Introduction to Convexity

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Subsection 1

Elementary Sets

Cells

- The basic elementary set is the *cell*.
- In $\mathbb{R}^1$ a **cell** is simply a bounded convex subset of the real line, i.e., a set of one of the following forms, in which $a, b \in \mathbb{R}$ with $a < b$:

$$\varnothing, \{a\}, [a,b], [a,b), (a,b], (a,b).$$

- A **cell** I in $\mathbb{R}^n$ is a set of the form

$$I = I_1 \times \cdots \times I_n = \{(x_1, \ldots, x_n) : x_1 \in I_1, \ldots, x_n \in I_n\},$$

where $I_1, \ldots, I_n$ are cells in $\mathbb{R}^1$.
- The empty set and singletons are examples of degenerate cells in $\mathbb{R}^n$.
- A typical cell in $\mathbb{R}^2$ is a closed rectangle with sides parallel to the coordinate axes, possibly having some or all of its sides removed.

Properties of Cells in $\mathbb{R}^1$

- Let I and J be cells in $\mathbb{R}^1$.
- Then I and J are bounded convex sets, whence so too are $\text{cl}I$, $\text{int}I$, $I \cap J$, and $I + J$.
- Thus, in $\mathbb{R}^1$ the closure and the interior of a cell are cells, as too are the intersection and the vector sum of two cells.
- In general, the set difference $I \setminus J$ is not a cell;
- It is, however, easily verified that $I \setminus J$ can be expressed as the union of two disjoint cells (one or both of which may be empty).

  Example:

$$[3,7) \setminus (4,5] = [3,4] \cup (5,7).$$

Properties of Cells in $\mathbb{R}^n$

- Now let I and J be cells in $\mathbb{R}^n$ specified by the equations $I = I_1 \times \cdots \times I_n$ and $J = J_1 \times \cdots \times J_n$, where $I_1, \ldots, I_n$, $J_1, \ldots, J_n$ are cells in $\mathbb{R}^1$.
- It is easily verified that

$$\mathrm{cl}\, I = \mathrm{cl}\, I_1 \times \cdots \times \mathrm{cl}\, I_n \quad \text{and} \quad \mathrm{int}\, I = \mathrm{int}\, I_1 \times \cdots \times \mathrm{int}\, I_n,$$

  whence the closure and the interior of a cell in $\mathbb{R}^n$ are also cells.
- The readily established relations

$$\begin{aligned} I \cap J &= (I_1 \cap J_1) \times \cdots \times (I_n \cap J_n), \\ I + J &= (I_1 + J_1) \times \cdots \times (I_n + J_n) \end{aligned}$$

  show that the intersection and the vector sum of two cells in $\mathbb{R}^n$ are themselves cells.

Set Difference of Cells

- We show that the set difference $I \backslash J$ can be expressed as a finite union of pairwise disjoint cells.
- For $i = 1, \ldots, n$, $I_i \cap J_i$ is a cell contained in the cell I_i.
- Since cells in $\mathbb{R}^1$ are simply intervals, there exist cells P_i and Q_i in $\mathbb{R}^1$ such that the equation $I_i = P_i \cup Q_i \cup (I_i \cap J_i)$ expresses I_i as a union of three pairwise disjoint cells.
- It follows that

$$I = (P_1 \cup Q_1 \cup (I_1 \cap J_1)) \times \cdots \times (P_n \cup Q_n \cup (I_n \cap J_n)).$$

- Hence, by elementary set theory, I can be written as a union of 3^n pairwise disjoint cells in $\mathbb{R}^n$, one of which is

$$(I_1 \cap J_1) \times \cdots \times (I_n \cap J_n) = I \cap J.$$

- Thus $I \backslash J$, which equals $I \backslash (I \cap J)$, can be written as the union of $3^n - 1$ pairwise disjoint cells.

Elementary Sets and Properties

- A set which can be expressed as a finite union of pairwise disjoint cells in $\mathbb{R}^n$ is called an **elementary set**.
- Every cell is an elementary set, as also is the set difference of two cells.

Theorem

Let A and B be elementary sets in $\mathbb{R}^n$. Then $A \cap B$, $A \backslash B$, $A \cup B$ and $A + B$ are elementary sets.

- Suppose that the equations $A = \bigcup_{i=1}^m I_i$ and $B = \bigcup_{j=1}^p J_j$ express A and B as finite unions of pairwise disjoint cells in $\mathbb{R}^n$.

  Then the equation $A \cap B = \bigcup_{i=1}^m \bigcup_{j=1}^p (I_i \cap J_j)$ expresses $A \cap B$ as a finite union of pairwise disjoint cells in $\mathbb{R}^n$. Hence $A \cap B$ is an elementary set.

  This result easily implies that the intersection of any finite non-zero number of elementary sets is again an elementary set.

Elementary Sets and Properties (Cont'd)

- Now, for $i=1,\ldots,m$, $I_i \backslash B = I_i \backslash \bigcup_{j=1}^{p} J_j = \bigcap_{j=1}^{p}(I_i \backslash J_j)$. So $A \backslash B = (\bigcup_{i=1}^{m} I_i) \backslash B = \bigcup_{i=1}^{m} \bigcap_{j=1}^{p}(I_i \backslash J_j)$. Thus $A \backslash B$ is a finite union of pairwise disjoint elementary sets. So it is itself an elementary set.

  The equation $A \cup B = (A \backslash B) \cup (B \backslash A) \cup (A \cap B)$ shows that $A \cup B$ is a finite union of pairwise disjoint elementary sets. Hence it is itself an elementary set.

  This result easily implies that the union of any finite number of elementary sets is an elementary set.

  The equation $A+B = \bigcup_{i=1}^{m} \bigcup_{j=1}^{p}(I_i + J_j)$ exhibits $A+B$ as a finite union of elementary sets. So $A+B$ is an elementary set.

Corollary

Every union of a finite number, and every intersection of a finite non-zero number of elementary sets in $\mathbb{R}^n$ is an elementary set.

Closure, Interior and Boundary of Elementary Sets

Corollary

The closure, the interior, and the boundary of an elementary set in $\mathbb{R}^n$ are elementary sets.

- Suppose that in $\mathbb{R}^n$ the elementary set A is the union of the pairwise disjoint cells $I_1,\ldots,I_m$. Then $\mathrm{cl}A=(\mathrm{cl}I_1)\cup\cdots\cup(\mathrm{cl}I_m)$. This shows that $\mathrm{cl}A$ is a union of the cells $\mathrm{cl}I_1,\ldots,\mathrm{cl}I_m$. So it is an elementary set by the preceding corollary.

  Let I be an open cell in $\mathbb{R}^n$ containing A. It can be shown that $\mathrm{int}A=I\setminus\mathrm{cl}(I\setminus A)$. Hence, by the theorem and the first part of this corollary, $\mathrm{int}A$ is an elementary set.

  Finally, $\mathrm{bd}A=\mathrm{cl}A\setminus\mathrm{int}A$. So $\mathrm{bd}A$ is an elementary set by the theorem, since $\mathrm{cl}A$ and $\mathrm{int}A$ are elementary sets.

Length and Volume

- The **length** $\ell(I)$ of a cell I in $\mathbb{R}^1$ is defined to be zero when I is empty or a singleton, and to be $b-a$ when I is a cell of one of the forms $[a,b]$, $[a,b)$, $(a,b]$ or (a,b), where $a<b$.
- Suppose next that I is the cell $I_1 \times \cdots \times I_n$ in $\mathbb{R}^n$, where $I_1,\ldots,I_n$ are cells in $\mathbb{R}^1$.
- Then the **volume** $v(I)$ of I is (uniquely) defined by the equation

$$v(I) = \ell(I_1)\cdots\ell(I_n),$$

  i.e., $v(I)$ is the product of the lengths of the cells from which I is constructed.
- This is a natural generalization of the definition of the area of a rectangle and the volume of a rectangular block as encountered in elementary geometry.
- When I is a cell in $\mathbb{R}^1$, we have $v(I) = \ell(I)$.

Pairwise Disjoint Cells

Theorem

Let $I_0, I_1, \ldots, I_m$, where $m \geq 1$, be cells in $\mathbb{R}^n$ with $I_1, \ldots, I_m$ pairwise disjoint and having union I_0. Then $v(I_0) = \sum_{i=1}^m v(I_i)$.

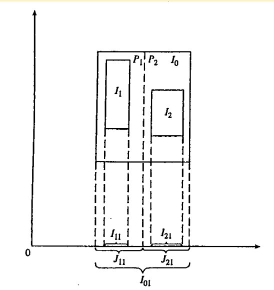
- We argue by induction on m.
  The assertion is trivially true when $m = 1$.
  Suppose, then, that $m > 1$ and that the assertion is true for all partitions of a cell into fewer than m cells.
  If one of the cells $I_1, \ldots, I_m$ is empty, the assertion follows from the induction hypothesis and the fact that the empty cell has volume zero.
  Assume, then, that none of $I_1, \ldots, I_m$ is empty. For $i = 0, 1, \ldots, m$, let $I_i = I_{i1} \times \cdots \times I_{in}$, where $I_{i1}, \ldots, I_{in}$ are cells in $\mathbb{R}^1$. By hypothesis,

  $$I_1 \cap I_2 = (I_{11} \cap I_{21}) \times \cdots \times (I_{1n} \cap I_{2n}) = \emptyset.$$

  So one of the cells $I_{11} \cap I_{21}$, ..., $I_{1n} \cap I_{2n}$ must be empty. Suppose that $I_{11} \cap I_{21}$ is empty.

Pairwise Disjoint Cells (Cont'd)

- Since $I_1 \cup I_2 \subseteq I_0$ and neither of I_1 and I_2 is empty, $I_{11} \cup I_{21} \subseteq I_{01}$. It now follows easily that there exist cells J_{11}, J_{21} in $\mathbb{R}^1$ such that $I_{11} \subseteq J_{11}$, $I_{21} \subseteq J_{21}$, $J_{11} \cup J_{21} = I_{01}$, $J_{11} \cap J_{21} = \emptyset$.
  Define cells P_1 and P_2 in $\mathbb{R}^n$ by the equations $P_1 = J_{11} \times I_{02} \times \cdots \times I_{0n}$ and $P_2 = J_{21} \times I_{02} \times \cdots \times I_{0n}$.



  Then $P_1 \cup P_2 = I_0$, $P_1 \cap P_2 = \emptyset$, and

$$v(P_1) + v(P_2) = (\ell(J_{11}) + \ell(J_{21}))\ell(I_{02}) \cdots \ell(I_{0n}) = \ell(I_{01}) \cdots \ell(I_{0n}) = v(I_0).$$

  Since the cells $P_1 \cap I_2$ and $P_2 \cap I_1$ are empty,

$$P_1 = P_1 \cap I_0 = \bigcup_{i=1}^m (P_1 \cap I_i) = (P_1 \cap I_1) \cup (P_1 \cap I_3) \cup \cdots \cup (P_1 \cap I_m);$$
$$P_2 = P_2 \cap I_0 = \bigcup_{i=1}^m (P_2 \cap I_i) = (P_2 \cap I_2) \cup (P_2 \cap I_3) \cup \cdots \cup (P_2 \cap I_m).$$

Pairwise Disjoint Cells (Cont'd)

- We deduce, using the induction hypothesis, that

$$v(P_1) = v(P_1 \cap I_1) + v(P_1 \cap I_3) + \cdots + v(P_1 \cap I_m) = \textstyle\sum_{i=1}^m v(P_1 \cap I_i);$$
$$v(P_2) = v(P_2 \cap I_2) + v(P_2 \cap I_3) + \cdots + v(P_2 \cap I_m) = \textstyle\sum_{i=1}^m v(P_2 \cap I_i).$$

For $i = 1, \ldots, m$,

$$\begin{array}{rcl} v(P_1 \cap I_i) + v(P_2 \cap I_i) & = & (\ell(J_{11} \cap I_{i1}) + \ell(J_{21} \cap I_{i1}))\ell(I_{i2}) \cdots \ell(I_{in}) \\ & = & \ell(I_{i1})\ell(I_{i2}) \cdots \ell(I_{in}) \\ & = & v(I_1). \end{array}$$

Thus,

$$v(I_0) = v(P_1) + v(P_2) = \sum_{i=1}^n v(P_1 \cap I_i) + \sum_{i=1}^m v(P_2 \cap I_i) = \sum_{i=1}^m v(I_i).$$

This shows that the assertion is true for a partition of a cell into m cells.

Uniqueness of the Volume

Corollary

Suppose that $I_1, \ldots, I_m$ and $J_1, \ldots, J_m$ are partitions of an elementary set A in $\mathbb{R}^n$ into cells. Then

$$\sum_{i=1}^{m} v(I_i) = \sum_{j=1}^{p} v(J_j).$$

- For $i = 1, \ldots, m$, the cell I_i is the union of the pairwise disjoint cells $I_i \cap J_1, \ldots, I_i \cap J_p$. Thus, by the theorem, $v(I_i) = \sum_{j=1}^{p} v(I_i \cap J_j)$. So

$$\sum_{i=1}^{m} v(I_i) = \sum_{i=1}^{m} \sum_{j=1}^{p} v(I_i \cap J_j) = \sum_{j=1}^{p} \sum_{i=1}^{m} v(I_i \cap J_j) = \sum_{j=1}^{p} v(J_j).$$

  Here we have deduced the last equation from the previous ones by interchanging the roles of the I's and the J's.

Volume of Elementary Sets

- Let A be an elementary set which is the union of pairwise disjoint cells $I_1, \ldots, I_m$ in $\mathbb{R}^n$.
- Then the **volume** $v(A)$ of A is defined by the equation

$$v(A) = \sum_{i=1}^{m} v(I_i).$$

- The preceding corollary shows that $v(A)$ is uniquely determined by A, i.e., that it is independent of the particular choice of the pairwise disjoint cells $I_1, \ldots, I_m$ whose union is A.
- A cell I in $\mathbb{R}^n$ is also an elementary set.

  So it is assigned a volume in two ways.

  By the preceding corollary the two definitions attach the same volume to I. So the volume $v(I)$ of the cell I is unambiguous.

Volume of Union of Pairwise Disjoint Elementary Sets

- An immediate consequence of the definition of volume is that, if $A_1, \ldots, A_m$ are pairwise disjoint elementary sets in $\mathbb{R}^n$, then

$$v(A_1 \cup \cdots \cup A_m) = v(A_1) + \cdots + v(A_m).$$

- Suppose now that A and B are elementary sets in $\mathbb{R}^n$ such that $A \subseteq B$. Then A and $B \backslash A$ are disjoint elementary sets whose union is B. Thus, we obtain:
  - $v(B) = v(A) + v(B \backslash A)$;
  - $v(B \backslash A) = v(B) - v(A)$;
  - $v(A) \leq v(B)$.

Property of Volume of Elementary Sets

Theorem

Let A and B be elementary sets in $\mathbb{R}^n$. Then

$$v(A \cup B) + v(A \cap B) = v(A) + v(B).$$

- The set $A \cup B$ is the union of the pairwise disjoint elementary sets $A \setminus (A \cap B)$, $B \setminus (A \cap B)$ and $A \cap B$.

  So by the comments preceding the theorem,

$$\begin{array}{rcl} v(A \cup B) & = & v(A \setminus (A \cap B)) + v(B \setminus (A \cap B)) + v(A \cap B) \\ & = & v(A) - v(A \cap B) + v(B) - v(A \cap B) + v(A \cap B) \\ & = & v(A) + v(B) - v(A \cap B). \end{array}$$

Union of Elementary Sets

Corollary

Let $A_1, \ldots, A_m$ be elementary sets in $\mathbb{R}^n$. Then

$$v(A_1 \cup \cdots \cup A_m) \leq v(A_1) + \cdots + v(A_m).$$

- We argue by induction with respect to m. The case $m = 1$ is trivial. Suppose that $m > 1$ and that the assertion is true for families of fewer than m elementary sets. Then, by the preceding theorem and the induction hypothesis,

$$\begin{aligned} v(A_1 \cup \cdots \cup A_m) &= v((A_1 \cup \cdots \cup A_{m-1}) \cup A_m) \\ &\leq v(A_1 \cup \cdots \cup A_{m-1}) + v(A_m) \\ &\leq v(A_1) + \cdots + v(A_{m-1}) + v(A_m). \end{aligned}$$

  This completes the proof by induction.

Interior and Closure and Boundary of Elementary Sets

Corollary

Let A be an elementary set in $\mathbb{R}^n$. Then

$$v(\text{int}A) = v(A) = v(\text{cl}A) \quad \text{and} \quad v(\text{bd}A) = 0.$$

- We make use of the trivial result that a cell, its interior and its closure all have the same volume. Let $A = I_1 \cup \cdots \cup I_m$, where $I_1, \ldots, I_m$ are pairwise disjoint cells. Then, by the preceding corollary,

$$\begin{array}{rcl} v(\text{cl}A) & = & v(\text{cl}I_1 \cup \cdots \cup \text{cl}I_m) \leq v(\text{cl}I_1) + \cdots + v(\text{cl}I_m) \\ & = & v(I_1) + \cdots + v(I_m) = v(A) \leq v(\text{cl}A); \\ v(\text{int}A) & \geq & v(\text{int}I_1 \cup \cdots \cup \text{int}I_m) = v(\text{int}I_1) + \cdots + v(\text{int}I_m) \\ & = & v(I_1) + \cdots + v(I_m) = v(A) \geq v(\text{int}A). \end{array}$$

Hence $v(\text{cl}A) = v(\text{int}A) = v(A)$.
Finally, $v(\text{bd}A) = v(\text{cl}A \setminus \text{int}A) = v(\text{cl}A) - v(\text{int}A) = 0$.

Subsection 2

Volume

Inner and Outer Volumes

- Denote by $\mathscr{E}$ the class of elementary sets in $\mathbb{R}^n$.
- Let A be the bounded set in $\mathbb{R}^n$ whose volume we wish to define.
- We should expect the (as yet undefined) volume of A to be an upper bound for the set of volumes of elementary sets contained in A.
- This observation leads us to define an **inner-volume** $\underline{v}(A)$ for A by the equation
$$\underline{v}(A) = \sup\{v(E) : E \subseteq A \text{ and } E \in \mathscr{E}\}.$$
- The assumption that A is bounded ensures that $\underline{v}(A)$ is a well-defined non-negative real number.
- Similarly, by considering the volumes of elementary sets containing A, we are led to define an **outer-volume** $\overline{v}(A)$ by the equation
$$\overline{v}(A) = \inf\{v(E) : A \subseteq E \text{ and } E \in \mathscr{E}\}.$$

Basic Properties of Inner and Outer Volumes

Theorem

Let A and B be bounded sets in $\mathbb{R}^n$. Then:

(i) $\underline{v}(A) \leq \overline{v}(A)$;

(ii) $\underline{v}(A) = v(A) = \overline{v}(A)$ when A is an elementary set;

(iii) $\underline{v}(A) \leq \underline{v}(B)$ and $\overline{v}(A) \leq \overline{v}(B)$ whenever $A \subseteq B$;

(iv) $\underline{v}(A) = \underline{v}(\text{int}A)$ and $\overline{v}(A) = \overline{v}(\text{cl}A)$;

(v) $\underline{v}(A \cup B) + \underline{v}(A \cap B) \geq \underline{v}(A) + \underline{v}(B)$ and $\overline{v}(A \cup B) + \overline{v}(A \cap B) \leq \overline{v}(A) + \overline{v}(B)$.

- Both (i) and (ii) follow immediately from the fact that $v(E) \leq v(F)$ whenever E and F are elementary sets with $E \subseteq F$.
  (iii) is clear from the definitions of $\underline{v}$ and $\overline{v}$.
  Suppose now that E is an elementary set with $E \subseteq A$. Then, by previous corollaries, int$E$ is an elementary set with $v(\text{int}E) = v(E)$.

Basic Properties of Inner and Outer Volumes (Cont'd)

- Also $\mathrm{int}E \subseteq \mathrm{int}A$. So

$$\begin{array}{rcl}\underline{v}(\mathrm{int}A) & = & \sup\{v(E) : E \subseteq \mathrm{int}A \text{ and } E \in \mathscr{E}\} \\ & \geq & \sup\{v(\mathrm{int}E) : E \subseteq A \text{ and } E \in \mathscr{E}\} \\ & = & \sup\{v(E) : E \subseteq A \text{ and } E \in \mathscr{E}\} \\ & = & \underline{v}(A).\end{array}$$

  But, by (iii), $\underline{v}(\mathrm{int}A) \leq \underline{v}(A)$. Hence $\underline{v}(\mathrm{int}A) = \underline{v}(A)$.
  Similarly, $\overline{v}(\mathrm{cl}A) = \overline{v}(A)$.
  Finally, let E and F be elementary sets with $E \subseteq A$ and $F \subseteq B$. Then $E \cup F$ and $E \cap F$ are elementary with $E \cup F \subseteq A \cup B$ and $E \cap F \subseteq A \cap B$. By a previous theorem and (ii), (iii) above,

$$\underline{v}(A \cup B) + \underline{v}(A \cap B) \geq v(E \cup F) + v(E \cap F) = v(E) + v(F).$$

  Since this inequality holds for all elementary sets E and F with $E \subseteq A$ and $F \subseteq B$, we can deduce that $\underline{v}(A \cup B) + \underline{v}(A \cap B) \geq \underline{v}(A) + \underline{v}(B)$.
  The last part of (v) is proved similarly.

Sets that Have Volume

- It can happen that $\underline{v}(A) < \overline{v}(A)$.

  Example: Suppose that A is the set of rational numbers in the interval $[0,1]$ of the real line. Then $\text{int}A = \emptyset$ and $\text{cl}A = [0,1]$. Hence, by Parts (ii) and (iv) of the theorem, $\underline{v}(A) = 0$, whereas $\overline{v}(A) = 1$.
- Fortunately, however, for all the sets A in which we are interested the numbers $\underline{v}(A)$ and $\overline{v}(A)$ are equal.
- In particular, this is true when A is a bounded convex set.
- We say that a set A in $\mathbb{R}^n$ **has volume** if it is bounded and $\underline{v}(A) = \overline{v}(A)$.
- Part (ii) of the theorem shows that every elementary set in $\mathbb{R}^n$ has volume.
- For each set A in $\mathbb{R}^n$ which has volume, we write $v(A)$ for the equal numbers $\underline{v}(A)$ and $\overline{v}(A)$, and we say that A has volume $v(A)$.
- In this way, we have extended the volume function v from the class of elementary sets to the class of all sets which have volume.

Criterion for Having Volume

Theorem

The set A in $\mathbb{R}^n$ has volume if and only if, for each $\varepsilon > 0$, there are elementary sets E and F in $\mathbb{R}^n$ such that $E \subseteq A \subseteq F$ and $v(F \setminus E) < \varepsilon$.

- Suppose that A has volume and that $\varepsilon > 0$. Then there exist elementary sets E and F in $\mathbb{R}^n$ with $E \subseteq A \subseteq F$ such that
$$v(E) > \underline{v}(A) - \frac{1}{2}\varepsilon = v(A) - \frac{1}{2}\varepsilon \quad \text{and} \quad v(F) < \overline{v}(A) + \frac{1}{2}\varepsilon = v(A) + \frac{1}{2}\varepsilon.$$
Hence $v(F \setminus E) = v(F) - v(E) < \varepsilon$.
Conversely, suppose that, for each $\varepsilon > 0$, there are elementary sets E and F in $\mathbb{R}^n$ such that $E \subseteq A \subseteq F$ and $v(F \setminus E) < \varepsilon$. This implies that A is bounded. Let ε, E, F be as described. Then
$$0 \le \overline{v}(A) - \underline{v}(A) \le v(F) - v(E) = v(F \setminus E) < \varepsilon.$$
Since $\varepsilon > 0$ is arbitrary, $\underline{v}(A) = \overline{v}(A)$ and A has volume.

Union, Intersection and Complementation

Theorem

Let A and B be sets in $\mathbb{R}^n$ which have volume. Then the sets $A \cup B$, $A \cap B$, and $A \backslash B$ have volume.

- We show that $A \backslash B$ has volume. The other two proofs are similar. Let $\varepsilon > 0$. Then there exist elementary sets E, F, G, H in $\mathbb{R}^n$ with $E \subseteq A \subseteq F$, $G \subseteq B \subseteq H$ such that $v(F \backslash E) < \frac{1}{2}\varepsilon$ and $v(H \backslash G) < \frac{1}{2}\varepsilon$. Now $E \backslash H$ and $F \backslash G$ are elementary sets with $E \backslash H \subseteq A \backslash B \subseteq F \backslash G$ and

$$(F \backslash G) \backslash (E \backslash H) \subseteq (F \backslash E) \cup (H \backslash G).$$

  Hence $v((F \backslash G) \backslash (E \backslash H)) \leq v(F \backslash E) + v(H \backslash G) < \varepsilon$. Thus $A \backslash B$ has volume.

Corollary

All unions of a finite number, and all intersections of a finite non-zero number, of sets in $\mathbb{R}^n$ which have volume also have volume.

Interior, Closure and Volume

Theorem

Let A be a set in $\mathbb{R}^n$ which has volume. Then the sets $\text{int}A$ and $\text{cl}A$ have volume with

$$v(\text{int}A) = v(A) = v(\text{cl}A).$$

- By a previous theorem,

$$\overline{v}(\text{int}A) \leq \overline{v}(A) = v(A) = \underline{v}(A) = \underline{v}(\text{int}A) \leq \overline{v}(\text{int}A).$$

Also

$$\overline{v}(\text{cl}A) = \overline{v}(A) = v(A) = \underline{v}(A) \leq \underline{v}(\text{cl}A) \leq \overline{v}(\text{cl}A).$$

Relation of Volumes of Union and Intersection

Theorem

Let A and B be sets in $\mathbb{R}^n$ which have volume. Then

$$v(A \cup B) + v(A \cap B) = v(A) + v(B).$$

- By previous theorems,

$$\begin{array}{rcl} v(A \cup B) + v(A \cap B) & = & \overline{v}(A \cup B) + \overline{v}(A \cap B) \\ & \leq & \overline{v}(A) + \overline{v}(B) \\ & = & v(A) + v(B); \\ v(A \cup B) + v(A \cap B) & = & \underline{v}(A \cup B) + \underline{v}(A \cap B) \\ & \geq & \underline{v}(A) + \underline{v}(B) \\ & = & v(A) + v(B). \end{array}$$

Relative Complements and Volume

Corollary

Let A, B be sets in $\mathbb{R}^n$ which have volume and are such that $A \subseteq B$. Then

$$v(B \setminus A) = v(B) - v(A) \quad \text{and} \quad v(A) \leq v(B).$$

- The first assertion follows by applying the theorem to the sets $B \setminus A$ and A.

$$v(B) = v(B) + 0 = v((B \setminus A) \cup A) + v((B \setminus A) \cap A) = v(B \setminus A) + v(A).$$

The second assertion follows immediately from the first.

Arbitrary Unions and Volume

Corollary

Let $A_1, \ldots, A_m$ be sets in $\mathbb{R}^n$ which have volume. Then

$$v(A_1 \cup \cdots \cup A_m) \leq v(A_1) + \cdots + v(A_m),$$

with equality holding when $v(A_1 \cap A_j) = 0$, for $1 \leq i < j \leq m$.

- We argue by induction on m. The case $m = 1$ is trivial. Let A_1, A_2 be sets in $\mathbb{R}^n$ which have volume. By the theorem,

$$v(A_1 \cup A_2) + v(A_1 \cap A_2) = v(A_1) + v(A_2).$$

So $v(A_1 \cup A_2) \leq v(A_1) + v(A_2)$, with equality if $v(A_1 \cap A_2) = 0$.

Suppose that $m > 1$ and that the assertion is true for all sequences of $m - 1$ sets.

Arbitrary Unions and Volume (Cont'd)

- Then, by the induction hypothesis and the case $m=2$ just established,

$$\begin{array}{rcl} v(A_1\cup\cdots\cup A_m) & = & v((A_1\cup\cdots\cup A_{m-1})\cup A_m) \\ & \leq & v(A_1\cup\cdots\cup A_{m-1})+v(A_m) \\ & \leq & v(A_1)+\cdots+v(A_{m-1})+v(A_m). \end{array}$$

If $v(A_i\cap A_j)=0$ when $1\leq i<j\leq m$, then

$$\begin{array}{rcl} v((A_1\cup\cdots\cup A_{m-1})\cap A_m) & = & v((A_1\cap A_m)\cup\cdots\cup(A_{m-1}\cap A_m)) \\ & \leq & v(A_1\cap A_m)+\cdots+v(A_{m-1}\cap A_m) \\ & = & 0. \end{array}$$

Hence

$$\begin{array}{rcl} v(A_1\cup\cdots\cup A_m) & = & v((A_1\cup\cdots A_{m-1})\cup A_m) \\ & = & v(A_1\cup\cdots\cup A_{m-1})+v(A_m) \\ & = & v(A_1)+\cdots+v(A_{m-1})+v(A_m). \end{array}$$

Thus the assertion is true for all sequences of m sets.

Volumes and Boundaries

Theorem

The bounded set A in $\mathbb{R}^n$ has volume if and only if its boundary bd$A$ has volume zero.

- Suppose that A has volume. Then the equation $\mathrm{bd}A = \mathrm{cl}A \setminus \mathrm{int}A$, together with previous results, shows that bd$A$ has volume zero. Conversely, suppose that bd$A$ has volume zero. Let $\varepsilon > 0$. Then there exists an elementary set E in $\mathbb{R}^n$ with $\mathrm{bd}A \subseteq E$ and $v(E) < \varepsilon$. Let I be a cell in $\mathbb{R}^n$ containing both E and A. Then $I \setminus E$ is an elementary set. Suppose it is the union of the pairwise disjoint cells $I_1, \ldots, I_m$ in $\mathbb{R}^n$. If an I_i meets A, then it must be contained in A, for otherwise, by the convexity of I_i, it would meet bd$A$, and hence E, which is impossible. Let F be the union of those I_i's which meet A. Then F is an elementary set contained in A, and $E \cup F$ is an elementary set containing A. Also $v((E \cup F) \setminus F) = v(E) < \varepsilon$. Hence, by a previous theorem, A has volume.

Subdivision of the Boundary of a Cube

Lemma

Let $a > 0$ and let I be the n-cube in $\mathbb{R}^n$ defined by the equation

$$I = \{(x_1, \ldots, x_n) : -a \leq x_i \leq a \text{ for } i = 1, \ldots, n\}.$$

Then, for each positive integer m, there exists a subset S of $2nm^{n-1}$ points of $\text{bd}I$ such that, for each $\boldsymbol{x} \in \text{bd}I$, there is $\boldsymbol{s} \in S$ with $\|\boldsymbol{x} - \boldsymbol{s}\| \leq a\frac{\sqrt{n-1}}{m}$.

- Let J denote the set of midpoints of the intervals obtained by subdividing the interval $[-a, a]$ on the real line into m equal subintervals in the obvious way. Then J is a subset of $[-a, a]$ which has m points. Moreover, for each x in $[-a, a]$, there is a point t of J such that $|x - t| < \frac{a}{m}$. Let S be that set in $\mathbb{R}^n$ consisting of all those points exactly one of whose coordinates is either a or $-a$ and whose remaining coordinates belong to the set J. Then S is a subset of $\text{bd}I$ having $2nm^{n-1}$ points.

Subdivision of the Boundary of a Cube (Cont'd)

- Now let $\boldsymbol{x} = (x_1, \ldots, x_n)$ be a point of bd$I$. Then one of the coordinates of $\boldsymbol{x}$ must be a or $-a$ and all of its coordinates must belong to $[-a, a]$. Suppose, without loss of generality, that $x_1 = a$. By the construction of I, there exist points $s_2, \ldots, s_n$ of I (supposing $n \geq 2$) such that $|s_i - x_i| \leq \frac{a}{m}$ for $i = 2, \ldots, n$. Put $\boldsymbol{s} = (a, s_2, \ldots, s_n)$. Then $\boldsymbol{s} \in S$ and

$$\begin{aligned} \|\boldsymbol{x} - \boldsymbol{s}\|^2 &= (a-a)^2 + (x_2 - s_2)^2 + \cdots + (x_n - s_n)^2 \\ &\leq (n-1)\frac{a^2}{m^2}. \end{aligned}$$

The desired result now follows.

Bounded Convex Sets Have Volume

Theorem

Every bounded convex set in $\mathbb{R}^n$ has volume.

- We show that the boundary of a bounded convex set has volume zero, whence the set has volume by a previous theorem. Since, by a previous corollary, a convex set and its closure have the same boundary, it will suffice to prove the theorem for a compact convex set.
  Let A be a non-empty compact convex set in $\mathbb{R}^n$ with projection operator f. Let $\varepsilon > 0$. Since A is bounded, there exists $a > 0$ such that A is contained in the cube I as defined in the statement of the lemma. Choose an integer m such that $m > \frac{2^{n+1} a^n n^{n+1}}{\varepsilon}$, and let the set S be as in the lemma. For each $\boldsymbol{s} \in S$, let $I(\boldsymbol{s})$ be the cube in $\mathbb{R}^n$ with center $f(\boldsymbol{s})$ defined by the equation
  $$I(\boldsymbol{s}) = \left\{(x_1, \ldots, x_n) : |x_i - y_i| \le \frac{na}{m} \text{ for } i = 1, \ldots, n\right\}.$$
  where $f(\boldsymbol{s}) = (y_1, \ldots, y_n)$. We show that the union of the cubes $f(\boldsymbol{s})$ for $\boldsymbol{s} \in S$ contains $\text{bd}A$.

Bounded Convex Sets Have Volume (Cont'd)

- To see why this is so, suppose that $\boldsymbol{a} \in \text{bd}A$. By a previous corollary, there exists $\boldsymbol{x} \in \text{bd}I$ with $f(\boldsymbol{x}) = \boldsymbol{a}$. The construction of S shows that there is $\boldsymbol{s} \in S$ such that

$$\|\boldsymbol{x} - \boldsymbol{s}\| \leq \frac{a\sqrt{n-1}}{m} \leq \frac{na}{m}.$$

Since f is a projection operator, we have

$$\|\boldsymbol{a} - f(\boldsymbol{s})\| = \|f(\boldsymbol{x}) - f(\boldsymbol{s})\| \leq \|\boldsymbol{x} - \boldsymbol{s}\| \leq \frac{na}{m}.$$

It follows that $\boldsymbol{a} \in I(\boldsymbol{s})$. Thus bd$A$ is contained in the union of the (at most) $2nm^{n-1}$ cubes $I(\boldsymbol{s})$, each of which has volume $(\frac{2na}{m})^n$.

By a previous corollary, the volume of this union does not exceed $2nm^{n-1}(\frac{2na}{m})^n = \frac{2^{n+1}a^n n^{n+1}}{m} < \varepsilon$. Hence $\overline{v}(\text{bd}A) < \varepsilon$. So $\overline{v}(\text{bd}A) = 0$.

Bounded Subsets of Hyperplanes

Corollary

Every bounded subset of a hyperplane in $\mathbb{R}^n$ has volume zero.

- Let A be a bounded subset of a hyperplane in $\mathbb{R}^n$. The theorem shows that conv$A$ has volume. By a previous theorem,

$$\mathrm{v}(\mathrm{conv}A) = v(\mathrm{int}(\mathrm{conv}A)) = v(\varnothing) = 0.$$

It now follows easily that A has volume zero.

Effects of Affine Transformations on Volume

- We consider the effect that an affine transformation has on volume.
- We will show that, if A is a set in $\mathbb{R}^n$ which has volume and $T:\mathbb{R}^n \to \mathbb{R}^n$ is an affine transformation with associated matrix $\boldsymbol{Q}$, then the image $T(A)$ of A under T has volume given by the formula

$$v(T(A)) = |\det \boldsymbol{Q}| v(A).$$

Effects of Translations on Volume

- The simplest type of affine transformation is the translation.
- Let A be a set in $\mathbb{R}^n$ which has volume and let $\boldsymbol{a}$ be a point of $\mathbb{R}^n$.
- If E is an elementary set contained in A, then it is easily verified that:
    - $E+\boldsymbol{a}$ is an elementary set contained in $A+\boldsymbol{a}$;
    - $v(E+\boldsymbol{a})=v(E)$.
- It follows that $\underline{v}(A+\boldsymbol{a})\geq\underline{v}(A)$.
- Similarly, we have $\overline{v}(A+\boldsymbol{a})\leq\overline{v}(A)$.
- Since A has volume,

$$\overline{v}(A+\boldsymbol{a})\geq\underline{v}(A+\boldsymbol{a})\geq\underline{v}(A)=\overline{v}(A)\geq\overline{v}(A+\boldsymbol{a}).$$

- This shows that $A+\boldsymbol{a}$ has volume $v(A)$.
- So every translate of a set having volume has a volume equal to that of the set itself.

Elementary Matrices

- A real $n \times n$ matrix is said to be an **elementary matrix** if it can be obtained from the identity matrix $\boldsymbol{I}_n$ by one of the following operations:
  - (i) The multiplication of a row by a non-zero scalar;
  - (ii) The interchange of two rows;
  - (iii) The addition of one row to another one.
- The matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -8 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$$

  illustrate Types (i), (ii) and (iii) of elementary matrices.
- We assume the result that every non-singular matrix can be expressed as a product of elementary matrices.

Elementary Transformations and Volumes

Lemma

Let $T:\mathbb{R}^n\to\mathbb{R}^n$ be the linear transformation given by $T(\boldsymbol{x})=\boldsymbol{Q}\boldsymbol{x}$ for $\boldsymbol{x}\in\mathbb{R}^n$, where $\boldsymbol{Q}$ is an elementary matrix. Then, for each cell I in $\mathbb{R}^n$, the set $T(I)$ has volume $|\det\boldsymbol{Q}|v(I)$.

- Let $I=I_1\times\cdots\times I_n$, where $I_1,\ldots,I_n$ are cells in $\mathbb{R}^1$.
  $T(I)$ has volume, since it is bounded and convex.
  The proof falls naturally into three parts, corresponding to the three types of elementary matrix.
  Suppose first that $\boldsymbol{Q}$ is an elementary matrix of Type (i); say $\boldsymbol{Q}$ is obtained from $\boldsymbol{I}_n$ by multiplying its rth row by a non-zero scalar λ.
  Then $T(I)=I_1\times\cdots\times\lambda I_r\times\cdots\times I_n$. So we get

$$v(T(I))=\ell(I_1)\cdots\ell(\lambda I_r)\cdots\ell(I_n)=|\lambda|\ell(I_1)\cdots\ell(I_n)=|\det\boldsymbol{Q}|v(I).$$

Elementary Transformations and Volumes (Cont'd)

- Suppose next that $\boldsymbol{Q}$ is an elementary matrix of Type (ii), say $\boldsymbol{Q}$ is obtained from $\boldsymbol{I}_n$ by interchanging its rth and sth rows, where $r < s$. Then $\det \boldsymbol{Q} = -1$. Further, $T(I) = I_1 \times \cdots \times I_s \times \cdots \times I_r \times \cdots \times I_n$, i.e., the cells I_r and I_s are transposed from their natural order. So

$$v(T(I)) = \ell(I_1)\cdots\ell(I_s)\cdots\ell(I_r)\cdots\ell(I_n) = \ell(I_1)\cdots\ell(I_n) = |\det \boldsymbol{Q}| v(I).$$

Suppose finally that $\boldsymbol{Q}$ is an elementary matrix of Type (iii), say $\boldsymbol{Q}$ is obtained from $\boldsymbol{I}_n$ by adding its second row to its first. Then $\det \boldsymbol{Q} = 1$. For notational simplicity, we assume that

$$I = \{(x_1, \ldots, x_n) : a_i \le x_i \le b_i \text{ for } i = 1, \ldots, n\}$$

where $a_i < b_i$, for $i = 1, \ldots, n$. Then

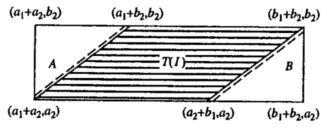
$$T(I) = \{(x_1, \ldots, x_n) : a_1 + x_2 \le x_1 \le b_1 + x_2 \text{ and } a_i \le x_i \le b_i, \text{ for } i = 2, \ldots, n\}.$$

Elementary Transformations and Volumes (Cont'd)

- Let bounded convex sets A and B be defined by the equations

$$A = \{(x_1,\ldots,x_n) : a_1 + a_2 \le x_1 < a_1 + x_2 \text{ and } a_i \le x_i \le b_i, i = 2,\ldots,n\};$$
$$B = \{(x_1,\ldots,x_n) : b_1 + x_2 < x_1 \le b_1 + b_2 \text{ and } a_i \le x_i \le b_i, i = 2,\ldots,n\}.$$

Then the cell $[a_1+a_2, b_1+b_2] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ is the pairwise disjoint union of the sets $A, T(I)$ and B.



So we get

$$(b_1 - a_1 + b_2 - a_2)(b_2 - a_2)\cdots(b_n - a_n) = v(A) + v(T(I)) + v(B).$$

The cell $[a_1+a_2, a_1+b_2] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$ is the disjoint union of the bounded convex sets $\mathrm{cl}A$ and $B - (b_1 - a_1)\boldsymbol{e}_1$. Hence,

$$(b_2 - a_2)(b_2 - a_2)\cdots(b_n - a_n) = v(\mathrm{cl}A) + v(B - (b_1 - a_1)\boldsymbol{e}_1) = v(A) + v(B).$$

Subtracting the second from the first, $v(T(I)) = v(I) = |\det \boldsymbol{Q}| v(I)$.

Affine Transformations and Volumes

Theorem

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the affine transformation given by $T(\boldsymbol{x}) = \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{q}$ for $\boldsymbol{x} \in \mathbb{R}^n$, where $\boldsymbol{Q}$ is an $n \times n$ real matrix and $\boldsymbol{q} \in \mathbb{R}^n$. Then, for each set A in $\mathbb{R}^n$ that has volume, the set $T(A)$ has volume $|\det \boldsymbol{Q}| v(A)$.

- We consider first the case when $\boldsymbol{Q}$ is an elementary matrix and $\boldsymbol{q} = \boldsymbol{0}$. Let $\varepsilon > 0$. Then there exist pairwise disjoint cells $I_1, \ldots, I_m$ in $\mathbb{R}^n$ such that $I_1 \cup \cdots \cup I_m \subseteq A$ and $v(I_1) + \cdots + v(I_m) = v(I_1 \cup \cdots \cup I_m) > v(A) - \varepsilon$. Now $T(I_1) \cup \cdots \cup T(I_m) = T(I_1 \cup \cdots \cup I_m) \subseteq T(A)$. Using the lemma and the fact that T is a bijection (as $\boldsymbol{Q}$ is non-singular), we deduce that

$$\begin{array}{rcl} \underline{v}(T(A)) & \geq & v(T(I_1) \cup \cdots \cup T(I_m)) \\ & = & v(T(I_1)) + \cdots + v(T(I_m)) \\ & = & |\det \boldsymbol{Q}|(v(I_1) + \cdots + v(I_m)) \\ & \geq & |\det \boldsymbol{Q}|(v(A) - \varepsilon). \end{array}$$

  Since $\varepsilon > 0$ is arbitrary, $\underline{v}(T(A)) \geq |\det \boldsymbol{Q}| v(A)$.

Affine Transformations and Volumes (Cont'd)

- Similarly, we have $\overline{v}(T(A)) \leq |\det \boldsymbol{Q}| v(A)$.
  Hence $|\det \boldsymbol{Q}| v(A) \leq \underline{v}(T(A)) \leq \overline{v}(T(A)) \leq |\det \boldsymbol{Q}| v(A)$.
  So $T(A)$ has volume $|\det \boldsymbol{Q}| v(A)$.
  Let now $\boldsymbol{Q}$ be an arbitrary $n \times n$ real matrix and $\boldsymbol{q} = \boldsymbol{0}$.
  The theorem is obvious when $\boldsymbol{Q}$ is singular.
  In that case, $T(A)$ is a bounded subset of some hyperplane of $\mathbb{R}^n$.
  So both $v(T(A))$ and $|\det \boldsymbol{Q}| v(A)$ are zero.
  Suppose, then, that $\boldsymbol{Q}$ is non-singular, and that $\boldsymbol{Q} = \boldsymbol{Q}_1 \cdots \boldsymbol{Q}_m$, where $\boldsymbol{Q}_1, \ldots, \boldsymbol{Q}_m$ are elementary matrices. By repeated applications of the special case of the theorem just proved, we deduce that

$$\begin{aligned} v(T(A)) &= |\det \boldsymbol{Q}_1| \cdots |\det \boldsymbol{Q}_m| v(A) \\ &= |\det(\boldsymbol{Q}_1 \cdots \boldsymbol{Q}_m)| v(A) \\ &= |\det \boldsymbol{Q}| v(A). \end{aligned}$$

  Finally, the case $\boldsymbol{q} \neq \boldsymbol{0}$ adds no difficulty, since translations leave volumes unchanged.

Scaling Translates, Congruence and Volume

Corollary

Let A be a set in $\mathbb{R}^n$ which has volume. Then, for all $\lambda \geq 0$ and $\boldsymbol{a} \in \mathbb{R}^n$,

$$v(\lambda A + \boldsymbol{a}) = \lambda^n v(A).$$

Corollary

Let A and B be congruent sets in $\mathbb{R}^n$ with A having volume. Then

$$v(B) = v(A).$$

- Since A and B are congruent, there exists an affine transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ given by $T(\boldsymbol{x}) = \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{q}$ for $\boldsymbol{x} \in \mathbb{R}^n$, where $\boldsymbol{Q}$ is an $n \times n$ orthogonal matrix and $\boldsymbol{q} \in \mathbb{R}^n$, such that $T(A) = B$. Since the determinant of an orthogonal matrix is ± 1, the result follows from the theorem.

Continuity With Respect to Hausforff Distance

Theorem

Let $A, A_1, \ldots, A_k, \ldots$ be non-empty compact convex sets in $\mathbb{R}^n$ such that $A_k \to A$ as $k \to \infty$. Then $v(A_k) \to v(A)$ as $k \to \infty$.

- Throughout the proof we denote the Hausdorff distance $\rho(A_k, A)$ between A_k and A by θ_k. Consider first the case when A has non-empty interior. Since both volume and the Hausdorff distance are unchanged by translations, we can assume that the origin is an interior point of A, say $rU \subseteq A$ for some $r > 0$. Choose k so large that $\theta_k < r$. Then, by the definition of ρ,

$$A_k \subseteq A + \theta_k U \subseteq A + \frac{\theta_k}{r} A = (1 + \frac{\theta_k}{r}) A;$$
$$(1 - \frac{\theta_k}{r}) A + \frac{\theta_k}{r} A = A \subseteq A_k + \theta_k U \subseteq A_k + \frac{\theta_k}{k} A.$$

By a previous (cancelation) theorem, $(1 - \frac{\theta_k}{r}) A \subseteq A_k$.

Continuity With Respect to Hausforff Distance (Cont'd)

- We showed that $(1-\frac{\theta_k}{r})A \subseteq A_k$. Thus, we have

$$(1-\frac{\theta_k}{r})A \subseteq A_k \subseteq (1+\frac{\theta_k}{r})A;$$
$$(1-\frac{\theta_k}{r})^n v(A) \le v(A_k) \le (1+\frac{\theta_k}{r})v(A).$$

So $v(A_k) \to v(A)$ as $k \to \infty$.

Suppose now that A has empty interior. Then A lies in some hyperplane of $\mathbb{R}^n$.

  - If $n=1$, then A is a singleton and $v(A_k) \le 2\theta_k$. So $v(A_k) \stackrel{k\to\infty}{\longrightarrow} 0 = v(A)$.
  - Suppose that $n \ge 2$. Since both volume and the Hausdorff distance are unchanged by congruence transformations, we can assume that, for some $R > 0$, $A \subseteq \{(x_1,\ldots,x_{n-1},0) : |x_1|,\ldots,|x_{n-1}| \le R\}$. Now

$$A_k \subseteq A + \theta_k U \subseteq \{(x_1,\ldots,x_n) : |x_1|,\ldots,|x_{n-1}| \le R+\theta_k, |x_n| \le \theta_k\}.$$

So $v(A_k) \le 2(2R+2\theta_k)^{n-1}\theta_k \stackrel{k\to\infty}{\longrightarrow} 0 = v(A)$.

A Limit Theorem

Theorem

Let A be a union of a finite number of bounded convex sets in $\mathbb{R}^n$ each of which has dimension at most $n-2$. Then $\frac{v((A)_\lambda)}{\lambda} \to 0$ as $\lambda \to 0^+$.

- Consider first the case of a bounded convex set A in $\mathbb{R}^n$ having dimension at most $n-2$. We can assume that $n \geq 3$ and that, for some $R > 0$, $A \subseteq \{(0,0,x_3,\ldots,x_n) : |x_3|,\ldots,|x_n| \leq R\}$. Thus, for $\lambda > 0$,

$$(A)_\lambda \subseteq \{(x_1,\ldots,x_n) : |x_1|,|x_2| \leq \lambda; |x_3|,\ldots,|x_n| \leq R+\lambda\}.$$

So

$$v((A)_\lambda) \leq 4\lambda^2(2R+2\lambda)^{n-2} = 2^n\lambda^2(R+\lambda)^{n-2}.$$

Hence,

$$\frac{v((A)_\lambda)}{\lambda} \leq 2^n\lambda(R+\lambda)^{n-2} \stackrel{\lambda\to 0^+}{\longrightarrow} 0.$$

A Limit Theorem (Cont'd)

- Consider now the general case when A is the union of bounded convex sets $A_1, \ldots, A_m$ in $\mathbb{R}^n$, each of which has dimension at most $n-2$. Then

$$v((A)_\lambda) = v((A_1)_\lambda \cup \cdots \cup (A_m)_\lambda) \leq v((A_1)_\lambda) + \cdots + v((A_m)_\lambda).$$

So, by what we have just proved,

$$\frac{v((A)_\lambda)}{\lambda} \leq \frac{v((A_1)_\lambda)}{\lambda} + \cdots + \frac{v((A_m)_\lambda)}{\lambda} \stackrel{\lambda \to 0^+}{\longrightarrow} 0.$$

Subsection 3

The Determination of Volume

Volumes of Parallelotopes

- A set in $\mathbb{R}^n$ is called a **parallelotope** if it is the image of the unit n-cube
$$\{(x_1,\ldots,x_n) : 0 \le x_1,\ldots,x_n \le 1\} = [0,1] \times \cdots \times [0,1]$$
under a non-singular affine transformation.
- We find the volume of the n-parallelotope
$$P = \{\lambda_1 \boldsymbol{a}_1 + \cdots + \lambda_n \boldsymbol{a}_n : 0 \le \lambda_1,\ldots,\lambda_n \le 1\} + \boldsymbol{a},$$
where $\boldsymbol{a},\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n \in \mathbb{R}^n$ with $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n$ linearly independent.
- Let $[\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n]$ be the matrix with columns $\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n$.
- Then P is the image of. the n-cube
$$\{(x_1,\ldots,x_n) : 0 \le x_1,\ldots,x_n \le 1\} = [0,1] \times \cdots \times [0,1]$$
under the affine transformation which maps $\boldsymbol{x}$ to $[\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n]\boldsymbol{x} + \boldsymbol{a}$.
- Hence, by a previous theorem, P has volume $|\det[\boldsymbol{a}_1,\ldots,\boldsymbol{a}_n]|$.

Extending the Volume Functions

- To indicate the dependence of the volume function v upon n, we write v_n for the volume function in $\mathbb{R}^n$ and refer to it as the n-**volume**.
- Thus $v_1(=\ell), v_2, v_3$ denote, respectively, length in $\mathbb{R}^1$, area in $\mathbb{R}^2$ and volume in $\mathbb{R}^3$.
- It turns out to be necessary to enlarge the domain of definition of v_{n-1}, for $n \geq 2$, to include those sets in $\mathbb{R}^n$ which are congruent to sets in $\mathbb{R}^{n-1}$ having $(n-1)$-volume.
- Let A be a set in $\mathbb{R}^n$ $(n \geq 2)$ that is congruent to some set B in $\mathbb{R}^{n-1}$ having $(n-1)$-volume $v_{n-1}(B)$.

  Then we define $v_{n-1}(A)$ to be $v_{n-1}(B)$.
- This defines $v_{n-1}(A)$ uniquely, for if A is also congruent to C in $\mathbb{R}^{n-1}$, then B and C are congruent, which shows that $v_{n-1}(B) = v_{n-1}(C)$.
- It is also helpful to define a volume function v_0 in $\mathbb{R}^1$ by putting $v_0(\emptyset) = 0$ and $v_0(\{a\}) = 1$, for each real number a.

Properties of Extended Volume Functions

- If A is a set in $\mathbb{R}^n$ for which $v_{n-1}(A)$ is defined, then A must be a bounded subset of some hyperplane of $\mathbb{R}^n$.
- Also, if A is a bounded subset of some $(n-2)$-flat in $\mathbb{R}^n$ $(n \geq 2)$, then it is congruent to some bounded subset of a hyperplane in $\mathbb{R}^{n-1}$, whence $v_{n-1}(A) = 0$.
- If $T : \mathbb{R}^n \to \mathbb{R}^n$ is a congruence transformation and $\lambda \geq 0$, then

$$v_{n-1}(T(A)) = v_{n-1}(A) \quad \text{and} \quad v_{n-1}(\lambda A) = \lambda^{n-1} v_{n-1}(A),$$

where A is a subset of $\mathbb{R}^n$ for which $v_{n-1}(A)$ is defined.

Review of Riemann Integrability

- Let $f:[a,b]\to\mathbb{R}$, where $a<b$, be a bounded real-valued function.
- For each subdivision $\mathscr{D}$ of $[a,b]$, where $\mathscr{D}$ is given by $a=\xi_0<\xi_1<\cdots<\xi_m=b$, **lower** and **upper sums** $s(\mathscr{D})$ and $S(\mathscr{D})$ of f with respect to $\mathscr{D}$ are defined by the equations

$$\begin{aligned} s(\mathscr{D}) &= \sum_{i=0}^{m-1}(\xi_{i+1}-\xi_i)\inf\{f(x):\xi_i\le x\le\xi_{i+1}\},\\ S(\mathscr{D}) &= \sum_{i=0}^{m-1}(\xi_{i+1}-\xi_i)\sup\{f(x):\xi_i\le x\le\xi_{i+1}\}.\end{aligned}$$

- **Lower** and **upper integrals** $\underline{\int}_a^b f(x)dx$ and $\overline{\int}_a^b f(x)dx$ of f on $[a,b]$ are defined by the equations:

$$\begin{aligned} \underline{\int}_a^b f(x)dx &= \sup\{s(\mathscr{D}):\mathscr{D}\text{ a subdivision of }[a,b]\},\\ \overline{\int}_a^b f(x)dx &= \inf\{S(\mathscr{D}):\mathscr{D}\text{ a subdivision of }[a,b]\}.\end{aligned}$$

- The inequality $\underline{\int}_a^b f(x)dx\le\overline{\int}_a^b f(x)dx$ always holds.
- If $\underline{\int}_a^b f(x)dx=\overline{\int}_a^b f(x)dx$, then f is said to be **Riemann integrable on** $[a,b]$ and the common value is denoted by $\int_a^b f(x)dx$.

Volumes and Integrals

Theorem

Let A be a bounded convex set in $\mathbb{R}^n$. For each real number x, denote by A_x the intersection of A with the hyperplane $x_1 = x$ in $\mathbb{R}^n$. Let a and b be real numbers such that $a < b$ and A_x is empty whenever $x < a$ or $x > b$. Then

$$v_n(A) = \int_a^b v_{n-1}(A_x)dx.$$

- Let E be any elementary set contained in A. For each real number x, denote by E_x the intersection of E with the hyperplane $x_1 = x$ in $\mathbb{R}^n$. The function $v_{n-1}(E_x)$ is a step function, and so is Riemann integrable. Clearly $\int_a^b v_{n-1}(E_x)dx = v_n(E)$ and $v_{n-1}(E_x) \leq v_{n-1}(A_x)$. Thus

$$v_n(E) = \underline{\int_a^b} v_{n-1}(E_x)dx \leq \underline{\int_a^b} v_{n-1}(A_x)dx.$$

Volumes and Integrals (Cont'd)

- Since E is any elementary set contained in A, $v_n(A) \leq \underline{\int}_a^b v_{n-1}(A_x)dx$.

  A similar argument shows that $\overline{\int}_a^b v_{n-1}(A_x)dx \leq v_n(A)$.

  Thus, $v_{n-1}(A_x)$ is Riemann integrable on $[a, b]$ and

$$v_n(A) = \int_a^b v_{n-1}(A_x)dx.$$

- The formula of the theorem can be written

$$v_n(A) = \int_{-\infty}^{\infty} v_{n-1}(A_x)dx,$$

  since $v_{n-1}(A_x) = 0$ when either $x < a$ or $x > b$.

Volumes and Integrals Along Hyperplanes

Corollary

Let A be a bounded convex set in $\mathbb{R}^n$ and let $\boldsymbol{u}$ be a unit vector in $\mathbb{R}^n$. For each real number x, denote by A_x the intersection of A with the hyperplane $\boldsymbol{u} \cdot \boldsymbol{x} = x$. Then

$$v_n(A) = \int_{-\infty}^{\infty} v_{n-1}(A_x) dx.$$

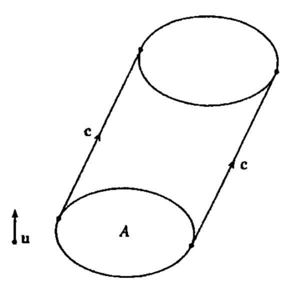
- Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a congruence transformation such that $T(\mathbf{0}) = \mathbf{0}$ and $T(\boldsymbol{u}) = \boldsymbol{e}_1$. By the theorem,

$$\begin{aligned} v_n(A) = v_n(T(A)) &= \int_{-\infty}^{\infty} v_{n-1}(T(A) \cap \{\boldsymbol{x} : \boldsymbol{x} \cdot \boldsymbol{e}_1 = x\}) dx \\ &= \int_{-\infty}^{\infty} v_{n-1}(T(A_x)) dx \\ &= \int_{-\infty}^{\infty} v_{n-1}(A_x) dx. \end{aligned}$$

Volume of a Cylindrical Set

- Let A be a bounded convex subset of a hyperplane $\boldsymbol{u} \cdot \boldsymbol{x} = u_0$ in $\mathbb{R}^n$, where $u_0 \in \mathbb{R}, \boldsymbol{u} \in \mathbb{R}^n$ and $\|\boldsymbol{u}\| = 1$. Let $\boldsymbol{c} \in \mathbb{R}^n$. We compute the volume of the **cylindrical set**

$$B = A + \{\lambda \boldsymbol{c} : 0 \le \lambda \le 1\}.$$



In calculating $v_n(B)$, we assume initially that $\boldsymbol{c} \cdot \boldsymbol{u} > 0$. The hyperplane $\boldsymbol{u} \cdot \boldsymbol{x} = x$ meets B in a translate of A if $u_0 \le x \le u_0 + \boldsymbol{c} \cdot \boldsymbol{u}$, and in the empty set for other real values of x. But each translate of A has the same $(n-1)$-volume as A itself. Thus, by the corollary,

$$v_n(B) = \int_{u_0}^{u_0 + \boldsymbol{c} \cdot \boldsymbol{u}} v_{n-1}(A) dx = (\boldsymbol{c} \cdot \boldsymbol{u}) v_{n-1}(A).$$

Volume of a Cylindrical Set (Cont'd)

- In the general case, i.e., when $\boldsymbol{c} \cdot \boldsymbol{u}$ is unrestricted, we have

$$v_n(B) = |\boldsymbol{c} \cdot \boldsymbol{u}| v_{n-1}(A).$$

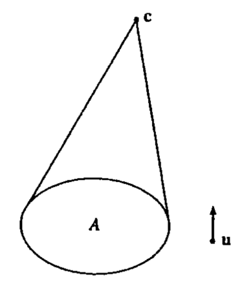
- This formula generalizes the result that the volume of a three dimensional cylinder is the product of the area of its base with its height.
- If $\boldsymbol{c}$ is normal to the hyperplane $\boldsymbol{u} \cdot \boldsymbol{x} = u_0$, then $\boldsymbol{c} \cdot \boldsymbol{u} = \pm \|\boldsymbol{c}\|$ and the above formula reduces to

$$v_n(B) = \|\boldsymbol{c}\| v_{n-1}(A).$$

Volume of a Conical Set

- Let A be a bounded convex subset of a hyperplane $\boldsymbol{u} \cdot \boldsymbol{x} = u_0$ in $\mathbb{R}^n$, where $u_0 \in \mathbb{R}, \boldsymbol{u} \in \mathbb{R}^n$ and $\|\boldsymbol{u}\| = 1$. Let $\boldsymbol{c} \in \mathbb{R}^n$. We compute the volume of the **conical set**

$$C = \text{conv}(A \cup \{\boldsymbol{c}\}).$$



Assume first that A is non-empty and that $u_0 < \boldsymbol{c} \cdot \boldsymbol{u}$. Clearly

$$C = \text{conv}(A \cup \{\boldsymbol{c}\}) = \bigcup(\lambda \boldsymbol{c} + (1 - \lambda)A : 0 \le \lambda \le 1).$$

The hyperplane $\boldsymbol{u} \cdot \boldsymbol{x} = x$ meets C in:
- The set $\lambda \boldsymbol{c} + (1 - \lambda)A$, for $u_0 \le x \le \boldsymbol{c} \cdot \boldsymbol{u}$, where $\lambda = \frac{x - u_0}{\boldsymbol{c} \cdot \boldsymbol{u} - u_0}$;
- The empty set for other real values of x.

Volume of a Conical Set (Cont'd)

- We have

$$v_{n-1}(\lambda \boldsymbol{c}+(1-\lambda)A)=v_{n-1}((1-\lambda)A)=(1-\lambda)^{n-1}v_{n-1}(A).$$

So, by the corollary,

$$v_n(C)=\int_{u_0}^{\boldsymbol{c}\cdot\boldsymbol{u}}\left(\frac{\boldsymbol{c}\cdot\boldsymbol{u}-x}{\boldsymbol{c}\cdot\boldsymbol{u}-u_0}\right)^{n-1}v_{n-1}(A)dx=\frac{1}{n}(\boldsymbol{c}\cdot\boldsymbol{u}-u_0)v_{n-1}(A).$$

In the general case, when $\boldsymbol{c}\cdot\boldsymbol{u}$ is unrestricted and A may be empty, we have

$$v_n(C)=\frac{1}{n}|\boldsymbol{c}\cdot\boldsymbol{u}-u_0|n_{n-1}(A).$$

- This formula generalizes the result that the volume of a three dimensional cone is one third the product of the area of its base with its height.

Volume of a Simplex

- Consider, first, the simplex S_n in $\mathbb{R}^n$ which is the polytope $\text{conv}\{\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$.

  Let α_n denote the n-volume of S_n.

  For $n \geq 2$,

  $$S_n = \text{conv}\{\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\} = \text{conv}\big(\text{conv}\{\mathbf{0}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}\} \cup \{\boldsymbol{e}_n\}\big).$$

  using the formula established above for the volume of a conical set,

  $$\alpha_n = \frac{\alpha_{n-1}}{n}.$$

  We also have $\alpha_1 = 1$.

  We conclude that

  $$\alpha_n = \frac{1}{n!}, \text{ for } n \geq 1.$$

Volume of a Simplex (Cont'd)

- Consider, next, the general n-simplex which is the convex hull of some affinely independent set $\{\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n\}$ in $\mathbb{R}^n$, where $\boldsymbol{a}_i = (a_{i1}, \ldots, a_{in})$ for $i = 0, \ldots, n$.

  This simplex is the image of S_n under the affine transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ defined by the equation $T(\boldsymbol{x}) = \boldsymbol{Q}\boldsymbol{x} + \boldsymbol{q}$ for $\boldsymbol{x} \in \mathbb{R}^n$, where $\boldsymbol{Q}$ is the $n \times n$ matrix with columns $\boldsymbol{a}_1 - \boldsymbol{a}_0, \ldots, \boldsymbol{a}_n - \boldsymbol{a}_0$ and $\boldsymbol{q} = \boldsymbol{a}_0$.

  Thus, $\mathrm{conv}\{\boldsymbol{a}_0, \ldots, \boldsymbol{a}_n\}$ has n-volume $\frac{|\det \boldsymbol{Q}|}{n!}$, i.e., the absolute value of

$$\frac{1}{n!}\det\begin{bmatrix} a_{11} - a_{01} & \cdots & a_{n1} - a_{01} \\ \vdots & & \vdots \\ a_{1n} - a_{0n} & \cdots & a_{nn} - a_{0n} \end{bmatrix} = \frac{1}{n!}\det\begin{bmatrix} a_{01} & a_{11} & \cdots & a_{n1} \\ \cdots & \vdots & & \vdots \\ a_{0n} & a_{1n} & \cdots & a_{nn} \\ 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Volume of a Closed Unit Ball

- We find a formula for ω_n, the n-volume of the closed unit ball U in $\mathbb{R}^n$.

  It is well known that $\omega_1 = 2$, $\omega_2 = \pi$, $\omega_3 = \frac{4\pi}{3}$.

  By the preceding theorem, $\omega_n = \int_{-1}^{1} v_{n-1}(U_x)dx$, where

  $$U_x = \{(x_1, x_2, \ldots, x_n) : x_2^2 + \cdots + x_n^2 = 1 - x^2\}, \quad -1 \le x \le 1.$$

  For $-1 < x < 1$, U_x is congruent to a closed ball in $\mathbb{R}^{n-1}$ of radius $\sqrt{1-x^2}$. So

  $$v_{n-1}(U_x) = v_{n-1}(\sqrt{1-x^2}U) = \omega_{n-1}(1-x^2)^{\frac{n-1}{2}}, \quad -1 \le x \le 1.$$

  Thus,

  $$\omega_n = \int_{-1}^{1} \omega_{n-1}(1-x^2)^{\frac{n-1}{2}} dx = 2\int_{0}^{1} \omega_{n-1}(1-x^2)^{\frac{n-1}{2}} dx.$$

Volume of a Closed Unit Ball (Cont'd)

- In the section on the Gamma and Beta Functions, it was shown that:
  - $B(\frac{n+1}{2},\frac{n+1}{2})=\frac{1}{2^{n-1}}\int_0^1(1-x^2)^{\frac{n-1}{2}}dx$;
  - $B(\frac{n+1}{2},\frac{n+1}{2})=\frac{1}{2^n}B(\frac{1}{2},\frac{n+1}{2})$.

  Using those, together with $B(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, we get

$$2\int_0^1(1-x^2)^{\frac{n-1}{2}}dx=B\left(\frac{1}{2},\frac{n+1}{2}\right)=\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}=\frac{\sqrt{\pi}\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}.$$

  Hence, since $\omega_n=(2\int_0^1(1-x^2)^{\frac{n-1}{2}}dx)\omega_{n-1}$,

$$\begin{aligned}\omega_n &= \frac{\sqrt{\pi}\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})}\omega_{n-1}=\frac{\sqrt{\pi}\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})}\cdot\frac{\sqrt{\pi}\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\omega_{n-2}\\ &= \cdots=\frac{\sqrt{\pi}\Gamma(\frac{n+1}{2})}{\Gamma(\frac{n+2}{2})}\cdot\frac{\sqrt{\pi}\Gamma(\frac{n}{2})}{\Gamma(\frac{n+1}{2})}\cdots\frac{\sqrt{\pi}\Gamma(\frac{3}{2})}{\Gamma(\frac{4}{2})}\omega_1=\frac{2\pi^{\frac{n}{2}}}{n\Gamma(\frac{n}{2})}.\end{aligned}$$

  We thus have $\omega_4=\frac{\pi^2}{2}$.

Volume of a Symmetric Ellipsoid

- An ellipse can be defined as the image of a closed circular disc under a non-singular affine transformation.
- A set in $\mathbb{R}^n$ is called an **ellipsoid** if it is the image of a closed ball under a non-singular affine transformation.
- Clearly every ellipsoid is a convex body.
- We find the volume of the symmetric ellipsoid

$$E = \{(x_1,\ldots,x_n) : (a_{11}x_1 + \cdots + a_{1n}x_n)^2 + \cdots + (a_{n1}x_1 + \cdots + a_{nn}x_n)^2 \leq r^2\},$$

  where $\boldsymbol{A} = [a_{ij}]$ is a real $n \times n$ matrix with non-zero determinant and $r > 0$.
- The image of E under the linear transformation that maps $\boldsymbol{x}$ in $\mathbb{R}^n$ to $\boldsymbol{Ax}$ is the closed ball rU.
- Thus, by a previous theorem, $|\det \boldsymbol{A}| v_n(E) = \omega_n r^n$.
- Hence $v_n(E) = \frac{\omega_n r^n}{|\det \boldsymbol{A}|}$.

Volume of a Polytope

Theorem

Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ be the outward unit normals to the facets of an n-polytope P in $\mathbb{R}^n$ corresponding to the facets $F_1, \ldots, F_m$. Let h be the support function of P. Then

$$v_n(P) = \frac{1}{n} \sum_{i=1}^{m} h(\boldsymbol{u}_i) v_{n-1}(F_i) \quad \text{and} \quad \sum_{i=1}^{m} v_{n-1}(F_i) \boldsymbol{u}_i = \mathbf{0}.$$

- Suppose first that the origin is an interior point of P. For each $i = 1, \ldots, m$, let $C_i = \text{conv}(\{\mathbf{0}\} \cup F_i)$. Then $P = C_1 \cup \cdots \cup C_m$ and $C_i \cap C_j = \text{conv}(\{\mathbf{0}\} \cup (F_i \cap F_j))$, for $i, j = 1, \ldots, m$. So $C_i \cap C_j$ $(i \neq j)$ is at most $(n-1)$-dimensional. Thus, $v_n(C_i \cap C_j) = 0$, $i \neq j$. By a previous corollary, $v_n(P) = v_n(C_1 \cup \cdots \cup C_m) = v_n(C_1) + \cdots + v_n(C_m)$. But, by the formula obtained earlier for the volume of a conical set, for $i = 1, \ldots, m$, $v_n(C_i) = \frac{1}{n} h(\boldsymbol{u}_i) v_{n-1}(F_i)$. Hence, $v_n(P) = \frac{1}{n} \sum_{i=1}^{m} h(\boldsymbol{u}_i) v_{n-1}(F_i)$.

Volume of a Polytope (Cont'd)

- Denote by $\boldsymbol{a}$ the vector $\sum_{i=1}^{m} v_{n-1}(F_i)\boldsymbol{u}_i$.

  Choose $\lambda > 0$ small enough to ensure that the origin is an interior point of the polytope $P + \lambda\boldsymbol{a}$.

  Applying the formula established above for the volume of a polytope having the origin as an interior point, we deduce that

$$\begin{array}{rcl} v_n(P) & = & v_n(P+\lambda\boldsymbol{a}) \\ & = & \frac{1}{n}\sum_{i=1}^{m}(h_{P+\lambda\boldsymbol{a}}(\boldsymbol{u}_i))v_{n-1}(F_i+\lambda\boldsymbol{a}) \\ & = & \frac{1}{n}\sum_{i=1}^{m}(h(\boldsymbol{u}_i)+\lambda\boldsymbol{a}\cdot\boldsymbol{u}_i)v_{n-1}(F_i) \\ & = & \frac{1}{n}\sum_{i=1}^{m} h(\boldsymbol{u}_i)v_{n-1}(F_i)+\frac{1}{n}\lambda\boldsymbol{a}\cdot(\sum_{i=1}^{m} v_{n-1}(F_i)\boldsymbol{u}_i) \\ & = & v_n(P)+\frac{\lambda}{n}\|\boldsymbol{a}\|^2. \end{array}$$

  This shows that $\boldsymbol{a} = \sum_{i=1}^{m} v_{n-1}(F_i)\boldsymbol{u}_i = \boldsymbol{0}$, as required.

Volume of a Polytope (Cont'd)

- Consider now the general case when it is not assumed that the origin is an interior point of P.
  With each n-polytope P associate the vector $\sum_{i=1}^{m} v_{n-1}(F_i)\boldsymbol{u}_i$.
  Clearly, this vector is the same for all translates of P.
  But for any translate of P which has the origin as an interior point, this associated vector is the zero vector, by what we have just proved.
  Thus, $\sum_{i=1}^{m} v_{n-1}(F_i)\boldsymbol{u}_i = \mathbf{0}$.
  Finally, let $\boldsymbol{c} \in \mathbb{R}^n$ be such that the polytope $P+\boldsymbol{c}$ has the origin in its interior. Then, by the first part of the proof,

$$\begin{array}{rcl} v_n(P) & = & v_n(P+\boldsymbol{c}) \\ & = & \frac{1}{n}\sum_{i=1}^{m}(h_{P+\boldsymbol{c}}(\boldsymbol{u}_i))v_{n-1}(F_i+\boldsymbol{c}) \\ & = & \frac{1}{n}\sum_{i=1}^{m}(h(\boldsymbol{u}_i)+\boldsymbol{c}\cdot\boldsymbol{u}_i)v_{n-1}(F_i) \\ & = & \frac{1}{n}\sum_{i=1}^{m} h(\boldsymbol{u}_i)v_{n-1}(F_i)+\frac{\boldsymbol{c}}{n}\cdot(\sum_{i=1}^{m} v_{n-1}(F_i)\boldsymbol{u}_i) \\ & = & \frac{1}{n}\sum_{i=1}^{m} h(\boldsymbol{u}_i)v_{n-1}(F_i). \end{array}$$

Subsection 4

Mixed Volumes and Surface Area

Blocks and Balls

- Consider the following simple problem:

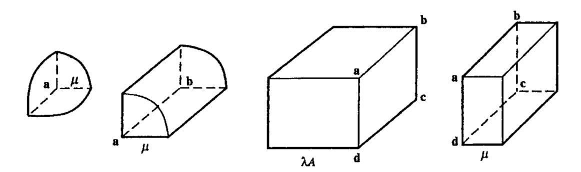
  What is the volume of the convex body $\lambda A + \mu B$, where A is the rectangular block (i.e., 3-orthotope) with edge lengths a, b, c defined by the equation

  $$A = \{(x, y, z) : 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c\},$$

  B is the closed unit ball in $\mathbb{R}^3$, and λ, μ are positive scalars?
- λA is a rectangular block with edge lengths $\lambda a, \lambda b, \lambda c$.
- μB is the closed ball of radius μ centered at the origin in $\mathbb{R}^3$.
- Thus $\lambda A + \mu B$ is the outer parallel body of λA at distance μ.

Blocks and Balls (Cont'd)

- We can see that $\lambda A + \mu B$ is the union of λA together with:



  - Six rectangular blocks (each with height μ and having a facet of λA as base);
  - Twelve quadrants of circular cylinders (each with base radius μ and having an edge of λA as axis);
  - Eight octants of balls (each with radius μ and having a vertex of λA as center).
- Any two different sets in this union meet in a set of volume zero.
- The figure shows one example of each, indicating their positions relative to λA.

Blocks and Balls (Volume)

- It is readily found that $v_3(\lambda A + \mu B)$ equals

$$(abc)\lambda^3 + 2(ab + bc + ca)\lambda^2\mu + \pi(a + b + c)\lambda\mu^2 + \frac{4\pi}{3}\mu^3,$$

  the four terms representing in order the volumes of:
  - λA;
  - the union of the six rectangular blocks;
  - the union of the twelve quadrants of circular cylinders;
  - the union of the eight octants of balls.
- Thus $v_3(\lambda A + \mu B)$ is a homogeneous polynomial of degree three in λ and μ with nonnegative coefficients.

Combinations of Polytopes and Outward Normals

Lemma

Let $C_1,\ldots,C_r$ be polytopes in $\mathbb{R}^n$ and let $\alpha_1,\ldots,\alpha_r>0$. Then $\alpha_1C_1+\cdots+\alpha_rC_r$ and $C_1+\cdots+C_r$ have the same dimension, and the sets of outward unit normals to the $(n-1)$-faces of the two polytopes are equal.

- The result is trivial when one of $C_1,\ldots,C_r$ is empty.

  Suppose, then, that $\boldsymbol{c}_1\in C_1$, ..., $\boldsymbol{c}_r\in C_r$. Let A be the flat $\mathrm{aff}(C_1+\cdots+C_r)$. Then $A-(\boldsymbol{c}_1+\cdots+\boldsymbol{c}_r)$ is a subspace of $\mathbb{R}^n$ containing each of the sets $C_1-\boldsymbol{c}_1,\ldots,C_r-\boldsymbol{c}_r$. Hence, it contains the set $\alpha_1(C_1-\boldsymbol{c}_1)+\cdots+\alpha_r(C_r-\boldsymbol{c}_r)$. It follows that $\alpha_1C_1+\cdots+\alpha_rC_r$ lies in the translate $A+(\alpha_1-1)\boldsymbol{c}_1+\cdots+(\alpha_r-1)\boldsymbol{c}_r$ of A. Hence the dimension of $\alpha_1C_1+\cdots+\alpha_rC_r$ does not exceed that of $C_1+\cdots+C_r$.

Combinations of Polytopes and Outward Normals (Cont'd)

- It follows, from what we have just proved, that the dimension of the set $\alpha_1^{-1}(\alpha_1 C_1) + \alpha_r^{-1}(\alpha_r C_r)$, i.e., $C_1 + \cdots + C_r$, does not exceed that of $\alpha_1 C_1 + \cdots + \alpha_r C_r$. Thus the polytopes $\alpha_1 C_1 + \cdots + \alpha_r C_r$ and $C_1 + \cdots + C_r$ have the same dimension.

  A unit vector $\boldsymbol{u}$ is an outward normal to some $(n-1)$-face of $\alpha_1 C_1 + \cdots + \alpha_r C_r$ if and only if the set $(\alpha_1 C_1 + \cdots + \alpha_r C_r)^{\boldsymbol{u}} = \alpha_1 C_1^{\boldsymbol{u}} + \cdots + \alpha_r C_r^{\boldsymbol{u}}$ has dimension $n-1$.

  By the first part of the proof, this occurs precisely when $C_1^{\boldsymbol{u}} + \cdots + C_r^{\boldsymbol{u}} = (C_1 + \cdots + C_r)^{\boldsymbol{u}}$ has dimension $n-1$.

  Therefore, $\boldsymbol{u}$ is an outward normal to some $(n-1)$-face of $\alpha_1 C_1 + \cdots + \alpha_r C_r$ if and only if it is an outward unit normal to some $(n-1)$-face of $C_1 + \cdots + C_r$.

Volume of Linear Combinations of Polytopes

Lemma

Let $A_1,\ldots,A_r$ be polytopes in $\mathbb{R}^n$. Then $v_n(\lambda_1 A_1+\cdots+\lambda_r A_r)$ is, for all $\lambda_1,\ldots,\lambda_r>0$, a homogeneous polynomial of degree n in $\lambda_1,\ldots,\lambda_r$, with nonnegative coefficients.

- We argue by induction on n. If $n=1$, then

$$v_1(\lambda_1 A_1+\cdots+\lambda_r A_r)=\lambda_1 v_1(A_1)+\cdots+\lambda_r v_r(A_r),\ \text{for } \lambda_1,\ldots,\lambda_r>0,$$

when none of $A_1,\ldots,A_r$ is empty, and is zero otherwise. This proves the lemma for the case $n=1$.

Suppose, then, that the assertion is true in $\mathbb{R}^{n-1}$, where $n\geq 2$.

If $A_1+\cdots+A_r$ has dimension less than n, then, by the preceding lemma, so too does $\lambda_1 A_1+\cdots+\lambda_r A_r$. Hence $v_n(\lambda_1 A_1+\cdots+\lambda_r A_r)$ is zero for all $\lambda_1,\ldots,\lambda_r>0$, and the assertion is true in this case.

Volume of Linear Combinations of Polytopes (Cont'd)

- Suppose now that $A_1+\cdots+A_r$ has dimension n.
  Since v_n-volumes are preserved by translations, we can assume that each of the polytopes $A_1,\ldots,A_r$ contains the origin.
  Let $\boldsymbol{u}_1,\ldots,\boldsymbol{u}_m$ be the outward unit normals to the facets of $A_1+\cdots+A_r$. For each $i=1,\ldots,m$, the polytopes $A_1^{\boldsymbol{u}_i},\ldots,A_r^{\boldsymbol{u}_i}$ lie in parallel hyperplanes of $\mathbb{R}^n$. Since v_{n-1}-volumes are preserved by translations, in calculating $v_{n-1}(\lambda_1 A_1^{\boldsymbol{u}_i}+\cdots+\lambda_r A_r^{\boldsymbol{u}_i})$, we can assume that $A_1^{\boldsymbol{u}_i},\ldots,A_r^{\boldsymbol{u}_i}$ lie in the same hyperplane of $\mathbb{R}^n$.
  By identifying this hyperplane with $\mathbb{R}^{n-1}$ and using the induction hypothesis, we deduce the existence of a homogeneous polynomial p_i of degree $n-1$ in $\lambda_1,\ldots,\lambda_r$ with non-negative coefficients such that, for all $\lambda_1,\ldots,\lambda_r>0$,

$$\begin{aligned} v_{n-1}((\lambda_1 A_1+\cdots+\lambda_r A_r)^{\boldsymbol{u}_i}) &= v_{n-1}(\lambda_1 A_1^{\boldsymbol{u}_i}+\cdots+\lambda_r A_r^{\boldsymbol{u}_i}) \\ &= p_i(\lambda_1,\ldots,\lambda_r). \end{aligned}$$

Volume of Linear Combinations of Polytopes (Cont'd)

- Let $\lambda_1,\ldots,\lambda_r > 0$.

  By the preceding lemma, the facets of $\lambda_1 A_1 + \cdots + \lambda_r A_r$ are $(\lambda_1 A_1 + \cdots + \lambda_r A_r)^{\boldsymbol{u}_i}$ with corresponding outward unit normals $\boldsymbol{u}_i$.

  A previous theorem shows that

$$\begin{aligned} & v_n(\lambda_1 A_1 + \cdots + \lambda_r A_r) \\ & = \tfrac{1}{n}\textstyle\sum_{i=1}^m (h_{\lambda_1 A_1 + \cdots + \lambda_r A_r}(\boldsymbol{u}_i)) v_{n-1}((\lambda_1 A_1 + \cdots + \lambda_r A_r)^{\boldsymbol{u}_i}) \\ & = \tfrac{1}{n}\textstyle\sum_{i=1}^m (\lambda_1 h_{A_1}(\boldsymbol{u}_i) + \cdots + \lambda_r h_{A_r}(\boldsymbol{u}_i)) p_i(\lambda_1,\ldots,\lambda_r). \end{aligned}$$

  Thus $v_n(\lambda_1 A_1 + \cdots + \lambda_r A_r)$ is a homogeneous polynomial of degree n in $\lambda_1,\ldots,\lambda_r$ with nonnegative coefficients.

  This completes the proof by induction.

Volume of Linear Combinations of Compact Convex Sets

Theorem

Let $A_1, \ldots, A_r$ be compact convex sets in $\mathbb{R}^n$. Then $v_n(\lambda_1 A_1 + \cdots + \lambda_r A_r)$ is, for all $\lambda_1, \ldots, \lambda_r \geq 0$, a homogeneous polynomial of degree n in $\lambda_1, \ldots, \lambda_r$ with non-negative coefficients.

- We assume that the sets $A_1, \ldots, A_r$ are non-empty. For each $i = 1, \ldots, r$, let $A_i^1, A_i^2, \ldots, A_i^j, \ldots$ be a sequence of polytopes converging to A_i. By the preceding lemma, for each $j = 1, 2, \ldots$, there exist non-negative scalars $a^j_{i_1 \ldots i_n}$ for $i_1, \ldots, i_n = 1, \ldots, r$, such that, for all $\lambda_1, \ldots, \lambda_r > 0$,

$$v_n(\lambda_1 A_1^j + \cdots + \lambda_r A_r^j) = \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} a^j_{i_1 \ldots i_n} \lambda_{i_1} \cdots \lambda_{i_n}.$$

Since the r sequences of polytopes considered above are convergent, there is a closed ball B in $\mathbb{R}^n$ such that $A_i^j \subseteq B$ for $i = 1, \ldots, r$ and $j = 1, 2, \ldots$. Setting $\lambda_1 = 1$, ..., $\lambda_r = 1$, we deduce $a^j_{i_1 \ldots i_n} \leq r^n v_n(B)$.

Linear Combinations of Compact Convex Sets (Cont'd)

- Since every bounded sequence of real numbers contains a convergent subsequence, it follows that there is a subsequence $k_1, k_2, \ldots, k_j, \ldots$ of $1, 2, \ldots, j, \ldots$ and nonnegative scalars $a_{i_1 \ldots i_n}$ for $i_1, \ldots, i_n = 1, \ldots, r$, such that $a^{k_j}_{i_1 \ldots i_n} \to a_{i_1 \ldots i_n}$ as $j \to \infty$ for $i_1, \ldots, i_n = 1, \ldots, r$. A previous result shows that, for $\lambda_1, \ldots, \lambda_r > 0$,

$$\lambda_1 A_1^{k_j} + \cdots + \lambda_r A_r^{k_j} \to \lambda_1 A_1 + \cdots + \lambda_r A_r \text{ as } j \to \infty.$$

The continuity of v_n now shows that

$$v_n(\lambda_1 A_1^{k_j} + \cdots + \lambda_r A_r^{k_j}) \to v_n(\lambda_1 A_1 + \cdots + \lambda_r A_r), \text{ as } j \to \infty.$$

But from the displayed equation of the preceding slide

$$v_n(\lambda_1 A_1^{k_j} + \cdots + \lambda_r A_r^{k_j}) \to \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} a_{i_1 \ldots i_n} \lambda_{i_1} \cdots \lambda_{i_n} \text{ as } j \to \infty.$$

Linear Combinations of Compact Convex Sets (Cont'd)

- Thus, for all $\lambda_1,\ldots,\lambda_r > 0$,

$$v_n(\lambda_1 A_1 + \cdots + \lambda_r A_r) = \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} a_{i_1 \ldots i_n} \lambda_{i_1} \cdots \lambda_{i_n}.$$

Suppose finally that $\lambda_1,\ldots,\lambda_r \geq 0$.

By what we have just proved, for each $\varepsilon > 0$,

$$v_n((\lambda_1+\varepsilon)A_1 + \cdots + (\lambda_r+\varepsilon)A_r) = \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} a_{i_1 \ldots i_n} (\lambda_{i_1}+\varepsilon) \cdots (\lambda_{i_n}+\varepsilon).$$

Letting $\varepsilon \to 0^+$ in the last equation, we find that

$$v_n(\lambda_1 A_1 + \cdots + \lambda_r A_r) = \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} a_{i_1 \ldots i_n} \lambda_{i_1} \cdots \lambda_{i_n}.$$

Homogeneous Polynomials

- Each homogeneous polynomial $p(\lambda_1,\ldots,\lambda_r)$ of degree n can be uniquely represented in the form

$$p(\lambda_1,\ldots,\lambda_r)=\sum_{\alpha_1+\cdots+\alpha_r=n}\frac{n!}{\alpha_1!\cdots\alpha_r!}a_{\alpha_1\cdots\alpha_r}\lambda_1^{\alpha_1}\cdots\lambda_r^{\alpha_r}.$$

- For integers $i_1,i_2,\ldots,i_n$ lying in $\{1,\ldots,r\}$, put

$$v_{i_1 i_2\ldots i_n}=a_{\alpha_1\alpha_2\ldots\alpha_r},\text{ where } \lambda_{i_1}\lambda_{i_2}\cdots\lambda_{i_n}=\lambda_1^{\alpha_1}\lambda_2^{\alpha_2}\cdots\lambda_r^{\alpha_r}.$$

- Then:
  (i) $v_{i_1 i_2\ldots i_n}$ remains unchanged when $i_1,i_2,\ldots,i_n$ are permuted;
  (ii) $p(\lambda_1,\ldots,\lambda_r)=\sum_{i_1=1}^r\cdots\sum_{i_n=1}^r v_{i_1\ldots i_n}\lambda_{i_1}\cdots\lambda_{i_n}$.
- Moreover, the $v_{i_1\ldots i_n}$ are uniquely determined by (i) and (ii).

Mixed Volumes

- When
$$p(\lambda_1,\ldots,\lambda_r)=v_n(\lambda_1A_1+\cdots+\lambda_rA_r),$$
where $A_1,\ldots,A_r$ are compact convex sets in $\mathbb{R}^n$ and $\lambda_1,\ldots,\lambda_r\geq 0$, the numbers $v_{i_1\ldots i_n}$ are called the **mixed volumes** of $A_1,\ldots,A_r$.
Example: Consider again the example studied previously in which A was a rectangular block with edge lengths a,b,c and B was the closed unit ball.
We found that, for $\lambda,\mu\geq 0$,
$$v_2(\lambda A+\mu B)=(abc)\lambda^3+2(ab+bc+ca)\lambda^2\mu+(a+b+c)\pi\lambda\mu^2+\frac{4\pi}{3}\mu^3.$$
It follows easily from this equation that the mixed volumes of A,B are:
$$\begin{aligned}&v_{111}=abc,\quad v_{222}=\tfrac{4\pi}{3},\\ &v_{112}=v_{121}=v_{211}=\tfrac{2}{3}(ab+bc+ca)\\ &v_{122}=v_{212}=v_{221}=\tfrac{1}{3}(a+b+c)\pi.\end{aligned}$$

Mixed Volumes' Dependence on the Sets

- The mixed volumes of compact convex sets $A_1,\ldots,A_r$ are determined by the function $v_n(\lambda_1 A_1+\cdots+\lambda_r A_r)$ of $\lambda_1,\ldots,\lambda_r$.
- It is tempting to assume that any particular mixed volume $v_{i_1\ldots i_n}$ depends only upon the sets $A_{i_1},\ldots,A_{i_n}$.
- For example, when none of the sets $A_1,\ldots,A_r$ is empty, it is easy to see that $v_{1\ldots 1}=v_n(A_1)$, which only depends upon A_1.
- If, however, even one of the sets $A_1,\ldots,A_r$ is empty, then all the mixed volumes $v_{i_1\ldots i_n}$ are zero.
- We will show that, when none of the sets $A_1,\ldots,A_r$ is empty, the mixed volume $v_{i_1\ldots i_n}$ does indeed depend only upon the sets $A_{i_1},\ldots,A_{i_n}$.

Scalar Associated with Tuple of Sets

- With each n-tuple $(A_1,\ldots,A_n)$ of non-empty compact convex sets in $\mathbb{R}^n$ we associate a non-negative scalar $v(A_1,\ldots,A_n)$ as follows.
- Suppose that there are exactly s distinct sets occurring in $(A_1,\ldots,A_n)$.
- We can assume, relabeling the A_i's if necessary, that the sets $A_1,\ldots,A_s$ are distinct.
- For $i=1,\ldots,s$, let α_i be the number of times which the set A_i occurs in $(A_1,\ldots,A_n)$. Then $\alpha_1+\cdots+\alpha_s=n$. For all $\lambda_1,\ldots,\lambda_s\geq 0$,

  $$v_n(\lambda_1 A_1+\cdots+\lambda_s A_s)=\sum_{i_1=1}^{s}\cdots\sum_{i_n=1}^{s} v_{i_1\ldots i_n}\lambda_{i_1}\ldots\lambda_{i_n},$$

  where the $v_{i_1\ldots i_n}$ are the mixed volumes of $A_1,\ldots,A_s$.
- We now define

  $$v(A_1,\ldots,A_n)=v_{\underbrace{1\ldots1}_{\alpha_1}\ldots\underbrace{s\ldots s}_{\alpha_s}}.$$
- This determines $v(A_1,\ldots,A_n)$ uniquely and in such a way that it remains unchanged when $A_1,\ldots,A_n$ are permuted.

The Scalar v as a Mixed Volume

Theorem

Let $A_1, \ldots, A_r$ be non-empty compact convex sets in $\mathbb{R}^n$. Then, for all $\lambda_1, \ldots, \lambda_r \geq 0$,

$$v_n(\lambda_1 A_1 + \cdots + \lambda_r A_r) = \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} v(A_{i_1}, \ldots, A_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}.$$

- We argue by induction on the **redundancy number** of the finite sequence $A_1, \ldots, A_r$; This is defined to be the non-negative integer $r - s$, where s is the number of distinct sets in the sequence.
  The sequence has redundancy number zero when all of its terms are different and $r - 1$ when all of its terms are the same.
  Suppose first that the sequence $A_1, \ldots, A_r$ has redundancy number zero, i.e., all of its terms are different. For all $\lambda_1, \ldots, \lambda_r \geq 0$,

  $$v_n(\lambda_1 A_1 + \cdots + \lambda_r A_r) = \sum_{i_1=1}^{r} \cdots \sum_{i_n=1}^{r} v_{i_1 \ldots i_n} \lambda_{i_1} \cdots \lambda_{i_n},$$

  where the $v_{i_1 \ldots i_n}$ are the mixed volumes of $A_1, \ldots, A_r$.

The Scalar v as a Mixed Volume (Cont'd)

- Consider a particular mixed volume $v_{i_1 \ldots i_n}$ and the corresponding n-tuple $(A_{i_1}, \ldots, A_{i_n})$. Suppose that there are exactly s distinct sets occurring in this last n-tuple, say $A_1, \ldots, A_s$. For $i = 1, \ldots, s$, let α_i be the number of times which the set A_i occurs in $(A_{i_1}, \ldots, A_{i_n})$. Then $\alpha_1 + \cdots + \alpha_s = n$. For all $\lambda_1, \ldots, \lambda_s \geq 0$,

$$v_n(\lambda_1 A_1 + \cdots + \lambda_s A_s) = \sum_{i_1=1}^{s} \cdots \sum_{i_n=1}^{s} v_{i_1 \ldots i_n} \lambda_{i_1} \cdots \lambda_{i_n}.$$

  By the definition of $v(A_{i_1}, \ldots, A_{i_n})$,

$$v(A_{i_1}, \ldots, A_{i_n}) = v_{\underbrace{1 \ldots 1}_{\alpha_1} \ldots \underbrace{s \ldots s}_{\alpha_s}} = v_{i_1 \ldots i_n}.$$

  Here we have used the fact that all the r sets $A_1, \ldots, A_r$ are different. Thus the assertion is true for the case of redundancy number zero.

The Scalar v as a Mixed Volume (Cont'd)

- Suppose next that the assertion is true for sequences with redundancy number m, where $m \geq 0$.

  Let the sequence $A_1, \ldots, A_r$ have redundancy number $m+1$. Then at least two terms of this sequence must be equal, say $A_{r-1} = A_r$. Since $A_1, \ldots, A_{r-1}$ has redundancy number m, the induction hypothesis shows that, for all $\lambda_1, \ldots, \lambda_{r-1} \geq 0$,

  $$v_n(\lambda_1 A_1 + \cdots + \lambda_{r-1} A_{r-1}) = \sum_{i_1=1}^{r-1} \cdots \sum_{i_n=1}^{r-1} v(A_{i_1}, \ldots, A_{i_n}) \lambda_{i_1} \cdots \lambda_{i_n}.$$

  Let $v_{i_1 \ldots i_n}$ be a typical mixed volume for the sequence $A_1, \ldots, A_r$. Let $\alpha_1, \ldots, \alpha_r$ be non-negative integers such that $i_1, \ldots, i_n$ is a rearrangement of the sequence $\underbrace{1, \ldots, 1}_{\alpha_1}, \ldots, \underbrace{r, \ldots, r}_{\alpha_r}$. Then the coefficient of the term $\lambda_1^{\alpha_1} \cdots \lambda_r^{\alpha_r}$ in the polynomial $v_n(\lambda_1 A_1 + \cdots + \lambda_r A_r)$ is $\frac{n!}{\alpha_1! \cdots \alpha_r!} v_{\underbrace{1 \ldots 1}_{\alpha_1} \ldots \underbrace{r \ldots r}_{\alpha_r}}$.

The Scalar v as a Mixed Volume (Cont'd)

- Now $v_n(\lambda_1 A_1 + \cdots + \lambda_r A_r)$ can be obtained from $v_n(\lambda_1 A_1 + \cdots + \lambda_{r-1} A_{r-1})$ by replacing λ_{r-1} with $\lambda_{r-1} + \lambda_r$.
  Thus the coefficient of $\lambda_1^{\alpha_1} \cdots \lambda_r^{\alpha_r}$ in $v_n(\lambda_1 A_1 + \cdots + \lambda_r A_r)$ is also the product of the coefficient of the term $\lambda_1^{\alpha_1} \cdots \lambda_{r-2}^{\alpha_{r-2}} \lambda_{r-1}^{\alpha_{r-1}+\alpha_r}$ in $v_n(\lambda_1 A_1 + \cdots + \lambda_{r-1} A_{r-1})$ with the coefficient of the term $\lambda_{r-1}^{\alpha_{r-1}} \lambda_r^{\alpha_r}$ in $(\lambda_{r-1} + \lambda_r)^{\alpha_{r-1}+\alpha_r}$, i.e., the product

$$\frac{n!}{\alpha_1! \cdots \alpha_{r-2}!(\alpha_{r-1}+\alpha_r)!} v(\underbrace{A_1}_{\alpha_1}, \ldots, \underbrace{A_{r-2}}_{\alpha_{r-2}}, \underbrace{A_{r-1}}_{\alpha_{r-1}+\alpha_r}) \frac{(\alpha_{r-1}+\alpha_r)!}{(\alpha_{r-1})!(\alpha_r)!}$$
$$= \frac{n!}{\alpha_1! \cdots \alpha_r!} v(\underbrace{A_1}_{\alpha_1}, \ldots, \underbrace{A_r}_{\alpha_r}).$$

  The two expressions which we have found for the coefficient of $\lambda_1^{\alpha_1} \cdots \lambda_r^{\alpha_r}$ in $v_n(\lambda_1 A_1 + \cdots + \lambda_r A_r)$ must be equal. So

$$v_{i_1 \ldots i_n} = v_{\underbrace{1 \ldots 1}_{\alpha_1}, \ldots, \underbrace{r \ldots r}_{\alpha_r}} = v(\underbrace{A_1}_{\alpha_1}, \ldots, \underbrace{A_r}_{\alpha_r}) = v(A_{i_1}, \ldots, A_{i_n}).$$

Restricted Linearity of Mixed Volume

Theorem

Let $A_1', A_1, A_2, \ldots, A_n$ be non-empty compact convex sets in $\mathbb{R}^n$. Let $\alpha_1', \alpha_1 \geq 0$. Then

$$\begin{aligned} v(\alpha_1' A_1' + \alpha_1 A_1, A_2, \ldots, A_n) &= \alpha_1' v(A_1', A_2, \ldots, A_n) \\ &+ \alpha_1 v(A_1, A_2, \ldots, A_n). \end{aligned}$$

- The coefficient of $\lambda_1 \cdots \lambda_n$ in $v_n(\lambda_1(\alpha_1' A_1' + \alpha_1 A_1) + \lambda_2 A_2 + \cdots + \lambda_n A_n)$ is $n! v(\alpha_1' A_1' + \alpha_1 A_1, A_2, \ldots, A_n)$, whereas in $v_n((\lambda_1 \alpha_1') A_1' + (\lambda_1 \alpha_1) A_1 + \lambda_2 A_2 + \cdots + \lambda_n A_n)$ it is $n! \alpha_1' v(A_1', A_2, \ldots, A_n) + n! \alpha_1 v(A_1, A_2, \ldots, A_n)$. Since the two polynomials are identical, the two coefficients must be equal, whence

$$\begin{aligned} v(\alpha_1' A_1' + \alpha_1 A_1, A_2, \ldots, A_n) &= \alpha_1' v(A_1', A_2, \ldots, A_n) \\ &+ \alpha_1 v(A_1, A_2, \ldots, A_n). \end{aligned}$$

Convergence and Coefficients of Polynomials

Lemma

Let m be a positive integer. For each $i = 0, 1, 2, \ldots$, let

$$P_i(x) = a_{im}x^m + \cdots + a_{i1}x + a_{i0}$$

be a real polynomial. Suppose that, for each $x \geq 0$, $P_i(x) \to P_0(x)$ as $i \to \infty$. Then $a_{ij} \to a_{0j}$ as $i \to \infty$, for $j = 0, 1, \ldots, m$.

- The $m+1$ vectors $\boldsymbol{a}_\lambda = (\lambda^m, \ldots, \lambda, 1)$ for $\lambda = 0, 1, \ldots, m$ are linearly independent. So they form a basis for $\mathbb{R}^{m+1}$. Thus, there are scalars $\mu_0, \mu_1, \ldots, \mu_m$ such that $(1, 0, \ldots, 0) = \mu_0 \boldsymbol{a}_0 + \mu_1 \boldsymbol{a}_1 + \cdots + \mu_m \boldsymbol{a}_m$. Writing those out, we get

$$\begin{array}{rrrlrcl}
\mu_0 & +\mu_1 & +\mu_2 & +\cdots & +\mu_m & = & 0; \\
0\mu_0 & +1\mu_1 & +2\mu_2 & +\cdots & +m\mu_m & = & 0; \\
\cdots & & & & & & \\
0^m\mu_0 & +1^m\mu_1 & +2^m\mu_2 & +\cdots & +m^m\mu_m & = & 1.
\end{array}$$

Convergence and Coefficients of Polynomials (Cont'd)

- For fixed i, multiplying the jth row by a_{ij},

$$\begin{array}{rrrrrcl} \mu_0 a_{i0} & +\mu_1 a_{i0} & +\mu_2 a_{i0} & +\cdots & +\mu_m a_{i0} & = & 0; \\ 0\mu_0 a_{i1} & +1\mu_1 a_{i1} & +2\mu_2 a_{i1} & +\cdots & +m\mu_m a_{i1} & = & 0; \\ \vdots & & & & & & \\ 0^m\mu_0 a_{im} & +1^m\mu_1 a_{im} & +2^m\mu_2 a_{im} & +\cdots & +m^m\mu_m a_{im} & = & a_{im}. \end{array}$$

Adding vertically, we get

$$\mu_0 P_i(0) + \mu_1 P_i(1) + \cdots + \mu_m P_i(m) = a_{im}.$$

By the hypothesis,

$$\begin{aligned} &\mu_0 P_i(0) + \mu_1 P_i(1) + \cdots + \mu_m P_i(m) \\ &\qquad \to \mu_0 P_0(0) + \mu_1 P_0(1) + \cdots + \mu_m P_0(m) \text{ as } i \to \infty. \end{aligned}$$

Thus, $a_{im} \to a_{0m}$ as $i \to \infty$.
Similarly, we can see that $a_{ij} \to a_{0j}$ as $i \to \infty$ for $j = 0, 1, \ldots, m-1$.

Continuity of Mixed Volumes

Theorem (Continuity of Mixed Volumes)

For each $j=1,\ldots,n$, let $A_j^1,A_j^2,\ldots,A_j^i,\ldots$ be a sequence of non-empty compact convex sets converging to a non-empty compact convex set A_j^0 in $\mathbb{R}^n$. Then $v(A_1^i,\ldots,A_n^i)\to v(A_1^0,\ldots,A_n^0)$ as $i\to\infty$.

- For $i=0,1,2,\ldots$, and for $\lambda_1,\ldots,\lambda_n\geq 0$, $v_n(\lambda_1 A_1^i+\cdots+\lambda_n A_n^i)=Q_i(\lambda_1,\ldots,\lambda_n)$ say, is a real homogeneous polynomial of degree n in $\lambda_1,\ldots,\lambda_n$. Since v_n is continuous, for all $\lambda_1,\ldots,\lambda_n\geq 0$, $Q_i(\lambda_1,\ldots,\lambda_n)\to Q_0(\lambda_1,\ldots,\lambda_n)$ as $i\to\infty$. Choose a positive integer r so large that the coefficient of $x^r x^{r^2}\cdots x^{r^n}$ in the real polynomial $P_i(x)=Q_i(x^r,x^{r^2},\ldots,x^{r^n})$ of the single variable x is $n!v(A_1^i,\ldots,A_n^i)$. For each $x\geq 0$, $P_i(x)\to P_0(x)$ as $i\to\infty$. We deduce from the lemma that $n!v(A_1^i,\ldots,A_n^i)\to n!v(A_1^0,\ldots,A_n^0)$ as $i\to\infty$. The desired result is immediate.

Volumes of Polytopes and Faces

Theorem

Let $\boldsymbol{u}_1,\ldots,\boldsymbol{u}_m$ be the outward unit normals to the $(n-1)$-faces of a polytope P in $\mathbb{R}^n$ corresponding to faces $F_1,\ldots,F_m$, respectively. Then, for any non-empty compact convex set A in $\mathbb{R}^n$ with support function h,

$$\lim_{\lambda\to 0^+}\frac{v_n(P+\lambda A)-v_n(P)}{\lambda}=\sum_{i=1}^m h(\boldsymbol{u}_i)v_{n-1}(F_i).$$

- Both sides of the above equation are unchanged in value if A is replaced by one of its translates. The non-trivial part of this assertion follows from a previous theorem.

  We can, therefore, assume that A contains the origin.

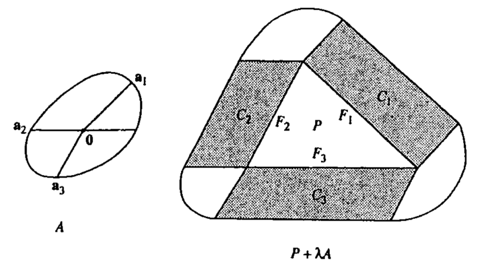
Volumes of Polytopes and Faces (Cont'd)

- Consider first the case when P has dimension n. We begin by showing that, for all $\lambda > 0$,

$$v_n(P+\lambda A) - v_n(P) \geq \sum_{i=1}^{m} \lambda h(\boldsymbol{u}_i) v_{n-1}(F_i).$$

For each $i = 1, \ldots, m$, let $\boldsymbol{a}_i \in A$ satisfy $\boldsymbol{u}_i \cdot \boldsymbol{a}_i = h(\boldsymbol{u}_i)$. Define a convex subset C_i of $P + \lambda A$ by the equation

$$C_i = \mathrm{ri} F_i + \lambda\{\mu \boldsymbol{a}_i : 0 \leq \mu \leq 1\}.$$



Volumes of Polytopes and Faces (Cont'd)

- The sets $C_1, \ldots, C_m$ are pairwise disjoint.

  If they were not, there would exist points $\boldsymbol{f}_i$ in $\text{ri}F_i$, $\boldsymbol{f}_j$ in $\text{ri}F_j$ and scalars $\theta_i, \theta_j \geq 0$, with $i \neq j$, satisfying $\boldsymbol{f}_i + \theta_i \boldsymbol{a}_i = \boldsymbol{f}_j + \theta_j \boldsymbol{a}_j$. Since P is n-dimensional, $\boldsymbol{f}_j \not\in F_i$. Hence $\boldsymbol{u}_i \cdot \boldsymbol{f}_j < \boldsymbol{u}_i \cdot \boldsymbol{f}_i$. It follows easily from the definition of $h(\boldsymbol{u}_i)$ that $\boldsymbol{u}_i \cdot \boldsymbol{a}_j \leq \boldsymbol{u}_i \cdot \boldsymbol{a}_i$. Hence

$$\boldsymbol{u}_i \cdot \boldsymbol{f}_i + \theta_i \boldsymbol{u}_i \cdot \boldsymbol{a}_i = \boldsymbol{u}_i \cdot \boldsymbol{f}_j + \theta_j \boldsymbol{u}_i \cdot \boldsymbol{a}_j < \boldsymbol{u}_i \cdot \boldsymbol{f}_i + \theta_j \boldsymbol{u}_i \cdot \boldsymbol{a}_i.$$

  Since A contains the origin, $h(\boldsymbol{u}_i) = \boldsymbol{u}_i \cdot \boldsymbol{a}_i \geq 0$. So $\theta_i < \theta_j$.

  By symmetry, $\theta_j < \theta_i$.

  This contradiction shows that the sets $C_1, \ldots, C_m$ are pairwise disjoint.

Volumes of Polytopes and Faces (Cont'd)

- For each $i = 1, \ldots, m$, $v_n(C_i \cap P) = 0$ and

$$v_n(C_i) = v_n(\text{cl}\, C_i) = |\lambda(\boldsymbol{u}_i \cdot \boldsymbol{a}_i) v_{n-1}(F_i)| = \lambda h(\boldsymbol{u}_i) v_{n-1}(F_i).$$

We can thus deduce that

$$v_n(P + \lambda A) - v_n(P) \geq v_n(C_1) + \cdots + v_n(C_m) = \sum_{i=1}^{m} \lambda h(\boldsymbol{u}_i) v_{n-1}(F_i).$$

We upper bound $v_n(P + \lambda A) - v_n(P)$ by showing that, for $\lambda > 0$,

$$P + \lambda A \subseteq P \cup \text{cl}\, C_1 \cup \cdots \cup \text{cl}\, C_m \cup (S)_{\lambda s},$$

where S is the union of the $(n-2)$-dimensional faces of P and s is the diameter of A.

To do this, we let $\boldsymbol{x}$ be in $P + \lambda A$, but not in any of P, $\text{cl}\, C_1$, ..., $\text{cl}\, C_m$, and show that it is in $(S)_{\lambda s}$. Let $\boldsymbol{f}$ be the nearest point of P to $\boldsymbol{x}$. Then $\boldsymbol{f} \in F_i$ for some $i = 1, \ldots, m$.

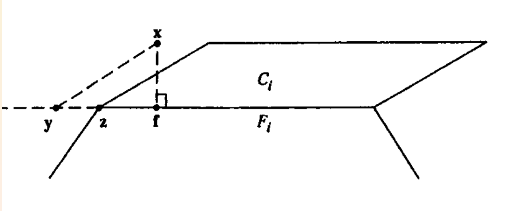
  - If $\boldsymbol{f} \in \text{rebd}\, F_i$, then $\boldsymbol{f} \in S$. So $\boldsymbol{x} \in (S)_{\lambda s}$.
  - If $\boldsymbol{f} \in \text{ri}\, F_i$, $\boldsymbol{x} = \boldsymbol{f} + \alpha \boldsymbol{u}_i$, for some $\alpha > 0$. Since $\boldsymbol{x} \in P + \lambda A$, $\alpha \leq \lambda h(\boldsymbol{u}_i)$.

Volumes of Polytopes and Faces (Cont'd)

- Let $\boldsymbol{y} = \boldsymbol{x} - \frac{\alpha \boldsymbol{a}_i}{h(\boldsymbol{u}_i)}$.

  Since $\alpha \le \lambda h(\boldsymbol{u}_i)$, $\boldsymbol{x} \notin \mathrm{cl} C_i$ shows that $\boldsymbol{y} \notin F_i$.

  Further, $\boldsymbol{u}_i \cdot \boldsymbol{y} = \boldsymbol{u}_i \cdot \boldsymbol{x} - \alpha = \boldsymbol{u}_i \cdot \boldsymbol{f} + \alpha - \alpha = \boldsymbol{u}_i \cdot \boldsymbol{f}$. So $\boldsymbol{y} \in \mathrm{aff} F_i$. Thus, for some β, $0 < \beta < 1$, $\boldsymbol{z} = (1-\beta)\boldsymbol{y} + \beta \boldsymbol{f}$ lies in $\mathrm{rebd} F_i$, and hence in S.



  Now we get

$$\begin{aligned} \|\boldsymbol{x} - \boldsymbol{z}\| &= \|\boldsymbol{x} - \boldsymbol{y} + \beta \boldsymbol{y} - \beta \boldsymbol{f}\| = \|(1-\beta)(\boldsymbol{x} - \boldsymbol{y}) + \beta(\boldsymbol{x} - \boldsymbol{f})\| \\ &= \le (1-\beta)\|\boldsymbol{x} - \boldsymbol{y}\| + \beta \|\boldsymbol{x} - \boldsymbol{f}\| \le (1-\beta)\lambda s + \beta \lambda s = \lambda s. \end{aligned}$$

  Hence $\boldsymbol{x} \in (S)_{\lambda s}$. This establishes the claim.

Volumes of Polytopes and Faces (Cont'd)

- Now we have

$$v_n(P+\lambda A) \le v_n(P) + v_n(\mathrm{cl}\, C_1) + \cdots + v_n(\mathrm{cl}\, C_m) + v_n((S)_{\lambda s}).$$

Combining this inequality with the one obtained previously, we deduce that, for $\lambda > 0$,

$$\sum_{i=1}^m h(\boldsymbol{u}_i) v_{n-1}(F_i) \le \frac{v_n(P+\lambda A) - v_n(P)}{\lambda} \le \sum_{i=1}^m h(\boldsymbol{u}_i) v_{n-1}(F_i) + \frac{1}{\lambda} v_n((S)_{\lambda s}).$$

By a previous theorem, $\frac{v_n((S)_{\lambda s})}{\lambda} \to 0$ as $\lambda \to 0^+$. Thus

$$\lim_{\lambda \to 0^+} \frac{v_n(P+\lambda A) - v_n(P)}{\lambda} = \sum_{i=1}^m h(\boldsymbol{u}_i) v_{n-1}(F_i).$$

This completes the proof for the case when P is n-dimensional.

Volumes of Polytopes and Faces (Cont'd)

- Suppose next that P has dimension $n-1$.

  Then $m=2$, $\boldsymbol{u}_1=\boldsymbol{u}$, $\boldsymbol{u}_2=-\boldsymbol{u}$, $F_1=P$ and $F_2=P$, where $\boldsymbol{u}$ is a unit normal to the hyperplane containing P. The proof in this case is the same as that just given, except that the sets C_1 and C_2 are not disjoint but meet in the set $\text{ri}P$, which has v_n-volume zero.

  When P has dimension less than $n-1$, the assertion of the theorem is assumed to mean that

$$\lim_{\lambda\to 0^+}\frac{v_n(P+\lambda A)-v_n(P)}{\lambda}=\lim_{\lambda\to 0^+}\frac{v_n(P+\lambda A)}{\lambda}=0.$$

  This is clear, since $P+\lambda A\subseteq(P)_{\lambda s}$, where s is the diameter of A and by a previous theorem $\lim_{\lambda\to 0^+}\frac{v_n((P)_{\lambda s})}{\lambda}=0$.

A Consequence

Corollary

Let $\boldsymbol{u}_1,\ldots,\boldsymbol{u}_m$ be the outward unit normals to the $(n-1)$-faces of a non-empty polytope P in $\mathbb{R}^n$ corresponding to faces $F_1,\ldots,F_m$, respectively. Then, for any non-empty compact convex set A in $\mathbb{R}^n$ with support function h,

$$v(A,P,\ldots,P)=\frac{1}{n}\sum_{i=1}^{m}h(\boldsymbol{u}_i)v_{n-1}(F_i).$$

- For all $\lambda>0$, $v_n(P+\lambda A)=\sum_{i=0}^{n}\binom{n}{i}v(\underbrace{A,\ldots,A}_{i},\underbrace{P,\ldots,P}_{n-i})\lambda^i$. So

$$v_n(P+\lambda A)-v_n(P)=nv(A,P,\ldots,P)\lambda+\sum_{i=2}^{n}\binom{n}{i}v(\underbrace{A,\ldots,A}_{i},\underbrace{P,\ldots,P}_{n-i})\lambda^i.$$

Thus $\lim_{\lambda\to 0^+}\frac{v_n(P+\lambda A)-v_n(P)}{\lambda}=nv(A,P,\ldots,P)$.
The corollary now follows from the theorem.

Expressing v In Terms of Normals

Theorem

Let $P_2, \ldots, P_n$ be non-empty polytopes in $\mathbb{R}^n$ $(n \geq 2)$. Let $\boldsymbol{u}_1, \ldots, \boldsymbol{u}_m$ be the outward unit normals to the $(n-1)$-faces of $P_2 + \cdots + P_n$. Then there are scalars $\alpha_1, \ldots, \alpha_m \geq 0$ such that, for every non-empty compact convex set A in $\mathbb{R}^n$ with support function h,

$$v(A, P_2, \ldots, P_n) = \frac{1}{n} \sum_{i=1}^{m} \alpha_i h(\boldsymbol{u}_i).$$

- Let $Q = \lambda_2 P_2 + \cdots + \lambda_n P_n$ for $\lambda_2, \ldots, \lambda_n > 0$. By repeated applications of a previous theorem,

$$v(A, Q, \ldots, Q) = \sum_{i_2=2}^{n} \cdots \sum_{i_n=2}^{n} v(A, P_{i_2}, \ldots, P_{i_n}) \lambda_{i_2} \cdots \lambda_{i_n},$$

which is a homogeneous polynomial of degree $n-1$ in $\lambda_2, \ldots, \lambda_n$, the coefficient of $\lambda_2 \cdots \lambda_n$ being $(n-1)! v(A, P_2, \ldots, P_n)$.

Expressing v In Terms of Normals (Cont'd)

- By a previous lemma, the set of outward unit normals to the $(n-1)$-faces of Q is $\{\boldsymbol{u}_1,\ldots,\boldsymbol{u}_m\}$. By the preceding corollary,

$$\begin{array}{rcl} v(A,Q,\ldots,Q) & = & \frac{1}{n}\sum_{i=1}^m h(\boldsymbol{u}_i)v_{n-1}(Q^{\boldsymbol{u}_i}) \\ & = & \frac{1}{n}\sum_{i=1}^m h(\boldsymbol{u}_i)v_{n-1}(\lambda_2 P_2^{\boldsymbol{u}_i}+\cdots+\lambda_n P_n^{\boldsymbol{u}_i}) \\ & = & \frac{1}{n}\sum_{i=1}^m h(\boldsymbol{u}_i)(\sum_{j_2=2}^n\cdots\sum_{j_n=2}^n u(P_{j_2}^{\boldsymbol{u}_i},\ldots,P_{j_n}^{\boldsymbol{u}_i})\lambda_{j_1}\cdots\lambda_{j_n}), \end{array}$$

where $u(P_{j_2}^{\boldsymbol{u}_i},\ldots,P_{j_n}^{\boldsymbol{u}_i})$ denotes an $(n-1)$-dimensional mixed volume. This shows again that $v(A,Q,\ldots,Q)$ is a homogeneous polynomial of degree $n-1$ in $\lambda_2,\ldots,\lambda_n$, the coefficient of $\lambda_2,\ldots,\lambda_n$ being

$$\frac{1}{n}\sum_{i=1}^m h(\boldsymbol{u}_i)(n-1)!u(P_2^{\boldsymbol{u}_i},\ldots,P_n^{\boldsymbol{u}_i}).$$

Equating the two coefficients that we have found for the term $\lambda_2\cdots\lambda_n$ in $v(A,Q,\ldots,Q)$, we get $v(A,P_2,\ldots,P_n)=\frac{1}{n}\sum_{i=1}^m h(\boldsymbol{u}_i)u(P_2^{\boldsymbol{u}_i},\ldots,P_n^{\boldsymbol{u}_i})$. The proof is completed by putting $\alpha_i=u(P_2^{\boldsymbol{u}_i},\ldots,P_n^{\boldsymbol{u}_i})$.

Monotonicity of v

Theorem

Let $A_1,\ldots,A_n$, $B_1,\ldots,B_n$ be non-empty compact convex sets in $\mathbb{R}^n$ with $A_1 \subseteq B_1$, ..., $A_n \subseteq B_n$. Then $v(A_1,\ldots,A_n) \leq v(B_1,\ldots,B_n)$.

- We assume that $n \geq 2$. Consider first the case when $A_1,\ldots,A_n$, $B_1,\ldots,B_n$ are polytopes. Let h_{A_1}, h_{B_1} be the support functions of A_1, B_1, respectively. Since $A_1 \subseteq B_1$, $h_{A_1}(\boldsymbol{u}) \leq h_{B_1}(\boldsymbol{u})$ for all $\boldsymbol{u}$ in $\mathbb{R}^n$. Using the preceding theorem and an obvious notation, we get

$$\begin{array}{rcl} v(A_1,A_2,\ldots,A_n) & = & \frac{1}{n}\sum_{i=1}^m h_{A_1}(\boldsymbol{u}_i)\alpha_i \\ & \leq & \frac{1}{n}\sum_{i=1}^m h_{B_1}(\boldsymbol{u}_i)\alpha_i \\ & = & v(B_1,A_2,\ldots,A_n). \end{array}$$

Repeating $n-1$ times, $v(A_1,A_2,\ldots,A_n) \leq v(B_1,B_2,\ldots,B_n)$.

Monotonicity of v (Cont'd)

- Consider now the general case.

  For each $i = 1, \ldots, n$, let $P_i^1, \ldots, P_i^j, \ldots$ and $Q^1, i, \ldots, Q_i^j, \ldots$ be sequences of non-empty polytopes in $\mathbb{R}^n$ such that $P_i^j \to A_i$, $Q_i^j \to B_i$ as $j \to \infty$, and $P_i^j \subseteq Q_i^j$, for $j = 1, 2, \ldots$.

  Using the first part of the proof and the continuity of the mixed volumes, we deduce that

$$\begin{array}{rcl} v(A_1, \ldots, A_n) & = & \lim_{j \to \infty} v(P_1^j, \ldots, P_n^j) \\ & \leq & \lim_{j \to \infty} v(Q_1^j, \ldots, Q_n^j) \\ & = & v(B_1, \ldots, B_n). \end{array}$$

Surface Area of a Compact Convex Set

- A previous theorem applied for the special case when P is an n-polytope and A is the closed unit ball U asserts that

$$\sum_{i=1}^{m} v_{n-1}(F_i) = \lim_{\lambda \to 0^+} \frac{v_n((P)_\lambda) - v_n(P)}{\lambda}.$$

- The left-hand side of this equation is what we intuitively regard as the surface area of the polytope P, i.e. the sum of the v_{n-1}-volumes of its facets.
- We define the **surface area** $s_n(A)$ of a compact convex set A in $\mathbb{R}^n$ by the equation

$$s_n(A) = \lim_{\lambda \to 0^+} \frac{v_n((A)_\lambda) - v_n(A)}{\lambda}.$$

Surface Area In Terms of v

- For non-empty A and $\lambda > 0$, we have

$$\begin{aligned} v_n((A)_\lambda) &= v_n(A+\lambda U) \\ &= v_n(A) + nv(A,\ldots,A,U)\lambda + \cdots + v_n(U)\lambda^n. \end{aligned}$$

- Hence, $s_n(A)$ is well defined and equals $nv(A,\ldots,A,U)$.
- In $\mathbb{R}^1$ this last assertion is taken to mean that $s_1(A) = v_1(U) = 2$.
- Thus, we can define the surface area $s_n(A)$ of a compact convex set A in $\mathbb{R}^n$ to be $nv(A,\ldots,A,U)$ when A is non-empty, and to be zero when A is empty.

Example

- We evaluate the surface area $s_n(U)$ of the closed unit ball U in $\mathbb{R}^n$.
- We know that, for any $\lambda > 0$,

$$v_n((U)_\lambda) = v_n((1+\lambda)U) = \omega_n(1+\lambda)^n.$$

- Hence

$$s_n(U) = \lim_{\lambda \to 0^+} \frac{\omega_n(1+\lambda)^n - \omega_n}{\lambda} = n\omega_n.$$

- Thus, we get:
  - $s_2(U) = 2\omega_2 = 2\pi$.
    The perimeter of a circle of unit radius is 2π.
  - $s_3(U) = 3\omega_3 = 4\pi$.
    The surface area of a closed ball of unit radius in $\mathbb{R}^3$ is 4π.

Properties of Surface Area

- The value of $s_n(A)$, when A is a compact convex set in $\mathbb{R}^n$ of dimension at most $n-1$, is $2v_{n-1}(A)$.
- Surface area is increasing in the sense that $s_n(A) \le s_n(B)$ whenever A, B are compact convex sets in $\mathbb{R}^n$ with $A \subseteq B$.
- Moreover it is continuous in the sense that $s_n(A_i) \to s_n(A)$ as $i \to \infty$, whenever $A_1, \ldots, A_i, \ldots$ is a sequence of non-empty compact convex sets which converges to the non-empty compact convex set A in $\mathbb{R}^n$.
- For obvious reasons, s_2 is referred to as the **perimeter function**.
- Let A, B be non-empty compact convex sets in $\mathbb{R}^2$.

  Then, by a previous theorem,

$$s_2(A+B) = 2v(A+B, U) = 2v(A, U) + 2v(B, U) = s_2(A) + s_2(B).$$

  So the perimeter of $A+B$ is the sum of the perimeters of A and B.

Subsection 5

The Brunn-Minkowski Theorem

Introduction

- The **Brunn-Minkowski Theorem** asserts that:

  If A, B are convex bodies in $\mathbb{R}^n$, then

  $$v_n^{1/n}(A+B) \geq v_n^{1/n}(A) + v_n^{1/n}(B),$$

  with equality holding if and only if A and B are *homothetic*, i.e., if and only if $B = \lambda A + \boldsymbol{a}$, for some $\lambda > 0$ and $\boldsymbol{a} \in \mathbb{R}^n$.
- We establish this important result, thereby also solving the most famous of extremum problems, the **Isoperimetric Problem:**

  Of all convex bodies in $\mathbb{R}^n$ with a given volume, which have the smallest surface area?

Sum and Volume

- We saw that the vector sum of two elementary sets is an elementary set.
- The vector sum of two sets each of which has volume, however, need not itself have volume.

  Example: Consider the sets A, B in $\mathbb{R}^2$ defined by the equations:

$$\begin{aligned} A &= \{(x,0), 0 \le x \le 1, x \text{ rational}\}; \\ B &= \{(0,y) : 0 \le y \le 1, y \text{ rational}\}. \end{aligned}$$

  Then the sets A and B have zero volume.

  But the set

$$A + B = \{(x,y) : 0 \le x, y \le 1; x, y \text{ rational}\}$$

  does not have volume: Its inner-volume is zero and its outer-volume is one.
- When $A, B, A+B$ are non-empty sets in $\mathbb{R}^n$, all of which do have volume, then we have: $v_n^{1/n}(A+B) \ge v_n^{1/n}(A) + v_n^{1/n}(B)$.

A Scaling Lemma

Lemma

Let A be a set in $\mathbb{R}^n$ which has volume. Let $\theta \geq 0$. Then, for $i = 1, \ldots, n$, there is a scalar λ_i, such that

$$v(\{(x_1, \ldots, x_n) \in A : x_i < \lambda_i\}) = \theta v(\{(x_1, \ldots, x_n) \in A : x_i > \lambda_i\}).$$

- Since A is bounded, there is $a > 0$ such that

$$A \subseteq \{(x_1, \ldots, x_n) : -a \leq x_i \leq a, i = 1, \ldots, n\}.$$

Define a function $f_i : \mathbb{R} \to \mathbb{R}$ by the equation

$$f_i(x) = v(\{(x_1, \ldots, x_n) \in A : x_i < x\}), \text{ for } x \in \mathbb{R}.$$

Then, for $x < y$,

$$0 \leq f_i(y) - f_i(x) = v(\{(x_1, \ldots, x_n) \in A : x \leq x_i < y\}) \leq 2^{n-1} a^{n-1} (y - x).$$

This shows that f_i is continuous.

A Scaling Lemma (Cont'd)

- The function

$$f_i(x) = v(\{(x_1,\ldots,x_n) \in A : x_i < x\}), \text{ for } x \in \mathbb{R}$$

is continuous. Moreover, $f_i(-a) = 0$, $f_i(a) = v(A)$. By the Intermediate Value Theorem, for some $\lambda_i \in [-a,a]$, $f_i(\lambda_i) = \frac{\theta v(A)}{1+\theta}$.

Now we get:

$$\begin{aligned}
\theta v(\{(x_1,\ldots,x_n) \in A : x_i > \lambda_i\}) &= \theta(v(A) - f_i(\lambda_i)) \\
&= \theta\left(v(A) - \frac{\theta v(A)}{1+\theta}\right) \\
&= \theta v(A)\left(1 - \frac{\theta}{1+\theta}\right) \\
&= \frac{\theta}{1+\theta} v(A) = f_i(\lambda_i) \\
&= v(\{(x_1,\ldots,x_n) \in A : x_i < \lambda_i\}).
\end{aligned}$$

Case of Pairwise Disjoint Cells

- A **non-degenerate cell** is a cell which has non-empty interior.

Lemma

Let $A = I_1 \cup \cdots \cup I_m$, $B = J_1 \cup \cdots \cup J_p$, where $I_1, \ldots, I_m$ and $J_1, \ldots, J_p$ are sequences of pairwise disjoint non-degenerate cells in $\mathbb{R}^n$. Then

$$v^{1/n}(A+B) \geq v^{1/n}(A) + v^{1/n}(B).$$

- We argue by induction on $m+p$. Suppose first that $m+p=2$, so that $m=1$, $p=1$. Let $A = S_1 \times \cdots \times S_n$, $B = T_1 \times \cdots \times T_n$, where $S_1, \ldots, S_n$, $T_1, \ldots, T_n$ are cells in $\mathbb{R}^1$ with positive lengths $a_1, \ldots, a_n$, $b_1, \ldots, b_n$, respectively. A previous corollary justifies the inequality

$$\begin{array}{rcl} v^{1/n}(A+B) & = & ((a_1+b_1)\cdots(a_n+b_n))^{1/n} \\ & \geq & (a_1 \cdots a_n)^{1/n} + (b_1 \cdots b_n)^{1/n} \\ & = & v^{1/n}(A) + v^{1/n}(B). \end{array}$$

This proves the lemma in the case $m+p=2$.

Case of Pairwise Disjoint Cells (Cont'd)

- Suppose next that $m+p>2$ and that the assertion is true for all cases in which the induction variable is less than $m+p$. We can assume that $m\geq 2$. Since the cells I_1 and I_2 are disjoint, there is some $i\in\{1,\ldots,n\}$ and some scalar μ such that I_1 lies in the closed halfspace $x_i\leq\mu$ and I_2 lies in the closed halfspace $x_i\geq\mu$, or vice versa. Denote by A^- and A^+ the intersections of A with the open halfspaces $x_i<\mu$ and $x_i>\mu$, respectively. Then each of A^- and A^+ is non-empty and is a union of fewer than m pairwise disjoint non-degenerate cells. Since A is the pairwise disjoint union of A^-,A^+ and a set of volume zero, $v(A)$ equals $v(A^-)+v(A^+)$. The preceding lemma shows that there is a scalar λ such that the hyperplane $x_i=\lambda$ divides B (in a fashion similar to that considered above for A) into disjoint sets B^-,B^+ and a set of volume zero such that
$$\frac{v(B^-)}{v(A^-)}=\frac{v(B^+)}{v(A^+)}=\alpha,\ \text{say}.$$

Case of Pairwise Disjoint Cells (Cont'd)

- Each of the sets B^- and B^+ is a union of p or fewer pairwise disjoint non-degenerate cells, and $v(B)$ equals $v(B^-)+v(B^+)$. The sets A^-+B^- and A^++B^+ lie in opposite open halfspaces bounded by the hyperplane $x_i=\lambda+\mu$, and so are disjoint. Their union is a subset of $A+B$. We deduce, applying the induction hypothesis to the pairs (A^-,B^-) and (A^+,B^+), that

$$\begin{array}{rcl} v(A+B) & \geq & v(A^-+B^-)+v(A^++B^+) \\ & \geq & (v^{1/n}(A^-)+v^{1/n}(B^-))^n+(v^{1/n}(A^+)+v^{1/n}(B^+))^n \\ & = & (v(A^-)+v(A^+))(1+\alpha^{1/n})^n \\ & = & v(A)(1+\alpha^{1/n})^n \\ & = & (v^{1/n}(A)+\alpha^{1/n}v^{1/n}(A))^n \\ & = & (v^{1/n}(A)+v^{1/n}(B))^n. \end{array}$$

Thus $v^{1/n}(A+B)\geq v^{1/n}(A)+v^{1/n}(B)$.

Brunn's Inequality

Theorem (Brunn's inequality)

Let $A, B, A+B$ be non-empty sets in $\mathbb{R}^n$ all of which have volume. Then

$$v^{1/n}(A+B) \geq v^{1/n}(A) + v^{1/n}(B).$$

- The inequality is trivial if either A or B has zero volume.
  We assume, therefore, that $v(A) > 0$ and $v(B) > 0$.
  There are sequences $A_1, \ldots, A_i, \ldots$ and $B_1, \ldots, B_i, \ldots$ of non-empty elementary sets in $\mathbb{R}^n$ such that $A_i \subseteq A$, $B_i \subseteq B$ for $i = 1, 2, \ldots$, and $v(A_i) \to v(A)$, $v(B_i) \to v(B)$ as $i \to \infty$. We can assume that all of the A_i's and B_i's are finite unions of pairwise disjoint non-degenerate cells.
  By the preceding lemma,
  $$v^{1/n}(A+B) \geq v^{1/n}(A_i + B_i) \geq v^{1/n}(A_i) + v^{1/n}(B_i).$$
  Letting $i \to \infty$, we deduce $v^{1/n}(A+B) \geq v^{1/n}(A) + v^{1/n}(B)$.

Volume of a Convex Combination

Corollary

Let A, B be non-empty bounded convex sets in $\mathbb{R}^n$. Then the function $f : [0,1] \to \mathbb{R}$ defined by the equation

$$f(t) = v^{1/n}((1-t)A + tB), \text{ for } 0 \le t \le 1,$$

is concave.

- Let $x, y \in [0,1]$. Let $\lambda, \mu \ge 0$ with $\lambda + \mu = 1$. We apply the theorem to the sets $\lambda((1-x)A + xB)$ and $\mu((1-y)A + yB)$ to deduce that

$$\begin{array}{rcl} f(\lambda x + \mu y) & = & v^{1/n}((1-(\lambda x + \mu y))A + (\lambda x + \mu y)B) \\ & = & v^{1/n}(\lambda((1-x)A + xB) + \mu((1-y)A + yB)) \\ & \ge & \lambda v^{1/n}((1-x)A + xB) + \mu v^{1/n}((1-y)A + yB) \\ & = & \lambda f(x) + \mu f(y). \end{array}$$

Thus the function f is concave.

Minkowski's Inequality for Mixed Volumes

Theorem (Minkowski's Inequality for Mixed Volumes)

Let A and B be convex bodies in $\mathbb{R}^n$. Then

$$v(A,\ldots,A,B) \geq v^{(n-1)/n}(A)v^{1/n}(B),$$

with equality holding if and only if $v^{1/n}(A+B) = v^{1/n}(A) + v^{1/n}(B)$.

- Define a function $f : [0,1] \to \mathbb{R}$ by the equation

  $$f(t) = v^{1/n}((1-t)A + tB), \text{ for } 0 \leq t \leq 1.$$

  Then

$$\begin{aligned} f^n(t) &= v(A)(1-t)^n + nv(A,\ldots,A,B)(1-t)^{n-1}t + \cdots + v(B)t^n \\ nf(t)^{n-1}f'(t) &= -nv(A)(1-t)^{n-1} + nv(A,\ldots,A,B)(1-t)^{n-1} - \cdots \\ nv^{(n-1)/n}(A)f'(0) &= -nv(A) + nv(A,\ldots,A,B) \quad (f^{n-1}(0) = v^{(n-1)/n}(A)) \\ f'(0) &= \frac{v(A,\ldots,A,B) - v(A)}{v^{(n-1)/n}(A)}. \end{aligned}$$

Minkowski's Inequality for Mixed Volumes (Cont'd)

- We set
$$f(t) = v^{1/n}((1-t)A + tB), \text{ for } 0 \le t \le 1,$$
and obtained
$$f'(0) = \frac{v(A,\dots,A,B) - v(A)}{v^{(n-1)/n}(A)}.$$
By the preceding corollary, f is concave.
A previous corollary shows $f'(0) \ge f(1) - f(0)$.
Thus,
$$\begin{aligned} \tfrac{v(A,\dots,A,B)-v(A)}{v^{(n-1)/n}(A)} &\ge v^{1/n}(B) - v^{1/n}(A) \\ v(A,\dots,A,B) - v(A) &\ge v^{(n-1)/n}(A)v^{1/n}(B) - v(A) \\ v(A,\dots,A,B) &\ge v^{(n-1)/n}(A)v^{1/n}(B). \end{aligned}$$
This inequality becomes an equality if and only if $f'(0) = f(1) - f(0)$.
So we must show $f'(0) = f(1) - f(0)$ iff $f(\frac{1}{2}) = \frac{1}{2}(f(0) + f(1))$.

Minkowski's Inequality for Mixed Volumes (Cont'd)

- We show $f'(0) = f(1) - f(0)$ if and only if $f(\frac{1}{2}) = \frac{1}{2}(f(0) + f(1))$.

  Suppose first that $f'(0) = f(1) - f(0)$.

  By a previous corollary, $\frac{f(x)-f(0)}{x} = f(1) - f(0)$, for $0 < x \le 1$.

  Setting $x = \frac{1}{2}$, we get $f(\frac{1}{2}) = \frac{1}{2}(f(0) + f(1))$.

  Suppose next that $f(\frac{1}{2}) = \frac{1}{2}(f(0) + f(1))$. Then

$$\frac{f(1) - f(\frac{1}{2})}{1 - \frac{1}{2}} = \frac{f(0) - f(\frac{1}{2})}{0 - \frac{1}{2}} = \frac{f(0) - \frac{1}{2}f(0) - \frac{1}{2}f(1)}{-\frac{1}{2}} = f(1) - f(0).$$

  Using the same corollary as above,

$$\frac{f(x) - f(\frac{1}{2})}{x - \frac{1}{2}} = f(1) - f(0), \text{ for } x \in [0,1] \backslash \{\frac{1}{2}\}.$$

  Hence, $f'(0) = f(1) - f(0)$.

Lemma for the Case of Equality

Lemma

(i) Let S be an n-simplex and let T be a convex body in $\mathbb{R}^n$. Suppose that $v(S) = v(T)$ and that $v(S,\ldots,S,T) = v^{(n-1)/n}(S)v^{1/n}(T)$. Then T is a translate of S.

(ii) Let A, B be convex bodies in $\mathbb{R}^n$ such that, for each n-simplex contained in either one of them, there is some translate of it which is contained in the other. Then B is a translate of A.

(i) Let $F_0,\ldots,F_n$ be the facets of S and let $\boldsymbol{u}_0,\ldots,\boldsymbol{u}_n$ be the corresponding outward unit normals.

Let h_T be the support function of T.

Let C be the simplex which is homothetic to S, and circumscribes T, i.e., $T \subseteq C$ and T meets each facet of C.

Suppose that $C = \lambda S + \boldsymbol{a}$, for some $\lambda > 0$ and $\boldsymbol{a} \in \mathbb{R}^n$.

Lemma for the Case of Equality (Cont'd)

- By a previous corollary,

$$v^{(n-1)/n}(S)v^{1/n}(T) = v(S,\ldots,S,T) = \frac{1}{n}\sum_{i=0}^{n} h_T(\boldsymbol{u}_i)v_{n-1}(F_i).$$

Also, if h_C is the support function of C,

$$\lambda^n v(S) = v(C) = \frac{1}{n}\sum_{i=0}^{n} h_C(\boldsymbol{u}_i)v_{n-1}(\lambda F_i + \boldsymbol{a}) = \frac{1}{n}\sum_{i=0}^{n} h_T(\boldsymbol{u}_i)v_{n-1}(F_i).$$

Thus

$$v^{(n-1)/n}(S)v^{1/n}(T) = \lambda v(S), \text{ or } v(T) = \lambda^n v(S) = v(C).$$

But $T \subseteq C$. So $T = C$.

Since $v(S) = v(T)$, λ must be 1, and T is the translate $S + \boldsymbol{a}$ of S.

Lemma for the Case of Equality (Cont'd)

(ii) Clearly A and B must have the same diameter, s, say. Let $\boldsymbol{a}, \boldsymbol{a}' \in A$ be such that $\|\boldsymbol{a} - \boldsymbol{a}'\| = s$. Then there is some $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{a} + \boldsymbol{x}, \boldsymbol{a}' + \boldsymbol{x} \in B$. Let $\boldsymbol{c} \in A$. Then $\boldsymbol{a}, \boldsymbol{a}', \boldsymbol{c}$ belong to some n-simplex of A. Hence, there is some $\boldsymbol{y} \in \mathbb{R}^n$ such that $\boldsymbol{a} + \boldsymbol{y}$, $\boldsymbol{a}' + \boldsymbol{y}$, $\boldsymbol{c} + \boldsymbol{y} \in B$. Now

$$\begin{array}{rcl} 2\|\boldsymbol{y} - \boldsymbol{x}\|^2 + 2s^2 & = & 2\|\boldsymbol{y} - \boldsymbol{x}\|^2 + 2\|\boldsymbol{a} - \boldsymbol{a}'\|^2 \\ & = & \|\boldsymbol{y} - \boldsymbol{x} + \boldsymbol{a} - \boldsymbol{a}'\|^2 + \|\boldsymbol{y} - \boldsymbol{x} - \boldsymbol{a} + \boldsymbol{a}'\|^2 \\ & = & \|(\boldsymbol{a} + \boldsymbol{y}) - (\boldsymbol{a}' + \boldsymbol{x})\|^2 + \|(\boldsymbol{a}' + \boldsymbol{y}) - (\boldsymbol{a} + \boldsymbol{x})\|^2 \\ & \le & 2s^2, \end{array}$$

since B has diameter s. This shows that $\boldsymbol{x} = \boldsymbol{y}$.

Hence $\boldsymbol{c} + \boldsymbol{x} \in B$ for all $\boldsymbol{c}$ in A. So $A + \boldsymbol{x} \subseteq B$.

A similar argument shows that $B - \boldsymbol{x} \subseteq A$.

Thus B is the translate $A + \boldsymbol{x}$ of A.

The Brunn-Minkowski Theorem

Theorem (Brunn-Minkowski Theorem)

Let A, B be convex bodies in $\mathbb{R}^n$. Then $v^{1/n}(A+B) \geq v^{1/n}(A) + v^{1/n}(B)$, with equality holding if and only if A and B are homothetic.

- We have already established the inequality.

  Now we establish the conditions under which equality occurs. If A and B are homothetic, say $B = \mu A + \boldsymbol{a}$, where $\mu > 0$ and $\boldsymbol{a} \in \mathbb{R}^n$, then equality holds, since both sides are equal to $(1+\mu)v^{1/n}(A)$.

  Conversely, suppose that A, B give equality. Choose $\lambda > 0$ so that λB and A have the same volume. The second assertion of the preceding theorem shows that the sets $A, \lambda B$ also give equality

$$v^{1/n}(A + \lambda B) = v^{1/n}(A) + v^{1/n}(\lambda B).$$

The Brunn-Minkowski Theorem (Cont'd)

- Let S be any n-simplex contained in A.
  Then $S = J_0 \cap \cdots \cap J_n$ for some closed halfspaces $J_0, \ldots, J_n$ in $\mathbb{R}^n$.
  Denote by K_0 the translate of J_0 which makes the volumes $v(A \cap J_0)$ and $v((\lambda B) \cap K_0)$ equal. We show that the sets $A \cap J_0$, $(\lambda B) \cap K_0$ give equality in Brunn's inequality.
  Consider $A^- = A \cap J_0$, $A^+ = A \backslash A^-$, $B^- = (\lambda B) \cap K_0$, $B^+ = (\lambda B) \backslash B^-$.
  Suppose that A^-, B^- do not give equality in Brunn's inequality.
  Then $v^{1/n}(A^- + B^-) > v^{1/n}(A^-) + v^{1/n}(B^-)$.
  The sets $A^- + B^-$, $A^+ B^+$ are disjoint and are contained in $A + \lambda B$.
  By Brunn's inequality and equalities $v(A^-) = v(B^-)$, $v(A^+) = v(B^+)$,

$$\begin{array}{rcl} v^{1/n}(A+\lambda B) & \geq & (v(A^- + B^-) + v(A^+ + B^+))^{1/n} \\ & > & ((v^{1/n}(A^-) + v^{1/n}(B^-))^n + (v^{1/n}(A^+) + v^{1/n}(B^+))^n)^{1/n} \\ & = & v^{1/n}(A) + v^{1/n}(\lambda B). \end{array}$$

  This contradicts that $A, \lambda B$ give equality in Brunn's inequality.
  Thus, $A \cap J_0$ and $(\lambda B) \cap K_0$ yield equality in Brunn's inequality.

The Brunn-Minkowski Theorem (Cont'd)

- We repeat the argument just given n more times to deduce the existence of closed halfspaces $K_1, \ldots, K_n$ in $\mathbb{R}^n$ such that the convex bodies

$$S = A \cap J_0 \cap \cdots \cap J_n, \quad T = (\lambda B) \cap K_0 \cap \cdots \cap K_n$$

have the same volume and produce equality in Brunn's inequality.

We deduce, from the second assertion of the preceding theorem and the first part of the lemma, that T must be a translate of S.

It follows, by symmetry, that, for each n-simplex contained in either one of A and λB, there is some translate of it which is contained in the other.

Hence, by the second part of the lemma, A is a translate of λB.

This shows that A and B are homothetic.

An Additional Inequality

Theorem

Let A, B be convex bodies in $\mathbb{R}^n$. Then

$$v(A, \ldots, A, B) \geq v^{(n-1)/n}(A) v^{1/n}(B)$$

with equality holding if and only if A and B are homothetic.

- The result follows immediately from the preceding theorems.

The Isoperimetric Inequality

Theorem (Isoperimetric Inequality)

Every convex body A in $\mathbb{R}^n$ has a surface area greater than that of a closed ball with the same volume, unless it is itself a closed ball. More specifically,

$$s^n(A) \geq \omega_n n^n v^{n-1}(A),$$

with equality holding if and only if A is a closed ball.

- Let B be the closed unit ball U in the theoem. Then that

$$s(A) = nv(A, \ldots, A, U) \geq nv^{(n-1)/n}(A)v^{1/n}(U) = nv^{(n-1)/n}(A)\omega_n^{1/n}.$$

  So $s^n(A) \geq \omega_n n^n v^{n-1}(A)$.

  Equality holds if and only if A is homothetic to U, i.e. if and only if A is a closed ball.

The Isodiametric Inequality

Theorem (Isodiametric Inequality)

Every convex body A in $\mathbb{R}^n$ with diameter d has a volume less than that of a closed ball with diameter d, unless it is itself a closed ball. More specifically,

$$v(A) \le \omega_n \left(\frac{1}{2}d\right)^n,$$

with equality holding if and only if A is a closed ball.

- Denote by $\mathscr{F}$ the family of all convex bodies in $\mathbb{R}^n$ with diameter d. Let $\alpha = \sup\{v(A) : A \in \mathscr{F}\}$. Then there is a sequence $A_1, \ldots, A_k, \ldots$ of members of $\mathscr{F}$ which lie in the closed ball dU such that $v(A_k) \to \alpha$ as $k \to \infty$. By the Blaschke Selection Theorem, there exists a subsequence $i_1, \ldots, i_k, \ldots$ of $1, \ldots, k, \ldots$ and a convex body A_0 such that $A_{i_k} \xrightarrow{k\to\infty} A_0$. Since both volume and diameter are continuous with respect to Hausdorff distance, it follows that $A_0 \in \mathscr{F}$, $v(A) = \alpha$. Thus A_0 is a member of $\mathscr{F}$ having maximal possible volume.

The Isodiametric Inequality (Cont'd)

- Let C be any member of $\mathscr{F}$ having maximal volume. It is easily verified that the convex body $C' = \frac{1}{2}(C - C)$ belongs to $\mathscr{F}$. The Brunn-Minkowski theorem shows that $v(C') \geq v(C)$ with equality holding if and only if C is homothetic to $-C$. By the choice of C, $v(C) \geq v(C')$. Thus $v(C') = v(C)$.

  Hence $C = -\lambda C + \boldsymbol{c}$, for some $\lambda > 0$ and $\boldsymbol{c} \in \mathbb{R}^n$. Since C has the same volume as $-C$, $\lambda = 1$ and $C = -C + \boldsymbol{c}$. Hence $C - \frac{1}{2}\boldsymbol{c} = -(C - \frac{1}{2}\boldsymbol{c})$, and $C - \frac{1}{2}\boldsymbol{c}$ is a symmetric member of $\mathscr{F}$ having maximal volume.

  The symmetry of $C - \frac{1}{2}\boldsymbol{c}$ together with the fact that its diameter is d shows that $C - \frac{1}{2}\boldsymbol{c} \subseteq \frac{1}{2}dU$. But $\frac{1}{2}dU \in \mathscr{F}$ and $v(C - \frac{1}{2}\boldsymbol{c}) \leq v(\frac{1}{2}dU)$.

  Hence $C - \frac{1}{2}\boldsymbol{c} = \frac{1}{2}dU$. Thus C is the closed ball $\frac{1}{2}\boldsymbol{c} + \frac{1}{2}dU$.

  The desired result is immediate.

The Schwarz Rotation-Symmetral: An Example

- Suppose that the given convex body is the square pyramid
$$A = \text{conv}\{(0,0,0),(1,1,1),(1,1,-1),(1,-1,1),(1,-1,-1)\}.$$
- Then A has for its base a square of side 2 lying in the plane $x_1 = 1$, for its vertex the origin, and its height is 1.
- For each x with $0 < x \le 1$, denote by A_x the intersection of A with the hyperplane $x_1 = x$.
- Denote by C_x the closed circular disc which lies in the hyperplane $x_1 = x$, has its center on the x_1-axis, and has the same area as A_x.
- Clearly A_x is a square of side $2x$ and C_x has radius $r_x = \frac{2x}{\sqrt{\pi}}$.
- We write $C_0 = \{(0,0,0)\}$ and $r_0 = 0$.
- The union $C = \bigcup(C_x : 0 \le x \le 1)$ of the circular discs C_x is called the *Schwarz rotation-symmetral* of A in the x_1-axis.
- Here C is a right circular cone with base a closed disc of radius $\frac{2}{\sqrt{\pi}}$ with axis the x_1-axis, and vertex the origin.

The Schwarz Rotation-Symmetral

- Let A be a convex body in $\mathbb{R}^n$, where $n \geq 2$.
- For simplicity of notation, we suppose that the line of rotation is the x_1-axis and that A lies between parallel support hyperplanes $x_1 = a$ and $x_1 = b$ to A, where $a < b$.
- For each x with $a \leq x \leq b$, denote by A_x the intersection of A with the hyperplane $x_1 = x$.
- Define r_x by the equation $\omega_{n-1} r_x^{n-1} = v_{n-1}(A_x)$.
- Thus, for $a < x < b$, r_x is the radius of an $(n-1)$-ball whose v_{n-1}-volume is the same as that of A_x.
- For each x with $a \leq x \leq b$, define a convex set C_x (indeed an $(n-1)$-ball when $a < x < b$) by the equation

$$C_x = \{(x, x_2, \ldots, x_n) : x_2^2 + \cdots + x_n^2 \leq r_x^2\}.$$

- Then the set $C = \bigcup (C_x : a \leq x \leq b)$ is called the **Schwarz rotation symmetral** of A in the x_1-axis.

Schwarz Construction for the Isoperimetric Problem

Theorem

Let A be a convex body in $\mathbb{R}^n$ ($n \geq 2$) whose Schwarz rotation-symmetral in the x_1-axis is C. Then C is a convex body having the same volume as A.

- We assume the notation introduced for the definition of the Schwarz rotation-symmetral. First we show that $r : [a,b] \to \mathbb{R}$ is a concave function. Let $x, y \in [a,b]$ and let $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$. By the convexity of A, $A_{\lambda x + \mu y} \supseteq \lambda A_x + \mu A_y$. Applying Brunn's inequality in $\mathbb{R}^{n-1}$, we find that

$$\begin{aligned} v_{n-1}^{1/(n-1)}(A_{\lambda x+\mu y}) &\geq v_{n-1}^{1/(n-1)}(\lambda A_x + \mu A_y) \\ &\geq \lambda v_{n-1}^{1/(n-1)}(A_x) + \mu v_{n-1}^{1/(n-1)}(A_y). \end{aligned}$$

Hence, $r_{\lambda x + \mu y} \geq \lambda r_x + \mu r_y$.

Schwarz Construction (Cont'd)

- We now establish the convexity of C, omitting the verification that it is compact with nonempty interior.
  Let $\boldsymbol{u}, \boldsymbol{v} \in C$ and let $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$. Then $\boldsymbol{u} \in C_x$, $\boldsymbol{v} \in C_y$, for some $x, y \in [a, b]$. Thus $\|\boldsymbol{u} - (x, 0, \ldots, 0)\| \leq r_x$, $\|\boldsymbol{v} - (y, 0, \ldots, 0)\| \leq r_y$.
  Now $a \leq \lambda x + \mu y \leq b$ and
  $$\begin{aligned} \|\lambda \boldsymbol{u} + \mu \boldsymbol{v} - (\lambda x + \mu y, 0, \ldots, 0)\| & \\ &\leq \lambda \|\boldsymbol{u} - (x, 0, \ldots, 0)\| + \mu \|\boldsymbol{v} - (y, 0, \ldots, 0)\| \\ &\leq \lambda r_x + \mu r_y \\ &\leq r_{\lambda x + \mu y}. \end{aligned}$$
  Hence, $\lambda \boldsymbol{u} + \mu \boldsymbol{v} \in C_{\lambda x + \mu y}$. So $\lambda \boldsymbol{u} + \mu \boldsymbol{v} \in C$. Thus, C is convex.
  It follows from a previous theorem that
  $$v_n(A) = \int_a^b v_{n-1}(A_x) dx = \int_a^b v_{n-1}(C_x) dx = v_n(C).$$

Example Revisited

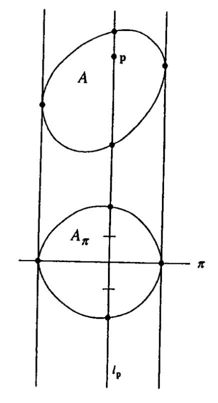
- Consider our earlier example in which A was a square pyramid in $\mathbb{R}^3$ and its Schwarz rotation-symmetral C was a circular cone.
  - A and C have the same volume $\frac{4}{3}$;
  - A has surface area $4+4\sqrt{2}$;
  - C has the smaller surface area $4+2\sqrt{\pi+4}$.
- It is a property of the Schwarz rotation-symmetral of a convex body that its surface area never exceeds that of the body itself.

Subsection 6

Steiner Symmetrization

Informal Description of the Steiner Symmetral

- Let A be a non-empty compact convex set and π a hyperplane in $\mathbb{R}^n$.
- Steiner's construction produces from A and π a convex set A_π in $\mathbb{R}^n$ called the *Steiner symmetral* of A about π.



- For each point p of A, denote by ℓ_p the line through p perpendicular to the hyperplane π.
- Translate the chord $A \cap \ell_p$ of A along ℓ_p until its midpoint lies on π.
- The union A_π of all such translated chords is called the *Steiner symmetral* of A about π.

Projection on a Hyperplane

- Let $\boldsymbol{u}$ be a unit normal vector to a hyperplane π.
- Then the **projection** $\pi(A)$ of A on π is the subset of π defined by the equation
$$\pi(A) = \{\boldsymbol{p} \in \pi : \boldsymbol{p} + \theta \boldsymbol{u} \in A, \text{ for some } \theta \in \mathbb{R}\}.$$
- We show that $\pi(A)$ is convex.

  Let $\boldsymbol{p}, \boldsymbol{q} \in \pi(A)$ and let $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$. Then there exist $\theta, \varphi \in \mathbb{R}$ such that $\boldsymbol{p} + \theta \boldsymbol{u}, \boldsymbol{q} + \varphi \boldsymbol{u} \in A$. Since A is convex,
$$\lambda \boldsymbol{p} + \mu \boldsymbol{q} + (\lambda\theta + \mu\varphi)\boldsymbol{u} = \lambda(\boldsymbol{p} + \theta \boldsymbol{u}) + \mu(\boldsymbol{q} + \varphi \boldsymbol{u}) \in A.$$
  Hence $\lambda \boldsymbol{p} + \mu \boldsymbol{q} \in \pi(A)$. So $\pi(A)$ is convex.

The Functions α, β, γ

- For each $\boldsymbol{p}$ in $\pi(A)$, denote by $I_{\boldsymbol{p}}$ the non-empty compact interval of $\mathbb{R}$ defined by the equation

$$I_{\boldsymbol{p}} = \{\theta \in \mathbb{R} : \boldsymbol{p} + \theta \boldsymbol{u} \in A\}.$$

- Define functions $\alpha, \beta, \gamma : \pi(A) \to \mathbb{R}$ as follows:

$$\alpha(\boldsymbol{p}) = \min I_{\boldsymbol{p}}, \ \beta(\boldsymbol{p}) = \max I_{\boldsymbol{p}}, \ \gamma(\boldsymbol{p}) = \beta(\boldsymbol{p}) - \alpha(\boldsymbol{p}), \ \boldsymbol{p} \in \pi(A).$$

- Thus $\gamma(\boldsymbol{p})$ is the length of the chord of A which is the intersection of A with the line through $\boldsymbol{p}$ normal to π.
- If we choose $-\boldsymbol{u}$ instead of $\boldsymbol{u}$ for a unit normal to π, then γ (unlike α and β) remains unchanged.
- Thus γ is uniquely determined by A and π.

Concavity of γ

Theorem

The function $\gamma : \pi(A) \to \mathbb{R}$ is concave.

- Let $\boldsymbol{p}, \boldsymbol{q} \in \pi(A)$ and let $\lambda, \mu \geq 0$ with $\lambda + \mu = 1$. Then $\boldsymbol{p} + \alpha(\boldsymbol{p})\boldsymbol{u}$, $\boldsymbol{q} + \alpha(\boldsymbol{q})\boldsymbol{u} \in A$. The convexity of A shows that

$$\lambda\boldsymbol{p} + \mu\boldsymbol{q} + (\lambda\alpha(\boldsymbol{p}) + \mu\alpha(\boldsymbol{q}))\boldsymbol{u} = \lambda(\boldsymbol{p} + \alpha(\boldsymbol{p})\boldsymbol{u}) + \mu(\boldsymbol{q} + \alpha(\boldsymbol{q})\boldsymbol{u}) \in A.$$

Hence, $\alpha(\lambda\boldsymbol{p} + \mu\boldsymbol{q}) \leq \lambda\alpha(\boldsymbol{p}) + \mu\alpha(\boldsymbol{q})$.
Similarly, $\beta(\lambda\boldsymbol{p} + \mu\boldsymbol{q}) \geq \lambda\beta(\boldsymbol{p}) + \mu\beta(\boldsymbol{q})$.
Thus,

$$\begin{array}{rcl}\gamma(\lambda\boldsymbol{p} + \mu\boldsymbol{q}) & = & \beta(\lambda\boldsymbol{p} + \mu\boldsymbol{q}) - \alpha(\lambda\boldsymbol{p} + \mu\boldsymbol{q}) \\ & \geq & \lambda(\beta(\boldsymbol{p}) - \alpha(\boldsymbol{p})) + \mu(\beta(\boldsymbol{q}) - \alpha(\boldsymbol{q})) \\ & = & \lambda\gamma(\boldsymbol{p}) + \mu\gamma(\boldsymbol{q}).\end{array}$$

So γ is concave.

The Steiner Symmetral

- We define the **Steiner symmetral** A_π of A about π by the equation
$$A_\pi = \left\{ \boldsymbol{p} + \theta \boldsymbol{u} : \boldsymbol{p} \in \pi(A), |\theta| \le \frac{1}{2}\gamma(\boldsymbol{p}) \right\}.$$
- Some easy consequences of the definition are:
  (i) A_π is (in an obvious sense) symmetric about π;
  (ii) If B is a closed ball with center on π, then $B_\pi = B$;
  (iii) If C is a compact convex set with $A \subseteq C$, then $A_\pi \subseteq C_\pi$.

Compactness, Convexity and Symmetral

Theorem

Let A be a non-empty compact convex set and let π be a hyperplane in $\mathbb{R}^n$. Then A_π is a non-empty compact convex set, which is a convex body when A is.

- Let $\boldsymbol{a},\boldsymbol{b} \in A_\pi$. Then there are $\boldsymbol{p},\boldsymbol{q} \in \pi(A)$ and scalars θ,φ such that $\boldsymbol{a} = \boldsymbol{p} + \theta\boldsymbol{u}$, $\boldsymbol{b} = \boldsymbol{q} + \varphi\boldsymbol{u}$, where $|\theta| \le \frac{1}{2}\gamma(\boldsymbol{p})$, $|\varphi| \le \frac{1}{2}\gamma(\boldsymbol{q})$. Let $\lambda,\mu \ge 0$ with $\lambda + \mu = 1$. Then

$$\lambda\boldsymbol{a} + \mu\boldsymbol{b} = \lambda\boldsymbol{p} + \mu\boldsymbol{q} + (\lambda\theta + \mu\varphi)\boldsymbol{u}.$$

By the concavity of γ,

$$|\lambda\theta + \mu\varphi| \le \lambda|\theta| + \mu|\varphi| \le \frac{1}{2}\lambda\gamma(\boldsymbol{p}) + \frac{1}{2}\mu\gamma(\boldsymbol{q}) \le \gamma(\lambda\boldsymbol{p} + \mu\boldsymbol{q}).$$

Thus $\lambda\boldsymbol{a} + \mu\boldsymbol{b} \in A_\pi$. This shows that A_π is convex.

Compactness, Convexity and Symmetral (Cont'd)

- We now show that A_π is closed. Let $\boldsymbol{x}_1,,\ldots,\boldsymbol{x}_k,\ldots$ be a sequence of points of A_π that converges to a point $\boldsymbol{x}$ of $\mathbb{R}^n$. For each $k=1,2,\ldots$, there exist $\boldsymbol{p}_k \in \pi(A)$, $\theta_k \in \mathbb{R}$ such that $\boldsymbol{x}_k = \boldsymbol{p}_k + \theta_k \boldsymbol{u}$, $|\theta_k| \le \frac{1}{2}\gamma(\boldsymbol{p}_k)$. The point $\boldsymbol{x}$ can be written in the form $\boldsymbol{p}+\theta\boldsymbol{u}$, $\boldsymbol{p}\in\pi$ and $\theta\in\mathbb{R}$.
  Since $\|\boldsymbol{p}_k - \boldsymbol{p}\| \le \|\boldsymbol{x}_k - \boldsymbol{x}\|$ and $\boldsymbol{x}_k \to \boldsymbol{x}$ as $k\to\infty$, $\boldsymbol{p}_k \to \boldsymbol{p}$ as $k\to\infty$. The points $\boldsymbol{y}_k = \boldsymbol{p}_k + \alpha(\boldsymbol{p}_k)\boldsymbol{u}$, $\boldsymbol{z}_k = \boldsymbol{p}_k + \beta(\boldsymbol{p}_k)\boldsymbol{u}$. lie in the compact set A. Hence there is a subsequence $i_1,\ldots,i_k,\ldots$ of $1,\ldots,k,\ldots$ and points $\boldsymbol{y},\boldsymbol{z}\in A$ such that $\boldsymbol{y}_{i_k}\to\boldsymbol{y}$, $\boldsymbol{z}_{i_k}\to\boldsymbol{z}$ as $k\to\infty$.
  A simple argument shows that $\boldsymbol{y}=\boldsymbol{p}+a\boldsymbol{u}$, $\boldsymbol{z}=\boldsymbol{p}+b\boldsymbol{u}$, where $a,b\in\mathbb{R}$ are such that $a\le b$ and $\alpha(\boldsymbol{p}_{i_k})\to a$, $\beta(\boldsymbol{p}_{i_k})\to b$ as $k\to\infty$.
  Thus, $\boldsymbol{p}\in\pi(A)$ and

$$|\theta| = \lim_{k\to\infty}|\theta_{i_k}| \le \lim_{k\to\infty}\frac{1}{2}(\beta(\boldsymbol{p}_{i_k}) - \alpha(\boldsymbol{p}_{i_k})) = \frac{1}{2}(b-a) \le \frac{1}{2}\gamma(\boldsymbol{p}).$$

  This shows that $\boldsymbol{x}\in A_\pi$. Hence A_π is closed.

Compactness, Convexity and Symmetral (Cont'd)

- Since A is bounded, it lies in some closed ball C.

  Hence, $A_\pi \subseteq C_\pi$.

  But C_π is clearly a closed ball.

  So A_π is bounded.

  We have thus shown that A_π is both closed and bounded.

  So A_π is compact.

  If A is a convex body, then it contains some closed ball B.

  Hence, $B_\pi \subseteq A_\pi$.

  But B_π is a closed ball.

  So the compact convex set A_π has a non-empty interior.

  Therefore, A_π is a convex body.

Sums and Symmetrals

Theorem

In $\mathbb{R}^n$ let A, B be non-empty compact convex sets and let π be a hyperplane passing through the origin. Then

$$A_\pi + B_\pi \subseteq (A+B)_\pi.$$

- Let $\boldsymbol{x} \in A_\pi + B_\pi$. Then $\boldsymbol{x} = \boldsymbol{a} + \boldsymbol{b}$ for some $\boldsymbol{a} \in A_\pi$, $\boldsymbol{b} \in B_\pi$. We can write, using an obvious notation, $\boldsymbol{a} = \boldsymbol{p} + \theta \boldsymbol{u}$, $\boldsymbol{b} = \boldsymbol{q} + \varphi \boldsymbol{u}$, where $\boldsymbol{p} \in \pi(A)$, $\boldsymbol{q} \in \pi(B)$ and $|\theta| \le \frac{1}{2}\gamma_A(\boldsymbol{p})$, $|\varphi| \le \frac{1}{2}\gamma_B(\boldsymbol{q})$. Since π is a subspace of $\mathbb{R}^n$, $\boldsymbol{p} + \boldsymbol{q} \in \pi$. From this follows that $\boldsymbol{p} + \boldsymbol{q} \in \pi(A+B)$. Clearly $\gamma_{A+B}(\boldsymbol{p} + \boldsymbol{q}) \ge \gamma_A(\boldsymbol{p}) + \gamma_B(\boldsymbol{q})$. Hence

$$|\theta + \varphi| \le |\theta| + |\varphi| \le \frac{1}{2}\gamma_A(\boldsymbol{p}) + \frac{1}{2}\gamma_B(\boldsymbol{q}) \le \frac{1}{2}\gamma_{A+B}(\boldsymbol{p} + \boldsymbol{q}).$$

Thus, $\boldsymbol{x} = \boldsymbol{p} + \boldsymbol{q} + (\theta + \varphi)\boldsymbol{u} \in (A+B)_\pi$. So $A_\pi + B_\pi \subseteq (A+B)_\pi$.

Comparing a Convex Set and its Symmetrization

- Let D and R denote, respectively, diameter and circumradius.

Theorem

In $\mathbb{R}^n$ let A be a non-empty compact convex set and let π be a hyperplane. Then $v_n(A_\pi) = v_n(A)$, $s_n(A_\pi) \leq s_n(A)$, $D(A_\pi) \leq D(A)$, $R(A_\pi) \leq R(A)$.

- We suppose throughout that π contains the origin.

  To show that $v_n(A_\pi) = v_n(A)$, we argue by induction on n.

  The case $n = 1$ is trivial. Suppose that $n \geq 2$ and that the assertion is known to be true in $\mathbb{R}^{n-1}$. Let $\boldsymbol{u}$ be a unit vector lying in π.

  A previous corollary shows that, in an obvious notation,

$$v_n(A) = \int_{-\infty}^{\infty} v_{n-1}(A_x)dx, \quad v_n(A_\pi) = \int_{-\infty}^{\infty} v_{n-1}((A_\pi)_x)dx.$$

  We can show that $(A_\pi)_x = (A_x)_\pi$. Using the induction hypothesis, $v_{n-1}((A_\pi)_x) = v_{n-1}((A_x)_\pi) = v_{n-1}(A_x)$. Hence, $v_n(A) = v_n(A_\pi)$.

Comparing a Convex Set and its Symmetrization (Cont'd)

- The preceding theorem shows that, for each $\lambda > 0$,

$$A_\pi + \lambda U = A_\pi + (\lambda U)_\pi \subseteq (A + \lambda U)_\pi.$$

Thus, by the first part of this proof,

$$v_n(A_\pi + \lambda U) \leq v_n((A + \lambda U)_\pi) = v_n(A + \lambda U).$$

Hence,

$$\lim_{\lambda \to 0^+} \frac{v_n(A_\pi + \lambda U) - v_n(A_\pi)}{\lambda} \leq \lim_{\lambda \to 0^+} \frac{v_n(A + \lambda U) - v_n(A)}{\lambda}.$$

That is, $s_n(A_\pi) \leq s_n(A)$.

Comparing a Convex Set and its Symmetrization (Cont'd)

- Suppose now that $\boldsymbol{u}$ is a unit normal to π. Let $\boldsymbol{x},\boldsymbol{y} \in A_\pi$. Then $\boldsymbol{x} = \boldsymbol{p} + \theta\boldsymbol{u}$, $\boldsymbol{y} = \boldsymbol{q} + \varphi\boldsymbol{u}$, for some $\boldsymbol{p},\boldsymbol{q} \in \pi(A)$ and $\theta,\varphi \in \mathbb{R}$ with $|\theta| \le \frac{1}{2}\gamma(\boldsymbol{p})$, $|\varphi| \le \frac{1}{2}\gamma(\boldsymbol{q})$. The points

$$\begin{array}{ll}\boldsymbol{x}_\alpha = \boldsymbol{p} + \alpha(\boldsymbol{p})\boldsymbol{u}, & \boldsymbol{x}_\beta = \boldsymbol{p} + \beta(\boldsymbol{p})\boldsymbol{u}, \\ \boldsymbol{y}_\alpha = \boldsymbol{q} + \alpha(\boldsymbol{q})\boldsymbol{u}, & \boldsymbol{y}_\beta = \boldsymbol{q} + \beta(\boldsymbol{q})\boldsymbol{u}\end{array}$$

belong to A. Moreover,

$$\begin{array}{lcl}\|\boldsymbol{x}_\alpha - \boldsymbol{y}_\beta\|^2 & = & \|\boldsymbol{p} - \boldsymbol{q}\|^2 + |\alpha(\boldsymbol{p}) - \beta(\boldsymbol{q})|^2, \\ \|\boldsymbol{x}_\beta - \boldsymbol{y}_\alpha\|^2 & = & \|\boldsymbol{p} - \boldsymbol{q}\|^2 + |\beta(\boldsymbol{p}) - \alpha(\boldsymbol{q})|^2.\end{array}$$

Comparing a Convex Set and its Symmetrization (Cont'd)

- Now

$$\begin{aligned}
\|\boldsymbol{x}-\boldsymbol{y}\|^2 &= \|\boldsymbol{p}-\boldsymbol{q}\|^2+|\theta-\varphi|^2 \\
&\le \|\boldsymbol{p}-\boldsymbol{q}\|^2+\tfrac{1}{4}(\gamma(\boldsymbol{p})+\gamma(\boldsymbol{q}))^2 \\
&= \|\boldsymbol{p}-\boldsymbol{q}\|^2+\tfrac{1}{4}(\beta(\boldsymbol{p})-\alpha(\boldsymbol{q})+\beta(\boldsymbol{q})-\alpha(\boldsymbol{p}))^2 \\
&\le \|\boldsymbol{p}-\boldsymbol{q}\|^2+\tfrac{1}{2}|\beta(\boldsymbol{p})-\alpha(\boldsymbol{q})|^2+\tfrac{1}{2}|\beta(\boldsymbol{q})-\alpha(\boldsymbol{p})|^2 \\
&= \tfrac{1}{2}\|\boldsymbol{x}_\alpha-\boldsymbol{y}_\beta\|^2+\tfrac{1}{2}\|\boldsymbol{x}_\beta-\boldsymbol{y}_\alpha\|^2 \\
&\le D^2(A).
\end{aligned}$$

This shows that $D(A_\pi)\le D(A)$.

Suppose C is a closed ball containing A.

Then C_π is a translate of C containing A_π.

Thus $R(A_\pi)\le R(A)$.

The Isodiametric Inequality

- The proof of a previous theorem shows that, for each $d > 0$, there exists among all convex bodies in $\mathbb{R}^n$ of diameter d some convex body C which has maximal volume.
- Let C_0 be the convex body obtained from C by successive Steiner symmetrizations in the hyperplanes $x_1 = 0, \ldots, x_n = 0$.
- It is a simple exercise to show that C_0 is a symmetric convex body, which has the same volume, and no larger diameter than C_0.
- Since C_0 is symmetric with diameter less than or equal to d, it must lie in the ball $\frac{1}{2}dU$.
- Thus, for any convex body A in $\mathbb{R}^n$ with diameter d, we have the isodiametric inequality:

$$v_n(A) \leq v_n(C) = v_n(C_0) \leq \omega_n \left(\frac{1}{2}d\right)^n.$$

Continuity of Steiner Symmetrization

Theorem

Let $A_1,\ldots,A_k,\ldots$ be a sequence of convex bodies that converges to a convex body A in $\mathbb{R}^n$. Then the sequence $(A_1)_\pi,\ldots,(A_k)_\pi,\ldots$ of its Steiner symmetrals about any hyperplane π of $\mathbb{R}^n$ converges to the Steiner symmetral A_π of A about it.

- We assume that the origin is an interior point of A lying on π. Thus there exist $r,s>0$ and a positive integer N_1, such that $rU\subseteq A\subseteq sU$ and $rU\subseteq A_k\subseteq sU$, for $k>N_1$. Hence, $rU\subseteq A_\pi\subseteq sU$ and $rU\subseteq(A_k)_\pi\subseteq sU$, for $k>N_1$. Let $\varepsilon>0$. Since $A_k\to A$ as $k\to\infty$, there is a positive integer N_2 such that, for $k>N_2$,
$$A_k\subseteq A+\frac{r\varepsilon}{s}U \quad \text{and} \quad A\subseteq A_k+\frac{r\varepsilon}{s}U.$$

Continuity of Steiner Symmetrization (Cont'd)

- Let $k > \max\{N_1, N_2\}$. Then

$$A_k \subseteq A + \frac{r\varepsilon}{s} U \subseteq A + \frac{\varepsilon}{s} A = \left(1 + \frac{\varepsilon}{s}\right) A.$$

So

$$(A_k)_\pi \subseteq \left(1 + \frac{\varepsilon}{s}\right) A_\pi = A_\pi + \frac{\varepsilon}{s} A_\pi \subseteq A_\pi + \varepsilon U.$$

Similarly, $A_\pi \subseteq (A_k)_\pi + \varepsilon U$.

Thus,

$$\rho((A_k)_\pi, A_\pi) \leq \varepsilon.$$

It follows that $(A_k)_\pi \to A_\pi$ as $k \to \infty$.

Symmetrization and Approximation by Balls

- Let A be a convex body in $\mathbb{R}^n$.
- Denote by $\mathscr{S}(A)$ the family of all convex bodies which can be obtained from A by a finite number of symmetrizations about hyperplanes through the origin.

Theorem

Let A be a convex body in $\mathbb{R}^n$. Then there is a sequence of members of $\mathscr{S}(A)$ which converges to the closed ball of volume $v_n(A)$ whose center is the origin.

- Let $r_0 = \inf\{r > 0 : \text{there is } C \text{ in } \mathscr{S}(A) \text{ such that } C \subseteq rU\}$.
  Then, for each $k = 1, 2, \ldots$, there exists A_k in $\mathscr{S}(A)$ such that $A_k \subseteq (r_0 + k^{-1})U$. By the Blaschke Selection Theorem, there is a subsequence of $A_1, A_2, \ldots$ which converges to some convex body, B, say. We assume that the sequence itself converges to B. Clearly $B \subseteq r_0 U$ and $v_n(B) = v_n(A)$.

Symmetrization and Approximation by Balls (Cont'd)

- We complete the proof by showing that $B = r_0 U$.

  Suppose that $B \neq r_0 U$. Then there exist $\boldsymbol{x}_0 \in \text{bd} r_0 U$ and $s > 0$ such that $B(\boldsymbol{x}_0; s) \cap B = \emptyset$. Since $\text{bd} r_0 U$ is compact, there exist distinct points $\boldsymbol{x}_0, \ldots, \boldsymbol{x}_m$ $(m \geq 1)$ of $\text{bd} r_0 U$ such that

  $$\text{bd} r_0 U \subseteq B(\boldsymbol{x}_0; s) \cup \cdots \cup B(\boldsymbol{x}_m; s).$$

  For $i = 0, \ldots, m$, set $C_i = B(\boldsymbol{x}_i; s) \cap \text{bd} r_0 U$. Then $\text{bd} r_0 U = C_0 \cup \cdots \cup C_m$.

  For $i = 1, \ldots, m$, let π_i be the hyperplane through the origin which has $\boldsymbol{x}_i - \boldsymbol{x}_0$ for a normal vector. Then C_0 and C_i are mirror images of one another in π_i. From $B(\boldsymbol{x}_0; s) \cap B = \emptyset$ and the definition of Steiner symmetrization, B_{π_1} is disjoint from $C_0 \cup C_1$.

  Similarly, $(B_{\pi_1})_{\pi_2}$ is disjoint from $C_0 \cup C_1 \cup C_2$.

Symmetrization and Approximation by Balls (Cont'd)

- Continuing, in this fashion, we find that the convex body $B^{\square}$ obtained from B by successive symmetrizations about $\pi_1, \ldots, \pi_m$ is disjoint from $C_0 \cup \cdots \cup C_m$. Hence, it is disjoint from $\text{bd} r_0 U$.

  Since $B^{\square}$ is a convex body lying in $r_0 U$, there exists ε, with $0 < \varepsilon < r_0$, such that $B^{\square} \subseteq (r_0 - \varepsilon) U$.

  For $k = 1, 2, \ldots$, denote by $A_k^{\square}$ the convex body obtained from A_k by successive symmetrizations about $\pi_1, \ldots, \pi_m$.

  Then $A_k^{\square} \in \mathscr{S}(A)$.

  By the preceding theorem, $A_k^{\square} \to B^{\square}$ as $k \to \infty$.

  But $B^{\square} \subseteq (r_0 - \varepsilon) U$.

  So there is a k such that $A_k^{\square} \subseteq \left(r_0 - \frac{1}{2}\varepsilon\right) U$.

  This, however, contradicts the definition of r_0.

  Thus $B = r_0 U$.

New Proof of Isoperimetric Inequality

- Let A be a convex body in $\mathbb{R}^n$ and let r be the radius of a ball having the same volume as that of A, i.e., $\omega_n r^n = v_n(A)$.
- The theorem shows the existence of a sequence of convex bodies converging to rU, each of whose members has surface area not exceeding $s_n(A)$.
- Hence
$$s_n(rU) = n\omega_n r^{n-1} \le s_n(A).$$
- We can thus deduce the isoperimetric inequality:
$$s_n^n(A) \ge (n\omega_n r^{n-1})^n = n^n \omega_n (\omega_n r^n)^{n-1} = \omega_n n^n v_n^{n-1}(A).$$

New Proof of Brunn's Inequality

- Let A, B be convex bodies in $\mathbb{R}^n$ and let $r, s > 0$ be such that $v_n(A) = \omega_n r^n$, $v_n(B) = \omega_n s^n$.
- Let $0 < k < 1$.
- It follows, by applying the theorem twice, that there exists a finite sequence of symmetrizations about hyperplanes through the origin which sends A, B to convex bodies $A^{\Box}, B^{\Box}$, respectively, such that $A^{\Box} \supseteq krU$, $B^{\Box} \supseteq ksU$.
- A previous theorem shows, that for any hyperplane π through the origin, $v_n(A_\pi + B_\pi) \leq v_n((A+B)_\pi) = v_n(A+B)$.
- We can deduce, by repeated applications of this result, that
$$\omega_n k^n (r+s)^n = v_n(k(r+s)U) \leq v_n(A^{\Box} + B^{\Box}) \leq v_n(A+B).$$
- Letting $k \to 1^-$, we deduce that $\omega_n (r+s)^n \leq v_n(A+B)$.
- Hence
$$v_n^{1/n}(A) + v_n^{1/n}(B) \leq v_n^{1/n}(A+B).$$