\boldsymbol{a} \in \operatorname{ri} A$. Let $\boldsymbol{z}_i$ be as in (i). Then $\boldsymbol{a}_i \cdot \boldsymbol{a} < \alpha_i < \boldsymbol{a}_i \cdot \boldsymbol{z}_i$. Write $\mu_i = \frac{\alpha_i - \boldsymbol{a}_i \cdot \boldsymbol{a}}{\boldsymbol{a}_i \cdot \boldsymbol{z}_i - \boldsymbol{a}_i \cdot \boldsymbol{a}}$. Then $0 < \mu_i < 1$. Write $\boldsymbol{b}_i = \mu_i \boldsymbol{z}_i + (1 - \mu_i)\boldsymbol{a}$.



Then (see next slide) $\boldsymbol{b}_i \in \operatorname{aff} A$, $\boldsymbol{a}_i \cdot \boldsymbol{b}_i = \alpha_i$ and $\boldsymbol{a}_j \cdot \boldsymbol{b}_i < \alpha_j$, for $j \neq i$. Hence, $\boldsymbol{b}_i \in A$. Thus, $\boldsymbol{b}_i \in F_i$ and $\boldsymbol{a}_i \cdot \boldsymbol{x} = \alpha_i$ is a support hyperplane to A at $\boldsymbol{b}_i$. It follows that F_i is a proper exposed face of A.

Proof of the Theorem (Part (iii) Cont'd)

- We set $\mu_i = \frac{\alpha_i - \boldsymbol{a}_i \cdot \boldsymbol{a}}{\boldsymbol{a}_i \cdot \boldsymbol{z}_i - \boldsymbol{a}_i \cdot \boldsymbol{a}}$ and $\boldsymbol{b}_i = \mu_i \boldsymbol{z}_i + (1 - \mu_i)\boldsymbol{a}$.
  Based on these and the inequalities $\boldsymbol{a}_i \cdot \boldsymbol{a} < \alpha_i < \boldsymbol{a}_i \cdot \boldsymbol{z}_i$, we get

$$\begin{aligned}
\boldsymbol{a}_i \cdot \boldsymbol{b}_i &= \mu_i \boldsymbol{a}_i \cdot \boldsymbol{z}_i + (1 - \mu_i)\boldsymbol{a}_i \cdot \boldsymbol{a} \\
&= \frac{\alpha_i - \boldsymbol{a}_i \cdot \boldsymbol{a}}{\boldsymbol{a}_i \cdot \boldsymbol{z}_i - \boldsymbol{a}_i \cdot \boldsymbol{a}} \boldsymbol{a}_i \cdot \boldsymbol{z}_i + \frac{\boldsymbol{a}_i \cdot \boldsymbol{z}_i - \alpha_i}{\boldsymbol{a}_i \cdot \boldsymbol{z}_i - \boldsymbol{a}_i \cdot \boldsymbol{a}} \boldsymbol{a}_i \cdot \boldsymbol{a} \\
&= \frac{\alpha_i}{\boldsymbol{a}_i \cdot \boldsymbol{z}_i - \boldsymbol{a}_i \cdot \boldsymbol{a}} (\boldsymbol{a}_i \cdot \boldsymbol{z}_i - \boldsymbol{a}_i \cdot \boldsymbol{a}) + \frac{(\boldsymbol{a}_i \cdot \boldsymbol{z}_i)(\boldsymbol{a}_i \cdot \boldsymbol{a}) - (\boldsymbol{a}_i \cdot \boldsymbol{a})(\boldsymbol{a}_i \cdot \boldsymbol{z}_i)}{\boldsymbol{a}_i \cdot \boldsymbol{z}_i - \boldsymbol{a}_i \cdot \boldsymbol{a}} \\
&= \alpha_i + 0 = \alpha_i; \\
\boldsymbol{a}_j \cdot \boldsymbol{b}_i &= \mu_i \boldsymbol{a}_j \cdot \boldsymbol{z}_i + (1 - \mu_i)\boldsymbol{a}_j \cdot \boldsymbol{a} \\
&< \mu_i \alpha_j + (1 - \mu_i)\alpha_j = \alpha_j.
\end{aligned}$$

Proof of the Theorem (Part (iii) Cont'd)

- We now show that $\operatorname{aff} F_i = (\operatorname{aff} A) \cap \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_i \cdot \boldsymbol{x} = \alpha_i\}$.
  Let $\boldsymbol{y}_i$ be a point belonging to the set on the right.
  Choose $\theta_i > 0$ such that $\theta_i(\boldsymbol{a}_j \cdot \boldsymbol{y}_i - \boldsymbol{a}_j \cdot \boldsymbol{b}_i) \le \alpha_j - \boldsymbol{a}_j \cdot \boldsymbol{b}_i$ when $j \neq i$.
  Write $\boldsymbol{c}_i = \theta_i \boldsymbol{y}_i + (1-\theta_i)\boldsymbol{b}_i$. Then $\boldsymbol{c}_i \in \operatorname{aff} A$ and we have, for $i \neq j$:

$$\begin{array}{rcl}\boldsymbol{a}_i \cdot \boldsymbol{c}_i & = & \theta_i \boldsymbol{a}_i \cdot \boldsymbol{y}_i + (1-\theta_i)\boldsymbol{a}_i \cdot \boldsymbol{b}_i \\ & = & \theta_i \alpha_i + (1-\theta_i)\alpha_i = \alpha_i; \\ \boldsymbol{a}_j \cdot \boldsymbol{c}_i & = & \theta_i \boldsymbol{a}_j \cdot \boldsymbol{y}_i + (1-\theta_i)\boldsymbol{a}_j \cdot \boldsymbol{b}_i \\ & = & \theta_i(\boldsymbol{a}_j \cdot \boldsymbol{y}_i - \boldsymbol{a}_j \cdot \boldsymbol{b}_i) + \boldsymbol{a}_j \cdot \boldsymbol{b}_i \le \alpha_j.\end{array}$$

  Hence, $\boldsymbol{c}_i \in F_i$. But $\boldsymbol{y}_i = \frac{1}{\theta_i}\boldsymbol{c}_i + (1-\frac{1}{\theta_i})\boldsymbol{b}_i \in \operatorname{aff} F_i$. So $(\operatorname{aff} A) \cap \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_i \cdot \boldsymbol{x} = \alpha_i\} \subseteq \operatorname{aff} F_i$. The opposite inclusion is trivial.
  This equality, together with a previous theorem, give

$$\dim F_i = \dim(\operatorname{aff} F_i) = \dim(\operatorname{aff} A) - 1 = \dim A - 1.$$

  So F_i is a facet of A.

Proof of the Theorem (Part (iii) Conclusion)

- We finally show that each facet of A is one of the F_is.

  Let F be a facet of A. Let $\boldsymbol{f} \in \mathrm{ri} F$.

  Since F is a proper face of A, $\boldsymbol{f} \notin \mathrm{ri} A$.

  Hence, by (ii), $\boldsymbol{f} \in F_{i_0}$ for some $i_0 \in \{1, \ldots, m\}$.

  Now the faces F and F_{i_0} of A have the same dimension and $\boldsymbol{f} \in \mathrm{ri} F$.

  Hence $F = F_{i_0}$.

Proof of the Theorem (Part (iv))

(iv) Suppose that B is a proper face of A. Let $\boldsymbol{b} \in \text{ri}B$. Denote by I the non-empty set of those i's in $\{1,\ldots,m\}$ for which $\boldsymbol{a}_i \cdot \boldsymbol{b} = \alpha_i$, i.e., $\boldsymbol{b} \in F_i$.

Denote by J the set of those j's in $\{1,\ldots,m\}$ for which $\boldsymbol{a}_j \cdot \boldsymbol{b} < \alpha_j$.

Let E be the intersection of all those facets of A which contain $\boldsymbol{b}$.

Since $\boldsymbol{b} \in F_i$ if and only if $B \subseteq F_i$, the set E is the intersection of all those facets of A which contain B. Hence E is a face of A which contains B. Choose $r > 0$ such that, for each $j \in J$,

$$B(\boldsymbol{b}; r) \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_j \cdot \boldsymbol{x} < \alpha_j\}.$$

This inclusion, together with the trivial inclusions $\text{aff}E \subseteq \text{aff}A$ and $\text{aff}E \subseteq \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}_i \cdot \boldsymbol{x} = \alpha_i\}$, for $i \in I$, show that $B(\boldsymbol{b}; r) \cap \text{aff}E \subseteq E$. Hence, $\boldsymbol{b} \in \text{ri}E$. Thus, $\boldsymbol{b} \in \text{ri}B \cap \text{ri}E$. So $B = E$.

Proof of the Theorem (Parts (v)-(vii))

(v) This follows easily from (iv), since A has only m facets and each of these is an exposed face of A.

(vi) This follows from the facts that each proper face of A is the intersection of A with one of its support hyperplanes, and the intersection of two polyhedral sets is itself a polyhedral set.

(vii) B_j is a proper face of the polyhedral set B_k.

By (iv), there is some facet B_{k-1} of B_k which contains B_j.

If $j = k-2$, then the proof is complete.

Otherwise, repeat this last argument $k-j-2$ more times to obtain the desired faces $B_{k-2}, \ldots, B_{j+1}$.

The General Case

- The preceding theorem concerns polyhedral sets which are not flats.
- It is convenient, however, to have a statement of the main properties of general polyhedral sets.

Theorem

Let A be a polyhedral set in $\mathbb{R}^n$. Then A has a finite number of faces, each of which is exposed and is a polyhedral set. Every proper face of A is the intersection of those facets of A that contain it, and $\text{rebd}A$ is the union of all the facets of A. If A has a non-empty face of dimension s, then A has faces of all dimensions from s to $\text{dim}A$.

- The theorem is trivially true when A is a flat.

  When A is not a flat, it follows easily from the preceding theorem.

Characterization of Polyhedral Sets

Theorem

Let A be a closed convex set in $\mathbb{R}^n$ which has only a finite number of exposed faces. Then A is a polyhedral set.

- If A has no proper exposed faces, then it must be a flat, which is polyhedral.

  Suppose, then, that A has proper exposed faces $B_1,\ldots,B_m$. Let $H_1,\ldots,H_m$ be support hyperplanes to A such that $B_1 = A\cap H_1$, ..., $B_m = A\cap H_m$. For each $i = 1,\ldots,m$, let J_i be the closed halfspace of $\mathbb{R}^n$ bounded by H_i, which contains A.

  Define a polyhedral set P by the equation

$$P = J_1 \cap \cdots \cap J_m \cap \operatorname{aff} A.$$

  We show that $A = P$.

Characterization of Polyhedral Sets (Cont'd)

- Clearly, $A \subseteq P$. Suppose that $P \nsubseteq A$. Then there is a point $\boldsymbol{p}$ lying in $P \backslash A$. Let $\boldsymbol{a} \in \text{ri} A$. Since A is closed and $\boldsymbol{p} \in \text{aff} A$, there exists $\lambda \in (0,1)$ such that the point $\boldsymbol{b} = \lambda \boldsymbol{p} + (1-\lambda)\boldsymbol{a}$ belongs to rebd$A$. By a previous theorem, there is some $i \in \{1, \ldots, m\}$ such that $\boldsymbol{b} \in B_i$. Now H_i is a face of J_i, $\boldsymbol{b} \in H_i$, and $\boldsymbol{p}, \boldsymbol{a} \in J_i$. Hence, $\boldsymbol{a} \in H_i$. Thus, $\boldsymbol{a} \in B_i$. This is impossible, since $\boldsymbol{a}$ cannot be both a relative interior point of A and a member of one of its proper faces! Hence $P \subseteq A$, and A is the polyhedral set P.

Corollary

A closed convex set in $\mathbb{R}^n$ which has only a finite number of faces is a polyhedral set.

Characterization of Polytopes

Theorem

A set in $\mathbb{R}^n$ is a polytope if and only if it is a bounded polyhedral set.

- Each polytope in $\mathbb{R}^n$ is compact and has a finite number of faces. So, by the preceding corollary, it must be a bounded polyhedral set.

  Conversely, every bounded polyhedral set in $\mathbb{R}^n$ is compact and has a finite number of faces. In particular, it has a finite number of extreme points. So, by a previous theorem, it must be a polytope.

Corollary

The intersection of two polytopes in $\mathbb{R}^n$ is a polytope.

- In view of the theorem, the corollary simply states the obvious fact that the intersection of two bounded polyhedral sets is a bounded polyhedral set.

Subsection 3

Pyramids, Bipyramids and Prisms

Number of k-Faces of a Polytope

- We denote by $f_k(P)$ the number of k-faces (faces of dimension k) of an r-polytope P.
- Then

$$f_{-1}(P) = f_r(P) = 1, \quad f_k(P) = 0 \text{ when } k < -1 \text{ or } k > r.$$

- Our results will lead us to anticipate **Euler's relation**, which asserts that,

$$f_{-1}(P) - f_0(P) + \cdots + (-1)^{r+1} f_r(P) = 0,$$

  for any non-empty r-polytope P.
- This will be proved in a later section.

The Case of Simplexes

- Let S be a non-empty r-simplex in $\mathbb{R}^n$.
- Then $S = \operatorname{conv} V$ for some affinely independent set V of $r+1$ points of $\mathbb{R}^n$.
- For each $k = -1, 0, \ldots, r$, the k-faces of S are precisely those sets of the form $\operatorname{conv} W$, where W is a subset of V having $k+1$ points.
- Thus, $f_k(S)$ equals the number of ways of choosing $k+1$ points from a set of $r+1$ points.
- Hence, using the standard notation for the binomial coefficients, we see that $f_k(S) = \binom{r+1}{k+1} = \frac{(r+1)!}{(k+1)!(r-k)!}$.
- By the Binomial Theorem, for all real x,
$$(1+x)^{r+1} = f_{-1}(S) + f_0(S)x + \cdots + f_r(S)x^{r+1}.$$
- Setting $x = -1$ in this equation, we deduce that
$$f_{-1}(S) - f_0(S) + \cdots + (-1)^{r+1} f_r(S) = 0.$$

Pyramids in $\mathbb{R}^n$

- Let Q be a nonempty $(r-1)$-polytope in $\mathbb{R}^n$.
- Let $\boldsymbol{x}$ be a point of $\mathbb{R}^n$ not lying in $\operatorname{aff} Q$.
- Then the r-**pyramid** P **with apex** $\boldsymbol{x}$ **and base** Q is defined to be the r-polytope $\operatorname{conv}(\{\boldsymbol{x}\} \cup Q)$.
- We say that P is obtained from Q by applying the **cone construction with apex** $\boldsymbol{x}$.

Numbers of Faces of a Pyramid

Theorem

Let P be an r-pyramid in $\mathbb{R}^n$ with apex $\boldsymbol{x}$ and base a non-empty $(r-1)$-polytope Q. Then

$$f_k(P) = f_k(Q) + f_{k-1}(Q), \text{ for } k = -1, \dots, r.$$

- We show first that, for $A, B \subseteq \operatorname{aff} Q$, $(\operatorname{aff}(\{\boldsymbol{x}\} \cup A)) \cap B = (\operatorname{aff} A) \cap B$.

  Consider the non-trivial case when A is non-empty. If $\boldsymbol{b}$ lies in the set on the left-hand side, then there exist $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m \in A$ and $\lambda, \lambda_1, \dots,$ $\lambda_m \in \mathbb{R}$ with $\lambda + \lambda_1 + \cdots + \lambda_m = 1$ such that $\boldsymbol{b} = \lambda \boldsymbol{x} + \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m$. If $\lambda \neq 0$, then the last equation can be rearranged to express $\boldsymbol{x}$ as an affine combination of points of $\operatorname{aff} Q$. This contradicts the (implied) hypothesis that $\boldsymbol{x} \notin \operatorname{aff} Q$. Thus, $\lambda = 0$. So $\boldsymbol{b} \in (\operatorname{aff} A) \cap B$. It follows that $(\operatorname{aff}(\{\boldsymbol{x}\} \cap A)) \cap B \subseteq (\operatorname{aff} A) \cap B$. The opposite inclusion is clear.

Numbers of Faces of a Pyramid (Cont'd)

- Denote by V the vertex set of Q. Then $P = \text{conv}(\{\boldsymbol{x}\} \cup V)$.

  By a previous corollary, $\{\boldsymbol{x}\}$ and Q are faces of P.

  Hence, each of the $f_k(Q)$ k-faces of Q is also a k-face of P.

  Thus, the set of extreme points of P is $\{\boldsymbol{x}\} \cup V$.

  Suppose that $W \subseteq V$ is such that $\text{conv} W$ is one of the $f_{k-1}(Q)$ $(k-1)$-faces of Q. Then by the equation just proved,

  $$(\text{aff}(\{\boldsymbol{x}\} \cup W)) \cap \text{conv}(V \setminus W) = (\text{aff} W) \cap \text{conv}(V \setminus W) = \emptyset.$$

  This shows that $\text{conv}(\{\boldsymbol{x}\} \cup W)$ is a k-face of P.

  It now follows that

  $$f_k(P) \geq f_k(Q) + f_{k-1}(Q).$$

Numbers of Faces of a Pyramid (Cont'd)

- Suppose next that $W \subseteq V$ is such that either $\text{conv}W$ or $\text{conv}(\{\boldsymbol{x}\} \cup W)$ is a face of P (every face of P must be of one of these two forms).

  Then either
  $$(\text{aff}\,W) \cap \text{conv}(\{\boldsymbol{x}\} \cup (V \setminus W)) = \emptyset,$$
  or
  $$(\text{aff}\,W) \cap \text{conv}(V \setminus W) = \emptyset.$$
  In both cases $(\text{aff}\,W) \cap \text{conv}(V \setminus W) = \emptyset$. This shows that $\text{conv}W$ is a face of Q. Thus, each face of P is either a face of Q or the convex hull of $\boldsymbol{x}$ and a face of Q. Hence,
  $$f_k(P) \leq f_k(Q) + f_{k-1}(Q).$$

  The conclusion follows.

Example

- The formula of the preceding theorem is easily verified for a 3-pyramid P in $\mathbb{R}^3$ which has for base an m-sided convex polygon.
- Here $f_0(Q) = m$, $f_1(Q) = m$;
- $f_0(P) = m+1$, $f_1(P) = 2m$, $f_2(P) = m+1$.
- We note that P satisfies Euler's relation:

$$\begin{aligned} f_{-1}(P) - f_0(P) + f_1(P) - f_2(P) + f_3(P) \\ = 1 - (m+1) + (2m) - (m+1) + 1 = 0. \end{aligned}$$

Two-Fold Pyramids

- Suppose now that P is an r-pyramid with base an $(r-1)$-polytope Q, and that Q is an $(r-1)$-pyramid with base an $(r-2)$-polytope S.
- So P is obtained from S by applying the cone construction twice.
- We say that P is:
  - a 2-**fold** r-**pyramid** with 2-**base** S, or
  - a 1-**fold** r-**pyramid** with 1-**base** Q.
- The preceding theorem shows that, for $k = -1, \ldots, r$,

$$\begin{aligned} f_k(P) &= f_k(Q) + f_{k-1}(Q) \\ &= f_k(S) + f_{k-1}(S) + f_{k-1}(S) + f_{k-2}(S) \\ &= f_k(S) + 2f_{k-1}(Q) + f_{k-2}(S). \end{aligned}$$

Multi-Fold Pyramids

- Let P be an r-polytope in $\mathbb{R}^n$ ($r = 1, \ldots, n$).
- Let Q be an $(r-s)$-polytope in $\mathbb{R}^n$ ($s = 1, \ldots, r$).
- Then P is said to be an s-**fold** r-**pyramid** with s-**base** Q if it can be obtained from Q by applying the cone construction s times.
- A simple induction argument, using the preceding theorem, shows that, for an s-fold r-pyramid P with s-base Q, we have

$$f_k(P) = \sum_{i=1}^{s} \binom{s}{i} f_{k-i}(Q), \quad k = -1, \ldots, r.$$

- Clearly, an r-fold r-pyramid is an r-simplex.
- An $(r-1)$-fold r-pyramid has a line segment for an $(r-1)$-base.

  A line segment is itself a 1-fold 1-pyramid.

  So each $(r-1)$-fold r-pyramid is an r-fold r-pyramid, i.e. an r-simplex.

Bipyramids in $\mathbb{R}^n$

- Let I be a line segment in $\mathbb{R}^n$ and let Q be an $(r-1)$-polytope in $\mathbb{R}^n$ such that $I \cap Q$ is a single point which is a relative interior point of both I and Q.
- Then the r-**bipyramid** P with **axis** I and **base** Q is defined to be the r-polytope $\text{conv}(I \cup Q)$.
- We say that P is obtained from Q by applying the **double-cone construction with axis** I.

Numbers of Faces of a Bipyramid

- Suppose that $I = \operatorname{conv}\{\boldsymbol{a}, \boldsymbol{b}\}$, where $\boldsymbol{a}$ and $\boldsymbol{b}$ are distinct points of $\mathbb{R}^n$.
- Then an argument similar to that used in the proof of the preceding theorem shows that:
  - The k-faces ($k = -1, \ldots, r-2$) of P are precisely the k-faces of Q and the k-polytopes of the form $\operatorname{conv}(\{\boldsymbol{a}\} \cup F)$ or $\operatorname{conv}(\{\boldsymbol{b}\} \cup F)$, where F is a $(k-1)$-face of Q.
  - The $(r-1)$-faces of P are simply the $(r-1)$-polytopes $\operatorname{conv}(\{\boldsymbol{a}\} \cup F)$ and $\operatorname{conv}(\{\boldsymbol{b}\} \cup F)$, where F is an $(r-2)$-face of Q.
- We thus arrive at the following result.

Theorem

Let P be an r-bipyramid in $\mathbb{R}^n$ with axis I and base a non-empty $(r-1)$-polytope Q. Then

$$\begin{aligned} f_k(P) &= f_k(Q) + 2f_{k-1}(Q), \text{ for } k = -1, \ldots, r-2, \\ f_{r-1}(P) &= 2f_{r-2}(Q). \end{aligned}$$

Example

- The formula of the preceding theorem is easily verified for a 3-bipyramid P in $\mathbb{R}^3$ which has for base an m-sided convex polygon Q.
- Here $f_0(Q) = m$, $f_1(Q) = m$;
- $f_0(P) = m + 2$, $f_1(P) = 3m$, $f_2(P) = 2m$.
- We note that P satisfies Euler's relation:

$$\begin{aligned} & f_{-1}(P) - f_0(P) + f_1(P) - f_2(P) + f_3(P) \\ & \qquad = 1 - (m+2) + (3m) - (2m) + 1 = 0. \end{aligned}$$

Multi-Fold Bipyramids

- Let P be an r-polytope in $\mathbb{R}^n$ ($r = 1, \ldots, n$).
- Let Q be an $(r-s)$-polytope in $\mathbb{R}^n$ ($s = 1, \ldots, r$).
- Then P is said to be an s-**fold** r-**bipyramid** with s-**base** Q if it can be obtained from Q by applying the double-cone construction s times.
- An $(r-1)$-fold r-bipyramid has a line segment for an $(r-1)$-base.

  A line segment is itself a 1-fold 1-bipyramid.

  So each $(r-1)$-fold r-bipyramid is also an r-fold r-bipyramid.

The r-Crosspolytope

- The simplest example of an r-fold r-bipyramid is the r-crosspolytope.
- Consider the r-crosspolytope P in $\mathbb{R}^n$ ($r = 1, \ldots, n$), which is the convex hull of r linearly independent line segments $\text{conv}\{\boldsymbol{a}_1, \boldsymbol{b}_1\}$, ..., $\text{conv}\{\boldsymbol{a}_r, \boldsymbol{b}_r\}$ (i.e., the vectors $\boldsymbol{a}_1 - \boldsymbol{b}_1$, ..., $\boldsymbol{a}_r - \boldsymbol{b}_r$ are linearly independent) whose midpoints coincide.
- The facial structure of P is easily described:
  - For each $k = 0, \ldots, r-1$, let $I = \{i_1, \ldots, i_{k+1}\}$ be a subset of $\{1, \ldots, r\}$ which has $k+1$ points and let $T = \{\boldsymbol{x}_{i_1}, \ldots, \boldsymbol{x}_{i_{k+1}}\}$ be such that each $\boldsymbol{x}_{i_j}$ is either $\boldsymbol{a}_{i_j}$ or $\boldsymbol{b}_{i_j}$ for $j = 1, \ldots, k+1$.
  - Then $\text{conv}\, T$ is a k-face of P and all k-faces of P arise in this way.
- Since there are $\binom{r}{k+1}$ possibilities for the set I and each I gives rise to 2^{k+1} possibilities for the set T, it follows that

$$f_k(P) = 2^{k+1} \binom{r}{k+1}, \quad k = 1, \ldots, r-1.$$

Prisms in $\mathbb{R}^n$

- Let Q be a non-empty $(r-1)$-polytope in $\mathbb{R}^n$.
- Let $\boldsymbol{x}$ be a point of $\mathbb{R}^n$ which does not lie in the subspace of $\mathbb{R}^n$ which is parallel to $\operatorname{aff} Q$.
- Let I be the line segment $\operatorname{conv}\{\mathbf{0}, \boldsymbol{x}\}$.
- Then the r-**prism** P with **axis** I and **base** Q is defined to be the r-polytope $Q + I$ or, equivalently, $\operatorname{conv}(Q \cup (Q + \boldsymbol{x}))$.
- We say that P is obtained from Q by applying the **prism construction with axis** I.

Numbers of Faces of Prisms

- An argument similar to that used in the proof of the preceding theorems shows that the k-faces $(k = 1, \ldots, r)$ of P are precisely the k-faces of Q and its translate $Q + \boldsymbol{x}$, together with k-polytopes of the form $F + I$, where F is a $(k-1)$-face of Q.
- We thus arrive at the following result.

Theorem

Let P be an r-prism in $\mathbb{R}^n$ with axis I and base a nonempty $(r-1)$-polytope Q. Then

$$\begin{array}{rcl} f_k(P) & = & 2f_k(Q) + f_{k-1}(Q), \quad k = 1, \ldots, r, \\ f_0(P) & = & 2f_0(Q). \end{array}$$

Example

- The formulas of the preceding theorem are easily verified for a 3-prism P in $\mathbb{R}^3$ which has for base an m-sided convex polygon Q.
- Here $f_0(Q) = m$, $f_1(Q) = m$;
- $f_0(P) = 2m$, $f_1(P) = 3m$, $f_2(P) = m + 2$.
- We note that P satisfies Euler's relation:

$$\begin{aligned} f_{-1}(P) - f_0(P) + f_1(P) - f_2(P) + f_3(P) \\ = 1 - 2m + 3m - (m + 2) + 1 = 0. \end{aligned}$$

Multi-Fold Prisms

- Let P be an r-polytope in $\mathbb{R}^n$ ($r = 1, \ldots, n$) and let Q be an ($r - s$)-polytope in $\mathbb{R}^n$ ($s = 1, \ldots, r$).
- Then P is said to be an s-**fold** r-**prism** with s-**base** Q if it can be obtained from Q by applying the prism construction s times.
- An ($r - 1$)-fold r-prism has a line segment for an ($r - 1$)-base.

  A line segment is itself a 1-fold 1-prism.

  So each ($r - 1$)-fold r-prism is also an r-fold r-prism.

Parallelotopes

- An r-fold r-prism P in $\mathbb{R}^n$ ($r = 1, \ldots, n$) is called an r-**parallelotope** and has the form
$$P = \boldsymbol{x} + \{\lambda_1 \boldsymbol{x}_1 + \cdots + \lambda_r \boldsymbol{x}_r : 0 \leq \lambda_1, \ldots, \lambda_r \leq 1\},$$
where $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r$ are linearly independent vectors in $\mathbb{R}^n$.
- Thus:
  - A 2-parallelotope in $\mathbb{R}^2$ is a parallelogram;
  - A 3-parallelotope in $\mathbb{R}^3$ is a parallelepiped.
- If $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r$ are pairwise orthogonal, P is known as an r-**orthotope**.
- If, in addition, $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_r$ have the same length, P is called an r-**cube**.
- A simple induction argument, using the preceding theorem, shows that, for any r-parallelotope P in $\mathbb{R}^n$, we have
$$f_k(P) = 2^{r-k} \binom{r}{k}, \quad k = 0, \ldots, r.$$

Subsection 4

Cyclic Polytopes

k-Neighborly Polytopes

- Any polytope having more than k vertices which is such that every k-membered subset of its vertex set determines one of its faces, is said to be k-**neighborly**.
- Thus n-simplexes ($n \geq 1$) are n-neighborly.

The Moment Curve

- The **moment curve** M_n in $\mathbb{R}^n$ is determined parametrically by the equation
$$\boldsymbol{x}(t) = (t, t^2, \ldots, t^n), \text{ for all real } t.$$
- Clearly, this sets up a bijection between the set $\mathbb{R}$ of real numbers and the set M_n of points on the moment curve.
- This bijection induces an ordering on M_n which is isomorphic to the standard ordering on $\mathbb{R}$.
- Having now made this remark, we shall in future refer to the ordering of points on M_n exactly as if they were real numbers.
- For example, if points $\boldsymbol{x}(t_1)$, $\boldsymbol{x}(t_2)$, $\boldsymbol{x}(t_3)$ on M_n are such that $t_1 < t_2 < t_3$, then we shall say that $\boldsymbol{x}(t_2)$ **lies between** $\boldsymbol{x}(t_1)$ and $\boldsymbol{x}(t_3)$.

Affine Independence of Points on Moment Curve

Theorem

Each set of $n+1$ or fewer points on the moment curve M_n in $\mathbb{R}^n$ is affinely independent.

- For $i=0,1,\ldots,n$, let $\boldsymbol{x}(t_i)=(t_i,t_i^2,\ldots,t_i^n)$, where $t_0<t_1<\cdots<t_n$. We must show that $\{\boldsymbol{x}(t_0),\boldsymbol{x}(t_1),\ldots,\boldsymbol{x}(t_n)\}$ is affinely independent. This is equivalent to the non-vanishing of the $(n+1)\times(n+1)$ determinant

$$\begin{vmatrix} 1 & t_0 & t_0^2 & \cdots & t_0^n \\ 1 & t_1 & t_1^2 & \cdots & t_1^n \\ \vdots & \vdots & \vdots & & \vdots \\ 1 & t_n & t_n^2 & \cdots & t_n^n \end{vmatrix}.$$

  It is a well-known result of elementary algebra that this determinant, called **Vandermonde's determinant**, equals $\prod_{0\le i<j\le n}(t_j-t_i)$. Hence, it is non-zero.

Cyclic Polytopes

- A **cyclic polytope** $C(v,n)$ is the convex hull of v ($v \geq n+1$) distinct points on the moment curve M_n in $\mathbb{R}^n$.
- Strictly speaking, $C(v,n)$ is a whole family of polytopes, all of the same combinatorial type.
- Our first result is that cyclic polytopes are **simplicial**.

  This means that all of their proper faces are simplexes.
- Examples of simplicial polytopes are:
  - simplexes;
  - bipyramids with simplicial bases;
  - crosspolytopes.

Cyclic Polytopes are Simplicial

Theorem

Cyclic polytopes are simplicial.

- Let F be a proper face of a cyclic polytope $C(v,n)$ in $\mathbb{R}^n$.

  Then $F = \text{conv}\{\mathbf{x}_1,\ldots,\mathbf{x}_m\}$ for some distinct points $\mathbf{x}_1,\ldots,\mathbf{x}_m$ $(1 \le m < v)$ on the moment curve M_n.

  Since the face F is proper, the set $\{\mathbf{x}_1,\ldots,\mathbf{x}_m\}$ cannot contain an affinely independent subset of more than n points.

  Hence, by the preceding theorem, $m \le n$ and $\{\mathbf{x}_1,\ldots,\mathbf{x}_m\}$ is affinely independent.

  Thus F is a simplex, showing that $C(v,n)$ is simplicial.

Points, Vertices and Faces

Theorem

Let $C(v,n)$ be the convex hull of the distinct points $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_v$ ($v \geq n+1 \geq 3$) on the moment curve M_n in $\mathbb{R}^n$. Let k be an integer satisfying $1 \leq k \leq \frac{1}{2}n$. Then each set of k points of $\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_v\}$, determines a $(k-1)$-face of $C(v,n)$ and $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_v$ are the vertices of $C(v,n)$.

- It suffices to show that $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k$ determine a $(k-1)$-face of $C(v,n)$. For each $i=1,\ldots,k$, let $\boldsymbol{x}_i=(t_i,t_i^2,\ldots,t_i^n)$. Define a polynomial p for real t by the equation

$$p(t)=(t-t_1)^2(t-t_2)^2\cdots(t-t_k)^2;$$

say $p(t)=t^{2k}+a_{2k-1}t^{2k-1}+\cdots+a_1t+a_0$, where $a_0,a_1,\ldots,a_{2k-1}\in\mathbb{R}$. Clearly, $p(t)\geq 0$, for all real t, and $p(t)=0$ if and only if t has one of the values $t_1,\ldots,t_k$.

Points, Vertices and Faces (Cont'd)

- It follows that the hyperplane with equation

$$a_0 + a_1x_1 + \cdots + a_{2k-1}x_{2k-1} + x_{2k} = 0$$

is a support hyperplane to $C(v,n)$ which meets $C(v,n)$ in the set $\text{conv}\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k\}$. Thus $\text{conv}\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k\}$ is a face of $C(v,n)$. By a previous theorem, $\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k\}$ is affinely independent. So $\text{conv}\{\boldsymbol{x}_1,\ldots,\boldsymbol{x}_k\}$ is a $(k-1)$-simplex.

That $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_v$ are vertices of $C(v,n)$ follows from the result just proved with $k=1$.

Number of Faces

Corollary

The cyclic polytope $C(v,n)$ in $\mathbb{R}^n$ $(v \geq n+1 \geq 3)$ has $\binom{v}{k}$ $(k-1)$-faces, when k is an integer satisfying $1 \leq k \leq \frac{1}{2}n$.

- By the preceding theorem, each set of k vertices of $C(v,n)$ determines one of its $(k-1)$-faces.

  Conversely, by the pre-preceding theorem, each $(k-1)$-face of $C(v,n)$ is the convex hull of some k of its vertices.

  Thus $C(v,n)$ has as many $(k-1)$-faces as there are ways of choosing a subset of k points from a set of v points, namely $\binom{v}{k}$.

Gale's Evenness Condition

- We saw that each proper face of a polytope is the intersection of those facets of the polytope which contain that face.
- Thus the facial structure of a polytope is completely determined by the vertex sets of its facets.
- We now give a simple criterion for determining which sets of vertices of a cyclic polytope determine one of its facets.

Theorem (Gale's Evenness Condition)

Let W be a set of n points of the vertex set V of a cyclic polytope $C(v,n)$ in $\mathbb{R}^n$ $(v \geq n+1)$. Then $\operatorname{conv} W$ is a facet of $C(v,n)$ if and only if each two points of $V \setminus W$ are separated on the moment curve M_n by an even number of points of W.

- Let W consist of the n points $(t_i, t_i^2, \ldots, t_i^n)$ for $i = 1, \ldots, n$.

Gale's Evenness Condition (Cont'd)

- Consider the real polynomial p defined (for real t) by the equation

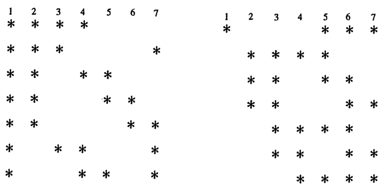
$$p(t)=(t-t_1)\cdots(t-t_n)=t^n+a_{n-1}t^{n-1}+\cdots+a_1t+a_0,$$

where $a_0,a_1,\ldots,a_{n-1}\in\mathbb{R}$. Then the hyperplane H in $\mathbb{R}^n$ that contains W has equation $a_0+a_1x_1+\cdots+a_{n-1}x_{n-1}+x_n=0$.

Now $\operatorname{conv}W$ will be a facet of $C(v,n)$ if and only if H is a support hyperplane to $C(v,n)$. This will be the case if and only if all the numbers $p(t)$, where t is such that $(t,t^2,\ldots,t^n)\in V\setminus W$, have the same sign. As t increases through all real values, the polynomial p changes sign precisely when t passes through one of the values $t_1,\ldots,t_n$. Thus $p(r)$ and $p(s)$, where r and s are unequal real numbers that are not equal to any of the values $t_1,\ldots,t_n$, will have the same sign if and only if an even number of $t_1,\ldots,t_n$ lie between r and s.

Example: Number of Facets of $C(7,4)$

- We use Gale's evenness condition to calculate the number of facets of the cyclic polytope $C(7,4)$.
- This is equivalent to finding how many subsets W of a totally ordered set V of seven elements there are having four elements, and which are such that between any two elements of $V \setminus W$ there is an even number of elements of W.
- The totality of such subsets W of V is illustrated in the figure, where V is represented by the numbers 1,2,3,4,5,6,7 on the real line with their usual ordering, and where the points of W are marked by asterisks.



- There are 14 such sets W, and so $C(7,4)$ has 14 facets.

Example (Cont'd)

- Since each proper face of $C(7,4)$ is an intersection of facets of $C(7,4)$, we find that $C(7,4)$ has 28 2-faces corresponding to the following subsets of V:

$$\begin{array}{cccc}
\{1,2,3\}, & \{1,2,4\}, & \{1,2,5\}, & \{1,2,6\}, \\
\{1,2,7\}, & \{1,3,4\}, & \{1,3,7\}, & \{1,4,5\}, \\
\{1,4,7\}, & \{1,5,6\}, & \{1,5,7\}, & \{1,6,7\}, \\
\{2,,3,4\}, & \{2,3,5\}, & \{2,3,6\}, & \{2,3,7\}, \\
\{2,4,5\}, & \{2,5,6\}, & \{2,6,7\}, & \{3,4,5\}, \\
\{3,4,6\}, & \{3,4,7\}, & \{3,5,6\}, & \{3,6,7\}, \\
\{4,,5,6\}, & \{4,5,7\}, & \{4,6,7\}, & \{5,6,7\}.
\end{array}$$

- By the upper bound theorem, no 4-polytope with 7 vertices has more than $C(7,4) = 28$ 2-faces.

Example (Cont'd)

- By a previous corollary, $C(7,4)$ has $\binom{7}{2} = 21$ 1-faces.
- Thus, denoting the polytope $C(7,4)$ by P, we find that

$$\begin{aligned} f_{-1}(P) - f_0(P) + f_1(P) - f_2(P) + f_3(P) - f_4(P) \\ = 1 - 7 + 21 - 28 + 14 - 1 = 0. \end{aligned}$$

This verifies Euler's relation for $C(7,4)$.

Number of Facets of $C(v,n)$

Theorem

The cyclic polytope $C(v,n)$ in $\mathbb{R}^n$ $(v \geq n+1)$ has $\frac{v}{v-d}\binom{v-d}{d}$ or $2\binom{v-d-1}{d}$ facets, according as $n=2d$ is even or $n=2d+1$ is odd.

- We first establish a simple combinatorial lemma. Let $A=\{1,\ldots,r\}$, $B=\{1,\ldots,r-s\}$, where r,s are integers satisfying $r\geq 1$ and $0\leq 2s\leq r$. Then a subset of A is said to be s-**paired** if it has the form

$$\{i_1,i_1+1,i_2,i_2+1,\ldots,i_s,i_s+1\}$$

where $i_1<i_1+1<i_2<i_2+1<\cdots<i_s<i_s+1$. The empty set (corresponding to $s=0$) is considered to be 0-paired. By associating with each such s-paired set the subset $\{i_1,i_2-1,\ldots,i_s-(s-1)\}$ of B, we set up a bijection between the s-paired subsets of A and the subsets of B having s elements. Thus A has $\binom{r-s}{s}$ s-paired subsets.

Number of Facets of $C(v,n)$ (Cont'd)

- By Gale's condition the number of facets of $C(v,n)$ is the number of subsets W of $V=\{1,\ldots,v\}$ with n elements, such that between any two integers of $V\backslash W$ there is an even number of integers of W.

  For this proof only, we refer to such a subset W of V as a **facet** of V. We need to determine the number of facets W of V.
  - Suppose $n=2d$ is even. Then the facets W of V are of two types:
    - W is a d-paired subset of V, or
    - $W\backslash\{1,v\}$ is a $(d-1)$-paired subset of $\{2,\ldots,v-1\}$.

    Conversely, each d-paired subset of V is a facet of V, and each $(d-1)$-paired subset of $\{2,\ldots,v-1\}$, when augmented with 1 and v, is a facet of V. By the combinatorial lemma, V has $\binom{v-d}{d}$ facets of the first type and $\binom{v-2-(d-1)}{d-1}=\binom{v-d-1}{d-1}$ facets of the second type.
    Thus the total number of the facets of V is

$$\binom{v-d}{d}+\binom{v-d-1}{d-1}=\frac{(v-d)!}{(v-2d)!d!}+\frac{(v-d-1)!}{(v-2d)!(d-1)!}=\frac{v}{v-d}\binom{v-d}{d}.$$

Number of Facets of $C(v,n)$ (Cont'd)

- Suppose $n = 2d + 1$ is odd.

  Again the facets W of V are of two types:
  - $W \setminus \{1\}$ is a d-paired subset of $\{2, \ldots, v\}$, or
  - $W \setminus \{v\}$ is a d-paired subset of $\{1, \ldots, v-1\}$.

  Conversely, each d-paired subset of $\{2, \ldots, v\}$, when augmented with 1, is a facet of V, and each d-paired subset of $\{1, \ldots, v-1\}$, when augmented with v, is a facet of V.

  The number of facets of V of either type is $\binom{v-1-d}{d}$.

  Hence, the total number of facets of V is $2\binom{v-1-d}{d}$.

k-Neighborly Polytopes

- Let k be a positive integer.
- Then a polytope in $\mathbb{R}^n$ (having more than k vertices) is said to be k-**neighborly** if every set of k of its vertices determines a face of the polytope.
- Thus:
  - Each r-polytope ($r \geq 1$) is 1-neighborly;
  - Each r-simplex ($r \geq 1$) is r-neighborly.
- A previous theorem shows that the cyclic polytope $C(v,n)$, where $v \geq n+1 \geq 3$, is $[\frac{1}{2}n]$-neighborly - here $[\frac{1}{2}n]$ denotes the greatest integer not exceeding $\frac{1}{2}n$.

Vertices of Neighborly Polytopes

Theorem

Let P be a k-neighborly polytope in $\mathbb{R}^n$. Then every set of k vertices of P is affinely independent and each $(k-1)$-face of P is a $(k-1)$-simplex.

- Suppose that $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k$ are k vertices of P which are affinely dependent, say $\boldsymbol{v}_k \in \mathrm{aff}\{\boldsymbol{v}_1,\ldots,\boldsymbol{v}_{k-1}\}$. Since P has more than k vertices, there is a vertex $\boldsymbol{v}_0$ of P different from $\boldsymbol{v}_1,\ldots,\boldsymbol{v}_k$.
  Since P is k-neighborly, $\mathrm{conv}\{\boldsymbol{v}_0,\ldots,\boldsymbol{v}_{k-1}\}$ is a face of P.
  By a previous theorem, $\boldsymbol{v}_k \notin \mathrm{aff}\{\boldsymbol{v}_0,\ldots,\boldsymbol{v}_{k-1}\}$, a contradiction.
  Thus every set of k vertices of P is affinely independent.
  Suppose now that F is a $(k-1)$-face of P. Then F must contain an affinely independent subset W consisting of precisely k vertices of P. Since P is k-neighborly, $\mathrm{conv}\,W$ is a $(k-1)$-face of P. Hence it is a face of F. But F has only one $(k-1)$-dimensional face, namely itself. Thus, $F = \mathrm{conv}\,W$. So F is a $(k-1)$-simplex.

k- and j-Neighborliness for $j \leq k$

Corollary

Let P be a k-neighborly polytope in $\mathbb{R}^n$ with v vertices. Let $j \in \{1, \ldots, k\}$. Then P is j-neighborly and has $\binom{v}{j}$ $(j-1)$-faces.

- Let X be a set of j vertices of P. Then $X \subseteq W$ for some set W of k vertices of P. Now $\operatorname{conv} W$ is a simplex and a face of P. Hence $\operatorname{conv} X$ is a face of $\operatorname{conv} W$, and hence of P. So P is k-neighborly.

  The k-neighborliness of P, together with the theorem, shows that P has as many $(j-1)$-faces as there are ways of choosing a set of j points from a set of v points. So P has $\binom{v}{j}$ $(j-1)$-faces.

Characterization of k-Neighborly Polytopes

- We now show that the only n-polytopes which are more neighbourly than the general cyclic polytope $C(v,n)$ are the n-simplexes.

Theorem

Let P be an n-polytope in $\mathbb{R}^n$ which is k-neighborly for some k with $k > [\frac{1}{2}n]$. Then P is an n-simplex.

- Suppose that P is not an n-simplex. Then the vertex set V of P must contain some subset W of $n+2$ points. By Radon's Theorem, W can be partitioned into two subsets X and Y with $(\text{conv} X) \cap (\text{conv} Y) \neq \emptyset$. One of X and Y, X say, has no more than $[\frac{1}{2}n]+1$ points. The corollary shows that conv$X$ is a face of P. Hence, by a previous theorem,

$$(\text{conv} X) \cap (\text{conv} Y) \subseteq (\text{aff} X) \cap (\text{conv}(V \setminus X)) = \emptyset.$$

This is a contradiction. Thus P is an n-simplex.

A Consequence

Corollary

Let P be an n-neighborly $2n$-polytope in $\mathbb{R}^{2n}$. Then P is simplicial.

- Let F be a facet of P. Then F is an n-neighborly $(2n-1)$-polytope. So, exactly as in the proof of the theorem, F is a simplex.

  But each proper face of P is a face of some facet of P.

  Thus, each proper face of P must be a simplex.

  So P is simplicial.

Subsection 5

Euler's Relation

Choice of Non-Perpendicular Vector

Lemma

Let $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ be a finite set of nonzero vectors in $\mathbb{R}^n$. There exists a vector $\boldsymbol{a}$ in $\mathbb{R}^n$, which is not perpendicular to any of $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$.

- We recursively construct reals α_k and vectors $\boldsymbol{x}_k$, such that, for all $k=1,\ldots,m$, $\boldsymbol{x}_k=\sum_{i=1}^k \alpha_i\boldsymbol{a}_i$ is not perpendicular to any of $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_k$.
  Set $\alpha_1=1$ and $\boldsymbol{x}_1=\alpha_1\boldsymbol{a}_1$. Clearly $\boldsymbol{x}_1\cdot\boldsymbol{a}_1\neq 0$.
  Assume $\boldsymbol{x}_k=\sum_{i=1}^k \alpha_i\boldsymbol{a}_i$ is not perpendicular to any of $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_k$.
  For $i=1,\ldots,k+1$, set $c_i=\boldsymbol{x}_k\cdot\boldsymbol{a}_i$. By hypothesis, $c_i\neq 0$, $i=1,\ldots,k$.
  - If $c_{k+1}\neq 0$, let $\alpha_{k+1}=0$. So $\boldsymbol{x}_{k+1}=\boldsymbol{x}_k$. Moreover, $\boldsymbol{x}_{k+1}$ is not perpendicular to any of $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{k+1}$.
  - If $c_{k+1}=0$, choose $\alpha_{k+1}\neq 0$, with $\alpha_{k+1}\boldsymbol{a}_{k+1}\cdot\boldsymbol{a}_i\neq -c_i$, $i=1,\ldots,k$.
    - For $i=1,\ldots,k$, $\boldsymbol{x}_{k+1}\cdot\boldsymbol{a}_i=\boldsymbol{x}_k\cdot\boldsymbol{a}_i+\alpha_{k+1}\boldsymbol{a}_{k+1}\cdot\boldsymbol{a}_i=c_i+\alpha_{k+1}\boldsymbol{a}_{k+1}\cdot\boldsymbol{a}_i\neq 0$.
    - For $i=k+1$, $\boldsymbol{x}_{k+1}\cdot\boldsymbol{a}_{k+1}=\boldsymbol{x}_k\cdot\boldsymbol{a}_{k+1}+\alpha_{k+1}\boldsymbol{a}_{k+1}\cdot\boldsymbol{a}_{k+1}=c_{k+1}+\alpha_{k+1}\boldsymbol{a}_{k+1}\cdot\boldsymbol{a}_{k+1}=\alpha_{k+1}\boldsymbol{a}_{k+1}\cdot\boldsymbol{a}_{k+1}\neq 0$.

  So $\boldsymbol{x}_{k+1}$ is not perpendicular to any of $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{k+1}$.

Choice of Vector With Distinct Inner Products

Corollary

Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ be a finite set of distinct vectors in $\mathbb{R}^n$. There exists a vector $\boldsymbol{a}$ in $\mathbb{R}^n$, such that, for all $1 \le i < j \le m$, $\boldsymbol{a} \cdot \boldsymbol{a}_i \ne \boldsymbol{a} \cdot \boldsymbol{a}_j$.

- Consider the collection

$$A = \{\boldsymbol{a}_j - \boldsymbol{a}_i : 1 \le i < j \le m\}$$

of $\frac{m(m-1)}{2}$ nonzero vectors.
By the lemma, there exists $\boldsymbol{a}$ in $\mathbb{R}^n$, such that

$$\boldsymbol{a} \cdot (\boldsymbol{a}_j - \boldsymbol{a}_i) \ne 0, \text{ for all } 1 \le i < j \le m.$$

Therefore, this $\boldsymbol{a}$ satisfies

$$\boldsymbol{a} \cdot \boldsymbol{a}_i \ne \boldsymbol{a} \cdot \boldsymbol{a}_j, \text{ for all } 1 \le i < j \le m.$$

Euler's Relation

Theorem (Euler's Relation)

Let P be a non-empty r-polytope in $\mathbb{R}^n$. Then

$$f_{-1}(P) - f_0(P) + \cdots + (-1)^{r+1} f_r(P) = 0,$$

where $f_k(P)$ denotes the number of k-faces of P.

- We argue by induction on r.

  The theorem is trivial when $r = 0$, since $f_{-1}(P) = 1$, $f_0(P) = 1$, and when $r = 1$, since $f_{-1}(P) = 1$, $f_0(P) = 2$, $f_1(P) = 1$.

  Suppose that the theorem has been established for polytopes of dimension $r - 1$, where $r \geq 2$.

  Let P be an r-polytope ($r \geq 2$) in $\mathbb{R}^n$ with vertices $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_v$.

  By the preceding corollary, we may choose a vector $\boldsymbol{a}$ in $\mathbb{R}^n$ such that the scalars $\boldsymbol{a} \cdot \boldsymbol{a}_1, \ldots, \boldsymbol{a} \cdot \boldsymbol{a}_v$ are distinct.

Euler's Relation (Cont'd)

- Suppose that the vertices of P are labeled so that $\boldsymbol{a}\cdot\boldsymbol{a}_1 < \cdots < \boldsymbol{a}\cdot\boldsymbol{a}_v$. Define hyperplanes $H_1, H_3, \ldots, H_{2v-1}$ in $\mathbb{R}^n$ by the equations

$$H_{2k-1} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}\cdot\boldsymbol{x} = \boldsymbol{a}\cdot\boldsymbol{a}_k\}, \quad k = 1, \ldots, v.$$

  Choose scalars $c_1, c_2, \ldots, c_{v-1}$ such that

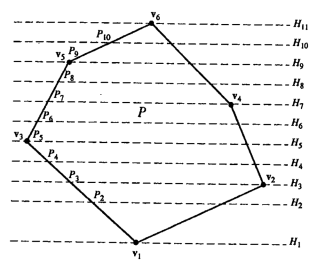
$$\boldsymbol{a}\cdot\boldsymbol{a}_1 < c_1 < \boldsymbol{a}\cdot\boldsymbol{a}_2 < c_2 < \cdots < c_{v-1} < \boldsymbol{a}\cdot\boldsymbol{a}_v.$$

  Define hyperplanes $H_2, H_4, \ldots, H_{2v-2}$ in $\mathbb{R}^n$ by the equations

$$H_{2k} = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{a}\cdot\boldsymbol{x} = c_k\}, \quad k = 1, \ldots, v-1.$$

Euler's Relation (Cont'd)

- This situation for a two-dimensional polytope with six vertices is illustrated on the right.
  The following observations about the hyperplanes $H_1, H_2, \ldots, H_{2v-1}$ are immediate:



(i) They are distinct and parallel to one another;

(ii) Each of the hyperplanes $H_1, H_3, \ldots, H_{2v-1}$, contains just one vertex of P;

(iii) H_1 and H_{2v-1} are support hyperplanes to P which meet P in a single point;

(iv) The set $P \cap H_k$, for $k = 2, 3, \ldots, 2v-2$, is an $(r-1)$-polytope, P_k say;

(v) None of the polytopes $P_2, P_4, \ldots, P_{2v-2}$ contains a vertex of P.

Euler's Relation (Cont'd)

- For each j-face F_j of P, where $j = 1,\ldots,r$, and for each polytope P_i, where $i = 2,3,\ldots,2v-2$, define an integer $\psi(F_j,P_i)$ to be 1 if $\mathrm{ri}F_j$ meets P_i, and 0 otherwise.
  For each j-face F_j of P, where $j = 1,\ldots,r$, denote by s and t, respectively, the smallest and largest integers i amongst $1,2,\ldots,2v-1$ for which H_i meets F_j.
  Clearly s and t are odd with $s < t$, and $\psi(F_j,P_i) = 1$ precisely when $s < i < t$. Thus, $\sum_{i=2}^{2v-2}(-1)^i\psi(F_j,P_i) = \sum_{i=s+1}^{t-1}(-1)^i = 1$. So, for each fixed $j = 1,\ldots,r$,
  $$\sum_{j\text{-faces}}\left(\sum_{i=2}^{2v-2}(-1)^i\psi(F_j,P_i)\right) = f_j(P),$$
  where the summation is over all the j-faces F_j of P_i. Hence
  $$\sum_{j=1}^{r}(-1)^j\left(\sum_{j\text{-faces}}\left(\sum_{i=2}^{2v-2}(-1)^i\psi(F_j,P_i)\right)\right) = \sum_{j=1}^{r}(-1)^j f_j(P).$$

Euler's Relation (Cont'd)

- We now find an alternative expression for the left-hand side.
  If i is one of $2,4,\ldots,2v-2$ or $1<j\le r$, then the number of $(j-1)$-faces of P_i is the same as the number of j-faces of P whose relative interiors meet P_i.
  If i is one of $1,3,\ldots,2v-1$, then the number of vertices of P, is one more than the number of edges of P whose relative interiors meet P_i.
  These observations are summarized in the following equations, where it is assumed that i is one of $2,3,\ldots,2v-2$; j is one of $1,\ldots,r$, and $f_k(P_j)$ denotes the number of k-faces of P_i:

$$\sum_{j\text{-faces}}\psi(F_j,P_i)=\begin{cases} f_{j-1}(P_i), & \text{if } i \text{ is even or } 1<j\le r,\\ -1+f_{j-1}(P_i), & \text{if } i \text{ is odd and } j=1.\end{cases}$$

  Hence,

$$\sum_{j=1}^{r}(-1)^j\left(\sum_{j\text{-faces}}\psi(F_j,P_i)\right)=\begin{cases} \sum_{j=1}^{r}(-1)^j f_{j-1}(P_i), & \text{if } i \text{ is even},\\ 1+\sum_{j=1}^{r}(-1)^j f_{j-1}(P_i), & \text{if } i \text{ is odd}.\end{cases}$$

Euler's Relation (Cont'd)

- By the induction hypothesis, $\sum_{j=-1}^{r-1}(-1)^j f_j(P_i) = 0$. So $1 + \sum_{j=1}^{r}(-1)^j f_{j-1}(P_i) = 0$. Hence,

$$\sum_{j=1}^{r}(-1)^j \left(\sum_{j\text{-faces}} \psi(F_j, P_i)\right) = \begin{cases} -1, & \text{if } i \text{ is even}, \\ 0, & \text{if } i \text{ is odd}. \end{cases}$$

So

$$\sum_{i=2}^{2v-2}(-1)^i \left(\sum_{j=1}^{r}(-1)^j \left(\sum_{j\text{-faces}} \psi(F_j, P_i)\right)\right) = 1 - v.$$

Comparing the two main equations, we deduce that

$$\sum_{j=1}^{r}(-1)^j f_j(P) = 1 - v = f_{-1}(P) - f_0(P).$$

So $\sum_{j=-1}^{r}(-1)^j f_j(P) = 0$.

Outline of a Generalization

- Suppose that F is a k-face of an r-polytope P ($-1 \leq k < r$) and that $h_i(F)$ denotes the number of i-faces of P containing F.
- For example, if F is a vertex of a cube P in $\mathbb{R}^3$, then this vertex belongs to three edges and three facets of P.
  So in this case: $h_0(F) = 1$, $h_1(F) = 3$, $h_2(F) = 3$, $h_3(F) = 1$.
  We note that
  $$h_0(F) - h_1(F) + h_2(F) - h_3(F) = 1 - 3 + 3 - 1 = 0.$$
- This suggests that we consider the alternating sum
  $$h_k(F) - h_{k+1}(F) + \cdots + (-1)^{r-k} h_r(F)$$
  in the general case.
- We will show that this alternating sum is always zero.
- This generalizes Euler's relation, which corresponds to the case when F is the empty face of P.

Polar Duality for Polytopes

- Let P be an r-polytope ($r \geq 1$) in $\mathbb{R}^r$ containing the origin as an interior point.
- Then the polar dual P^* of P is a compact convex set in $\mathbb{R}^r$ containing the origin as an interior point.
- Suppose that P has extreme points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$.
- Then $P = \text{conv}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ and P^* is the intersection of the m closed half spaces $\boldsymbol{a}_i \cdot \boldsymbol{x} \leq 1$ for $i = 1, \ldots, m$, whence P^* is a polyhedral set.
- Thus P^* is a bounded polyhedral set, i.e., a polytope.

Polar Duality for Polytopes (Cont'd)

- Suppose further that F_i is an i-face of P $(i=1,\ldots,r)$.
- Then a previous theorem shows that there exists a sequence $F_{-1},\ldots,F_i,\ldots,F_r$ of $r+2$ faces of P such that

$$F_{-1}\subset\cdots\subset F_i\subset\cdots\subset F_r.$$

- Denote by φ the polar face mapping of P.
- φ is an inclusion-reversing bijection from the family of faces of P to the family of faces of P^*.
- So $\varphi(F_{-1}),\ldots,\varphi(F_i),\ldots,\varphi(F_r)$ is a sequence of $r+2$ faces of P^* with

$$\varphi(F_r)\subset\cdots\subset\varphi(F_i)\subset\cdots\subset\varphi(F_{-1}).$$

- It follows from a previous corollary that $\dim\varphi(F_i)=r-i-1$.
- Hence, the number of i-faces of P is the same as the number of $(r-i-1)$-faces of P^*.

Generalization of Euler's Theorem

Theorem

Let F be a k-face of an r-polytope P ($k = 1, \ldots, r-1$) in $\mathbb{R}^n$. Then

$$h_k(F) - h_{k+1}(F) + \cdots + (-1)^{r-k} h_r(F) = 0,$$

where $h_i(F)$, $i = k, \ldots, r$, denotes the number of i-faces of P containing F.

- We may assume, without loss of generality, that $r = n$ and that P contains the origin as an interior point.
  Denote by φ the polar face mapping of P. Then the number $h_i(F)$ of i-faces of P containing F is the same as the number $f_{n-i-1}(\varphi(F))$ of $(n-i-1)$-faces of $\varphi(F)$. Euler's relation applied to the polytope $\varphi(F)$ shows that

$$\begin{aligned} h_n(F) - h_{n-1}(F) + \cdots + (-1)^{n-k} h_k(F) \\ = f_{-1}(\varphi(F)) - f_0(\varphi(F)) + \cdots + (-1)^{n-k} f_{n-1-k}(\varphi(F)) = 0. \end{aligned}$$

Linear Relation Between Numbers of Faces

- Euler's relation shows that, for every r-polytope P ($r \geq 1$), the numbers $f_0(P),\ldots,f_{r-1}(P)$ of faces of P of dimensions $0,\ldots,r-1$, respectively, satisfy the linear equation

$$f_0(P) - f_1(P) + \cdots + (-1)^{r-1} f_{r-1}(P) = 1 - (-1)^r.$$

- We now prove that this is essentially the only linear equation which is satisfied by the numbers $f_0(P),\ldots,f_{r-1}(P)$ for all r-polytopes P ($r \geq 1$).

Theorem

Let r be a positive integer. Suppose that $\alpha_0,\ldots,\alpha_r$ are real numbers such that the numbers $f_i(P)$ of the i-faces ($i = 0,\ldots,r-1$) of any r-polytope P satisfy the equation

$$\alpha_0 f_0(P) + \alpha_1 f_1(P) + \cdots + \alpha_{r-1} f_{r-1}(P) = \alpha_r.$$

Then $\alpha_1 = -\alpha_0, \alpha_2 = \alpha_0, \ldots, \alpha_{r-1} = (-1)^{r-1}\alpha_0, \alpha_r = (1 - (-1)^r)\alpha_0$.

Proof

- We argue by induction on r.

  The theorem is trivially true when $r=1$, for in this case $f_0(P)=2$ for all 1-polytopes.

  Suppose, then, that the theorem has been proved for the case when r is some positive integer k, and that $\alpha_0 \ldots, \alpha_{k+1}$ are real numbers such that

  $$\alpha_0 f_0(P)+\alpha_1 f_1(P)+\cdots+\alpha_k f_k(P)=\alpha_{k+1}$$

  for all $(k+1)$-polytopes P.

  Let Q be any k-polytope. Let S be a $(k+1)$-pyramid with base combinatorially equivalent to Q. Let T be a $(k+1)$-bipyramid with base combinatorially equivalent to Q. Previous theorems show that

  $$\begin{array}{rcll} f_i(S) & = & f_{i-1}(Q)+f_i(Q), & i=0,\ldots,k, \\ f_i(T) & = & 2f_{i-1}(Q)+f_i(Q), & i=0,\ldots,k-1, \\ f_k(T) & = & 2f_{k-1}(Q). & \end{array}$$

Proof (Cont'd)

- Write the equation above for S and T:

$$\alpha_0 f_0(S)+\alpha_1 f_1(S)+\cdots+\alpha_k f_k(S)=\alpha_{k+1} \text{ and}$$
$$\alpha_0 f_0(T)+\alpha_1 f_1(T)+\cdots+\alpha_k f_k(T)=\alpha_{k+1}.$$

Substituting the preceding values for $f_i(S)$ and $f_i(T)$,

$$\alpha_0(f_{-1}(Q)+f_0(Q))+\alpha_1(f_0(Q)+f_1(Q))+\cdots$$
$$+\alpha_k(f_{k-1}(Q)+f_k(Q))=\alpha_{k+1} \text{ and}$$
$$\alpha_0(2f_{-1}(Q)+f_0(Q))+\alpha_1(2f_0(Q)+f_1(Q))+\cdots$$
$$+\alpha_{k-1}(2f_{k-2}(Q)+f_{k-1}(Q))+\alpha_k 2f_{k-1}(Q)=\alpha_{k+1}.$$

Subtracting, we find $\alpha_0(f_{-1}(Q)+f_0(Q)-2f_{-1}(Q)-f_0(Q))+$ $\alpha_1(f_0(Q)+f_1(Q)-2f_0(Q)-f_1(Q))+\cdots+\alpha_{k-1}(f_{k-2}(Q)+f_{k-1}(Q)-$ $2f_{k-2}(Q)-f_{k-1}(Q))+\alpha_k(f_{k-1}(Q)+f_k(Q)-2f_{k-1}(Q))=0.$
Equivalently,

$$-\alpha_0 f_{-1}(Q)-\alpha_1 f_0(Q)-\cdots-\alpha_{k-1} f_{k-2}(Q)-\alpha_k f_{k-1}(Q)+\alpha_k f_k(Q)=0.$$

Proof (Conclusion)

- We got the equation

$$-\alpha_0 f_{-1}(Q) - \alpha_1 f_0(Q) - \cdots - \alpha_{k-1} f_{k-2}(Q) - \alpha_k f_{k-1}(Q) + \alpha_k f_k(Q) = 0.$$

Taking into account $f_{-1}(Q) = 1$ and $f_k(Q) = 1$, we get

$$\alpha_1 f_0(Q) + \alpha_2 f_1(Q) + \cdots + \alpha_k f_{k-1}(Q) = \alpha_k - \alpha_0.$$

This equation holds for all k-polytopes Q. By induction,

$$\alpha_2 = -\alpha_1, \alpha_3 = \alpha_1, \ldots, \alpha_k = (-1)^{k-1}\alpha_1, \alpha_k - \alpha_0 = (1 - (-1)^k)\alpha_1.$$

So $\alpha_1 = -\alpha_0$. Now the original equation can be written in the form

$$\alpha_0(f_0(P) - f_1(P) + \cdots + (-1)^k f_k(P)) = \alpha_{k+1}.$$

But Euler's relation applied to any $(k+1)$-polytope P shows that

$$f_0(P) - f_1(P) + \cdots + (-1)^k f_k(P) = 1 - (-1)^{k+1}.$$

Hence $\alpha_{k+1} = (1 - (-1)^{k+1})\alpha_0$.

Dehn-Sommerville Equations

- The Euler relation is the only linear equation satisfied by the numbers of faces of various dimensions of every polytope with a given dimension.
- The Dehn-Sommerville equations are satisfied by the numbers of faces of various dimensions of every simplicial polytope with a given dimension.

Theorem (Dehn-Sommerville Equations)

Let P be a simplicial r-polytope ($r \geq 1$) in $\mathbb{R}^n$. Then

$$\sum_{j=k}^{r-1} (-1)^j \binom{j+1}{k+1} f_j(P) = (-1)^{r-1} f_k(P), \quad k = -1, \ldots, r-2.$$

- For each k-face F of P ($k = -1, \ldots, r-2$), consider the equation $h_k(F) - h_{k+1}(F) + \cdots + (-1)^{r-k} h_r(F) = 0$, given in a previous theorem.

Dehn-Sommerville Equations

- We add together these equations corresponding to all the k-faces F of P to deduce that

$$h_k - h_{k+1} + \cdots + (-1)^{r-k} h_r = 0,$$

where h_j $(j = k, \ldots, r)$ denotes the total number of inclusions of the form $F_k \subseteq F_j$, where F_k and F_j are, respectively, k- and j-faces of P.
  - If $j < r$, then each of the $f_j(P)$ j-faces of P is a j-simplex. So it has $\binom{j+1}{k+1}$ k-faces. Hence $h_j = \binom{j+1}{k+1} f_j(P)$.
  - If $j = r$, then the only j-face of P is P itself. P has $f_k(P)$ k-faces. So $h_r = f_k(P)$.

We now get $\binom{k+1}{k+1} f_k(P) - \binom{k+2}{k+1} f_{k+1}(P) + \cdots + (-1)^{r-k-1} \binom{r}{k+1} f_{r-1}(P)$ $+(-1)^{r-k} f_k(P) = 0$, i.e., $\sum_{j=k}^{r-1} (-1)^{j-k} \binom{j+1}{k+1} f_j(P) = (-1)^{r-k-1} f_k(P)$. Multiplying both sides by $(-1)^k$,

$$\sum_{j=k}^{r-1} (-1)^j \binom{j+1}{k+1} f_j(P) = (-1)^{r-1} f_k(P).$$

Special Cases

- The Dehn-Sommerville equation corresponding to $k = -1$ is simply the Euler relation.
- We derive the Dehn-Sommerville equations corresponding to $k = 0, \ldots, r-1$ for simplicial r-polytopes P with $r = 2, 3, 4$.
- For $r = 2$ and $k = 0$, we get:

  $$f_0(P) - 2f_1(P) = -f_0(P).$$

  This is the same as the Euler relation.
- For $r = 3$ and $k = 0$, we get:

  $$f_0(P) - 2f_1(P) + 3f_2(P) = f_0(P).$$

  For $r = 3$ and $k = 1$, we get:

  $$-f_1(P) + 3f_2(P) = f_1(P).$$

  These are the same as one another, but essentially different from the Euler relation.

Special Cases (Cont'd)

- For $r = 4$ and $k = 0$, we get:

$$f_0(P) - 2f_1(P) + 3f_2(P) - 4f_3(P) = -f_0(P).$$

For $r = 4$ and $k = 1$, we get:

$$-f_1(P) + 3f_2(P) - 6f_3(P) = -f_1(P).$$

For $r = 4$ and $k = 2$, we get:

$$f_2(P) - 4f_3(P) = -f_2(P).$$

The last two of these are the same.

The first one can be deduced from Euler's relation and the second (or third) equation.

Regular 3-Polytopes

- A 3-polytope P is said to be **regular of type** $(p|q)$ if there exist positive integers p, q with $p, q \geq 3$ such that:
  - Each facet of P is a regular p-gon;
  - Each vertex of P belongs to q such facets.
- Suppose now that P is a regular 3-polytope of type $(p|q)$ which has:
  - v vertices;
  - e edges;
  - f facets.
- It follows immediately from a previous theorem that:
  - Each edge of a 3-polytope is contained in precisely two of its facets;
  - Each vertex of P belongs to precisely q of its edges.

Regular 3-Polytopes (Cont'd)

- Counting the edges of P by (i) vertices, and (ii) facets, in an obvious way, we find that $qv = 2e$ and $pf = 2e$.
- Now, using Euler's relation, we get

$$\begin{aligned} & 1 - v + e - f + 1 = 0 \quad \Rightarrow \quad 2 - \frac{2e}{q} + e - \frac{2e}{p} = 0 \\ \Rightarrow \quad & \frac{2pq}{e} - 2p + 2pq - 2q = 0 \quad \Rightarrow \quad 2pq - 2p - 2q + 4 = 4 - \frac{2pq}{e} \\ \Rightarrow \quad & (p-2)(q-2) = 4 - \frac{2pq}{e} < 4. \end{aligned}$$

- The only possible types of regular 3-polytopes are: (3|3), (3|4), (4|3), (3|5) and (5|3).
- These types do indeed exist:
  - The regular tetrahedron;
  - The regular octahedron;
  - The cube;
  - The regular icosahedron;
  - The regular dodecahedron.

Subsection 6

Gale Transforms

Affine Dependence and Cofaces

- An **affine dependence** of a sequence of points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ in $\mathbb{R}^n$ is a point $(\lambda_1, \ldots, \lambda_m)$ of $\mathbb{R}^m$ such that

$$\lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m = \mathbf{0} \quad \text{and} \quad \lambda_1 + \cdots + \lambda_m = 0.$$

- Clearly the zero vector of $\mathbb{R}^m$ is an affine dependence of any sequence of points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ in $\mathbb{R}^n$.
- A subset W of the vertex set V of a polytope P in $\mathbb{R}^n$ is called a **coface** of P if $\text{conv}(V \setminus W)$ is a face of P.
- For example, every set comprising three vertices of a square in $\mathbb{R}^2$ is a coface of that square.

Characterization of Cofaces

Theorem

Let $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ be the vertices of a polytope P in $\mathbb{R}^n$. Then $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r\}$, where $1 \le r \le m$, is a coface of P if and only if there is no affine dependence $\{\lambda_1, \ldots, \lambda_m\}$ of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ such that $\lambda_1, \ldots, \lambda_r \ge 0$ with at least one of $\lambda_1, \ldots, \lambda_r$ positive.

- Suppose that $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r\}$ is not a coface of P. Then, by a previous theorem, $\mathrm{conv}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r\} \cap \mathrm{aff}\{\boldsymbol{a}_{r+1}, \ldots, \boldsymbol{a}_m\} \neq \emptyset$. Hence, there exist scalars $\mu_1, \ldots, \mu_m$, with $\mu_1, \cdots, \mu_r \ge 0$, $\mu_1 + \cdots + \mu_r = 1$ and $\mu_{r+1} + \cdots + \mu_m = 1$ such that

$$\mu_1 \boldsymbol{a}_1 + \cdots + \mu_r \boldsymbol{a}_r = \mu_{r+1} \boldsymbol{a}_{r+1} + \cdots + \mu_m \boldsymbol{a}_m.$$

  Let $\lambda_1 = \mu_1$, ..., $\lambda_r = \mu_r$ and $\lambda_{r+1} = -\mu_{r+1}$, ..., $\lambda_m = -\mu_m$. Then $(\lambda_1, \ldots, \lambda_m)$ is an affine dependence of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ with $\lambda_1, \ldots, \lambda_r \ge 0$ and at least one of $\lambda_1, \ldots, \lambda_r$ positive.

Characterization of Cofaces (Cont'd)

- Conversely, suppose that $(\lambda_1,\ldots,\lambda_m)$ is an affine dependence of $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ such that $\lambda_1,\ldots,\lambda_r \geq 0$ and at least one of $\lambda_1,\ldots,\lambda_r$ is positive.

  Then
  $$\frac{\lambda_1\boldsymbol{a}_1+\cdots+\lambda_r\boldsymbol{a}_r}{\lambda_1+\cdots+\lambda_r}=\frac{(-\lambda_{r+1})\boldsymbol{a}_{r+1}+\cdots+(-\lambda_m)\boldsymbol{a}_m}{(-\lambda_{r+1})+\cdots+(-\lambda_m)}.$$
  Hence,
  $$\text{conv}\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_r\}\cap\text{aff}\{\boldsymbol{a}_{r+1},\ldots,\boldsymbol{a}_m\}\neq\emptyset.$$
  So, by a previous theorem, $\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_r\}$ is not a coface of P.

Set of Affine Dependencies

- We denote the set of all affine dependencies of a sequence $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ in $\mathbb{R}^n$ by $\mathfrak{a}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)$.
- By the theorem, an exact description of $\mathfrak{a}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)$ might be helpful in studying the facial structure of the polytope $\mathrm{conv}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$.
- Such a description is given in the following result, in which the statement that

  the sequence $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ in $\mathbb{R}^n$ is n-**dimensional**

  means that the affine hull of its points is $\mathbb{R}^n$.

Dimensions of Sequences and Subspaces

Theorem

Let $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ be an n-dimensional sequence in $\mathbb{R}^n$. Then $\mathfrak{a}(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)$ is an $(m-n-1)$-dimensional subspace of $\mathbb{R}^m$.

- Denote the rows of the $(n+1)\times m$ matrix $\begin{bmatrix} \boldsymbol{a}_1 & \cdots & \boldsymbol{a}_m \\ 1 & \cdots & 1 \end{bmatrix}$ in which $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ are considered column vectors, by $\boldsymbol{b}_1,\ldots,\boldsymbol{b}_{n+1}$, considered as points of $\mathbb{R}^m$. Denote by S the row space of the matrix, i.e., the set of all linear combinations of its rows. Then

$$\mathfrak{a}(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m) = \{\boldsymbol{\lambda}\in\mathbb{R}^m : \boldsymbol{\lambda}\cdot\boldsymbol{b}_i = 0 \text{ for } i=1,\ldots,n+1\} = S^\perp.$$

  Since $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ is n-dimensional, the column space of the matrix has dimension $n+1$. Hence, so too does S. Since $\dim S + \dim S^\perp = m$, $\mathfrak{a}(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m) = S^\perp$ is an $(m-n-1)$-dimensional subspace of $\mathbb{R}^m$.

Finding All Affine Dependencies

- We now show how to find all the affine dependencies of an n-dimensional sequence $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ in $\mathbb{R}^n$ $(m > n+1)$.
- It follows from the theorem that $\mathfrak{a}(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)$ has a basis consisting of $m-n-1$ vectors of $\mathbb{R}^m$, say

$$\boldsymbol{x}_1 = (x_{11},\ldots,x_{1m}),\ldots,\boldsymbol{x}_{m-n-1} = (x_{m-n-11},\ldots,x_{m-n-1m}).$$

(The condition $m > n+1$ avoids an exceptional, but trivial case.)

- $\boldsymbol{\lambda} \in \mathfrak{a}(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)$ if and only if there exist scalars $c_1,\ldots,c_{m-n-1}$, such that

$$\begin{aligned}\boldsymbol{\lambda} &= c_1(x_{11},\ldots,x_{1m}) + \cdots + c_{m-n-1}(x_{m-n-11},\ldots,x_{m-n-1m}) \\ &= (c_1x_{11} + \cdots + c_{m-n-1}x_{m-n-11},\ldots,c_1x_{1m} + \cdots + c_{m-n-1}x_{m-n-1m}).\end{aligned}$$

- Write $\overline{\boldsymbol{a}}_1 = (x_{11},\ldots,x_{m-n-11})$, ..., $\overline{\boldsymbol{a}}_m = (x_{1m},\ldots,x_{m-n-1m})$.

Gale Transform

- Then we see that $\boldsymbol{\lambda}$ lies in $\mathfrak{a}(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m)$ if and only if there exists a vector $\boldsymbol{c}=(c_1,\ldots,c_{m-n-1})$ in $\mathbb{R}^{m-n-1}$ such that $\boldsymbol{\lambda}=(\boldsymbol{c}\cdot\overline{\boldsymbol{a}}_1,\ldots,\boldsymbol{c}\cdot\overline{\boldsymbol{a}}_m)$.
- We have thus found a simple way of expressing all of the affine dependencies of $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ in terms of $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$.
- The sequence of vectors $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$ in $\mathbb{R}^{m-n-1}$ is called a **Gale transform** of the sequence $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ of vectors in $\mathbb{R}^n$.

Example

- We find a Gale transform of the sequence $\boldsymbol{a}_1 = (1,0,0)$, $\boldsymbol{a}_2 = (0,1,0)$, $\boldsymbol{a}_3 = (0,0,1)$, $\boldsymbol{a}_4 = (-1,0,0)$, $\boldsymbol{a}_5 = (0,-1,0)$, $\boldsymbol{a}_6 = (0,0,-1)$, which lists the vertices of a regular octahedron in $\mathbb{R}^3$.
- The subspace $\mathfrak{a}(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_6)$ of $\mathbb{R}^6$ consists of those points $(\lambda_1,\ldots,\lambda_6)$, which satisfy the simultaneous equations

$$\begin{array}{rcl} \lambda_1-\lambda_4 & = & 0 \\ \lambda_2-\lambda_5 & = & 0 \\ \lambda_3-\lambda_6 & = & 0 \\ \lambda_1+\lambda_2+\lambda_3+\lambda_4+\lambda_5+\lambda_6 & = & 0. \end{array}$$

- The general solution to this system of linear equations can be expressed in the form

$$(\lambda_1,\lambda_2,\lambda_3,\lambda_4,\lambda_5,\lambda_6) = (\alpha,\beta,-(\alpha+\beta),\alpha,\beta,-(\alpha+\beta)),$$

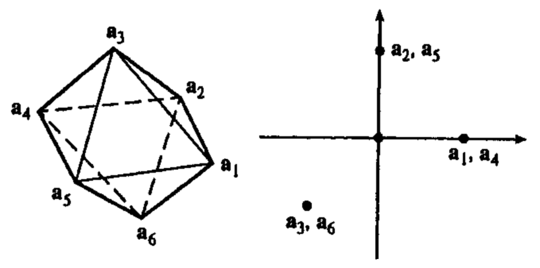
where $\alpha,\beta \in \mathbb{R}$.

Example (Cont'd)

- Thus $\boldsymbol{x}_1 = (1,0,-1,1,0,-1)$, $\boldsymbol{x}_2 = (0,1,-1,0,1,-1)$ form a basis for $\mathfrak{a}(\boldsymbol{a}_1,\ldots,\boldsymbol{a}_6)$, which has dimension $m-n-1 = 6-3-1 = 2$.
- The Gale transform derived from the above basis is the sequence

$$\overline{\boldsymbol{a}}_1 = (1,0),\ \overline{\boldsymbol{a}}_2 = (0,1),\ \overline{\boldsymbol{a}}_3 = (-1,-1),$$
$$\overline{\boldsymbol{a}}_4 = (1,0),\ \overline{\boldsymbol{a}}_5 = (0,1),\ \overline{\boldsymbol{a}}_6 = (-1,-1).$$

- We note that although the six points $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_6$ are distinct, the points $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_6$ are not.



Properties of Gale Transforms

Theorem

Let $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$ be a Gale transform in $\mathbb{R}^{m-n-1}$ of an n-dimensional sequence $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ in $\mathbb{R}^n$ $(m > n+1)$. Then:

(i) A vector in $\mathbb{R}^m$ is an affine dependence of $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ if and only if it has the form $(\boldsymbol{c}\cdot\overline{\boldsymbol{a}}_1,\ldots,\boldsymbol{c}\cdot\overline{\boldsymbol{a}}_m)$ for some $\boldsymbol{c}\in\mathbb{R}^{m-n-1}$;

(ii) The sequence $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$ is $(m-n-1)$-dimensional;

(iii) $\overline{\boldsymbol{a}}_1+\cdots+\overline{\boldsymbol{a}}_m=\boldsymbol{0}$;

(iv) The origin of $\mathbb{R}^{m-n-1}$ is an interior point of $\text{conv}\{\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m\}$;

(v) Every open halfspace of $\mathbb{R}^{m-n-1}$ whose bounding hyperplane passes through the origin contains at least one of the points $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$.

(i) This result was established in the discussion following the preceding theorem, which motivated the definition of a Gale transform.

Properties of Gale Transforms ((ii) and (iii))

(ii(i)) Let

$$\boldsymbol{x}_1 = (x_{11}, \ldots, x_{1m}), \ldots, \boldsymbol{x}_{m-n-1} = (x_{m-n-11}, \ldots, x_{m-n-1m})$$

be the basis for $\mathfrak{a}(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m)$ for which

$$\overline{\boldsymbol{a}}_1 = (x_{11}, \ldots, x_{m-n-11}), \ldots, \overline{\boldsymbol{a}}_m = (x_{1m}, \ldots, x_{m-n-1m}).$$

Since $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{m-n-1}$ are affine dependencies of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$, we have

$$x_{11} + \cdots + x_{1m} = \cdots = x_{m-n-11} + \cdots + x_{m-n-1m} = 0.$$

Hence $\overline{\boldsymbol{a}}_1 + \cdots + \overline{\boldsymbol{a}}_m = \boldsymbol{0}$. The $\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m$ can be identified with the rows of the matrix whose columns are $\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{m-n-1}$. The latter are linearly independent. Thus, $\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m$ span $\mathbb{R}^{m-n-1}$.
Now $\boldsymbol{0} = \frac{1}{m}(\overline{\boldsymbol{a}}_1 + \cdots + \overline{\boldsymbol{a}}_m)$. Hence, $\boldsymbol{0} \in \text{aff}\{\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m\}$. Thus, $\text{aff}\{\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m\}$ is a subspace of $\mathbb{R}^{m-n-1}$ containing $\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m$.
Hence, it must be $\mathbb{R}^{m-n-1}$. So $\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m$ is $(m-n-1)$-dimensional.

Properties of Gale Transforms ((iv) and (v))

(iv) A previous theorem and the equation $\mathbf{0} = \frac{1}{m}(\overline{\boldsymbol{a}}_1 + \cdots + \overline{\boldsymbol{a}}_m)$ show that $\mathbf{0} \in \text{ri}(\text{conv}\{\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m\})$. Hence from (ii), $\mathbf{0} \in \text{int}(\text{conv}\{\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m\})$.

(v) Let H be a hyperplane in $\mathbb{R}^{m-n-1}$ passing through the origin.

Denote by H^- and H^+ the open halfspaces determined by H.

Suppose that H^- contains none of the points $\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m$.

Then $\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m$ lie in the closed half space $H \cup H^+$.

Hence $\text{conv}\{\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m\} \subseteq H \cup H^+$.

This, however, is incompatible with (iv).

Thus H^- must contain at least one of the points $\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m$.

Relative Interior of the Convex Hull

Lemma

Let $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_r \in \mathbb{R}^n$. Then $\mathbf{0} \in \text{ri}(\text{conv}\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_r\})$ if and only if there exists no $\boldsymbol{c} \in \mathbb{R}^n$ such that $\boldsymbol{c}\cdot\boldsymbol{a}_1 \geq 0,\ldots,\boldsymbol{c}\cdot\boldsymbol{a}_r \geq 0$, with at least one of the inequalities being strict.

- Suppose that $\mathbf{0} \in \text{ri}(\text{conv}\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_r\})$. Then, by a previous theorem, there exist $\lambda_1,\ldots,\lambda_r > 0$ such that $\mathbf{0} = \lambda_1\boldsymbol{a}_1 + \cdots + \lambda_r\boldsymbol{a}_r$. Clearly, there exists no $\boldsymbol{c} \in \mathbb{R}^n$ for which $\boldsymbol{c}\cdot\boldsymbol{a}_1 \geq 0$, ..., $\boldsymbol{c}\cdot\boldsymbol{a}_r \geq 0$, with at least one of these inequalities being strict.

  Conversely, suppose that $\mathbf{0} \notin \text{ri}(\text{conv}\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_r\})$. Then $\{\mathbf{0}\}$ and $\text{conv}\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_r\}$ can be properly separated. So there exist $\boldsymbol{c} \in \mathbb{R}^n$, $c_0 \in \mathbb{R}$ such that $\boldsymbol{c}\cdot\mathbf{0} = 0 \leq c_0$ and $\boldsymbol{c}\cdot\boldsymbol{a}_1 \geq c_0$, ..., $\boldsymbol{c}\cdot\boldsymbol{a}_r \geq c_0$, where at least one of these $r+1$ inequalities is strict. If $c_0 = 0$, then at least one of the inequalities $\boldsymbol{c}\cdot\boldsymbol{a}_1 \geq 0$, ..., $\boldsymbol{c}\cdot\boldsymbol{a}_r \geq 0$ must be strict. If $c_0 > 0$, then all of the inequalities $\boldsymbol{c}\cdot\boldsymbol{a}_1 \geq 0$, ..., $\boldsymbol{c}\cdot\boldsymbol{a}_r \geq 0$ are strict. Thus, in every case, the required condition is met.

Cofaces and Gale Transforms

- Let $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$ be a Gale transform of a vertex sequence $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ of some n-polytope in $\mathbb{R}^n$ $(m > n+1)$.
- Then, for each subset W of $\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m\}$, we define a set $\overline{W}$ by the equation $\overline{W} = \{\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m\}$.

Theorem

Let $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$ be a Gale transform in $\mathbb{R}^{m-n-1}$ of a vertex sequence $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ of an n-polytope P in $\mathbb{R}^n$ $(m > n+1)$. Then a subset W of $\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m\}$ is a coface of P iff either $\overline{W}$ is empty or $\mathbf{0} \in \mathrm{ri}(\mathrm{conv}\,\overline{W})$.

- We assume throughout the proof that $W = \{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_r\}$ for some r with $1 \le r \le m$. Suppose first that W is not a coface of P. By a previous theorem, there exists an affine dependence $(\lambda_1,\ldots,\lambda_m)$ of $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ such that $\lambda_1,\ldots,\lambda_r \ge 0$, with at least one of $\lambda_1,\ldots,\lambda_r$ positive. By Part (i) of the preceding theorem, $\lambda_1 = \boldsymbol{c}\cdot\overline{\boldsymbol{a}}_1$, ..., $\lambda_m = \boldsymbol{c}\cdot\overline{\boldsymbol{a}}_m$ for some $\boldsymbol{c}$ in $\mathbb{R}^{m-n-1}$. The lemma now shows that $\mathbf{0} \notin \mathrm{ri}(\mathrm{conv}\,\overline{W})$.

Cofaces and Gale Transforms (Cont'd)

- Suppose next that $\mathbf{0} \notin \text{ri}(\text{conv}\overline{W})$.

  Then the lemma shows the existence of $\boldsymbol{c}$ in $\mathbb{R}^{m-n-1}$ such that $\boldsymbol{c} \cdot \overline{\boldsymbol{a}}_1 \geq 0, \ldots, \boldsymbol{c} \cdot \overline{\boldsymbol{a}}_r \geq 0$, with at least one of the inequalities being strict.

  Let $\lambda_1 = \boldsymbol{c} \cdot \overline{\boldsymbol{a}}_1, \ldots, \lambda_m = \boldsymbol{c} \cdot \overline{\boldsymbol{a}}_m$.

  Again by Part (i) of the preceding theorem, $(\lambda_1, \ldots, \lambda_m)$ is an affine dependence of $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$.

  It now follows from a previous theorem that W is not a coface of P.

Gale Transforms and Open Halfspaces

Corollary

Let $\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m$ be a Gale transform in $\mathbb{R}^{m-n-1}$ of a vertex sequence $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ of an n-polytope P in $\mathbb{R}^n$ $(m > n+1)$. Every open halfspace in $\mathbb{R}^{m-n-1}$ whose bounding hyperplane passes through the origin contains at least two terms of the sequence $\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m$.

- Let H be a hyperplane in $\mathbb{R}^{m-n-1}$ passing through the origin. Denote by H^- and H^+ the open halfspaces determined by H. Suppose that H^- contains fewer than two terms of $\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m$. Part (v) of a previous theorem shows that H^- must contain precisely one term of $\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m$, say the first one. Since $\boldsymbol{a}_1$ is a vertex of P, the theorem shows that $\boldsymbol{0} \in \mathrm{ri}(\mathrm{conv}\{\overline{\boldsymbol{a}}_2, \ldots, \overline{\boldsymbol{a}}_m\})$. This is impossible, because $\overline{\boldsymbol{a}}_2, \ldots, \overline{\boldsymbol{a}}_m$ lie in the closed halfspace $H \cup H^+$ with at least one of them being in H^+, again by Part (v) of the same theorem. Thus, H^- must contain at least two terms of $\overline{\boldsymbol{a}}_1, \ldots, \overline{\boldsymbol{a}}_m$.

Example

- Consider again the Gale transform of the octahedron with vertices $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_6$ discussed in the preceding example.
- By the preceding theorem, a subset $\underline{W}$ of $\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_6\}$ is a coface of the octahedron if and only if $\mathbf{0} \in \mathrm{ri}(\mathrm{conv}\overline{W})$.
- But this is the case if and only if W contains at least one of $\boldsymbol{a}_1,\boldsymbol{a}_4$, at least one of $\boldsymbol{a}_1,\boldsymbol{a}_5$, and at least one of $\boldsymbol{a}_3,\boldsymbol{a}_6$.
- Thus a non-empty subset of $\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_6\}$ determines a proper face of the octahedron if and only if contains at most one of $\boldsymbol{a}_1,\boldsymbol{a}_4$, at most one of $\boldsymbol{a}_2,\boldsymbol{a}_5$ and at most one of $\boldsymbol{a}_3,\boldsymbol{a}_6$.

Characterization of Gale Transforms

Theorem

A sequence $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$ of points in $\mathbb{R}^{m-n-1}$ $(m > n+1)$ is a Gale transform of a vertex sequence of some n-polytope in $\mathbb{R}^n$ if and only if:

(i) $\overline{\boldsymbol{a}}_1 + \cdots + \overline{\boldsymbol{a}}_m = \mathbf{0}$;

(ii) Every open halfspace in $\mathbb{R}^{m-n-1}$ whose bounding hyperplane passes through the origin contains at least two terms of the sequence $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$.

- The only if part of the theorem follows from a previous theorem and the preceding corollary.
  Suppose, then, that $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$ is a sequence of points in $\mathbb{R}^{m-n-1}$ $(m > n+1)$ which satisfies conditions (i) and (ii) of the theorem. First, we find an n-dimensional sequence $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ in $\mathbb{R}^n$ of which $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$ is a Gale transform. This we do by reversing the procedure whereby the Gale transform of a sequence was constructed.

Characterization of Gale Transforms (Cont'd)

- Let
$$\overline{\boldsymbol{a}}_1 = (x_{11},\ldots,x_{m-n-11}),\ldots,\overline{\boldsymbol{a}}_m = (x_{1m},\ldots,x_{m-n-1m}).$$
Define points $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{m-n-1}$ in $\mathbb{R}^m$ by the equations
$$\boldsymbol{x}_1 = (x_{11},\ldots,x_{1m}),\ldots,\boldsymbol{x}_{m-n-1} = (x_{m-n-11},\ldots,x_{m-n-1m}).$$
Condition (ii) ensures that $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$ span $\mathbb{R}^{m-n-1}$. Hence, $\boldsymbol{x}_1,\ldots,\boldsymbol{x}_{m-n-1}$ form a basis for some $(m-n-1)$-dimensional subspace of $\mathbb{R}^m$, S say. Thus, $S^\perp$ has dimension $m-(m-n-1)=n+1$. Condition (i) shows that $(1,\ldots,1)\in S^\perp$. Hence $(1,\ldots,1)$ can be extended by vectors $(a_{11},\ldots,a_{m1})$, ..., $(a_{1n},\ldots,a_{mn})$ in $\mathbb{R}^m$ to form a basis for $S^\perp$. Write
$$\boldsymbol{a}_1 = (a_{11},\ldots,a_{1n}),\ldots,\boldsymbol{a}_m = (a_{m1},\ldots,a_{mn}).$$
Then $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ is an n-dimensional sequence in $\mathbb{R}^n$ that has $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$ for a Gale transform.

Characterization of Gale Transforms (Cont'd)

- We complete the proof by showing that $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ is a vertex sequence of the n-polytope $\text{conv}\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m\}$.
  To do this, we show that, for $i=1,\ldots,m$,
  $$\boldsymbol{a}_i \notin \text{conv}\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{i-1},\boldsymbol{a}_{i+1},\ldots,\boldsymbol{a}_m\}.$$
  Suppose that this is not so. Then, for some i in $\{1,\ldots,m\}$, there exists an affine dependence $(\lambda_1,\ldots,\lambda_m)$ of $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ with $\lambda_i=-1$ and $\lambda_j\geq 0$ for $j\in\{1,\ldots,m\}\backslash\{i\}$. By a previous theorem, there is $\boldsymbol{c}$ in $\mathbb{R}^{m-n-1}$ such that $\boldsymbol{c}\cdot\overline{\boldsymbol{a}}_i<0$ and $\boldsymbol{c}\cdot\overline{\boldsymbol{a}}_j=\lambda_j$ for $j\in\{1,\ldots,m\}\backslash\{i\}$.
  Thus, the open halfspace
  $$\{\boldsymbol{z}\in\mathbb{R}^{m-n-1}:\boldsymbol{c}\cdot\boldsymbol{z}<0\}$$
  in $\mathbb{R}^{m-n-1}$ has the origin on its boundary and contains only one term of the sequence $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$, contradicting condition (ii). Therefore, $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ is a vertex sequence of the n-polytope $\text{conv}\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m\}$.

Gale Transforms and Simplicial Polytopes

Theorem

Let $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$ be a Gale transform in $\mathbb{R}^{m-n-1}$ of a vertex sequence $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ of an n-polytope P in $\mathbb{R}^n$ $(m>n+1)$. Then P is simplicial if and only if the origin of $\mathbb{R}^{m-n-1}$ cannot be expressed as a positive convex combination of fewer than $m-n$ terms of $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$.

- P is simplicial if and only if it has no proper face with more than n vertices.

  I.e., P is simplicial if and only if it has no non-empty coface with fewer than $m-n$ vertices.

  Thus, by a previous theorem, P is simplicial if and only if the origin of $\mathbb{R}^{m-n-1}$ cannot be expressed as a positive convex combination of fewer than $m-n$ terms of $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$.

Gale Transforms and Combinatorial Types

- Since a Gale transform of a polytope contains full information about its combinatorial structure, the combinatorial type of a polytope can be determined from any one of its Gale transforms.
- Suppose that $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ and $\boldsymbol{b}_1,\ldots,\boldsymbol{b}_m$ are, respectively, vertex sequences of n-polytopes P and Q in $\mathbb{R}^n$ $(m>n+1)$.
- Suppose that $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_m$ and $\overline{\boldsymbol{b}}_1,\ldots,\overline{\boldsymbol{b}}_m$ are, respectively, Gale transforms of $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m$ and $\boldsymbol{b}_1,\ldots,\boldsymbol{b}_m$.
- By the definition of combinatorial equivalence and a previous theorem, P and Q are combinatorially equivalent if and only if there is a permutation θ of $\{1,\ldots,m\}$ such that, for every subset J of $\{1,\ldots,m\}$,

$$\mathbf{0}\in\text{ri}(\text{conv}\{\overline{\boldsymbol{a}}_j : j\in J\})\quad\text{if and only if}\quad\mathbf{0}\in\text{ri}(\text{conv}\{\overline{\boldsymbol{b}}_{\theta(j)} : j\in I\}).$$

Number of Combinatorial Types of Polytopes

Theorem

There are $\left[\frac{1}{4}n^2\right]$ combinatorial types of n-polytopes with $n+2$ vertices and $\left[\frac{1}{2}n\right]$ of these are simplicial.

- Let $\overline{\boldsymbol{a}}_1,\ldots,\overline{\boldsymbol{a}}_{n+2}$ be a Gale transform in $\mathbb{R}^1$ of a vertex sequence $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_{n+2}$ of an n-polytope P in $\mathbb{R}^n$. By a previous theorem, this transform is a sequence of $n+2$ real numbers whose sum is zero. Suppose that this sequence has r positive terms and s negative ones, so that $r \geq 2$, $s \geq 2$ and $r+2 \leq n+2$. We call such a sequence a **G-sequence of type** (r,s).

Number of Combinatorial Types of Polytopes (Cont'd)

- Suppose next that $\overline{\boldsymbol{b}}_1,\ldots,\overline{\boldsymbol{b}}_{n+2}$ is a Gale transform in $\mathbb{R}^1$ of a vertex sequence $\boldsymbol{b}_1,\ldots,\boldsymbol{b}_{n+2}$ of an n-polytope Q in $\mathbb{R}^n$.

  Suppose $\overline{\boldsymbol{b}}_1,\ldots,\overline{\boldsymbol{b}}_{n+2}$ is a G-sequence of type (r',s').

  In view of our preceding remarks on combinatorial equivalence, P and Q are combinatorially equivalent if and only if either $r=r'$ and $s=s'$ or $r=s'$ and $s=r'$.

  A previous theorem shows that every G-sequence of $n+2$ terms of $\mathbb{R}^1$ is a Gale transform of some n-polytope in $\mathbb{R}^n$ with $n+2$ vertices.

Number of Combinatorial Types of Polytopes (Even n)

- Thus, the number of combinatorial types of n-polytopes with $n+2$ vertices equals the number of ordered pairs (r,s) of integers satisfying $s \geq r \geq 2$ and $r+s \leq n+2$.
  We now calculate this number.
  - When n is even, these ordered pairs are:

$$\begin{array}{l}(2,n),\ (2,n-1),\ \ldots,\ (2,3),\ (2,2);\\ (3,n-1),\ (3,n-2),\ \ldots,\ (3,3);\\ \vdots\\ (\frac{1}{2}(n+2),\frac{1}{2}(n+2)).\end{array}$$

    The total number is

$$\begin{array}{rcl}(n-1)+(n-3)+\cdots+1 & = & 1+2+\cdots+(n-1)-(2+4+\cdots+(n-2))\\ & = & 1+2+\cdots+(n-1)-2\left(1+2+\cdots\frac{n-2}{2}\right)\\ & = & \frac{n(n-1)}{2}-2\frac{\frac{n-2}{2}\frac{n}{2}}{2}=\frac{n^2-n}{2}-\frac{n^2-2n}{4}=\frac{1}{4}n^2.\end{array}$$

Number of Combinatorial Types of Polytopes (Odd n)

- The number of combinatorial types of n-polytopes with $n+2$ vertices equals the number of ordered pairs (r,s) of integers satisfying $s \geq r \geq 2$ and $r+s \leq n+2$.
  - When n is odd, these ordered pairs are:

$$\begin{array}{l}(2,n),\ (2,n-1),\ \ldots,\ (2,3),\ (2,2);\\ (3,n-1),\ (3,n-2),\ \ldots,\ (3,3);\\ \vdots\\ (\frac{1}{2}(n+1),\frac{1}{2}(n+3)),\ (\frac{1}{2}(n+1),\frac{1}{2}(n+1)).\end{array}$$

The total number is

$$\begin{array}{rcl}(n-1)+(n-3)+\cdots+2 & = & 2\left(1+2+\cdots\frac{n-1}{2}\right)\\ & = & 2\frac{\frac{n-1}{2}\frac{n+1}{2}}{2}=\frac{1}{4}(n^2-1).\end{array}$$

In both cases, the required number is $\left[\frac{1}{4}n^2\right]$.

Number of Combinatorial Types of Polytopes (Cont'd)

- The preceding theorem shows that a G-sequence of $n+2$ terms which is of type (r,s) corresponds to a simplicial n-polytope with $n+2$ vertices if and only if $\mathbf{0}$ is not one of its terms, i.e., if and only if $r+s=n+2$.

  Thus the number of combinatorial types of simplicial n-polytopes with $n+2$ vertices equals the number of ordered pairs (r,s) of integers such that $s \geq r \geq 2$ and $r+s=n+2$.

  This number is $\frac{1}{2}n$ when n is even, and $\frac{1}{2}(n-1)$ when n is odd.

  In both cases it equals $[\frac{1}{2}n]$.

Applications on Combinatorial Types

- The last theorem with $n=3$ shows that there are precisely two combinatorial types of 3-polytopes with five vertices, only one type being simplicial.
- We have already seen examples of these two types:
  - A square pyramid (non-simplicial);
  - The polytope formed by taking the union of a regular tetrahedron and its reflection in one of its triangular faces (simplicial).
- Possible Gale transforms for these two examples: $1,-1,1,-1,0$ and $2,2,2,-3,-3$, themselves make it clear why the two examples are of different combinatorial types, and that the first one (the square pyramid) is non-simplicial, as 0 occurs in its Gale transform.
- This example serves to show the power and potential of Gale transform techniques in studying the combinatorial properties of polytopes.