Introduction to Convexity

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Subsection 1

The Finite Basis Theorem

Finitely Generated Convex Cones and Polyhedral Cones

- A **finitely generated convex cone** is one that is generated by a finite set, i.e., a convex cone of the form

$$\mathrm{cone}\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m\} = \{\lambda_1\boldsymbol{a}_1+\cdots+\lambda_m\boldsymbol{a}_m : \lambda_1,\ldots,\lambda_m \geq 0\},$$

  where $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m \in \mathbb{R}^n$.
- A convex cone in $\mathbb{R}^n$ which is also a polyhedral set is called a **polyhedral cone**.
- Clearly, a set in $\mathbb{R}^n$ is a polyhedral cone if and only if it is a finite intersection of closed halfspaces whose bounding hyperplanes pass through the origin.

Characterization of Finitely Generated Convex Cones

Theorem

A convex cone in $\mathbb{R}^n$ is finitely generated if and only if it is polyhedral.

- Suppose first that C is a polyhedral cone in $\mathbb{R}^n$. Let P be a polytope in $\mathbb{R}^n$ such that $\mathbf{0} \in \operatorname{int} P$. Then $C \cap P$ is a bounded polyhedral set, and hence a polytope. Thus $C \cap P$ is $\operatorname{conv}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$ for some points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ of $C \cap P$. We show that C is the finitely generated convex cone $\operatorname{cone}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$.

  Since C is a convex cone containing $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$, $\operatorname{cone}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\} \subseteq C$.

  If $\boldsymbol{c} \in C$, then, since $\mathbf{0} \in \operatorname{int} P$, there is some $\lambda > 0$ such that $\lambda \boldsymbol{c} \in P$. Thus, $\lambda \boldsymbol{c} \in C \cap P = \operatorname{conv}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\} \subseteq \operatorname{cone}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$. So $\boldsymbol{c} \in \frac{1}{\lambda}\operatorname{cone}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\} = \operatorname{cone}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$. Hence, we have $C \subseteq \operatorname{cone}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$. Therefore, $C = \operatorname{cone}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\}$.

Characterization (Cont'd)

- Suppose next that C is the finitely generated convex cone $\text{cone}\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m\}$, where $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m \in \mathbb{R}^n$. Since polytopes are polyhedral sets, $\text{conv}\{\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m\}$ can be written as the intersection of some closed halfspaces $J_1,\ldots,J_r$ in $\mathbb{R}^n$. We show that C is the polyhedral cone A formed by the intersection of those J_i's which have the origin on their boundaries. Since $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m \in A$, $C \subseteq A$. If $\boldsymbol{a} \in A$, then:
  - $\lambda\boldsymbol{a} \in J_i$, for all $\lambda > 0$ when $\mathbf{0}$ lies on the boundary of J_i;
  - $\lambda\boldsymbol{a} \in J_i$ for all sufficiently small $\lambda > 0$ when $\mathbf{0}$ does not lie on the boundary of J_i.

  It follows that there is a $\lambda > 0$ such that

$$\lambda\boldsymbol{a} \in J_1 \cap \cdots \cap J_r = \text{conv}\{\mathbf{0},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_m\} \subseteq C.$$

  Hence $\boldsymbol{a} \in \frac{1}{\lambda}C = C$. Thus, $A \subseteq C$, and $C = A$ as desired.

Corollary

Finitely generated convex cones in $\mathbb{R}^n$ are closed.

Finite Basis Theorem

Theorem (Finite Basis Theorem)

A set in $\mathbb{R}^n$ is a polyhedral set if and only if it can be expressed as a vector sum of a polytope and a finitely generated convex cone.

- Suppose first that $P = \text{conv} S + \text{cone} T$, where S and T are finite sets in $\mathbb{R}^n$. Then P is the vector sum of the compact polytope $\text{conv} S$ and the closed finitely generated convex cone $\text{cone} T$. So it is closed by a previous theorem. Since $\text{conv} S$ and $\text{cone} T$ are polyhedral sets, they only have a finite number of faces. It follows from a previous theorem that P has only a finite number of exposed faces. Thus, P is a closed convex set which has only a finite number of exposed faces. So it must be a polyhedral set by a previous theorem.

Finite Basis Theorem (Cont'd)

- Suppose now that P is a polyhedral set in $\mathbb{R}^n$ which contains no lines.

  If P is bounded, then it is a polytope. So it is trivially the vector sum of a polytope and the zero cone.

  Assume, then, that P is unbounded. By a previous corollary, $P = \text{conv}(S \cup L_1 \cup \cdots \cup L_r)$, where S is the set of extreme points of P and $L_1, \ldots, L_r$ are its extreme halflines. Let $T = \{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r\}$, where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$ are non-zero vectors belonging to the directions of $L_1, \ldots, L_r$, respectively. A previous theorem shows that T lies in the recession cone of P. Hence, so too does $\text{cone}\, T$. By a previous theorem, $\text{conv}\, S + \text{cone}\, T \subseteq P + \text{cone}\, T \subseteq P$.

  On the other hand, $\text{conv}\, S + \text{cone}\, T$ is a convex set containing $S \cup L_1 \cup \cdots \cup L_r$. This shows that $P \subseteq \text{conv}\, S + \text{cone}\, T$.

  Thus, P is the vector sum of the polytope $\text{conv}\, S$ and the finitely generated convex cone $\text{cone}\, T$.

Finite Basis Theorem (Cont'd)

- Suppose, finally, that P is a polyhedral set in $\mathbb{R}^n$ which contains a line. A previous theorem shows that $P = (P \cap L^\perp) + L$, where L is the (non-zero) lineality space of P, and $P \cap L^\perp$ is a polyhedral set containing no lines. By what we have just proved, $P \cap L^\perp$ can be expressed as $\operatorname{conv} S + \operatorname{cone} T$ for some finite sets S and T in $\mathbb{R}^n$. Let T' be a basis for the subspace L. Then $L = \operatorname{cone}(T' \cup (-T'))$ and

$$\begin{array}{rcl} P & = & \operatorname{conv} S + \operatorname{cone} T + \operatorname{cone}(T' \cup (-T')) \\ & = & \operatorname{conv} S + \operatorname{cone}(T \cup T' \cup (-T')). \end{array}$$

  Thus we have expressed P as a vector sum of a polytope and a finitely generated convex cone.

Finite Bases

- The finite basis theorem shows that, given any polyhedral set P in $\mathbb{R}^n$, there exist $\boldsymbol{a}_1, \dots, \boldsymbol{a}_m \in \mathbb{R}^n$ and $k \in \{0, 1, \dots, m\}$ such that

$$P = \{\lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m : \lambda_1, \dots, \lambda_m \ge 0 \text{ and } \lambda_1 + \cdots + \lambda_k = 1\};$$

  it being understood that P is empty when $m = 0$, and $\operatorname{cone}\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_m\}$ when $k = 0$ and $m > 0$.
- The ordered pair $(\{\boldsymbol{a}_1, \dots, \boldsymbol{a}_k\}, \{\boldsymbol{a}_{k+1}, \dots, \boldsymbol{a}_m\})$ is sometimes referred to as a **finite basis** for P, and P is said to be **finitely generated** by $\boldsymbol{a}_1, \dots, \boldsymbol{a}_k; \boldsymbol{a}_{k+1}, \dots, \boldsymbol{a}_m$.

Polyhedral Sets Under Addition and Scalar Multiplication

Theorem

Let A, B be polyhedral sets in $\mathbb{R}^n$ and let α be a scalar. Then $A+B$ and αA are polyhedral sets.

- By the finite basis theorem, there are finite sets C, D, E, F in $\mathbb{R}^n$ such that $A = \text{conv}\, C + \text{cone} D$ and $B = \text{conv}\, E + \text{cone} F$.

  Now $\text{conv}\, C + \text{conv}\, E = \text{conv}(C+E)$ and $\text{cone} D + \text{cone} F = \text{cone}(D \cup F)$. The first of these equations can be established by the argument used in the proof for the sum of polytopes, and the second is trivial. Thus, $A+B = \text{conv}(C+E) + \text{cone}(D \cup F)$. This shows that $A+B$ is a polyhedral set.

  We also have $\alpha A = \text{conv}(\alpha C) + \text{cone}(\alpha D)$.

  Hence, αA is a polyhedral set.

Strict Separability of Polyhedral Sets

Theorem

Each pair A and B of disjoint non-empty polyhedral sets in $\mathbb{R}^n$ can be strictly separated.

- The preceding theorem shows that $A - B$ is a polyhedral set. Since $A - B$ is closed and does not contain the origin, it can be strictly separated from the origin. Thus there exist $\boldsymbol{c} \in \mathbb{R}^n$ and $c_0 \in \mathbb{R}$ such that

  $$\boldsymbol{c} \cdot (\boldsymbol{a} - \boldsymbol{b}) > c_0 > 0, \text{ for } \boldsymbol{a} \in A, \boldsymbol{b} \in B.$$

  It follows easily that there is a scalar d satisfying

  $$\inf \{\boldsymbol{c} \cdot \boldsymbol{a} : \boldsymbol{a} \in A\} > d > \sup \{\boldsymbol{c} \cdot \boldsymbol{b} : \boldsymbol{b} \in B\}.$$

  So the hyperplane with equation $\boldsymbol{c} \cdot \boldsymbol{z} = d$ strictly separates A and B.

Bounds of Functions Defined on Polyhedral Sets

Theorem

Suppose that $f : P \to \mathbb{R}$ is a linear function which is bounded above on a non-empty line-free polyhedral set P in $\mathbb{R}^n$. Then f attains its upper bound at an extreme point of P.

- We can write

$$P = \text{conv}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m\} + \text{cone}\{\boldsymbol{b}_1, \ldots, \boldsymbol{b}_p\},$$

  where $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m, \boldsymbol{b}_1, \ldots, \boldsymbol{b}_p \in \mathbb{R}^n$ and $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ are the extreme points of P.

  A typical point $\boldsymbol{x}$ of P can be written in the form

$$\boldsymbol{x} = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m + \mu_1 \boldsymbol{b}_1 + \cdots + \mu_p \boldsymbol{b}_p,$$

  where $\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_p \geq 0$ and $\lambda_1 + \cdots + \lambda_m = 1$.

Bounds of Functions Defined on Polyhedral Sets (Cont'd)

- Since f is linear,

$$f(\boldsymbol{x}) = \lambda_1 f(\boldsymbol{a}_1) + \cdots + \lambda_m f(\boldsymbol{a}_m) + \mu_1 f(\boldsymbol{b}_1) + \cdots + \mu_p f(\boldsymbol{b}_p).$$

  Since f is bounded above and $\mu_1, \ldots, \mu_p$ may assume any positive values, $f(\boldsymbol{b}_1), \ldots, f(\boldsymbol{b}_p) \leq 0$.

  It is now easy to see that f assumes its upper bound (maximal value) at any extreme point $\boldsymbol{a}_i$ of P for which

$$f(\boldsymbol{a}_i) = \max\{f(\boldsymbol{a}_1), \ldots, f(\boldsymbol{a}_m)\}.$$

Subsection 2

Linear Inequalities

Matrix Notation

- We denote matrices by square brackets and we identify the points of $\mathbb{R}^n$ with column matrices:

$$\boldsymbol{x} = (x_1, \ldots, x_n) = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

- We denote the transpose of a matrix A by A^T and we identify a square matrix of order one with the number which determines it.
- Thus, we write

$$\boldsymbol{x}^T = (x_1, \ldots, x_n)^T = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix},$$
$$\boldsymbol{x} \cdot \boldsymbol{y} = (x_1, \ldots, x_n) \cdot (y_1, \ldots, y_n) = \boldsymbol{x}^T \boldsymbol{y}.$$

Matrix Notation (Cont'd)

- When real $m \times n$ matrices $\boldsymbol{A} = [a_{ij}]$, $\boldsymbol{B} = [b_{ij}]$ are such that $a_{ij} < b_{ij}$, for $i = 1, \ldots, m$ and $j = 1, \ldots, n$, we write $\boldsymbol{A} < \boldsymbol{B}$;
- Similar definitions apply to the inequalities $\boldsymbol{A} \leq \boldsymbol{B}$, $\boldsymbol{A} > \boldsymbol{B}$, $\boldsymbol{A} \geq \boldsymbol{B}$.
- In the following discussion:
  - $\boldsymbol{A}$ denotes a real $m \times n$ matrix $[a_{ij}]$;
  - $\boldsymbol{b}$ denotes a point $(b_1, \ldots, b_m)$ of $\mathbb{R}^m$;
  - $\boldsymbol{x}$ denotes a point $(x_1, \ldots, x_n)$ of $\mathbb{R}^n$;
  - $\boldsymbol{y}$ denotes a point $(y_1, \ldots, y_m)$ of $\mathbb{R}^m$;
  - $\mathbf{0}$ denotes a zero vector, whose size can be determined from the context.

Closed Convex Cones and Points

Theorem

In $\mathbb{R}^n$ let $\boldsymbol{a}$ be a point not lying in a closed convex cone C. Then there exists a point $\boldsymbol{u}$ in $\mathbb{R}^n$ such that $\boldsymbol{u} \cdot \boldsymbol{a} < 0$, and $\boldsymbol{u} \cdot \boldsymbol{c} \geq 0$ for all points $\boldsymbol{c}$ in C.

- Since $\boldsymbol{a}$ does not belong to the closed convex set C, $\boldsymbol{a}$ can be strictly separated from C. Thus, there exist $\boldsymbol{u} \in \mathbb{R}^n$ and $u_0 \in \mathbb{R}$ such that

$$\boldsymbol{u} \cdot \boldsymbol{a} < u_0 < \boldsymbol{u} \cdot \boldsymbol{c}, \text{ for } \boldsymbol{c} \in C.$$

Since $\mathbf{0} \in C$, we have $\boldsymbol{u} \cdot \boldsymbol{a} < u_0 < 0$. Let $\boldsymbol{c} \in C$ and $\lambda > 0$. Then $\lambda \boldsymbol{c} \in C$. So $\boldsymbol{u} \cdot (\lambda \boldsymbol{c}) > u_0$. Hence, $\boldsymbol{u} \cdot \boldsymbol{c} > \frac{u_0}{\lambda}$. Letting $\lambda \to \infty$ in the last inequality, we deduce that $\boldsymbol{u} \cdot \boldsymbol{c} \geq 0$.

Systems of Linear Inequalities

- Consider the following system of m linear inequalities in n variables:

$$\begin{gathered} a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1, \\ \vdots \\ a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m. \end{gathered}$$

- In matrix notation, this system of inequalities assumes the form

$$\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}.$$

- If there exist real numbers $x_1, \ldots, x_m$ which simultaneously satisfy the m linear inequalities of the system, then the system is said to be **consistent**.
- Otherwise it is said to be **inconsistent**.
- To show that a system is consistent, we only have to find $x_1, \ldots, x_n$ which satisfy it.

Example: Showing Inconsistency

- Consider the following system of three inequalities in three variables:

$$\begin{array}{rcr} x - y + 2z & \leq & -1 \\ -2x + y - 3z & \leq & 4 \\ 5x - y + 6z & \leq & -14. \end{array}$$

- After making several unsuccessful attempts at finding x, y, z which simultaneously satisfy these three inequalities, we might correctly conclude that the system is inconsistent.
- Such a lack of success does not, of course, prove the inconsistency.
- Suppose, arguing for a contradiction, that the real numbers x, y, z satisfy the given inequalities.
- After multiplying the inequalities by $3, 4, 1$, respectively, and adding the resulting inequalities together, we find $0x + 0y + 0z = 0 \leq -1$.
- This contradiction shows that the given system is inconsistent.

The General Method

- Suppose that $x_1, \ldots, x_n$ satisfy the system of linear inequalities and that $y_1, \ldots, y_m \geq 0$.
- Then
$$(a_{11}y_1 + \cdots + a_{m1}y_m)x_1 + \cdots + (a_{1n}y_1 + \cdots + a_{mn}y_m)x_n \\ \leq b_1y_1 + \cdots + b_my_m.$$
- In matrix form this is $\boldsymbol{y}^T\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{y}^T\boldsymbol{b}$.
- This last inequality cannot be satisfied if
$$a_{11}y_1 + \cdots + a_{m1}y_m = 0, \ldots, a_{1n}y_1 + \cdots + a_{mn}y_m = 0, \\ b_1y_1 + \cdots + b_my_m < 0.$$
- In matrix form, if $\boldsymbol{y}^T\boldsymbol{A} = \boldsymbol{0}^T$, $\boldsymbol{y}^T\boldsymbol{b} < 0$.
- We have thus shown that, if there exists $\boldsymbol{y} \geq 0$ such that $\boldsymbol{y}^T\boldsymbol{A} = \boldsymbol{0}^T$, $\boldsymbol{y}^T\boldsymbol{b} < 0$, then the system $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ is inconsistent.
- We will see that the converse is true, i.e., if the system $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ is inconsistent, then there exists $\boldsymbol{y} \geq \boldsymbol{0}$ such that $\boldsymbol{y}^T\boldsymbol{A} = \boldsymbol{0}^T$, $\boldsymbol{y}^T\boldsymbol{b} < 0$.

An Auxiliary Lemma

Lemma

Suppose that there exists no $\boldsymbol{x} \geq 0$ in $\mathbb{R}^n$ such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$. Then there exists $\boldsymbol{y}$ in $\mathbb{R}^m$ such that $\boldsymbol{y}^T\boldsymbol{A} \geq \boldsymbol{0}^T$, $\boldsymbol{y}^T\boldsymbol{b} < 0$.

- Denote the columns of $\boldsymbol{A}$ by $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$. By the hypothesis of the lemma, there is no $\boldsymbol{x} = (x_1, \ldots, x_n) \geq \boldsymbol{0}$ such that

  $$\boldsymbol{A}\boldsymbol{x} = x_1\boldsymbol{c}_1 + \cdots + x_n\boldsymbol{c}_n = \boldsymbol{b},$$

  i.e., $\boldsymbol{b} \notin \operatorname{cone}\{\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n\}$. The preceding theorem shows that there exists $\boldsymbol{y}$ in $\mathbb{R}^m$ such that $\boldsymbol{y}^T\boldsymbol{b} < 0$ and $\boldsymbol{y}^T\boldsymbol{c}_1 \geq 0, \ldots, \boldsymbol{y}^T\boldsymbol{c}_n \geq 0$. The latter can be rewritten as $\boldsymbol{y}^T[\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n] = \boldsymbol{y}^T\boldsymbol{A} \geq \boldsymbol{0}^T$.

Characterization of Inconsistency

Theorem

The system of inequalities $\boldsymbol{Ax} \leq \boldsymbol{b}$ is inconsistent if and only if there exists $\boldsymbol{y} \geq \boldsymbol{0}$ such that $\boldsymbol{y}^T\boldsymbol{A} = \boldsymbol{0}^T$, $\boldsymbol{y}^T\boldsymbol{b} < 0$.

- We have already seen that, if there exists $\boldsymbol{y} \geq \boldsymbol{0}$ such that $\boldsymbol{y}^T\boldsymbol{A} = \boldsymbol{0}^T$, $\boldsymbol{y}^T\boldsymbol{b} < 0$, then the system $\boldsymbol{Ax} \leq \boldsymbol{b}$ is inconsistent.
  Suppose, then, that the system $\boldsymbol{Ax} \leq \boldsymbol{b}$ is inconsistent. Consider the system of m linear equations in $2n+m$ variables represented by the matrix equation

  $$[\boldsymbol{A}, -\boldsymbol{A}, \boldsymbol{I}_m]\boldsymbol{z} = \boldsymbol{b}, \text{ where } \boldsymbol{z} = (z_1, \ldots, z_{2n+m}).$$

  This system cannot have a solution $\boldsymbol{z}$ for which $\boldsymbol{z} \geq \boldsymbol{0}$.
  The existence of such a solution would imply that

  $$\boldsymbol{A}(z_1 - z_{n+1}, \ldots, z_n - z_{2n}) \leq \boldsymbol{b}.$$

  This would contradict the assumed inconsistency of $\boldsymbol{Ax} \leq \boldsymbol{b}$.

Characterization of Inconsistency (Cont'd)

- By the preceding lemma, there exists $\boldsymbol{y}$ such that $\boldsymbol{y}^T\boldsymbol{b}<0$ and

$$\boldsymbol{y}^T[\boldsymbol{A},-\boldsymbol{A},\boldsymbol{I}]\geq\boldsymbol{0}^T.$$

  The latter can be rewritten as

$$\boldsymbol{y}^T\boldsymbol{A}\geq\boldsymbol{0}^T,\quad -\boldsymbol{y}^T\boldsymbol{A}\geq\boldsymbol{0}^T,\quad \boldsymbol{y}^T\boldsymbol{I}_m\geq\boldsymbol{0}^T.$$

  Hence

$$\boldsymbol{y}\geq 0,\quad \boldsymbol{y}^T\boldsymbol{A}=\boldsymbol{0}^T,\quad \boldsymbol{y}^T\boldsymbol{b}<0.$$

Dual Pairs of Inequalities

- An immediate consequence of the last theorem is that precisely one of the following systems of inequalities in $\boldsymbol{x}$ and $\boldsymbol{y}$ has a solution:
  (i) $\boldsymbol{Ax} \leq \boldsymbol{b}$;
  (ii) $\boldsymbol{y}^T\boldsymbol{A} = \boldsymbol{0}^T$, $\boldsymbol{y}^T\boldsymbol{b} < 0$, $\boldsymbol{y} \geq \boldsymbol{0}$.
- Two finite systems of linear inequalities such as (i) and (ii), precisely one of which has a solution, are said to form a **dual pair** or to be **dual** to each other.
- It follows easily from the preceding lemma that the following systems are dual to each other:
  (i) $\boldsymbol{Ax} = \boldsymbol{b}$, $\boldsymbol{x} \geq \boldsymbol{0}$;
  (ii) $\boldsymbol{y}^T\boldsymbol{A} \geq \boldsymbol{0}^T$, $\boldsymbol{y}^T\boldsymbol{b} < 0$.

Dual Pairs and Linear Equations

- One interesting application of dual pairs is to the theory of linear equations.
- It is an easy exercise to show that the following systems form a dual pair:
  (i) $\boldsymbol{Ax} = \boldsymbol{b}$;
  (ii) $\boldsymbol{y}^T \boldsymbol{A} = \boldsymbol{0}^T$, $\boldsymbol{y}^T \boldsymbol{b} \neq 0$.
- Thus, if the system of linear equations $\boldsymbol{Ax} = \boldsymbol{b}$ is inconsistent, then some linear combination of its equations yields the contradiction $0 \neq 0$.

  This result is often tacitly assumed, but rarely proved.

A Mixed System of Weak and Strict Inequalities

- Let $\boldsymbol{A}$ be an $m \times n$ matrix, $\boldsymbol{C}$ a $p \times n$ matrix, $\boldsymbol{x}$ an $n \times 1$ matrix, $\boldsymbol{b}$ an $m \times 1$ matrix, and $\boldsymbol{d}$ a $p \times 1$ matrix.
- Consider the system comprising the m weak inequalities $\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}$ and the p strict inequalities $\boldsymbol{C}\boldsymbol{x} < \boldsymbol{d}$.
- When is this mixed system of linear inequalities inconsistent?
- Two possibilities immediately suggest themselves:
  (i) Suppose there are $\boldsymbol{u} = (u_1, \ldots, u_m) \ge \boldsymbol{0}$, $\boldsymbol{v} = (v_1, \ldots, v_p) \ge \boldsymbol{0}$ with $\boldsymbol{v} \ne \boldsymbol{0}$ such that $\boldsymbol{u}^T\boldsymbol{A} + \boldsymbol{v}^T\boldsymbol{C} = \boldsymbol{0}^T$ and $\boldsymbol{u}^T\boldsymbol{b} + \boldsymbol{v}^T\boldsymbol{d} \le 0$. Then, if $\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}$, $\boldsymbol{C}\boldsymbol{x} < \boldsymbol{d}$, we may conclude that $(\boldsymbol{u}^T\boldsymbol{A} + \boldsymbol{v}^T\boldsymbol{C})\boldsymbol{x} = 0 < \boldsymbol{u}^T\boldsymbol{b} + \boldsymbol{v}^T\boldsymbol{d} \le 0$. This shows that the mixed system is inconsistent.
  (ii) Suppose there is $\boldsymbol{u} = (u_1, \ldots, u_m) \ge \boldsymbol{0}$ such that $\boldsymbol{u}^T\boldsymbol{A} = \boldsymbol{0}^T$ and $\boldsymbol{u}^T\boldsymbol{b} < 0$. Then, if $\boldsymbol{A}\boldsymbol{x} < \boldsymbol{b}$, we conclude that $\boldsymbol{u}^T\boldsymbol{A}\boldsymbol{x} = 0 \le \boldsymbol{u}^T\boldsymbol{b} < 0$. This shows that the mixed system is inconsistent.
- We show that these are the only ways in which the mixed system can be inconsistent.

Characterization of Inconsistency

Theorem

Suppose that the mixed system of inequalities $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{C}\boldsymbol{x} < \boldsymbol{d}$ is inconsistent. Then either:

(i) there exist $\boldsymbol{u} \geq \mathbf{0}$, $\boldsymbol{v} \geq \mathbf{0}$ with $\boldsymbol{v} \neq \mathbf{0}$ such that $\boldsymbol{u}^T\boldsymbol{A} + \boldsymbol{v}^T\boldsymbol{C} = \mathbf{0}^T$ and $\boldsymbol{u}^T\boldsymbol{b} + \boldsymbol{v}^T\boldsymbol{d} \leq 0$ or

(ii) there exixts $\boldsymbol{u} \geq \mathbf{0}$ such that $\boldsymbol{u}^T\boldsymbol{A} = \mathbf{0}^T$ and $\boldsymbol{u}^T\boldsymbol{b} < 0$.

- Consider the following system of $m+p+1$ weak inequalities in the $n+1$ variables $z_1, \ldots, z_n, z$:

$$\begin{bmatrix} \boldsymbol{A} & -\boldsymbol{b} \\ \boldsymbol{C} & -\boldsymbol{d} \\ \mathbf{0}^T & -1 \end{bmatrix} \begin{bmatrix} z_1 \\ \vdots \\ z_n \\ z \end{bmatrix} \leq \begin{bmatrix} \mathbf{0}_m \\ -\mathbf{1}_p \\ -1 \end{bmatrix},$$

  where $\mathbf{0}_m$ is the column vector consisting of m 0's, and $-\mathbf{1}_p$ is the column vector consisting of p -1's.

Characterization of Inconsistency (Cont'd)

- This system is inconsistent, for if it were satisfied by $z_1, \ldots, z_n, z$, then
$$x_1 = \frac{z_1}{z}, \ldots, x_n = \frac{z_n}{z}$$
would satisfy the inconsistent system of the theorem.
By the preceding theorem, there exist $\boldsymbol{u} = (u_1, \ldots, u_m) \geq \mathbf{0}$, $\boldsymbol{v} = (v_1, \ldots, v_p) \geq \mathbf{0}$, $w \geq 0$ such that
$$\boldsymbol{u}^T \boldsymbol{A} + \boldsymbol{v}^T \boldsymbol{C} = \mathbf{0}^T, \ -\boldsymbol{u}^T \boldsymbol{b} - \boldsymbol{v}^T \boldsymbol{d} - w = 0, \ -v_1 - \cdots - v_p - w < 0.$$
The alternatives (i) and (ii) of the theorem correspond to the cases $\boldsymbol{v} \neq \mathbf{0}$ and $\boldsymbol{v} = \mathbf{0}$, respectively.

Corollary

Suppose that the system of strict inequalities $\boldsymbol{C}\boldsymbol{x} < \boldsymbol{d}$ is inconsistent. Then there exists $\boldsymbol{v} \geq \mathbf{0}$ with $\boldsymbol{v} \neq \mathbf{0}$ such that $\boldsymbol{v}^T \boldsymbol{C} = \mathbf{0}^T$ and $\boldsymbol{v}^T \boldsymbol{d} \leq 0$.

- Take $\boldsymbol{A}$ and $\boldsymbol{b}$ to be zero matrices in the theorem.

Solutions and Consequences

- Consider the following system $\mathscr{S}$ of m linear inequalities in n variables $x_1,\ldots,x_n$:
$$\begin{array}{ccc} a_{11}x_1+\cdots+a_{1n}x_n & r_1 & b_1 \\ & \vdots & \\ a_{m1}x_1+\cdots+a_{mn}x_n & r_m & b_m \end{array}$$
where each r_i is either $\leq$ or $<$.
- By a **solution** to $\mathscr{S}$ is meant an n-tuple $(x_1,\ldots,x_n)$ whose coordinates simultaneously satisfy all the inequalities of $\mathscr{S}$.
- The system $\mathscr{S}$ is said to be **consistent** it it has a solution.
- Otherwise it is said to be **inconsistent**.
- An inequality $e_1x_1+\cdots+e_nx_n\ r\ f$ where r is either $\leq$ or $<$ is called a **consequence** of $\mathscr{S}$ if it is satisfied by all solutions of $\mathscr{S}$.
- If $\mathscr{S}$ is inconsistent, then every linear inequality in $x_1,\ldots,x_n$ is (vacuously) a consequence of $\mathscr{S}$.

Legal Linear Combinations

- Let $y_1, \ldots, y_m \geq 0$ and let b be a scalar such that $b_1 y_1 + \cdots + b_m y_m \leq b$.
- Consider an inequality of the form

$$(a_{11} y_1 + \cdots + a_{m1} y_m) x_1 + \cdots + (a_{1n} y_1 + \cdots + a_{mn} y_m) x_n \ r \ b,$$

  where one of the following holds:
  - r is $\leq$;
  - r is $<$ and for some $i \in \{1, \ldots, m\}$, $y_i > 0$ and r_i is $<$;
  - r is $<$ and $b_1 y_1 + \cdots + b_m y_m < b$.

  Such an inequality is a consequence of $\mathscr{S}$ called a **legal linear combination** of the inequalities of $\mathscr{S}$.
- The reason for this choice of name should be clear.
- We shall prove later in the section that every consequence of a consistent system $\mathscr{S}$ must be a legal linear combination of its inequalities.

Consistency and Legal Linear Combinations

Theorem

The finite system $\mathscr{S}$ of linear inequalities is consistent if and only if the inequality $0x_1 + \cdots + 0x_n < 0$ is not a legal linear combination of the inequalities of $\mathscr{S}$.

- If $\mathscr{S}$ is consistent, then clearly $0x_1 + \cdots + 0x_n < 0$ is not a consequence of $\mathscr{S}$. So it is not a legal linear combination of the inequalities of $\mathscr{S}$.

  If $\mathscr{S}$ is inconsistent, then the inequality $0x_1 + \cdots + 0x_n < 0$ can be expressed as a legal linear combination of the inequalities of $\mathscr{S}$ by means of one of the preceding theorems according as the inequalities of $\mathscr{S}$ are weak, mixed or strict.

  We give the details for the case when $\mathscr{S}$ is an inconsistent system of weak inequalities, $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$, say. By the first theorem in the series, there exists $\boldsymbol{y} \geq \boldsymbol{0}$ such that $\boldsymbol{y}^T\boldsymbol{A} = \boldsymbol{0}^T$, $\boldsymbol{y}^T\boldsymbol{b} < 0$. So $0x_1 + \cdots + 0x_n < 0$ is a legal linear combination of the inequalities of $\mathscr{S}$.

Consequences of a Consistent System

Theorem

Let $\mathscr{S}$ be a finite consistent system of linear inequalities. Then every consequence of $\mathscr{S}$ is a legal linear combination of the inequalities of $\mathscr{S}$.

- Suppose first that $\boldsymbol{e}^T\boldsymbol{x} \le f$ is a consequence of $\mathscr{S}$. We consider three cases.
  (i) Suppose that (in the notation used earlier) $\mathscr{S}$ is the system of inequalities $\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}$. Since $\boldsymbol{e}^T\boldsymbol{x} \le f$ is a consequence of $\mathscr{S}$, the mixed system of inequalities $\begin{cases} \boldsymbol{A}\boldsymbol{x} & \le & \boldsymbol{b} \\ -\boldsymbol{e}^T\boldsymbol{x} & < & -f \end{cases}$ must be inconsistent. By a previous theorem, there exist $\boldsymbol{u} \ge \boldsymbol{0}$, $v > 0$, such that $\boldsymbol{u}^T\boldsymbol{A} - v\boldsymbol{e}^T = \boldsymbol{0}^T$ and $\boldsymbol{u}^T\boldsymbol{b} - vf \le 0$. The possibility (ii) of the theorem cannot occur here for it would imply that $\mathscr{S}$ was inconsistent. Thus $\boldsymbol{e}^T = (\frac{\boldsymbol{u}}{v})^T\boldsymbol{A}$ and $(\frac{\boldsymbol{u}}{v})^T\boldsymbol{b} \le f$. This shows that $\boldsymbol{e}^T\boldsymbol{x} \le f$ is a legal linear combination of the inequalities of $\mathscr{S}$.

Consequences of a Consistent System (Case (ii))

(ii) Suppose that $\mathscr{S}$ is the mixed system of inequalities $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$, $\boldsymbol{C}\boldsymbol{x} < \boldsymbol{d}$. Since $\boldsymbol{e}^T\boldsymbol{x} \leq f$ is a consequence of $\mathscr{S}$, the mixed system of inequalities $\left\{\begin{array}{rcl} \boldsymbol{A}\boldsymbol{x} & \leq & \boldsymbol{b} \\ \boldsymbol{C}\boldsymbol{x} & < & \boldsymbol{d} \\ -\boldsymbol{e}^T\boldsymbol{x} & < & -f \end{array}\right.$ must be inconsistent. By a previous theorem, there exist $\boldsymbol{u} \geq \boldsymbol{0}$, $\boldsymbol{v} \geq \boldsymbol{0}$, $w > 0$, such that $\boldsymbol{u}^T\boldsymbol{A} + \boldsymbol{v}^T\boldsymbol{C} - w\boldsymbol{e}^T = \boldsymbol{0}^T$ and $\boldsymbol{u}^T\boldsymbol{b} + \boldsymbol{v}^T\boldsymbol{d} - wf \leq 0$. Neither possibility (ii) of the theorem nor $w = 0$ can occur for each would imply that $\mathscr{S}$ was inconsistent. Thus

$$\boldsymbol{e}^T = \left(\frac{\boldsymbol{u}}{w}\right)^T \boldsymbol{A} + \left(\frac{\boldsymbol{u}}{w}\right)^T \boldsymbol{C} \quad \text{and} \quad \left(\frac{\boldsymbol{u}}{w}\right)^T \boldsymbol{b} + \left(\frac{\boldsymbol{v}}{w}\right)^T \boldsymbol{d} \leq f.$$

This exhibits $\boldsymbol{e}^T\boldsymbol{x} \leq f$ as a legal linear combination of the inequalities of $\mathscr{S}$.

Consequences of a Consistent System (Case (iii))

(iii) Suppose that $\mathscr{S}$ is the system of inequalities $\boldsymbol{C}\boldsymbol{x} < \boldsymbol{d}$. That $\boldsymbol{e}^T\boldsymbol{x} \le f$ is a legal linear combination of the inequalities of $\mathscr{S}$ follows from case (ii) by taking $\boldsymbol{A}$ and $\boldsymbol{b}$ to be zero matrices.

In the same manner, we can prove that every consequence of $\mathscr{S}$ of the form $\boldsymbol{e}^T\boldsymbol{x} < f$ is a legal linear combination of the inequalities of $\mathscr{S}$.

Corollary (Farkas' Lemma)

Let $\boldsymbol{a}, \boldsymbol{a}_1, \ldots, \boldsymbol{a}_m \in \mathbb{R}^n$ be such that $\boldsymbol{a} \cdot \boldsymbol{x} \ge 0$ whenever $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{a}_1 \cdot \boldsymbol{x} \ge 0$, …, $\boldsymbol{a}_m \cdot \boldsymbol{x} \ge 0$. Then there exist $\lambda_1, \ldots, \lambda_m \ge 0$ such that

$$\boldsymbol{a} = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_m \boldsymbol{a}_m.$$

Connections with Convexity

- Many of the results on inequalities have simple geometric interpretations in terms of the separation of polyhedral sets.
- To illustrate this point, consider the dual pair:
  (i) $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$;
  (ii) $\boldsymbol{y}^T\boldsymbol{A} = \boldsymbol{0}^T$, $\boldsymbol{y}^T\boldsymbol{b} \neq 0$.
- Suppose that the equations $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ are inconsistent.
- Geometrically, this means that in $\mathbb{R}^m$ the point $\boldsymbol{b}$ does not belong to the subspace $\mathscr{S}$ spanned by the columns of $\boldsymbol{A}$.
- The existence of a $\boldsymbol{y}$ with $\boldsymbol{y}^T\boldsymbol{A} = \boldsymbol{0}^T$ and $\boldsymbol{y}^T\boldsymbol{b} \neq 0$ means that the hyperplane $\boldsymbol{y}^T\boldsymbol{z} = 0$, which has $\boldsymbol{y}$ as a normal vector and passes through the origin, contains $\mathscr{S}$ but not $\boldsymbol{b}$.
- Thus the existence of this dual pair is equivalent to the following result:

  A point belongs to a subspace of $\mathbb{R}^m$ if and only if there exists no hyperplane containing the subspace but not the point.

Subsection 3

Linear Programming

The Standard Maximum Problem

- Suppose that a manufacturer produces n products and that he produces, and sells, x_j units of the jth product, $x_j \geq 0$.
- If c_j denotes his income from the sale of one unit of the jth product, then his total income is $c_1 x_1 + \cdots + c_n x_n$.
- Suppose further that each of the n products is made from m raw materials, there being available b_i units of the ith raw material.
- If the amount of the ith raw material used in producing a unit of the jth product is a_{ij}, then $a_{i1} x_1 + \cdots + a_{in} x_n \leq b_i$, $i = 1, \ldots, m$.
- We are led to define the **standard maximum problem** P:

$$\begin{array}{ll} \text{maximize} & c_1 x_1 + \cdots + c_n x_n \\ \text{subject to} & a_{11} x_1 + \cdots + a_{1n} x_n \leq b_1 \\ & \cdots \\ & a_{m1} x_1 + \cdots + a_{mn} x_n \leq b_m \\ & \quad x_1 \geq 0, \ldots, x_n \geq 0. \end{array}$$

Feasible Vectors and Feasible Sets

- This standard maximum problem P can be expressed in matrix notation as follows:

$$\text{maximize } \boldsymbol{c}^T\boldsymbol{x} \text{ subject to } \boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}, \boldsymbol{x} \ge \boldsymbol{0},$$

  where $\boldsymbol{A}$ is the real $m \times n$ matrix $[a_{ij}]$, $\boldsymbol{b} = (b_1, \ldots, b_m)$, $\boldsymbol{c} = (c_1, \ldots, c_n)$, and $\boldsymbol{x} = (x_1, \ldots, x_n)$.
- A vector $\boldsymbol{x}$ satisfying the constraints of the standard maximum problem P, i.e., $\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}$ and $\boldsymbol{x} \ge \boldsymbol{0}$, is called a **feasible vector** for P.
- The set of all such feasible vectors is called the **feasible set** for P.
- The problem P is called **feasible** or **infeasible** according as its feasible set is non-empty or empty.
- The feasible set for P is the intersection of the m closed halfspaces represented by the inequalities $\boldsymbol{A}\boldsymbol{x} \le \boldsymbol{b}$ and the n closed halfspaces represented by the inequalities $\boldsymbol{x} \ge \boldsymbol{0}$, and so is a polyhedral set.

Optimal Vectors and Solubility

- A feasible vector $\boldsymbol{x}_0$ for P which satisfies $\boldsymbol{c}^T\boldsymbol{x} \le \boldsymbol{c}^T\boldsymbol{x}_0$ for all feasible vectors $\boldsymbol{x}$ for P is called an **optimal vector** for P.
- The scalar $\boldsymbol{c}^T\boldsymbol{x}_0$ is called the **value** of P.
- The problem P is said to be **soluble** or **insoluble** according to whether it has an optimal vector or not.

Finding an Optimal Vector

Theorem

Suppose that the feasible set for the standard maximum problem P is a non-empty polytope with extreme points $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k$. Let $i \in \{1, \ldots, k\}$ be such that $\boldsymbol{c}^T \boldsymbol{a}_i = \max\{\boldsymbol{c}^T \boldsymbol{a}_1, \ldots, \boldsymbol{c}^T \boldsymbol{a}_k\}$. Then P is soluble having $\boldsymbol{a}_i$ as an optimal vector and value $\boldsymbol{c}^T \boldsymbol{a}_i$.

- Let $\boldsymbol{x}$ lie in the feasible set $\operatorname{conv}\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_k\}$ of P. Then

$$\boldsymbol{x} = \lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_k \boldsymbol{a}_k,$$

for some $\lambda_1, \ldots, \lambda_k \geq 0$ with $\lambda_1 + \cdots + \lambda_k = 1$. Then

$$\boldsymbol{c}^T \boldsymbol{x} = \lambda_1 \boldsymbol{c}^T \boldsymbol{a}_1 + \cdots + \lambda_k \boldsymbol{c}^T \boldsymbol{a}_k \leq \lambda_1 \boldsymbol{c}^T \boldsymbol{a}_i + \cdots + \lambda_k \boldsymbol{c}^T \boldsymbol{a}_i = \boldsymbol{c}^T \boldsymbol{a}_i.$$

Example

- A tailor has 16 units of material A, 11 units of material B and 15 units of material C from which he cuts suits and dresses.
  - Each suit requires 2 units of A, 1 unit of B, 1 unit of C.
  - Each dress requires 1 unit of A, 2 units of B, 3 units of C.
- Suits sell at 30 units, dresses at 50 units.
- How can the tailor maximize his income?
- Suppose that the tailor makes x_1 suits and x_2 dresses.
- Then the tailor's problem is to

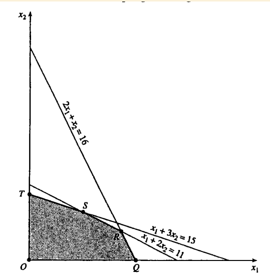
$$\begin{array}{ll}\text{maximize} & 30x_1+50x_2\\ \text{subject to} & 2x_1+x_2\leq 16\\ & x_1+2x_2\leq 11\\ & x_1+3x_2\leq 15\\ & x_1\geq 0, x_2\geq 0.\end{array}$$

Example (Cont'd)

- Perhaps we should add the constraints that x_1 and x_2 are integers!
- We will, however, suppose that our tailor can produce, and sell, any non-negative number of suits and dresses, subject only to the amount of materials he has at his disposal.
- We refer to this example as the **tailor's problem**.
- The feasible set F for the problem is the intersection of the closed halfplanes

$$\begin{aligned} 2x_1 + x_2 &\leq 16, \\ x_1 + 2x_2 &\leq 11, \\ x_1 + 3x_2 &\leq 15 \end{aligned}$$

with the nonnegative quadrant.

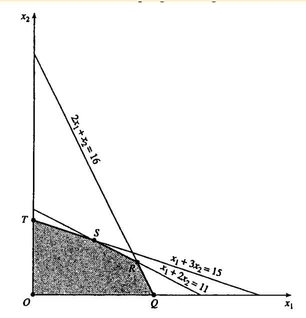


- It is readily verified that F is the pentagon whose extreme points are

$$O = (0,0), \quad Q = (8,0), \quad R = (7,2), \quad S = (3,4), \quad T = (0,5).$$

Example (Cont'd)

- The feasible set is the pentagon with extreme points $O = (0,0)$, $Q = (8,0)$, $R = (7,2)$, $S = (3,4)$, $T = (0,5)$.



- The values of $30x_1 + 50x_2$ at the points O, Q, R, S, T are, respectively: $0, 240, 310, 290, 250$.
- By the theorem, the problem has optimal vector $(7,2)$ and value 310:

  The tailor should make 7 suits and 2 dresses so as to give him a maximal possible income of 310 units.

Sufficient Condition for Solubility

Theorem

Suppose that the function $\boldsymbol{c}^T\boldsymbol{x}$ is bounded above as $\boldsymbol{x}$ ranges over a non-empty feasible set F for the standard maximum problem P. Then P is soluble and at least one of its optimal vectors is an extreme point of F.

- F is a subset of the non-negative orthant of $\mathbb{R}^n$.

  So it is a nonempty line-free polyhedral set in $\mathbb{R}^n$.

  The result now follows from the last theorem of the first section.

Obtaining the Upper Bound

- Suppose now that the standard maximum problem P has a non-empty feasible set F.
- The preceding theorem shows that P is soluble when the set $\{\boldsymbol{c}^T\boldsymbol{x} : \boldsymbol{x} \in F\}$ of real numbers has an upper bound.
- Suppose that, for some $\boldsymbol{y} = (y_1, \ldots, y_m) \geq \mathbf{0}$, $\boldsymbol{y}^T\boldsymbol{A} \geq \boldsymbol{c}^T$.
- Then $\boldsymbol{c}^T\boldsymbol{x} \leq \boldsymbol{y}^T\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{y}^T\boldsymbol{b}$ for $\boldsymbol{x} \in F$, and P is soluble with value not exceeding $\boldsymbol{y}^T\boldsymbol{b}$.
- The smaller the number $\boldsymbol{y}^T\boldsymbol{b}$, the more information we can deduce about the value of P.
- We are thus led to consider the following problem:

$$\text{minimize } \boldsymbol{y}^T\boldsymbol{b} \text{ subject to } \boldsymbol{y}^T\boldsymbol{A} \geq \boldsymbol{c}^T, \boldsymbol{y} \geq \mathbf{0}.$$

- This problem turns out to be closely related to the standard maximum problem P.

The Standard Minimum Problem

- The **standard minimum problem** is:

$$\text{minimize } \boldsymbol{c}^T\boldsymbol{x} \text{ subject to } \boldsymbol{A}\boldsymbol{x} \ge \boldsymbol{b}, \boldsymbol{x} \ge \boldsymbol{0};$$

- That is,

$$\begin{array}{ll} \text{mimmize} & c_1x_1 + \cdots + c_nx_n \\ \text{subject to} & a_{11}x_1 + \cdots + a_{1n}x_n \ge b_1 \\ & \cdots \\ & a_{m1}x_1 + \cdots + a_{mn}x_n \ge b_m \\ & \quad x_1 \ge 0, \ldots, x_n \ge 0. \end{array}$$

- The definitions of **feasible vector**, **feasible set**, **feasible**, **infeasible**, **optimal vector**, **value**, **soluble** and **insoluble** are modified in the obvious way so as to apply to the standard minimum problem.

Duality

- With each standard maximum problem P, we associate a standard minimum problem P^* called the **dual** of P as follows:

maximize	$\mathbf{c}^T\mathbf{x}$	minimize	$\mathbf{b}^T\mathbf{y}$
subject to	$\mathbf{A}\mathbf{x} \le \mathbf{b}$	subject to	$\mathbf{A}^T\mathbf{y} \ge \mathbf{c}$
	$\mathbf{x} \ge \mathbf{0}$		$\mathbf{y} \ge \mathbf{0}$
that is		that is	
maximize	$c_1x_1 + \cdots + c_nx_n$	minimize	$b_1y_1 + \cdots + b_my_m$
subject to	$a_{11}x_1 + \cdots + a_{1n}x_n \le b_1$	subject to	$a_{11}y_1 + \cdots + a_{m1}y_m \ge c_1$
	$\cdots$		$\cdots$
	$a_{m1}x_1 + \cdots + a_{mn}x_n \le b_m$		$a_{1n}y_1 + \cdots + a_{mn}y_m \ge c_n$
	$x_1 \ge 0, \ldots, x_n \ge 0$		$y_1 \ge 0, \ldots, y_m \ge 0$

- In the context that we are considering, the problem P is referred to as the **primal problem**.
- We note that, ignoring the non-negativity constraints on $\mathbf{x}$ and $\mathbf{y}$, the primal problem has m constraints in n variables, whereas the dual problem has n constraints in m variables.

Example (Tailor's Problem Cont'd)

- Recall the tailor's problem

$$\begin{array}{ll} \text{maximize} & 30x_1 + 50x_2 \\ \text{subject to} & 2x_1 + x_2 \leq 16 \\ & x_1 + 2x_2 \leq 11 \\ & x_1 + 3x_2 \leq 15 \\ & x_1 \geq 0, x_2 \geq 0. \end{array}$$

- The dual of the tailor's problem is:

$$\begin{array}{ll} \text{minimize} & 16y_1 + 11y_2 + 15y_3 \\ \text{subject to} & 2y_1 + y_2 + y_3 \geq 30 \\ & y_1 + 2y_2 + 3y_3 \geq 50 \\ & y_1 \geq 0, y_2 \geq 0, y_3 \geq 0. \end{array}$$

Equilibrium or Complementary Slackness Theorem

Theorem (Complementary Slackness Theorem)

Let $\boldsymbol{x}, \boldsymbol{y}$ be feasible vectors for the problems P, P^*, respectively. Then $\boldsymbol{c}^T \boldsymbol{x} \leq \boldsymbol{b}^T \boldsymbol{y}$, with equality if and only if:

(i) $x_j > 0$ implies $a_{1j} y_1 + \cdots + a_{mj} y_m = c_j$;

(ii) $y_i > 0$ implies $a_{i1} x_1 + \cdots + a_{in} x_n = b_i$.

Moreover, if $\boldsymbol{c}^T \boldsymbol{x} = \boldsymbol{b}^T \boldsymbol{y}$, then $\boldsymbol{x}, \boldsymbol{y}$ are optimal vectors for their respective problems.

- Since $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$ and $\boldsymbol{y} \geq 0$, we have $\boldsymbol{y}^T \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{y}^T \boldsymbol{b}$, with equality holding if and only if (ii) is true.

  Similarly, $\boldsymbol{c}^T \boldsymbol{x} \leq \boldsymbol{y}^T \boldsymbol{A}\boldsymbol{x}$, with equality holding if and only if (i) is true.

  Thus $\boldsymbol{c}^T \boldsymbol{x} \leq \boldsymbol{y}^T \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{y}^T \boldsymbol{b}$ and $\boldsymbol{c}^T \boldsymbol{x} = \boldsymbol{b}^T \boldsymbol{y}$ if and only if both (i) and (ii) hold.

Complementary Slackness Theorem (Cont'd)

- Suppose now that $\boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{y}$.

  Let $\boldsymbol{x}', \boldsymbol{y}'$ be feasible vectors for the problems P, P^*, respectively.

  Then, by what we have just proved,

$$\boldsymbol{c}^T\boldsymbol{x}' \leq \boldsymbol{b}^T\boldsymbol{y} = \boldsymbol{c}^T\boldsymbol{x} \quad \text{and} \quad \boldsymbol{b}^T\boldsymbol{y}' \geq \boldsymbol{c}^T\boldsymbol{x} = \boldsymbol{b}^T\boldsymbol{y}.$$

  This shows that $\boldsymbol{x}, \boldsymbol{y}$ are optimal for their respective problems.

Duality Theorem of Linear Programming

Theorem (Duality Theorem of Linear Programming)

Denote by P the standard maximum problem, and by P^* its dual.

(i) If either one of P and P^* is soluble, then so too is the other, and both problems have the same value.

(ii) If both P and P^* are feasible, then they are both soluble.

(i) Suppose first that P is soluble with optimal vector $\boldsymbol{x}_0$ and value v. Then the inequality $\boldsymbol{c}^T\boldsymbol{x} \leq v$ is a consequence of the consistent combined system of inequalities $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$, $-\boldsymbol{I}_n\boldsymbol{x} \leq \boldsymbol{0}$. By a previous theorem, there exist $\boldsymbol{y}_0 \geq \boldsymbol{0}$, $\boldsymbol{u} \geq \boldsymbol{0}$, such that $\boldsymbol{y}_0^T\boldsymbol{A} - \boldsymbol{u}^T = \boldsymbol{c}^T$ and $\boldsymbol{y}_0^T\boldsymbol{b} \leq v$. This shows that $\boldsymbol{y}_0$ is a feasible vector for P^*. By the preceding theorem, $\boldsymbol{c}^T\boldsymbol{x}_0 = v \geq \boldsymbol{y}_0^T\boldsymbol{b} \geq \boldsymbol{c}^T\boldsymbol{x}_0$. This proves that $\boldsymbol{c}^T\boldsymbol{x}_0 = \boldsymbol{y}_0^T\boldsymbol{b}$. So $\boldsymbol{y}_0$ is an optimal vector for P^*. Thus, P^* is soluble and has the same value as P, namely v.

Duality Theorem of Linear Programming (Cont'd)

- A similar argument shows that, if P^* is soluble with value v, then so too is P.

(ii) Suppose that both P and P^* are feasible.

Let $\boldsymbol{y}_0$ be a feasible vector for P^*.

By the preceding theorem, for any feasible vector $\boldsymbol{x}$ of P,

$$\boldsymbol{c}^T\boldsymbol{x} \leq \boldsymbol{b}^T\boldsymbol{y}_0.$$

A previous theorem shows that P is soluble.

Now the desired result follows from part (i) of this theorem.

Tailor's Problem Revisited

- We use the complementary slackness theorem to confirm that the vector $(x_1, x_2) = (7, 2)$, obtained earlier by graphical means, is optimal for the tailor's problem, and to obtain an optimal vector for its dual.
- Certainly (x_1, x_2) is a feasible vector for the problem.
- Suppose that there is a feasible vector (y_1, y_2, y_3) for the dual which, together with (x_1, x_2), satisfies conditions (i) and (ii) of the complementary slackness theorem.
- Since $x_1, x_2 > 0$, we have from (i) that:

$$2y_1 + y_2 + y_3 = 30 \text{ and } y_1 + 2y_2 + 3y_3 = 50.$$

- Since the third constraint of the primal, i.e., $x_1 + 3x_2 \leq 15$, is strictly satisfied, we have from (ii) that $y_3 = 0$.
- Thus $2y_1 + y_2 = 30$, $y_1 + 2y_2 = 50$.
- So $y_1 = \frac{10}{3}$, $y_2 = \frac{70}{3}$.

Tailor's Problem Revisited (Cont'd)

- A routine verification now shows $(\frac{10}{3},\frac{70}{3},0)$ is feasible for the dual with

$$30\cdot 7+50\cdot 2=16\cdot\frac{10}{3}+11\cdot\frac{70}{3}+15\cdot 0=310.$$

- The last statement of the complementary slackness theorem now enables us to conclude that:
  - $(7,2)$ is optimal for the tailor's problem;
  - $(\frac{10}{3},\frac{70}{3},0)$ is optimal for its dual;
  - Both problems have value 310.

Subsection 4

Basic Solutions of Linear Equations

System of Linear Equations

- Consider the following system of m linear equations in n variables:

$$\begin{array}{rcl} a_{11}x_1 + \cdots + a_{1n}x_n & = & b_1 \\ \vdots & & \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = & b_m \end{array}$$

- In matrix notation it is $\boldsymbol{Ax} = \boldsymbol{b}$, where $\boldsymbol{A}$ is a real $m \times n$ matrix $[a_{ij}]$, $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{b} = (b_1, \ldots, b_m)$.
- To avoid a vacuous discussion, we shall assume throughout, unless stated otherwise, that some m of the columns of $\boldsymbol{A}$ form a linear basis for $\mathbb{R}^m$, i.e., that $\boldsymbol{A}$ has rank m.
- In particular, we have $m \leq n$.

Basic Solutions of the System

- Denote the columns of $\boldsymbol{A}$ by $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n$.
- Then the system of equations can be written in the form

$$x_1 \boldsymbol{a}_1 + x_2 \boldsymbol{a}_2 + \cdots + x_n \boldsymbol{a}_n = \boldsymbol{b}.$$

- Suppose that the columns $\boldsymbol{a}_{i_1}, \ldots, \boldsymbol{a}_{i_m}$ form a linear basis for $\mathbb{R}^m$.
- Then there exist unique scalars $x_{i_1}, \ldots, x_{i_m}$ such that

$$x_{i_1} \boldsymbol{a}_{i_1} + \cdots + x_{i_m} \boldsymbol{a}_{i_m} = \boldsymbol{b}.$$

- If we put the remaining $n-m$ x_i's equal to zero, we obtain a solution $\boldsymbol{x} = (x_1, \ldots, x_n)$ of $\boldsymbol{Ax} = \boldsymbol{b}$.
- A solution obtained in this way is called a **basic solution** of $\boldsymbol{Ax} = \boldsymbol{b}$.

Example

- Find the basic solutions of the system of equations:

$$\begin{array}{rcl} x_1+x_2+x_3 & = & 3 \\ 3x_1+2x_2+4x_3 & = & 10. \end{array}$$

- Every two of the columns of the matrix of coefficients on the left-hand side of this system of equations form a basis for $\mathbb{R}^2$, and so the system has three basic solutions.
- First we put $x_1=0$ to obtain the basic solution $(0,1,2)$.
- Next we put $x_2=0$ to obtain the basic solution $(2,0,1)$.
- Lastly we put $x_3=0$ to obtain the basic solution $(4,-1,0)$.

Geometry of Basic Solutions

Theorem

The extreme points of the polyhedral set $C = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}\}$ are precisely the non-negative basic solutions of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

- Suppose that $\boldsymbol{x}_0$ is a non-negative basic solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$, say $\boldsymbol{x}_0 = (x_1, \ldots, x_m, 0, \ldots, 0)$, where the first m columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ of $\boldsymbol{A}$ are linearly independent.

  Let $\boldsymbol{x}_0 = \lambda \boldsymbol{y} + \mu \boldsymbol{z}$, where $\lambda, \mu > 0$ with $\lambda + \mu = 1$, and $\boldsymbol{y}, \boldsymbol{z} \in C$.

  Since $\boldsymbol{y}, \boldsymbol{z} \geq \boldsymbol{0}$ and $\lambda, \mu > 0$, we deduce, on equating the last $n - m$ coordinates on each side of the last expression for $\boldsymbol{x}_0$, that $\boldsymbol{y}$ and $\boldsymbol{z}$ must have the forms $\boldsymbol{y} = (y_1, \ldots, y_m, 0, \ldots, 0)$, $\boldsymbol{z} = (z_1, \ldots, z_m, 0, \ldots, 0)$.

  Since $\boldsymbol{y}, \boldsymbol{z} \in C$, we have $y_1 \boldsymbol{a}_1 + \cdots + y_m \boldsymbol{a}_m = \boldsymbol{b}$ and $z_1 \boldsymbol{a}_1 + \cdots + z_m \boldsymbol{a}_m = \boldsymbol{b}$.

  But $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ are linearly independent, whence $y_1 = z_1, \ldots, y_m = z_m$.

  Thus $\boldsymbol{x}_0 = \boldsymbol{y} = \boldsymbol{x}$, which shows that $\boldsymbol{x}_0$ is an extreme point of C.

Geometry of Basic Solutions (Converse)

- Suppose next that $\boldsymbol{x}_0$ is an extreme point of C.
  If $\boldsymbol{x}_0 = \boldsymbol{0}$, certainly $\boldsymbol{x}_0$ is a non-negative basic solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.
  Assume, then, that $\boldsymbol{x}_0 \neq \boldsymbol{0}$; say $\boldsymbol{x}_0 = (x_1, \ldots, x_r, 0, \ldots, 0)$ for some $r \in \{1, \ldots, n\}$, where $x_1, \ldots, x_r > 0$. Then the first r columns of $\boldsymbol{A}$, say $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$ must be linearly independent. To see why this is so, let the scalars $\lambda_1, \ldots, \lambda_r$ be such that $\lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_r \boldsymbol{a}_r = \boldsymbol{0}$. Choose $\theta > 0$ so small that the points

$$\begin{array}{rcl} \boldsymbol{y} & = & (x_1 + \theta\lambda_1, \ldots, x_r + \theta\lambda_r, 0, \ldots, 0), \\ \boldsymbol{z} & = & (x_1 - \theta\lambda_1, \ldots, x_r - \theta\lambda_r, 0, \ldots, 0), \end{array}$$

  belong to C. Then $\boldsymbol{x}_0 = \frac{1}{2}(\boldsymbol{y} + \boldsymbol{z})$. But $\boldsymbol{x}_0$ is an extreme point of C. So $\boldsymbol{y} = \boldsymbol{z}$. Hence, $\lambda_1 = 0, \ldots, \lambda_r = 0$. Thus, $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r$ are linearly independent.
  By extending $\{\boldsymbol{a}_1, \ldots, \boldsymbol{a}_r\}$ to a linear basis for $\mathbb{R}^m$ using the columns of $\boldsymbol{A}$, we deduce that $\boldsymbol{x}_0$ is a non-negative basic solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

The Canonical Maximum problem

- The **canonical maximum problem** is to

$$\text{maximize } \boldsymbol{c}^T\boldsymbol{x} \text{ subject to } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0}.$$

- Note that now we also assume that some m columns of $\boldsymbol{A}$ are linearly independent.
- A vector $\boldsymbol{x} \geq \boldsymbol{0}$ satisfying $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is said to be a **feasible vector** for the problem.
- The set of all such feasible vectors is called the **feasible set** for the problem.
- A feasible vector $\boldsymbol{x}_0$ such that $\boldsymbol{c}^T\boldsymbol{x} \leq \boldsymbol{c}^T\boldsymbol{x}_0$, for all feasible vectors $\boldsymbol{x}$, is called an **optimal vector** for the problem.
- An optimal vector which is also a basic solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ is called a **basic optimal vector** for the problem.

Existence of Basic Optimal Vectors

Theorem

Suppose that the canonical maximum problem has an optimal vector. Then it has a basic optimal vector.

- We consider the non-trivial case when $\boldsymbol{c} \neq \mathbf{0}$.

  Suppose that the canonical maximum problem has feasible set C and optimal vector $\boldsymbol{x}_0$.

  The hyperplane H with equation $\boldsymbol{c} \cdot \boldsymbol{x} = \boldsymbol{c} \cdot \boldsymbol{x}_0$, supports C at $\boldsymbol{x}_0$.

  The non-empty polyhedral set $C \cap H$ contains no lines.

  So it possesses an extreme point, $\boldsymbol{x}^*$ say.

  By a previous theorem, $\boldsymbol{x}^*$ is an extreme point of C.

  By the preceding theorem, $\boldsymbol{x}^*$ is a basic solution of $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

  Since $\boldsymbol{x}^* \in C$ and $\boldsymbol{c} \cdot \boldsymbol{x}^* = \boldsymbol{c} \cdot \boldsymbol{x}_0$, $\boldsymbol{x}^*$ is a basic optimal vector for the canonical maximum problem.

Relation Between Standard and Canonical Problems

- Let $\boldsymbol{A}$ be any real $m \times n$ matrix, not necessarily with columns forming a basis for $\mathbb{R}^m$.
- Recall that the standard maximum problem P

$$\begin{array}{ll} \text{maximize} & c_1x_1 + \cdots + c_nx_n \\ \text{subject to} & a_{11}x_1 + \cdots + a_{1n}x_n \leq b_1, \ldots, a_{m1}x_1 + \cdots + a_{mn}x_n \leq b_m \\ & x_1 \geq 0, \ldots, x_n \geq 0. \end{array}$$

- We pass from this problem involving m inequalities (excluding the non-negativity constraints on $x_1, \ldots, x_n$) to an equivalent problem involving m equations by introducing m new variables $x_{n+1}, \ldots, x_{n+m}$:

$$\begin{array}{rcl} x_{n+1} & = & b_1 - a_{11}x_1 - \cdots - a_{1n}x_n \\ & & \ldots \\ x_{n+m} & = & b_m - a_{m1}x_1 - \cdots - a_{mn}x_n \end{array}$$

- Since each x_{n+i} $(i = 1, \ldots, m)$ measures the amount of slack in $a_{i1}x_1 + \cdots + a_{in}x_n \leq b_i$, $x_{n+1}, \ldots, x_{n+m}$ are called **slack variables**.

The Standard and Canonical Problems (Cont'd)

- It is now easy to see that the above standard maximum problem P is equivalent to the following related canonical problem P_R:

$$\begin{array}{ll} \text{maximize} & c_1x_1+\cdots+c_nx_n+0x_{n+1}+\cdots+0x_{n+m} \\ \text{subject to} & a_{11}x_1+\cdots+a_{1n}x_n+x_{n+1}=b_1 \\ & \cdots \\ & a_{m1}x_1+\cdots+a_{mn}x_n+x_{n+m}=b_m \\ & x_1\geq 0,\ldots,x_{n+m}\geq 0. \end{array}$$

- Denote by F the feasible set for the standard maximum problem P, and by F_R the feasible set for the related canonical maximum problem P_R.
- Then there is a natural bijection $f:F\to F_R$ defined by the equation $f(\boldsymbol{x})=f(x_1,\ldots,x_n)=(x_1,\ldots,x_{n+m})=(\boldsymbol{x},\boldsymbol{b}-\boldsymbol{A}\boldsymbol{x})$.
- Clearly f preserves convex combinations of points.
- So the extreme points of F and F_R correspond under f.

Example

- We solve the tailor's problem using the preceding ideas.
- The canonical maximum problem related to the tailor's problem is:

$$\begin{array}{ll} \text{maximize} & 30x_1+50x_2+0x_3+0x_4+0x_5 \\ \text{subject to} & 2x_1+x_2+x_3=16 \\ & x_1+2x_2+x_4=11 \\ & x_1+3x_2+x_5=15 \\ & x_1\geq 0,\ldots,x_5\geq 0. \end{array}$$

- Clearly this canonical problem has an optimal vector, and hence a basic optimal vector.
- Thus to solve the problem, we find at which nonnegative basic solutions of the above system of equations the function $30x_1+50x_2$ has its maximum.

Example (Cont'd)

- We construct the following table:

Columns	Basic Solution	Extreme F_R	Extreme F	$30x_1+50x_2$
1,2,3	$(3,4,6,0,0)$	$(3,4,6,0,0)$	$(3,4)$	290
1,2,4	$(\frac{33}{5},\frac{14}{5},0,-\frac{6}{5},0)$			
1,2,5	$(7,2,0,0,2)$	$(7,2,0,0,2)$	$(7,2)$	310
1,3,4	$(15,0,-14,-4,0)$			
1,3,5	$(11,0,-6,0,4)$			
1,4,5	$(8,0,0,3,7)$	$(8,0,0,3,7)$	$(8,0)$	240
2,3,4	$(0,5,11,1,0)$	$(0,5,11,1,0)$	$(0,5)$	250
2,3,5	$(0,\frac{11}{2},\frac{21}{2},0,-\frac{3}{2})$			
2,4,5	$(0,16,0,-21,-33)$			
3,4,5	$(0,0,16,11,15)$	$(0,0,16,11,15)$	$(0,0)$	0

- The optimal vector for the canonical is $(7,2,0,0,2)$ and for the tailor's problem $(7,2)$.

Drawbacks of the Method

- The method just outlined for solving a linear programming problem is rarely used in practice.
  - The method gives no indication as to whether or not the problem has a solution.
  - The amount of work in finding a solution is often prohibitive.
    A system of m equations in $m+n$ unknowns can have as many as $\frac{(m+n)!}{m!n!}$ basic solutions, each one obtained as the solution of a system of m linear equations in m unknowns.
- A more practical method of solving linear programming problems is required.
- The most well-known of such methods, the *simplex algorithm*, is discussed in the next section.

Subsection 5

The Simplex Algorithm

Pivoting

- Consider the following system of equations:

$$\begin{array}{rcl} a_{11}x_1 + \cdots + a_{1n}x_n & = & b_1 \\ & \vdots & \\ a_{m1}x_1 + \cdots + a_{mn}x_n & = & b_m. \end{array}$$

- Suppose that $a_{ij} \neq 0$. Then we obtain a new system equivalent to the given one as follows:
  (i) Divide the ith equation by a_{ij};
  (ii) Subtract multiples of the ith equation from the remaining ones in such a way as to remove their x_j term.

Pivoting (Cont'd)

- The new system that we obtain is:

$$\begin{array}{rcl}
(a_{11}-\frac{a_{1j}}{a_{ij}}a_{i1})x_1+\cdots+0x_j+\cdots+(a_{1n}-\frac{a_{1j}}{a_{ij}}a_{in})x_n & = & b_1-\frac{a_{1j}}{a_{ij}}b_i \\
\vdots & & \\
\frac{a_{i1}}{a_{ij}}x_1+\cdots+x_j+\cdots+\frac{a_{in}}{a_{ij}}x_n & = & \frac{b_i}{a_{ij}} \\
\vdots & & \\
(a_{m1}-\frac{a_{mj}}{a_{ij}}a_{i1})x_1+\cdots+0x_j+\cdots+(a_{mn}-\frac{a_{mj}}{a_{ij}}a_{in})x_n & = & b_m-\frac{a_{mj}}{a_{ij}}b_i.
\end{array}$$

- We say that this new system has been obtained from the original one by **pivoting** about a_{ij}.
- This a_{ij} is called the **pivot**.

Tailor's Problem Revisited

- The canonical form of the problem is to maximize $\widehat{x}$, subject to the constraints:

$$\begin{array}{lllllcl}
30x_1 & +50x_2 & +0x_3 & +0x_4+ & 0x_5 & = & \widehat{x} \\
\hline
2x_1 & +x_2 & +x_3 & & & = & 16 \\
x_1 & +2x_2 & & +x_4 & & = & 11 \\
x_1 & +3x_2 & & & +x_5 & = & 15
\end{array}$$

  and the non-negativity constraints $x_1 \geq 0, \ldots, x_5 \geq 0$.
- Here we have added the defining equation of the objective function $\widehat{x}$ to the constraint equations of the problem.

Tailor's Problem Revisited (Cont'd)

- We seek a basic optimal vector, beginning at the extreme point (non-negative basic solution) $(0,0,16,11,15)$, where $\widehat{x}$ is 0.
- Can we find an extreme point where $\widehat{x} > 0$?
- Yes, we can increase $\widehat{x}$ by increasing x_1 from 0, while keeping x_2 at 0 and adjusting x_3, x_4, x_5 as required by the equations.
- As x_1 increases in this way to $8, 11, 15$, x_3, x_4, x_5 decrease, respectively, to 0.
- Since x_3, x_4, x_5 must be non-negative, we can only increase x_1 to 8, while keeping x_2 at 0, when x_3, x_4, x_5 are 0,3,7 respectively.
- We have thus arrived at the extreme point $(8,0,0,3,7)$, where $x = 240$.

Tailor's Problem Revisited (Cont'd)

- We now express $\hat{x}$ in terms of the new zero variables x_2, x_3.
- This we do by pivoting the whole system of equations about the 2 in the second row and the first column to obtain the following system of equations:

$$\begin{array}{cccccccc}
 & 35x_2 & -15x_3 & & & = & \hat{x}-240 \\
\hline
x_1 & +\frac{1}{2}x_2 & +\frac{1}{2}x_3 & & & = & 8 \\
 & \frac{3}{2}x_2 & -\frac{1}{2}x_3 & +x_4 & & = & 3 \\
 & \frac{5}{2}x_2 & -\frac{1}{2}x_3 & & +x_5 & = & 7
\end{array}$$

- It is clear from this system of equations that $\hat{x} = 240$ at the extreme point $(8,0,0,3,7)$.

Tailor's Problem Revisited (Cont'd)

- Can we find an extreme point where $\widehat{x} > 240$?
- Yes, we can increase $\widehat{x}$ by increasing x_2 from 0, while keeping x_3 at 0 and adjusting x_1, x_4, x_5.
- In fact x_2 can be increased to 2, when $x_1 = 7, x_4 = 0, x_5 = 2$.
- We have thus arrived at the extreme point $(7,2,0,0,2)$, where $\widehat{x} = 310$.
- We now express $\widehat{x}$ in terms of the new zero variables x_3, x_4.
- This we do by pivoting the whole system of equations about the $\frac{3}{2}$ in the third row and the second column to obtain the following system:

$$\begin{array}{cccccccc}
 & & -\frac{10}{3}x_3 & -\frac{70}{3}x_4 & & = & \widehat{x} - 310 \\ \hline
x_1 & & +\frac{2}{3}x_3 & -\frac{1}{3}x_4 & & = & 7 \\
 & x_2 & -\frac{1}{3}x_3 & +\frac{2}{3}x_4 & & = & 2 \\
 & & \frac{1}{3}x_3 & -\frac{5}{3}x_4 & +x_5 & = & 2.
\end{array}$$

- It is clear from this system of equations that $\widehat{x} = 310$ at the extreme point $(7,2,0,0,2)$.

Tailor's Problem Revisited (Cont'd)

- Can we find an extreme point where $\hat{x} > 310$?
- No, we cannot, for the first equation shows that

$$\hat{x} = 310 - \frac{10}{3}x_3 - \frac{70}{3}x_4 \leq 310,$$

  since $x_3, x_4 \geq 0$ for all feasible vectors $(x_1, x_2, x_3, x_4, x_5)$ for the canonical problem.
- This ends the search:
  - $(7,2,0,0,2)$ is a basic optimal vector for the canonical problem.
  - Hence, the tailor's problem has optimal vector $(7,2)$ and value 310.

Tailor's Problem Revisited (Cont'd)

- We interpret the above solution to the tailor's problem geometrically:
  - The search for an optimal vector began at the origin, where $\widehat{x}$ was 0.
  - It then moved to the adjacent extreme point $A=(8,0)$, where $\widehat{x}$ was 240.
  - Finally, it moved to the adjacent extreme point $(7,2)$, where $\widehat{x}$ assumed its maximum of 310.
- To summarize:

  The search started at an extreme point of the feasible set and then moved along the edges of the feasible set, passing from one extreme point to an adjacent one, in such a way that $\widehat{x}$ increased at each successive extreme point until it reached its maximum on the feasible set.
- This is the basic principle that underlies the simplex algorithm.

Canonical Form of the Standard Maximum Problem

- Consider the standard maximum problem, which, in canonical form, is to maximize $\widehat{x}$ subject to the constraints:

$$\begin{array}{cccccccc}
c_1x_1 & +\cdots & +c_nx_n & +0x_{n+1} & +\cdots & +0x_{n+m} & = & \widehat{x} \\
\hline
a_{11}x_1 & +\cdots & +a_{1n}x_n & +x_{n+1} & & & = & b_1 \\
\cdots & & & & & & & \\
a_{m1}x_1 & +\cdots & +a_{mn}x_n & & & +x_{n+m} & = & b_m
\end{array}$$

  and $x_1 \geq 0, \ldots, x_{n+m} \geq 0$.
- We denote by F the feasible set for the problem.
- To simplify our initial discussion, we make two assumptions about the system of equations $[\boldsymbol{A}, \boldsymbol{I}_m](x_1, \ldots, x_{n+m}) = \boldsymbol{b}$.
  (i) $\boldsymbol{b} = (b_1, \ldots, b_m) \geq \mathbf{0}$;
  (ii) Every non-negative solution $(x_1, \ldots, x_{n+m})$ of the system has at least m positive coordinates. (**non-degeneracy**)
- Since $(0, \ldots, 0, b_1, \ldots, b_m)$ is a solution, assumptions (i) and (ii) together imply that $b_1, \ldots, b_m > 0$.

Tableau Form

- The canonical problem can be usefully summarized in tableau form:

x_1	x_2	$\cdots$	x_n	x_{n+1}	x_{n+2}	$\cdots$	x_{n+m}	
c_1	c_2	$\cdots$	c_n	0	0	$\cdots$	0	$\widehat{x}$
a_{11}	a_{12}	$\cdots$	a_{1n}	1	0	$\cdots$	0	b_1
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
a_{m1}	a_{m2}	$\cdots$	a_{mn}	0	0	$\cdots$	1	b_m

- It is clear from this initial tableau that $(0,\ldots,0,b_1,\ldots,b_m)$ is an extreme point of the feasible set F at which $\widehat{x}=0$.
- The variables $x_1,\ldots,x_n$ which are zero at this point are called the **non-basic variables** and the non-zero variables $x_{n+1},\ldots,x_{n+m}$ are called the **basic variables**.

Exploring the Solubility of the Problem

- Can we increase $\widehat{x}$ and continue to satisfy the constraints of the problem?
- Certainly not if $c_1 \leq 0$, ..., $c_n \leq 0$, for then

$$\widehat{x} = c_1 x_1 + \cdots + c_n x_n \leq 0,$$

  as $x_1, \ldots, x_n \geq 0$ for all vectors $(x_1, \ldots, x_{n+m})$ in the feasible set F.
- Thus in this case $(0, \ldots, 0, b_1, \ldots, b_m)$ is an optimal vector for the problem and the problem has value 0.
- Suppose, then, that at least one of $c_1, \ldots, c_n$ is positive, say $c_1 > 0$.
- If all the numbers in the column below c_1 in the initial tableau are non-positive, then, for any $x_1 \geq 0$,

$$(x_1, 0, \ldots, 0, b_1 - a_{11} x_1, \ldots, b_m - a_{m1} x_1) \in F$$

  and $\widehat{x} = c_1 x_1$ at this point.
- Thus $\widehat{x}$ is not bounded above on F and the problem is insoluble.

Pivoting

- Suppose, then, that at least one of $a_{11},\ldots,a_{m1}$ is positive.
- For each i such that $a_{i1} > 0$, find $\frac{b_i}{a_{i1}}$ and choose an i which minimizes these quotients; say $a_{11} > 0$ and that $\frac{b_1}{a_{11}}$ is the minimum of the quotients.
- We now increase x_1 from 0 to $\frac{b_1}{a_{11}}$, while keeping $x_2,\ldots,x_n$ at 0 and adjusting $x_{n+1},\ldots,x_{n+m}$ as required by the constraints of the problem.
- We thus arrive at the extreme point

  $$\left(\frac{b_1}{a_{11}},0,\ldots,0,b_2-\frac{b_1}{a_{11}}a_{21},\ldots,b_m-\frac{b_1}{a_{11}}a_{m1}\right)$$

  of F, where $\widehat{x} = \frac{b_1}{a_{11}}c_1 > 0$.
- We now express $\widehat{x}$ in terms of the new non-basic (zero) variables $x_2,\ldots,x_{n+1}$ by pivoting about the number a_{11} in the first tableau.

Second Tableau

- We obtain a second tableau with the following form:

0	c'_2	$\cdots$	c'_n	c'_{n+1}	0	$\cdots$	0	$\widehat{x}-\frac{b_1}{a_{11}}c_1$
1	a'_{12}	$\cdots$	a'_{1n}	a'_{1n+1}	0	$\cdots$	0	$\frac{b_1}{a_{11}}$
0	a'_{22}	$\cdots$	a'_{2n}	a'_{2n+1}	1	$\cdots$	0	$b_2-\frac{b_1}{a_{11}}a_{21}$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
0	a'_{m2}	$\cdots$	a'_{mn}	a'_{mn+1}	0	$\cdots$	1	$b_m-\frac{b_1}{a_{11}}a_{m1}$

- This new tableau shows immediately that $\widehat{x}=\frac{b_1}{a_{11}}c_1$ at the new extreme point of F, for here the variables $x_2,\ldots,x_{n+1}$ are zero.
- Because of non-degeneracy, the elements in the last column of the tableau under $\widehat{x}-\frac{b_1}{a_{11}}c_1$ cannot be zero - they must be positive.
- The non-zero coordinates $x_1,x_{n+2},\ldots,x_{n+m}$ of the new extreme point can be read off immediately from the above tableau.
- Since $\frac{b_1}{a_{11}}c_1>0$, the value of $\widehat{x}$ at the new extreme point is strictly larger than its value at the initial extreme point.

Second Pivoting

- If $c'_2 \leq 0$, ..., $c'_{n+1} \leq 0$, the extreme point just found will be an optimal vector for the problem.
- Suppose, then, that at least one of $c'_2, \ldots, c'_{n+1}$ is positive, say $c'_{j_0} > 0$.
- If all the numbers in the column below c'_{j_0} in this second tableau are non-positive, then $\widehat{x}$ is not bounded above on F and the problem is insoluble.
- Suppose, then, that at least one of $a'_{1j_0}, \ldots, a'_{mj_0}$ is positive.
- For each i such that $a'_{ij_0} > 0$, consider $\frac{b'_i}{a'_{ij_0}}$, where b'_i is the number in the same row as a'_{ij_0} and in the last column of the tableau; say $a'_{i_0j_0} > 0$ and that $\frac{b'_{i_0}}{a'_{i_0j_0}}$ is the minimum of these quotients.
- Now pivot about the number $a'_{i_0j_0}$ in the second tableau to obtain a third tableau, which will indicate a third extreme point, where the value of $\widehat{x}$ exceeds its value at the second extreme point.

Final Tableau

- We now repeat the procedure.
- Since F, being a polyhedral set, has only a finite number of extreme points and $\widehat{x}$ strictly increases in value at each stage in the algorithm, one of two possibilities must occur:
  - (i) A tableau is reached in which the first $m+n$ numbers on the top row are non-positive;
  - (ii) A tableau is reached which has one of its first $m+n$ numbers on the top row positive with all the numbers below it non-positive.
- In Case (i), the tableau, which is called a **final tableau**, will yield an optimal vector when the non-basic variables are put equal to zero and the values of the basic variables are read off from the tableau;

  The value v of the problem is to be found from the last entry on the first row of the tableau which is $\widehat{x}-v$.
- In Case (ii), the problem is insoluble.

Example

- We use the simplex algorithm to solve the problem:

$$\begin{array}{ll} \text{maximize} & 2x_1 - 3x_2 + x_3 \\ \text{subject to} & 3x_1 + 6x_2 + x_3 \le 6 \\ & 4x_1 + 2x_2 + x_3 \le 4 \\ & x_1 - x_2 + x_3 \le 3 \\ & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0. \end{array}$$

- We convert this standard problem to a canonical problem in the usual way to obtain the following initial tableau in the simplex algorithm.

$$\begin{array}{|rrr|rrr|r|} \hline 2 & -3 & 1 & 0 & 0 & 0 & 0 \\ \hline 3 & 6 & 1 & 1 & 0 & 0 & 6 \\ 4 & 2 & 1 & 0 & 1 & 0 & 4 \\ 1 & -1 & 1 & 0 & 0 & 1 & 3 \\ \hline \end{array}$$

- We have omitted the $\widehat{x}$ in the top right-hand corner of the tableau; the number in this position is the negative of the value of $\widehat{x}$.

Example (Cont'd)

- The first tableau is

$$\begin{array}{|ccc|ccc|c|}
\hline
2 & -3 & 1 & 0 & 0 & 0 & 0 \\
\hline
3 & 6 & 1 & 1 & 0 & 0 & 6 \\
\color{red}{4} & 2 & 1 & 0 & 1 & 0 & 4 \\
1 & -1 & 1 & 0 & 0 & 1 & 3 \\
\hline
\end{array}
\qquad
\begin{array}{|ccc|ccc|c|}
\hline
0 & -4 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & -2 \\
\hline
0 & \frac{9}{2} & \frac{1}{4} & 1 & -\frac{3}{4} & 0 & 3 \\
1 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 1 \\
0 & -\frac{3}{2} & \frac{3}{4} & 0 & -\frac{1}{4} & 1 & 2 \\
\hline
\end{array}$$

- We examine the top row of the tableau for positive entries, selecting the 2 in the first column (although the 1 in the third column would serve equally well).
- According to the simplex algorithm, we next choose the least of the ratios $\frac{6}{3}$, $\frac{4}{4}$, $\frac{3}{1}$, i.e., $\frac{4}{4}$.
- So we pivot about the 4 in the first column, indicating this by marking the 4 in the initial tableau.
- We thus obtain the second tableau (shown on the right).

Example (Cont'd)

- We have the tableau

$$\begin{array}{|ccc|ccc|c|}
\hline
0 & -4 & \frac{1}{2} & 0 & -\frac{1}{2} & 0 & -2 \\
\hline
0 & \frac{9}{2} & \frac{1}{4} & 1 & -\frac{3}{4} & 0 & 3 \\
1 & \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 1 \\
0 & -\frac{3}{2} & {\color{red}\frac{3}{4}} & 0 & -\frac{1}{4} & 1 & 2 \\
\hline
\end{array}
\qquad
\begin{array}{|ccc|ccc|c|}
\hline
0 & -3 & 0 & 0 & -\frac{1}{3} & -\frac{2}{3} & -\frac{10}{3} \\
\hline
0 & 5 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{7}{3} \\
1 & 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\
0 & -2 & 1 & 0 & -\frac{1}{3} & \frac{4}{3} & \frac{8}{3} \\
\hline
\end{array}$$

- We examine the top row of the tableau for positive entries, selecting the $\frac{1}{2}$ in the third column.
- The least of the ratios to be considered, viz. $\frac{3}{1/4}$, $\frac{1}{1/4}$, $\frac{2}{3/4}$, is the last one.
- Thus we pivot about the $\frac{3}{4}$ in the third column to obtain the third tableau (shown on the right).

Example (Cont'd)

- We obtained the tableau

$$\begin{array}{|ccc|ccc|c|}\hline 0 & -3 & 0 & 0 & -\frac{1}{3} & -\frac{2}{3} & -\frac{10}{3} \\ \hline 0 & 5 & 0 & 1 & -\frac{2}{3} & -\frac{1}{3} & \frac{7}{3} \\ 1 & 1 & 0 & 0 & \frac{1}{3} & -\frac{1}{3} & \frac{1}{3} \\ 0 & -2 & 1 & 0 & -\frac{1}{3} & \frac{4}{3} & \frac{8}{3} \\ \hline \end{array}$$

- There are no positive entries on the top row here, so we have a final tableau.
- The non-basic variables indicated by this tableau are x_2, x_5, x_6, which are zero.
- The basic variables x_1, x_3, x_4 have values $\frac{1}{3}$, $\frac{8}{3}, \frac{7}{3}$, respectively, which can be easily read off from the above tableau.
- Hence $(\frac{1}{3}, 0, \frac{8}{3}, \frac{7}{3}, 0, 0)$ is an optimal vector for the canonical problem.
- Thus the standard problem has optimal vector $(\frac{1}{3}, 0, \frac{8}{3})$ and value $\frac{10}{3}$.

Relation to the Dual Problem

- Suppose that, in the usual notation, the initial and final tableaux corresponding to the solution of the standard maximum problem by the simplex algorithm are as follows:

$$\begin{bmatrix} \boldsymbol{c}^T & \boldsymbol{0}^T & 0 \\ \boldsymbol{A} & \boldsymbol{I}_m & \boldsymbol{b} \end{bmatrix}, \quad \begin{bmatrix} -z_1,\ldots,-z_n & -y_1,\ldots,-y_m & -v \\ * & * & * \end{bmatrix},$$

  where $z_1,\ldots,z_n,y_1,\ldots,y_m \geq 0$ and v is the value of the problem.
- The method of operation of the simplex algorithm shows that the first row of the final tableau is obtained from the initial tableau by adding multiples of its last m rows to its first row.
- In particular, $[-y_1,\ldots,-y_m]$ is a linear combination of the rows of $\boldsymbol{I}_m$. So the multiples referred to above are $-y_1,\ldots,-y_m$, in that order.

Relation to the Dual Problem (Cont'd)

- Thus, writing $\boldsymbol{y} = (y_1, \ldots, y_m)$, we deduce that

$$\begin{array}{rcl} [-z_1, \ldots, -z_n] & = & \boldsymbol{c}^T + [-y_1, \ldots, -y_m]\boldsymbol{A}, \\ -v & = & [-y_1, \ldots, -y_m]\boldsymbol{b}. \end{array}$$

- Thus

$$\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{c} + (z_1, \ldots, z_n) \geq \boldsymbol{c}$$

  and

$$v = \boldsymbol{b}^T\boldsymbol{y}.$$

- This shows that:
  - $\boldsymbol{y}$ is a feasible vector for the dual problem;
  - $\boldsymbol{b}^T\boldsymbol{y} = v$, where v is the value of the primal problem.
- By the Complementary Slackness Theorem, $\boldsymbol{y}$ is an optimal vector for the dual.

Significance of Hypothesis

- In our discussion of the simplex algorithm we made two assumptions:
  - (i) The vector $\boldsymbol{b}$ was non-negative;
  - (ii) The system of equations $[\boldsymbol{A}, \boldsymbol{I}_m](x_1, \ldots, x_{n+m}) = \boldsymbol{b}$ was non-degenerate.
- The first assumption was needed at the outset of the algorithm to show that $(0, \ldots, 0, b_1, \ldots, b_m)$ was an initial extreme point.
- Without this assumption, it would not have been clear how to find an extreme point with which to begin the simplex algorithm - indeed such an extreme point might not exist.
- We now describe a method which will tell us:
  - If the feasible set of the canonical problem has an extreme point;
  - If it does, how to find it.

The Augmented Problem

- Consider the canonical maximum problem:

$$\begin{array}{ll} \text{maximize} & c_1x_1+\cdots+c_nx_n=\widehat{x} \\ \text{subject to} & [\boldsymbol{A},\boldsymbol{I}_m](x_1,\ldots,x_{n+m})=\boldsymbol{b};\ x_1,\ldots,x_{n+m}\geq 0, \end{array}$$

  under the single assumption of non-degeneracy.
- Since we have discussed the case when $\boldsymbol{b}\geq\mathbf{0}$, we suppose that at least one of $b_1,\ldots,b_m$ is negative.
- Consider now the following **augmented problem**:

$$\begin{array}{ll} \text{maximize} & -x_0=\widetilde{x} \\ \text{subject to} & -x_0+a_{11}x_1+\cdots+a_{1n}x_n+x_{n+1}=b_1 \\ & \cdots \\ & -x_0+a_{m1}x_1+\cdots+a_{mn}x_n+x_{n+m}=b_m \\ & x_0\geq 0,\ x_1\geq 0,\ \ldots,\ x_{n+m}\geq 0. \end{array}$$

Properties of the Augmented Problem

- We have the following properties
  (i) The problem is feasible, for if x_0 is chosen so that $b_1 + x_0 \geq 0, \ldots,$ $b_m + x_0 \geq 0$, then $(x_0, 0, \ldots, 0, b_1 + x_0, \ldots, b_m + x_0)$ is a feasible vector.
  (ii) The objective function $\widetilde{x} = -x_0$ is bounded above by 0, so, in view of (i), the problem is soluble.
  (iii) Suppose that the unaugmented problem has a feasible vector $(x_1, \ldots, x_{n+m})$. Then $(0, x_1, \ldots, x_{n+m})$ is an optimal vector for the augmented problem, which has value 0.
  Conversely, if $(x_0, x_1, \ldots, x_{n+m})$ is a basic optimal vector for the augmented problem giving it value 0, then $x_0 = 0$ and $(x_1, \ldots, x_{n+m})$ is an extreme point of the feasible set for the canonical problem.
- Thus what we need first is to solve the augmented problem.
  - If its value is negative, then the canonical problem is insoluble.
  - If its value is zero and $(0, x_1, \ldots, x_{n+m})$ is one of its optimal vectors, then $(x_1, \ldots, x_{n+m})$ is the sought-for extreme point of the feasible set for the canonical problem.

Solving the Augmented Problem

- We cannot initially solve the augmented problem by the simplex algorithm, for at least one of $b_1, \ldots, b_m$ is negative.
- Suppose, without loss of generality, that b_1 is less than or equal to each of $b_2, \ldots, b_m$, and hence negative.
- We pivot about the -1 in the first row and the first column of the system of equations to obtain the following problem, which is equivalent to the augmented problem:

$$\begin{array}{ll} \text{maximize} & -a_{11}x_1 - \cdots - a_{1n}x_n - x_{n+1} = \widetilde{x} - b_1 \\ \text{subject to} & x_0 - a_{11}x_1 - \cdots - a_{1n}x_n - x_{n+1} = -b_1 \\ & a'_{21}x_1 + \cdots + a'_{2n}x_n - x_{n+1} + x_{n+2} = b_2 - b_1 \\ & \ldots \\ & a'_{m1}x_1 + \cdots + a'_{mn}x_n - x_{n+1} + x_{n+m} = b_m - b_1 \\ & x_0 \geq 0, x_1 \geq 0, \ldots, x_{n+m} \geq 0, \end{array}$$

where $a'_{21}, \ldots, a'_{mn}$ are real numbers whose specific values do not interest us here.

Solving the Augmented Problem (Cont'd)

- Since $b_1 < 0, b_1 \leq b_2, \ldots, b_1 \leq b_m$,

$$-b_1 > 0, b_2 - b_1 \geq 0, \ldots, b_m - b_1 \geq 0.$$

- We have a problem to which we can apply the simplex algorithm.
- Hence we can find an extreme point of the feasible set of the unaugmented problem (should such an extreme point exist).
- This procedure for finding an extreme point by solving the augmented problem is known as the **method of the artificial variable**.
- The name comes from the artificial introduction of variable x_0, which disappears before the final solution of the original problem is obtained.

Example

- We use the method of the artificial variable to solve the following problem P, which can also be solved graphically:

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & 2x_1 + 3x_2 \le 18 \\ & 4x_1 + x_2 \le 13 \\ & -x_1 - 2x_2 \le -5 \\ & x_1 \ge 0, x_2 \ge 0. \end{array}$$

- Denote by P_R the canonical maximum problem related to P, and by F_R the feasible set for P_R.

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 + 0x_3 + 0x_4 + 0x_5 \\ \text{subject to} & 2x_1 + 3x_2 + x_3 = 18 \\ & 4x_1 + x_2 + x_4 = 13 \\ & -x_1 - 2x_2 + x_5 = -5 \\ & x_1 \ge 0, \ldots, x_5 \ge 0. \end{array}$$

Example (Cont'd)

- The augmented problem associated with P_R is:

$$\begin{array}{ll} \text{maximize} & -x_0 \\ \text{subject to} & -x_0 + 2x_1 + 3x_2 + x_3 = 18 \\ & -x_0 + 4x_1 + x_2 + x_4 = 13 \\ & -x_0 - x_1 - 2x_2 + x_5 = -5 \\ & x_0 \geq 0, x_1 \geq 0, \ldots, x_5 \geq 0. \end{array}$$

- The initial tableau for this problem is:

−1	0	0	0	0	0	0
−1	2	3	1	0	0	18
−1	4	1	0	1	0	13
−1	−1	−2	0	0	1	−5

Example (Cont'd)

- Since -5 is the smallest element in the right-hand column, we pivot about the element in the same row as this and in the column of the artificial variable, to obtain the following tableau:

$$\begin{array}{c|cc|ccc|r}
0 & 1 & 2 & 0 & 0 & -1 & 5 \\
\hline
0 & 3 & 5 & 1 & 0 & -1 & 23 \\
0 & 5 & 3 & 0 & 1 & -1 & 18 \\
1 & 1 & 2 & 0 & 0 & -1 & 5
\end{array}$$

- This tableau is not final, because of the 1 and 2 in its top row.
- We choose a pivot in the column headed by the 2, which is easily seen to be 2:

$$\begin{array}{r|cc|ccr|r}
-1 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & \frac{1}{2} & 0 & 1 & 0 & \frac{3}{2} & \frac{21}{2} \\
-\frac{3}{2} & \frac{7}{2} & 0 & 0 & 1 & \frac{1}{2} & \frac{21}{2} \\
\frac{1}{2} & \frac{1}{2} & 1 & 0 & 0 & -\frac{1}{2} & \frac{5}{2}
\end{array}$$

Example (Cont'd)

- We see that the augmented problem has value 0 and optimal vector $(0,0,\frac{5}{2},\frac{21}{2},\frac{21}{2},0)$.
- So $(0,\frac{5}{2},\frac{21}{2},\frac{21}{2},0)$ is an extreme point of F_R.
- The non-basic variables at this last extreme point are x_1 and x_5.
- We now express the objective function x_1+x_2 of P_R in terms of x_1 and x_5.
- Since $-x_1-2x_2+x_5=-5$, it follows that $x_1+x_2=\frac{1}{2}x_1+\frac{1}{2}x_5+\frac{5}{2}$.
- This enables us to write down an initial tableau for P_R with starting point $(0,\frac{5}{2},\frac{21}{2},\frac{21}{2},0)$:

$$
\begin{array}{|cc|ccc|c|}
\hline
\frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & -\frac{5}{2} \\
\hline
\frac{1}{2} & 0 & 1 & 0 & \frac{3}{2} & \frac{21}{2} \\
\frac{7}{2} & 0 & 0 & 1 & \frac{1}{2} & \frac{21}{2} \\
\frac{1}{2} & 1 & 0 & 0 & -\frac{1}{2} & \frac{5}{2} \\
\hline
\end{array}
$$

Example (Cont'd)

- Starting from that tableau we proceed to a final one as follows:

$$\begin{array}{|cc|ccc|c|}\hline \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} & -\frac{5}{2} \\ \hline \frac{1}{2} & 0 & 1 & 0 & \frac{3}{2} & \frac{21}{2} \\ \frac{7}{2} & 0 & 0 & 1 & \frac{1}{2} & \frac{21}{2} \\ \frac{1}{2} & 1 & 0 & 0 & -\frac{1}{2} & \frac{5}{2} \\ \hline \end{array} \qquad \begin{array}{|cc|ccc|c|}\hline 0 & 0 & 0 & -\frac{1}{7} & \frac{3}{7} & -4 \\ \hline 0 & 0 & 1 & -\frac{1}{7} & \frac{10}{7} & 9 \\ 1 & 0 & 0 & \frac{2}{7} & \frac{1}{7} & 3 \\ 0 & 1 & 0 & -\frac{1}{7} & -\frac{4}{7} & 1 \\ \hline \end{array}$$

$$\begin{array}{|cc|ccc|c|}\hline 0 & 0 & -\frac{3}{10} & -\frac{1}{10} & 0 & -\frac{67}{10} \\ \hline 0 & 0 & \frac{7}{10} & -\frac{1}{10} & 1 & \frac{63}{10} \\ 1 & 0 & -\frac{1}{10} & \frac{3}{10} & 0 & \frac{21}{10} \\ 0 & 1 & \frac{2}{3} & -\frac{1}{5} & 0 & \frac{23}{5} \\ \hline \end{array}$$

- Thus, P has an optimal vector $(\frac{21}{10}, \frac{23}{5})$ and value $\frac{67}{10}$.

Non-Degeneracy and Cycling

- Throughout our discussion we have only considered non-degenerate problems.
- We even made a tacit assumption of non-degeneracy in our account of the method of the artificial variable just described.
- We need this non-degeneracy assumption in showing that the algorithm terminated after a finite number of steps.
- Without the assumption, it would be possible to enter into an infinite sequence of pivoting operations without ever reaching a solution (even when one exists!).
- Such a phenomenon is called **cycling**.
- This difficulty is more apparent than real, for it can be shown that for any problem, there is a sequence of pivots which will ensure that the simplex algorithm is completed in a finite number of steps.
- In practice, cycling rarely occurs, although problems have been specially constructed to demonstrate its existence.

Subsection 6

Game Theory

Matrix Games

- A **matrix game** consists of the following:
- Two players compete against each other:
  - A **row player** R;
  - A **column player** C.
- A game is determined by a real $m \times n$ matrix $A = [a_{ij}]$, called the **pay-off matrix** of the game.
  - The row player chooses a row of A (i.e., one of the numbers $1, \ldots, m$);
  - The column player chooses a column of A (i.e., one of the numbers $1, \ldots, n$).
- Each players acts in ignorance of his opponent.
- If R chooses i and C chooses j, then R receives an amount a_{ij} from C.
- This procedure constitutes one play of the game, and the game consists of a large number of plays.
- The object of each player is to maximize/minimize his gains/losses.

Example

- Player R selects two of the numbers 1,2,4, while C independently selects one of them.
  - For each number chosen by R, but not by C, C pays R that number.
  - For each number chosen by both R and C, R pays C that number.
- This is essentially a matrix game, since we can construct its pay-off matrix.
  - R has three choices: (i) 1,2; (ii) 1,4; (iii) 2,4;
  - C has three choices: (i) 1; (ii) 2; (iii) 4.
- Suppose that both players play their first choices. Then R pays 1 to C and C pays 2 to R. The net result of this play is a gain of 1 to R.

  So the element in row 1 and column 1 of the pay-off matrix is 1.
- The completed matrix is $\begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix}$.

Game Determined by a Matrix: Informal Discussion

- Consider a large number N of plays of the game.
- Suppose that R chooses $1,\ldots,m$, respectively, $N_1,\ldots,N_m$ times.
- Then $N_1+\cdots+N_m=N$, and R has made the choice i $(i=1,\ldots,m)$ with relative frequency $x_i=\frac{N_i}{N}$.
- Clearly, $x_1,\ldots,x_m\geq 0$ and $x_1+\cdots+x_m=1$.
- Suppose, similarly, that C has made the choice j $(j=1,\ldots,n)$ with relative frequency y_j.
- Then $y_1,\ldots,y_n\geq 0$ and $y_1+\cdots+y_n=1$.
- We say that:
  - R employs strategy $\boldsymbol{x}=(x_1,\ldots,x_m)$;
  - C employs strategy $\boldsymbol{y}=(y_1,\ldots,y_n)$.
- How much can R expect to receive from C during the game?
- We assume that the players, within their preferred strategies, make their choices in a random way.

Game Determined by a Matrix (Cont'd)

- R chooses i with relative frequency x_i.
- C chooses j with relative frequency y_j.
- The relative frequency with which both R chooses i and C chooses j is $x_i y_j$, the number of times this occurring being about $x_i y_j N$.
- The amount which R receives from C as a result is $a_{ij} x_i y_j N$.
- Thus the total amount R receives from C after N plays is
$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j N.$$
- The average amount R can expect to receive from C for a single play is
$$\sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij} x_i y_j.$$
- This last expression, denoted by $E(\boldsymbol{x}, \boldsymbol{y})$, is called R's **expected gain** and C's **expected loss**.

Strategy

- Consider again the game determined by a real $m \times n$ matrix $\boldsymbol{A} = [a_{ij}]$.
- A **strategy for** R is a vector $\boldsymbol{x} = (x_1, \ldots, x_m)$ for which $x_1, \ldots, x_m \geq 0$ and $x_1 + \cdots + x_m = 1$.
- A **strategy for** C is a vector $\boldsymbol{y} = (y_1, \ldots, y_n)$ for which $y_1, \ldots, y_n \geq 0$ and $y_1 + \cdots + y_n = 1$.
- The set of all strategies for R is denoted by S_m.
- The set of all strategies for C is denoted by S_n.
- The simplest strategies are the **pure strategies** in which a player consistently chooses a given row or column.
  - The ith **pure strategy** for R is the m-vector $(0, \ldots, 1, \ldots, 0)$, which has a 1 in the ith place and zeros elsewhere;
  - The jth **pure strategy** for C is the n-vector $(0, \ldots, 1, \ldots, 0)$, which has a 1 in the jth place and zeros elsewhere.
- Clearly, the set S_m of all strategies for R is a polytope in $\mathbb{R}^m$ whose extreme points are R's pure strategies.
- Similar remarks apply to S_n.

Expected Gain and Expected Loss

- Suppose that R and C employ strategies $\boldsymbol{x}$ and $\boldsymbol{y}$, respectively.
- Then R's **expected gain** (which is C's **expected loss**), denoted by $E(\boldsymbol{x},\boldsymbol{y})$, is defined by

$$E(\boldsymbol{x},\boldsymbol{y}) = \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}x_iy_j = \boldsymbol{x}^T\boldsymbol{A}\boldsymbol{y}.$$

- We observe that $E(\boldsymbol{x},\boldsymbol{y})$ is simply a particular value of the bilinear form associated with the matrix $\boldsymbol{A}$.

Maximization of Gain; Minimization of Loss

- Let $\underline{a}$ and $\overline{a}$ denote, respectively, the minimal and maximal elements of the matrix $\boldsymbol{A}$.
- Then it is easily seen that, whatever the strategies adopted by the two players:
  - R's expected gain is at least $\underline{a}$;
  - Cs expected loss is at most $\overline{a}$.
- Consider the use of pure strategies by both players.
- If R plays his ith pure strategy against a pure strategy of C, his expected gain will be one of the numbers $a_{i1}, \ldots, a_{in}$.
- So he can be certain of receiving at least $\min_j a_{ij}$.
- Clearly, R should choose i in such a way as to make this minimum as large as possible.

Maximization of Gain; Minimization of Loss (Cont'd)

- Suppose that the minimum $\min_j a_{ij}$ is maximized when $i = i_0$, say.
- Then we have shown that, by suitable choice of a pure strategy, R can guarantee an expected gain of at least
$$\max_i \min_j a_{ij}$$
against any pure strategy of C.
- Similarly, for some $j = j_0$, the j_0th pure strategy of C will keep his expected loss to at most $\min_j \max_i a_{ij}$ against any pure strategy of R.
- By considering R's expected gain (C's expected loss) when R chooses his i_0th pure strategy and C chooses his j_0th pure strategy, we can deduce that
$$\underline{a} \le \max_i \min_j a_{ij} \le \min_j \max_i a_{ij} \le \overline{a}.$$

Example

- Consider the matrix

$$\begin{bmatrix} 3 & 7 & 2 \\ 8 & 1 & 6 \\ 4 & 9 & 5 \end{bmatrix}$$

- Here

$$1 = \underline{a} \leq \max_i \min_j a_{ij} = 4 < 6 = \min_j \max_i a_{ij} \leq \overline{a} = 9.$$

- R's best pure strategy is to play his third row.
- C's best pure strategy is to play his third column.
- When both players choose their best pure strategies, the expected gain (loss) is 5, which lies strictly between the max-min and min-max.

Security Levels

- Consider the general matrix game, determined by a real $m \times n$ matrix $\boldsymbol{A} = [a_{ij}]$, from the point of view of the row player R.
- Suppose that he decides on some strategy $\boldsymbol{x} \in S_m$.
- Denote by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ the pure strategies of C.
- Let $\boldsymbol{y} = (y_1, \ldots, y_n) \in S_n$.
- Then
$$\begin{array}{rcl} E(\boldsymbol{x},\boldsymbol{y}) & = & E(\boldsymbol{x}, y_1\boldsymbol{e}_1 + \cdots + y_n\boldsymbol{e}_n) \\ & = & y_1 E(\boldsymbol{x},\boldsymbol{e}_1) + \cdots + y_n E(\boldsymbol{x},\boldsymbol{e}_n) \\ & \geq & \min\{E(\boldsymbol{x},\boldsymbol{e}_1), \ldots, E(\boldsymbol{x},\boldsymbol{e}_n)\}. \end{array}$$
- So R can be sure that his expectation is at least equal to $u_R(\boldsymbol{x})$, where
$$u_R(\boldsymbol{x}) = \min_{\boldsymbol{y} \in S_n} E(\boldsymbol{x},\boldsymbol{y}) = \min\{E(\boldsymbol{x},\boldsymbol{e}_1), \ldots, E(\boldsymbol{x},\boldsymbol{e}_n)\}.$$
- The number $u_R(\boldsymbol{x})$ is called R's **security level** for his strategy $\boldsymbol{x}$.
- $E(\boldsymbol{x},\boldsymbol{e}_1), \ldots, E(\boldsymbol{x},\boldsymbol{e}_n)$ are linear in $\boldsymbol{x}$, so they are continuous.
- Hence $u_R : S_m \to \mathbb{R}$, being the minimum of a finite number of continuous functions, is itself continuous.

Value and Optimal Strategy for R

- Player R will naturally choose a strategy $\boldsymbol{x}$ in such a way as to make his security level $u_R(\boldsymbol{x})$ as large as possible.
- Since u_R is a continuous real-valued function defined on the compact set S_m, its maximal value, v_R say, will be attained at some point $\boldsymbol{x}_R$ of S_m.
- Thus
$$v_R = u_R(\boldsymbol{x}_R) = \max_{\boldsymbol{x}\in S_m} u_R(\boldsymbol{x}) = \max_{\boldsymbol{x}\in S_m} \min_{\boldsymbol{y}\in S_n} E(\boldsymbol{x},\boldsymbol{y}).$$
- The number v_R is called the **value** of R's game.
- Any strategy such as $\boldsymbol{x}_R$ which gives R a security level of v_R is called an **optimal strategy** for R.

Value and Optimal Strategy for C

- Player C can see his objective as minimizing R's expectation.
- Suppose C decides on a strategy $\boldsymbol{y} \in S_n$.
- Then he can be sure that R's expectation is at most

$$u_C(\boldsymbol{y}) = \max_{\boldsymbol{x} \in S_m} E(\boldsymbol{x}, \boldsymbol{y}).$$

- In perfect analogy to R maximizing $u_R(\boldsymbol{x})$, C tries to minimize $u_C(\boldsymbol{y})$.
- There exist $\boldsymbol{y}_C \in S_n$ and $v_C \in \mathbb{R}$ such that

$$v_C = u_C(\boldsymbol{y}_C) = \min_{\boldsymbol{y} \in S_n} u_C(\boldsymbol{y}) = \min_{\boldsymbol{y} \in S_n} \max_{\boldsymbol{x} \in S_m} E(\boldsymbol{x}, \boldsymbol{y}).$$

- The number v_C is called the **value** of C's game.
- Any strategy such as $\boldsymbol{y}_C$ which gives $u_C(\boldsymbol{y})$ the value v_C is called an **optimal strategy** for C.

Introducing the Minimax Theorem

- Suppose now that R and C have optimal strategies $\boldsymbol{x}_R, \boldsymbol{y}_C$ and that the values of their games are v_R, v_C.
- If R uses $\boldsymbol{x}_R$, he can guarantee himself an expectation of at least v_R;
- If C uses $\boldsymbol{y}_C$, he guarantees that R's expectation will not exceed v_C.
- Thus,
$$v_R \le E(\boldsymbol{x}_R, \boldsymbol{y}_C) \le v_C.$$
- The minimax theorem, proved below, asserts the equality of the values v_R and v_C.
- The theorem, therefore, shows that every matrix game is soluble in the sense that there exists a number v for which:
  - R can play so that his expectation is at least v;
  - C can play so that R's expectation is at most v.

Von Neumann's Minimax Theorem

Theorem (Von Neumann's Minimax Theorem)

In the matrix game determined by a real $m \times n$ matrix $\boldsymbol{A}$, the value of R's game is equal to the value of C's game, i.e.,

$$v_R = \max_{\boldsymbol{x} \in S_m} \min_{\boldsymbol{y} \in S_n} \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{y} = \min_{\boldsymbol{y} \in S_n} \max_{\boldsymbol{x} \in S_m} \boldsymbol{x}^T \boldsymbol{A} \boldsymbol{y} = v_C.$$

- Suppose first that the elements a_{ij} of $\boldsymbol{A}$ are all positive. Consider the following linear programming problem P and its dual P^*:

$$\begin{array}{ll} \text{maximize} & y_1 + \cdots + y_n \\ \text{subject to} & a_{11} y_1 + \cdots + a_{1n} y_n \leq 1 \\ & \cdots \\ & a_{m1} y_1 + \cdots + a_{mn} y_n \leq 1 \\ & y_1 \geq 0, \ldots, y_n \geq 0 \end{array} \qquad \begin{array}{ll} \text{minimize} & x_1 + \cdots + x_m \\ \text{subject to} & a_{11} x_1 + \cdots + a_{m1} x_m \geq 1 \\ & \cdots \\ & a_{1n} x_1 + \cdots + a_{mn} x_m \geq 1 \\ & x_1 \geq 0, \ldots, x_m \geq 0 \end{array}$$

Since A has all positive elements, both P and P^* are feasible. Hence, by the Duality Theorem of linear programming, both P and P^* are soluble and have the same value, v say.

Von Neumann's Minimax Theorem (Cont'd)

- Let $\boldsymbol{X} = (X_1, \ldots, X_m)$, $\boldsymbol{Y} = (Y_1, , \ldots, Y_n)$ be optimal vectors for P^*, P, respectively. Then $\boldsymbol{X} \geq \boldsymbol{0}$, $\boldsymbol{Y} \geq \boldsymbol{0}$, $\boldsymbol{X}^T\boldsymbol{A} \geq [1, \ldots, 1]$, $\boldsymbol{AY} \leq (1, \ldots, 1)$, and

$$X_1 + \cdots + X_m = v = Y_1 + \cdots + Y_n.$$

Write $\boldsymbol{x}_R = \frac{1}{v}\boldsymbol{X}$ and $\boldsymbol{y}_C = \frac{1}{v}\boldsymbol{Y}$. Then $\boldsymbol{x}_R \in S_m$, $\boldsymbol{y}_C \in S_n$, and, for all $\boldsymbol{x} \in S_m$, $\boldsymbol{y} \in S_n$,

$$\begin{array}{rcl} E(\boldsymbol{x}_R, \boldsymbol{y}) & = & \boldsymbol{x}_R^T\boldsymbol{Ay} \geq \frac{1}{v}[1, \ldots, 1]\boldsymbol{y} = \frac{1}{v}; \\ E(\boldsymbol{x}, \boldsymbol{y}_C) & = & \boldsymbol{x}^T\boldsymbol{Ay}_C \leq \frac{1}{v}\boldsymbol{x}^T(1, \ldots, 1) = \frac{1}{v}. \end{array}$$

Thus $v_C \leq \frac{1}{v} \leq v_R$. But $v_R \leq v_C$. So $v_R = \frac{1}{v} = v_C$. Hence $\boldsymbol{x}_R, \boldsymbol{y}_C$ are optimal strategies for R, C, respectively.

Von Neumann's Minimax Theorem (Cont'd)

- Consider now the general case when $\boldsymbol{A}$ is not assumed to be positive. Let k be any real number such that the matrix $\boldsymbol{B}$ obtained by adding k to each element of $\boldsymbol{A}$ is positive. By what we have just proved,

$$\max_{\boldsymbol{x}\in S_m}\min_{\boldsymbol{y}\in S_n}\boldsymbol{x}^T\boldsymbol{B}\boldsymbol{y}=\min_{\boldsymbol{y}\in S_n}\max_{\boldsymbol{x}\in S_m}\boldsymbol{x}^T\boldsymbol{B}\boldsymbol{y}.$$

Equivalently,

$$k+\max_{\boldsymbol{x}\in S_m}\min_{\boldsymbol{y}\in S_n}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{y}=k+\min_{\boldsymbol{y}\in S_n}\max_{\boldsymbol{x}\in S_m}\boldsymbol{x}^T\boldsymbol{A}\boldsymbol{y}.$$

This proves the result.

Value and Solution of a Game

- Since, in a matrix game, the values v_R and v_C are the same, either of them is referred to simply as the **value** of the game.
- By a **solution** to a matrix game is meant:
  - An optimal strategy for R;
  - An optimal strategy for C;
  - The value of the game.
- In the course of proving the minimax theorem, we showed how the solution of a matrix game could be found by solving a certain linear programming problem and its dual.

Example

- Consider the matrix game that has pay-off matrix

$$\begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix} \qquad \begin{bmatrix} 5 & 3 & 7 \\ 7 & 9 & 1 \\ 10 & 6 & 2 \end{bmatrix}$$

- There are some non-positive elements in this matrix.
- So we add 4 to each of its elements, and discuss the game with pay-off matrix the one on the right.
- To find a solution to this game, we solve the following linear programming problem and its dual:

$$\begin{array}{ll} \text{maximize} & y_1 + y_2 + y_3 \\ \text{subject to} & 5y_1 + 3y_2 + 7y_3 \le 1 \\ & 7y_1 + 9y_2 + y_3 \le 1 \\ & 10y_1 + 6y_2 + 2y_3 \le 1 \\ & y_1, y_2, y_3 \ge 0. \end{array}$$

Example (Cont'd)

- This we do using the simplex algorithm as follows.

$$\begin{array}{|ccc|ccc|c|}\hline 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ \hline 5 & 3 & \color{red}{7} & 1 & 0 & 0 & 1 \\ 7 & 9 & 1 & 0 & 1 & 0 & 1 \\ 10 & 6 & 2 & 0 & 0 & 1 & 1 \\ \hline \end{array}$$

$$\begin{array}{|ccc|ccc|c|}\hline \frac{2}{7} & \frac{4}{7} & 0 & -\frac{1}{7} & 0 & 0 & -\frac{1}{7} \\ \hline \frac{5}{7} & \frac{3}{7} & 1 & \frac{1}{7} & 0 & 0 & \frac{1}{7} \\ \frac{44}{7} & \color{red}{\frac{60}{7}} & 0 & -\frac{1}{7} & 1 & 0 & \frac{6}{7} \\ \frac{60}{7} & \frac{36}{7} & 0 & -\frac{2}{7} & 0 & 1 & \frac{5}{7} \\ \hline \end{array}$$

$$\begin{array}{|ccc|ccc|c|}\hline -\frac{2}{15} & 0 & 0 & -\frac{2}{15} & -\frac{1}{15} & 0 & -\frac{1}{5} \\ \hline \frac{2}{5} & 0 & 1 & \frac{3}{20} & -\frac{1}{20} & 0 & \frac{1}{10} \\ \frac{11}{15} & 1 & 0 & -\frac{1}{60} & \frac{7}{60} & 0 & \frac{1}{10} \\ \frac{24}{5} & 0 & 0 & -\frac{1}{5} & -\frac{3}{5} & 1 & \frac{1}{5} \\ \hline \end{array}$$

Example (Cont'd)

- We obtained

$$\begin{array}{|ccc|ccc|c|}\hline -\frac{2}{15} & 0 & 0 & -\frac{2}{15} & -\frac{1}{15} & 0 & -\frac{1}{5} \\ \hline \frac{2}{5} & 0 & 1 & \frac{3}{20} & -\frac{1}{20} & 0 & \frac{1}{10} \\ \frac{11}{15} & 1 & 0 & -\frac{1}{60} & \frac{7}{60} & 0 & \frac{1}{10} \\ \frac{24}{5} & 0 & 0 & -\frac{1}{5} & -\frac{3}{5} & 1 & \frac{1}{5} \\ \hline \end{array}$$

- Thus $(0, \frac{1}{10}, \frac{1}{10})$ and $(\frac{2}{15}, \frac{1}{15}, 0)$ are optimal vectors for the problem and its dual respectively, the value of both problems is $\frac{1}{5}$.
- Referring back to the proof of the minimax theorem, we deduce that, for the modified game: an optimal row strategy is $(\frac{2}{3}, \frac{1}{3}, 0)$, an optimal column strategy is $(0, \frac{1}{2}, \frac{1}{2})$, and the value of the game is 5.
- For the original game: an optimal row strategy is $(\frac{2}{3}, \frac{1}{3}, 0)$, an optimal column strategy is $(0, \frac{1}{2}, \frac{1}{2})$ and its value is $5 - 4 = 1$.

Essential Strategies

- In a matrix game, the ith pure strategy for the row player is said to be **essential** if there is an optimal strategy $(x_1,\ldots,x_m)$ for R in which $x_i > 0$.
- A similar definition applies to the pure strategies for the column player.
- In the example, $(\frac{2}{3},\frac{1}{3},0)$ and $(0,\frac{1}{2},\frac{1}{2})$ were shown to be optimal strategies for the row and column players, respectively.

  Thus the first two pure strategies for the row player and the last two pure strategies for the column player are essential.

Property of Essential Strategies

Theorem

Suppose that some pure strategy for a player in a matrix game is essential. Then this strategy achieves the value of the game against each optimal strategy of the opponent.

- Suppose that, in a game with $m \times n$ pay-off matrix $\boldsymbol{A} = [a_{ij}]$ and value v, R's ith pure strategy is essential and $\boldsymbol{x} = (x_1, \ldots, x_m)$ is an optimal strategy for R in which $x_i > 0$. Let $\boldsymbol{y} = (y_1, \ldots, y_n)$ be an optimal strategy for C. Then $E(\boldsymbol{x}, \boldsymbol{y}) = v$, i.e.

$$x_1(a_{11}y_1 + \cdots + a_{1n}y_n) + \cdots + x_m(a_{m1}y_1 + \cdots + a_{mn}y_n) = v.$$

  Since $(y_1, \ldots, y_n)$ is optimal for C, it will give C an expected loss of at most v against each pure strategy of R. Hence,

$$a_{11}y_1 + \cdots + a_{1n}y_n \leq v, \ldots, a_{m1}y_1 + \cdots + a_{mn}y_n \leq v.$$

Property of Essential Strategies (Cont'd)

- It now follows from the preceding together with the relations $x_1,\ldots,x_m \geq 0$, $x_1+\cdots+x_m=1$, $x_i>0$, that
$$a_{i1}y_1+\cdots+a_{in}y_n=v.$$
Thus, R's ith pure strategy achieves the value v of the game against any optimal strategy $(y_1,\ldots,y_n)$ of C.

Games With a Saddle Point

- Suppose that the (i_0, j_0)th position in a real $m \times n$ matrix $\boldsymbol{A} = [a_{ij}]$ is such that $a_{i_0 j_0}$ is the least element in its row and the greatest element in its column.
- Then $\boldsymbol{A}$ is said to have a **saddle point** at (i_0, j_0) with value $a_{i_0 j_0}$.
- Suppose that, in the game with pay-off matrix $\boldsymbol{A}$, R plays his i_0th pure strategy and C plays an arbitrary strategy $(y_1, \ldots, y_n)$.
- Then R's expected gain is $a_{i_0 1} y_1 + \cdots + a_{i_0 n} y_n \geq a_{i_0 j_0}$ and $v \geq a_{i_0 j_0}$, where v denotes the value of the game.
- Suppose next that R plays an arbitrary strategy $(x_1, \ldots, x_m)$ and C plays his j_0th pure strategy.
- Then C's expected loss is $a_{1 j_0} x_1 + \cdots + a_{m j_0} x_m \leq a_{i_0 j_0}$, and $v \leq a_{i_0 j_0}$.
- It follows that $v = a_{i_0 j_0}$, and that R's i_0th pure strategy and C's j_0th pure strategy are both optimal for their respective players.

Examples

- Not all matrices have saddle points:

$$\begin{bmatrix} 1 & -1 & 3 \\ 3 & 5 & -3 \\ 6 & 2 & -2 \end{bmatrix} \qquad \begin{bmatrix} 3 & 7 & 2 \\ 8 & 1 & 6 \\ 4 & 9 & 5 \end{bmatrix}.$$

- The matrix

$$\begin{bmatrix} 7 & 6 & 8 \\ 2 & 4 & 3 \\ 1 & -1 & 8 \end{bmatrix}$$

  has a saddle point at $(1,2)$ with value 6.

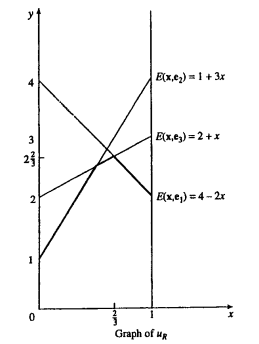
  The game defined by this matrix has value 6.

  Optimal strategies for the row and column players are $(1,0,0)$ and $(0,1,0)$, respectively.

Graphical Solution of Small Games

- We show how games whose pay-off matrices have either just two rows or just two columns can be solved graphically.
- We illustrate the general method by solving the game determined by the 2×3 matrix
$$\begin{bmatrix} 2 & 4 & 3 \\ 4 & 1 & 2 \end{bmatrix}.$$
- Suppose that R employs the strategy $\boldsymbol{x} = (x, 1-x)$, where $0 \le x \le 1$.
- Denoting the pure strategies of C by $\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3$, we find that:
$$\begin{array}{rcl} E(\boldsymbol{x}, \boldsymbol{e}_1) & = & 2x + 4(1-x) = 4 - 2x; \\ E(\boldsymbol{x}, \boldsymbol{e}_2) & = & 4x + 1(1-x) = 1 + 3x; \\ E(\boldsymbol{x}, \boldsymbol{e}_3) & = & 3x + 2(1-x) = 2 + x. \end{array}$$
- Thus, we see that R's security level for his strategy $\boldsymbol{x}$ is given by the equation
$$u_R(\boldsymbol{x}) = \min\{4 - 2x, 1 + 3x, 2 + x\}.$$

Graphical Solution of Small Games (Cont'd)



- The graphs of $E(\boldsymbol{x}, \boldsymbol{e}_1)$, $E(\boldsymbol{x}, \boldsymbol{e}_2)$ and $E(\boldsymbol{x}, \boldsymbol{e}_3)$ are shown on the right, where the graph of u_R is drawn with a thick line. It is clear from this figure that the value v of the game is given by the equations

$$v = \max\{u_R(\boldsymbol{x}) : 0 \leq x \leq 1\} = 2\frac{2}{3}$$

  and that this maximum occurs when $x = \frac{2}{3}$. Thus $(\frac{2}{3}, \frac{1}{3})$ is an optimal strategy for R.
- Suppose now that (y_1, y_2, y_3) is an optimal strategy for C. The figure shows that C's second pure strategy does not achieve the value of the game against R's optimal strategy $(\frac{2}{3}, \frac{1}{3})$. Thus, this strategy is not essential for C, and so $y_2 = 0$.

Graphical Solution of Small Games (Cont'd)

- Since both of R's pure strategies are essential, they must achieve the value of the game against C's optimal strategy $(y_1, 0, y_3)$. Hence

$$2y_1 + 3y_3 = 2\frac{2}{3} \quad \text{and} \quad 4y_1 + 2y_3 = 2\frac{2}{3}.$$

  It follows that $(\frac{1}{3}, 0, \frac{2}{3})$ is an optimal strategy for C.

Dominance

- Consider a game whose pay-off matrix has rows $\boldsymbol{r}_1, \ldots, \boldsymbol{r}_m$ and columns $\boldsymbol{c}_1, \ldots, \boldsymbol{c}_n$.
- Suppose that $\boldsymbol{r}_i \leq \boldsymbol{r}_j$ $(i \neq j)$.
- Then choosing the ith row offers no advantage to R over choosing the jth row.
- So R can exclude the ith row in his search for an optimal strategy.
- We say that the ith row is **dominated** by the jth row.
- In this case the ith row can be omitted from the game.
- Similarly, if $\boldsymbol{c}_i \leq \boldsymbol{c}_j$ $(i \neq j)$, then choosing the jth column offers no advantage to C over choosing the ith column.
- We say that the jth column is **dominated** by the ith column.
- In this case the jth column can be omitted from the game.

Example

- Consider the game with pay-off matrix is on the left:

$$\begin{bmatrix} 2 & 4 \\ 3 & 1 \\ 2 & 3 \end{bmatrix} \qquad \begin{bmatrix} 2 & 4 \\ 3 & 1 \end{bmatrix}$$

- Here the third row is dominated by the first row.
- Hence we exclude the third row from the game, and consider the reduced game determined by the matrix on the right.
- This game is easily solved graphically.
  - Its value is $2\frac{1}{2}$;
  - Optimal strategies for row and column are $(\frac{1}{2}, \frac{1}{2})$ and $(\frac{3}{4}, \frac{1}{4})$.
- Reverting to the original game, we see that:
  - Its value is $2\frac{1}{2}$;
  - Optimal strategies for row and column are $(\frac{1}{2}, \frac{1}{2}, 0)$ and $(\frac{3}{4}, \frac{1}{4})$.