# Introduction to Differential Geometry

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LSSU Math 600

1. Introduction to Manifolds
   - Preliminaries on $\mathbb{R}^n$
   - $\mathbb{R}^n$ and Euclidean Space
   - Topological Manifolds
   - Further Examples. Cutting and Pasting
   - Abstract Manifolds. Some Examples

Subsection 1

## Preliminaries on $\mathbb{R}^n$

# The $n$-Fold Cartesian Product $\mathbb{R}^n$

- Let $\mathbb{R}$ denote the set of real numbers.
- Let $\mathbb{R}^n$ be their $n$-fold Cartesian product

$$\overbrace{\mathbb{R} \times \cdots \times \mathbb{R}}^{n},$$

the set of all ordered $n$-tuples $(x^1, \ldots, x^n)$ of real numbers.
- Individual $n$-tuples may be denoted at times by a single letter.
- Thus $x = (x^1, \ldots, x^n)$, $a = (a^1, \ldots, a^n)$, and so on.
- We agree to use on $\mathbb{R}^n$ its topology as a metric space, with the metric defined by

$$d(x, y) = \left( \sum_{i=1}^{n} (x^i - y^i)^2 \right)^{1/2}.$$

# Neighborhoods

- The neighborhoods are then open balls $B^n_\varepsilon(x)$, or $B_\varepsilon(x)$, defined, for any $\varepsilon > 0$, as
$$B_\varepsilon(x) = \{y \in \mathbb{R}^n : d(x,y) < \varepsilon\}.$$
- One may take equivalently open cubes $C^n_\varepsilon(x)$, or $C_\varepsilon(x)$, of side $2\varepsilon$ and center $x$, defined by
$$C_\varepsilon(x) = \{y \in \mathbb{R}^n : |x^i - y^i| < \varepsilon, \ i = 1, \ldots, n\}.$$
- Note that $\mathbb{R}^1 = \mathbb{R}$.
- We define $\mathbb{R}^0$ to be a single point.

# Meanings of $\mathbb{R}^n$

- We shall invariably consider $\mathbb{R}^n$ with the topology defined by the metric.
- This space $\mathbb{R}^n$ is used in several senses, however, and we must usually decide from the context which one is intended.
  - Sometimes $\mathbb{R}^n$ means merely $\mathbb{R}^n$ as topological space;
  - Sometimes $\mathbb{R}^n$ denotes an $n$-dimensional vector space;
  - Sometimes it is identified with Euclidean space.

# The Issue of Naturality

- We assume familiarity with the definition and basic theorems of vector spaces over $\mathbb{R}$.
- Among these is the theorem which states that any two vector spaces over $\mathbb{R}$ which have the same dimension $n$ are isomorphic.
- It is important to note that this isomorphism depends on choices of bases in the two spaces.
- There is in general no natural or canonical isomorphism independent of these choices.
- However, there does exist one important example of an $n$-dimensional vector space over $\mathbb{R}$ which has a distinguished or canonical basis.
- By this we mean a basis given by the nature of the space itself.
- This is the vector space of $n$-tuples of real numbers with componentwise addition and scalar multiplication.

# The Space $\boldsymbol{V}^n$

- The vector space of $n$-tuples of real numbers with componentwise addition and scalar multiplication is, as a set at least, just $\mathbb{R}^n$.
- To avoid confusion, sometimes we will denote it by $\boldsymbol{V}^n$.
- We then use boldface for its elements (e.g., $\boldsymbol{x}$ instead of $x$).
- For this space the $n$-tuples

$$\boldsymbol{e}_1 = (1, 0, \ldots, 0), \ldots, \boldsymbol{e}_n = (0, 0, \ldots, 0, 1)$$

  form a basis, known as the **natural** or **canonical basis**.
- We may at times suppose that the $n$-tuples are written as rows, that is, $1 \times n$ matrices, and at other times as columns, that is, $n \times 1$ matrices.
- This only becomes important when we use matrix notation to simplify things, e.g., to describe linear mappings, equations, and so on.

# Euclidean Spaces

- $\mathbb{R}^n$ may denote a vector space of dimension $n$ over $\mathbb{R}$.
- We sometimes mean even more by $\mathbb{R}^n$.
- An abstract $n$-dimensional vector space over $\mathbb{R}$ is called **Euclidean** if it has defined on it a positive definite inner product.
- In general there is no natural way to choose such an inner product.
- In the case of $\mathbb{R}^n$ or $\boldsymbol{V}^n$, we have the natural inner product

$$(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i=1}^{n} x^i y^i.$$

- It is characterized by the fact that relative to this inner product the natural basis is orthonormal, $(\boldsymbol{e}_i, \boldsymbol{e}_j) = \delta_{ij}$.

# The Norm

- The metric in $\mathbb{R}^n$ discussed at the beginning can be defined using the inner product on $\mathbb{R}^n$.
- We define $\|\boldsymbol{x}\|$, the **norm** of the vector $\boldsymbol{x}$, by

$$\|\boldsymbol{x}\| = ((\boldsymbol{x}, \boldsymbol{x}))^{1/2}.$$

- Then we have

$$d(x, y) = \|\boldsymbol{x} - \boldsymbol{y}\|.$$

- This notation is frequently useful even when we are dealing with $\mathbb{R}^n$ as a metric space and not using its vector space structure.
- Note, in particular, that $\|\boldsymbol{x}\| = d(x, 0)$, the distance from the point $x$ to the origin.
- In this equality $\boldsymbol{x}$ is a vector on the left-hand side and $x$ is the corresponding point on the right-hand side.

Subsection 2

## $\mathbb{R}^n$ and Euclidean Space

# Euclidean Space

- Another role which $\mathbb{R}^n$ plays is that of a model for $n$-dimensional Euclidean space $\boldsymbol{E}^n$, in the sense of Euclidean geometry.
- Some texts refer to $\mathbb{R}^n$ with the metric $d(x, y)$ as Euclidean space.
- This identification is misleading in the same sense that it would be misleading to identify all $n$-dimensional vector spaces with $\mathbb{R}^n$.
- It is an identification that can hamper clarification of the concept of manifold and the role of coordinates.
- Euclid and the geometers before the seventeenth century did not think of the Euclidean plane $\boldsymbol{E}^2$ or three-dimensional space $\boldsymbol{E}^3$ as pairs or triples of real numbers.
- They were rather defined axiomatically.

# Coordinatization

- Consider the Euclidean plane $\boldsymbol{E}^2$ as studied in high school geometry.
- We later introduce coordinates using the notions of length and perpendicularity.
- We choose two mutually perpendicular "number axes".
- They are used to define a one-to-one mapping of $\boldsymbol{E}^2$ onto $\mathbb{R}^2$ by

$$p \mapsto (x(p), y(p)),$$

  where $x(p)$ and $y(p)$ are the coordinates of $p \in \boldsymbol{E}^2$.
- This mapping is (by design) an isometry, preserving distances of points of $\boldsymbol{E}^2$ and their images in $\mathbb{R}^2$.

# Coordinatization (Cont'd)

- Finally, we obtain further correspondences of essential geometric elements.
- E.g., lines of $\boldsymbol{E}^2$ correspond to subsets of $\mathbb{R}^2$ consisting of the solutions of linear equations.
- Thus, we carry each geometric object to a corresponding one in $\mathbb{R}^2$.
- It is the existence of such coordinate mappings which make the identification of $\boldsymbol{E}^2$ and $\mathbb{R}^2$ possible.

# Coordinate Systems

- An arbitrary choice of coordinates is involved.
- There is no natural, geometrically determined way to identify the two spaces.
- Thus, at best, we can say that $\mathbb{R}^2$ may be identified with $\boldsymbol{E}^2$ plus a coordinate system.
- Even then we need to define in $\mathbb{R}^2$ the notions of line, angle of lines and other attributes of the Euclidean plane, before thinking of it as Euclidean space.
- Thus, with qualifications, we may identify $\boldsymbol{E}^2$ and $\mathbb{R}^2$ or $\boldsymbol{E}^n$ and $\mathbb{R}^n$, especially remembering that they carry a choice of rectangular coordinates.

# Properties

- Using the analytic geometry approach to the study of a geometry makes it sometimes difficult to distinguish between:
  - Underlying geometric properties;
  - Properties which depend on the choice of coordinates.

  Example: Suppose we have identified $\boldsymbol{E}^2$ and $\mathbb{R}^2$.

  Suppose we identify lines with the graphs of linear equations.

  E.g., consider

  $$L = \{(x, y) : y = mx + b\}.$$

  We then define the slope $m$ and the $y$-intercept $b$.

# Properties (Cont'd)

- Neither slope nor $y$-intercept has geometric meaning.

  They both depend on the choice of coordinates.

  However, given two such lines of slope $m_1$, $m_2$, the expression

  $$\frac{m_2 - m_1}{1 + m_1 m_2}$$

  does have geometric meaning.

  This can be demonstrated in one of two ways:
  - Directly checking independence of the choice of coordinates.
  - Determining that its value is the tangent of the angle between the lines. This concept is indeed independent of coordinates.

# The Two Approaches

- It should be clear that it can be difficult to do geometry, even in the simplest case of Euclidean geometry, working with coordinates alone, that is, with the model $\mathbb{R}^n$.
- We need to develop both approaches:
  - The coordinate method;
  - The coordinate-free method.
- We shall often seek ways of looking at manifolds and their geometry, which do not involve coordinates.
- But will use coordinates as a useful computational device (and not only), when necessary.
- Being aware now of what is involved, we shall usually refer to $\mathbb{R}^n$ as Euclidean space and make the identification.
- This is especially true when we are interested only in questions involving topology or differentiability.

## Subsection 3

# Topological Manifolds

# Locally Euclidean Spaces

- Of all the spaces which one studies in topology the Euclidean spaces and their subspaces are the most important.
- As we have just seen, the metric spaces $\mathbb{R}^n$ serve as a **topological model** for Euclidean space $\boldsymbol{E}^n$, for finite-dimensional vector spaces over $\mathbb{R}$ or $\mathbb{C}$, and for other basic mathematical systems.
- It is natural enough that we are led to study those spaces which are **locally** like $\mathbb{R}^n$.
- These are the spaces for which each point $p$ has a neighborhood $U$ which is homeomorphic to an open subset $U'$ of $\mathbb{R}^n$, $n$ fixed.
- A space with this property is said to be **locally Euclidean of dimension** $n$.

# Manifolds

- In order to stay as close as possible to Euclidean spaces, we will consider spaces called manifolds.

**Definition**

A **manifold** $M$ of dimension $n$, or $n$-**manifold**, is a topological space with the following properties:

(i) $M$ is Hausdorff;

(ii) $M$ is locally Euclidean of dimension $n$.

(iii) $M$ has a countable basis of open sets.

- As a matter of notation $\dim M$ is used for the **dimension** of $M$;
- When $\dim M = 0$, then $M$ is a countable space with the discrete topology.

# The Locally Euclidean Property

- It follows from the homeomorphism of $U$ and $U'$ that **locally Euclidean** is equivalent to the requirement that each point $p$ have a neighborhood $U$ homeomorphic to an $n$-ball in $\mathbb{R}^n$.
- Thus a manifold of dimension 1 is locally homeomorphic to an open interval.
- Similarly, a manifold of dimension 2 is locally homeomorphic to an open disk, and so on.

# Example

- Our first examples will remove any lingering suspicion that an $n$-manifold is necessarily globally equivalent, that is, homeomorphic, to $\boldsymbol{E}^n$.

  Example: Let $M$ be an open subset of $\mathbb{R}^n$ with the subspace topology.

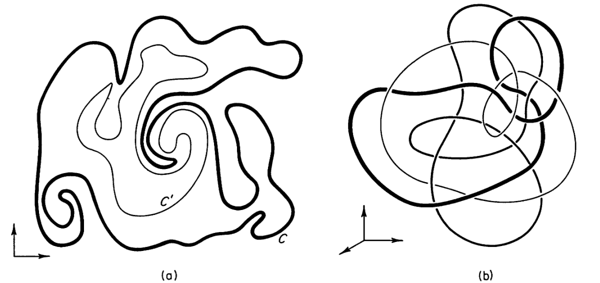
  Then $M$ is an $n$-manifold.

  Indeed Properties (i) and (iii) of the definition are hereditary, holding for any subspace of a space which possesses them.

  Property (ii) holds with $U = U' = M$ and with the homeomorphism of $U$ to $U'$ being the identity map.

# Example (Cont'd)

- We use some imagination, assisted perhaps by the figure.




(a) The manifold is the open set $M$ of $\boldsymbol{R}^2$ between the curves $C$ and $C'$. (b) The manifold is the open subset of $\boldsymbol{R}^3$ obtained by removing the knots.

- Even when $n = 2$ or 3 these examples can be rather complicated and certainly not equivalent to Euclidean space in general.
- They may be equivalent in special cases: e.g., trivially when $M = \boldsymbol{E}^n$.

# Example: Manifolds Not Homeomorphic to Open Subsets

- Consider the circle $S^1$ and the 2-sphere $S^2$, which may be defined to be all points of $\boldsymbol{E}^2$, or of $\boldsymbol{E}^3$, respectively, which are at unit distance from a fixed point 0.
- These are to be taken with the subspace topology so that (i) and (iii) are immediate.
- To see that they are locally Euclidean we introduce coordinate axes with 0 as origin in the corresponding ambient Euclidean space.

# Example (Cont'd)

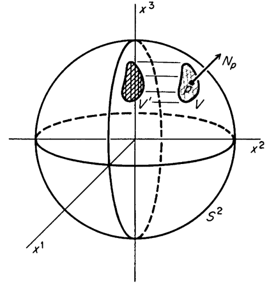
- Thus in the case of $S^2$ we identify $\mathbb{R}^3$ and $\boldsymbol{E}^3$, and $S^2$ becomes the unit sphere centered at the origin.
- At each point $p$ of $S^2$ we have:
  - A tangent plane;
  - A unit normal vector $N_p$.
- There will be a coordinate axis which is not perpendicular to $N_p$.
- Some neighborhood $U$ of $p$ on $S^2$ will then project in a continuous and one-to-one fashion onto an open set $U'$ of the coordinate plane perpendicular to that axis.

# Example (Cont'd)

- Consider the figure.
- $N_p$ is not perpendicular to the $x_2$-axis.
- So for $q \in U$, the projection is given quite explicitly by

$$\varphi(q) = (x^1(q), 0, x^3(q)),$$

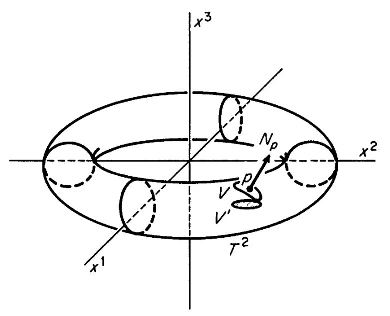
where $(x^1(q), x^2(q), x^3(q))$ are the coordinates of $q$ in $\boldsymbol{E}^3$.




- Similar considerations show that $S^1$ is locally Euclidean.
- Note that $S^2$ and $\mathbb{R}^2$ cannot be homeomorphic since one is compact while the other is not.

# Example: Torus

- Our final example is that of the surface of revolution obtained by revolving a circle around an axis which does not intersect it.
- The figure we obtain is the torus or "inner tube" (denoted $T^2$).




- This figure can be studied analytically.
- We may write down an equation whose locus is $T^2$ if we introduce coordinates in $\boldsymbol{E}^3$ as shown in the figure.

# Example: Torus (Cont'd)

- $T^2$ is indeed locally Euclidean.
- Consider once more the normal vector $N_p$ at $p \in T^2$.
- There will be at least one coordinate axis to which it is not perpendicular, say $x^3$.
- Then some neighborhood $U$ of $p$ projects homeomorphically onto a neighborhood $U'$ in the $x^1x^2$-plane.
- We use the relative topology derived from $\boldsymbol{E}^3$.
- So the space $T^2$ is necessarily Hausdorff and has a countable basis of open sets.
- Thus Conditions (i)-(iii) of the manifold definition are satisfied.

# Remark

- It should be clear from the last two examples that certain subspaces $M$ of $\boldsymbol{E}^3$ are easily seen to be 2-manifolds.
- They are surfaces which are "smooth", i.e., without corners or edges.
- So they have at each $p \in M$:
  - A (unit) normal vector $N_p$, which varies continuously as we move from point to point;
  - A tangent plane $T_p(M)$.
- Continuity means that the components of the unit normal vector depend continuously on the point $p$.
- This smoothness allows us to prove the locally Euclidean property by projection of a neighborhood of $p$ onto a plane, as in the preceding examples.
- The other properties are immediate since we use the subspace topology.

# A Little Topology

- Recall that a topological space is called **normal** if, for each disjoint pair of closed sets $A$ and $B$, there are disjoint open sets $U$ and $V$, such that $A \subseteq U$ and $B \subseteq V$.
- Lindelöf's Theorem asserts that in a topological space whose topology has a countable base, every open cover of a subset of the space has a countable subcover.
- A $T_1$-**space** is one in which singletons are closed.
- A space is **regular** if, for each point $x$ and each neighborhood $U$ of $x$, there is a closed neighborhood $V$ of $x$, such that $V \subseteq U$.
- Urysohn's Metrization Theorem asserts that a regular $T_1$-space whose topology has a countable base is homeomorphic to a subspace of the unit cube $Q^\omega$ and is, hence, metrizable.

# Properties of Topological Manifolds

**Theorem**

A topological manifold $M$ is locally connected, locally compact and a union of a countable collection of compact subsets. Furthermore, it is normal and metrizable.

- These are all immediate consequences of the definition and standard theorems of general topology.

  Let $p$ be a point of $M$ and $U$ a neighborhood of $p$ homeomorphic to an open ball $B_\varepsilon(x)$ of radius $\varepsilon$ in $\mathbb{R}^n$.

  We denote the homeomorphism by $\varphi$, and suppose $\varphi(p) = x$.

  Interior to any neighborhood $V$ of $p$, there is a neighborhood $W$ whose closure $\overline{W}$ is in $V$, for which $\varphi(W) = B_\delta(x)$, $\varepsilon > \delta > 0$.

  Now $B_\delta(x)$ and hence $W$, to which it is homeomorphic by $\varphi^{-1}$, is connected. It follows that $M$ is locally connected.

# Properties of Topological Manifolds (Cont'd)

- Similarly $\overline{W}$ is compact since $\overline{B}_\delta(x)$ is compact.

  Thus, $M$ is locally compact.

  By hypothesis, $M$ has a countable base of open sets.

  So we may now suppose that it has a countable base of relatively compact open sets $\{V_i\}$.

  Obviously $M = \bigcup \overline{V}_i$.

  Normality follows from Lindelöf's theorem.

  Metrizability is a consequence of Urysohn's Metrization Theorem.

# Different Dimensions

- There is one difficulty in our concept of manifold.
- It concerns Euclidean spaces and their topology and arises even before consideration of manifolds.
- It is the question of dimension.
- Could it be that $\boldsymbol{E}^n$ and $\boldsymbol{E}^m$ are homeomorphic, or locally homeomorphic, so that an open set $U$ of $\boldsymbol{E}^n$ is homeomorphic to some open set $U'$ of $\boldsymbol{E}^m$, with $m \neq n$?
- The answer is no, but the proof requires algebraic topology.
- The result is known as Brouwer's Theorem on Invariance of Domain.
- Later we will give a differentiable version of this theorem.
- For now we assume the theorem.

# Coordinate Neighborhoods and Coordinates

- The notion of coordinates plays an important role in manifold theory, just as it does in the study of the geometry of $\boldsymbol{E}^n$.
- In $\boldsymbol{E}^n$, however, it is possible to find a single system of coordinates for the entire space, that is, to establish a correspondence between all of $\boldsymbol{E}^n$ and $\mathbb{R}^n$.
- Built into the definition of $n$-manifold $M$ is a correspondence of a neighborhood $U$ of each $p \in M$ and an open subset $U'$ of $\mathbb{R}^n$.
- Let $\varphi : U \to U'$ be this correspondence.
- The pair $U, \varphi$ is called a **coordinate neighborhood** of $p \in M$.
- The numbers $x^1(p), \ldots, x^n(p)$, given by

$$\varphi(p) = (x^1(p), \ldots, x^n(p)),$$

are called the **coordinates** of $p \in M$.

# Coordinate Functions

- The assumption is that $\varphi$ is a homeomorphism, i.e., it is one-to-one and both $\varphi$ and $\varphi^{-1}$ are continuous.
- Thus each $q \in U$ has $n$ uniquely determined real coordinates, which vary continuously with $q$.
- For each $1 \leq i \leq n$, the function

$$q \mapsto x^i(q),$$

  is called the ***i*th coordinate function**.
- It is, by definition, continuous.
- There is obviously nothing unique about our choice of coordinates.
- Finally, note that even in the case of Euclidean space it is often useful to use local coordinates.
- E.g., the domain of a polar coordinate system on $\boldsymbol{E}^2$ must omit a ray if it is to be one-to-one.

Subsection 4

# Further Examples. Cutting and Pasting

# Introducing Manifolds With Boundary

- Typical examples of manifolds with boundary:
  - A hemispherical cap (including the equator);
  - A right circular cylinder (including the circles at the ends).
- Except for the equator, or the end-circles, they are 2-manifolds.
- The boundary sets are themselves manifolds of dimension one less.
- In fact, they are homeomorphic to $S^1$ or to $S^1 \cup S^1$ in these two cases.

# Introducing Manifolds With Boundary (Cont'd)

- An even simpler example is the upper half-plane $H^2$.
- More generally we may consider $H^n$, the subspace of $\mathbb{R}^n$ defined by

$$H^n = \{(x^1, \ldots, x^n) \in \mathbb{R}^n : x^n \geq 0\}.$$

- Every point $p \in H^n$ has a neighborhood $U$ which is homeomorphic to an open subset $U'$ of $\mathbb{R}^n$ except the set of points $(x^1, \ldots, x^{n-1}, 0)$.
- This set forms a subspace homeomorphic to $\mathbb{R}^{n-1}$
- It is called the **boundary** of $H^n$ and denoted by $\partial H^n$.

# Manifolds With Boundary

- We shall define a **manifold with boundary** to be a Hausdorff space $M$ with a countable basis of open sets which has the property that each $p \in M$ is contained in an open set $U$, with a homeomorphism

$$\varphi : U \to U',$$

where $U'$ is one of the following:

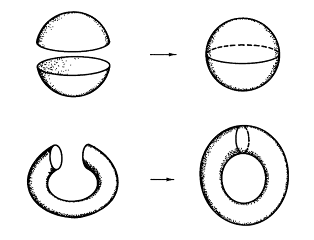
(a) An open set of $H^n - \partial H^n$;
(b) An open set of $H^n$ with $\varphi(p) \in \partial H^n$, i.e., a boundary point of $H^n$.

# Interior and Boundary

- Let $M$ be a manifold with boundary.
- It can be shown (as a consequence of invariance of domain) that every $p \in M$ satisfies exactly one of (a) or (b).
  - Those $p$ of the first type are called **interior points** of $M$.
  - Those $p$ mapped onto the boundary of $H^n$ by one, and hence by all, homeomorphisms of their neighborhoods into $H^n$ are called **boundary points**.
- The collection of boundary points is denoted by $\partial M$ and is called the **boundary of** $M$.
- The boundary $\partial M$ of $M$ is a manifold of dimension $n-1$.

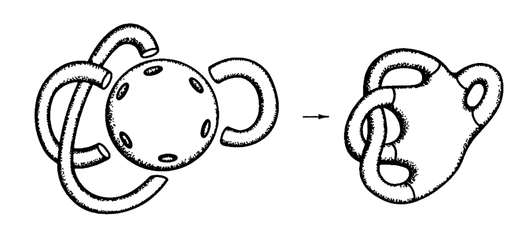
# Pasting Manifolds Along Boundaries

- Our interest is in pointing out that new surfaces, that is, 2-manifolds, can be formed by fastening together manifolds with boundary along their boundaries.
- This involves identifying points of various boundary components by a homeomorphism, assuming, of course, the necessary condition that such components are homeomorphic.
- The simplest examples are:
  - $S^2$, which is obtained by pasting two disks (or hemispheres) together so as to form the equator;
  - $T^2$, formed by pasting the two end-circles of a cylinder together.



# Pretzel-Like Surfaces

- One can go much further and paste any number of cylinders onto a sphere $S^2$ with "holes", that is, with circular disks removed.
- This gives various pretzel-like surfaces as illustrated below.



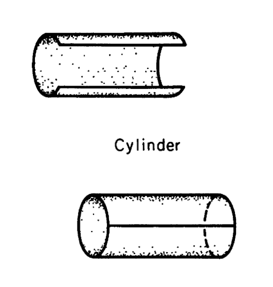
- One can prove that these are manifolds.

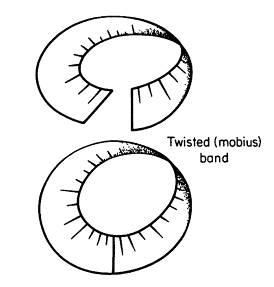
# Cutting and Pasting

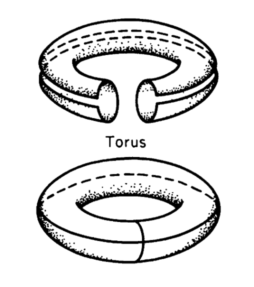
- To generate new 2-manifolds from old ones we may:
  (1) Cut out two disks, leaving a manifold $M$ whose boundary $\partial M$ is the disjoint union of two circles;
  (2) Paste on a cylinder or "handle" so that each end-circle is identified with one of the boundary circles of $M$.

# More on Cutting and Pasting

- The pasting on of handles is not the only way in which we can generate examples of 2-manifolds.
- It is also possible to do so by identifying or pasting together the edges of certain polygons.
- For example, the sides of a square may be identified in various ways in order to obtain surfaces.

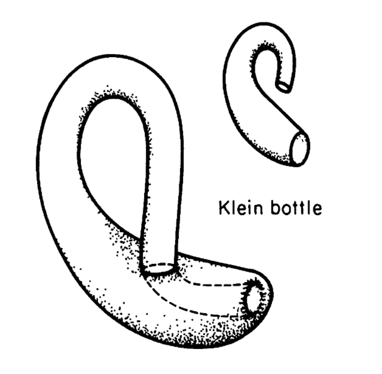


# More on Cutting and Pasting: The Klein Bottle

- The Klein bottle cannot be pictured as a surface in $\boldsymbol{E}^3$ unless we allow it to cut itself as shown.




- Thus as a subspace of $\boldsymbol{E}^3$ it is not a manifold.
- It is possible to identify the sides of the square, as shown, and obtain a manifold, but it is not possible to put it inside $\boldsymbol{E}^3$.

# Orientable Manifolds with Boundary

- Let $M$ be a connected 2-manifold, which lies smoothly inside $\boldsymbol{E}^3$.
- That is, at each point $p$, there is a tangent plane and normal line $L_p$.
- We may ask whether it is possible to choose a unit normal vector $N_p$ (on $L_p$), for every $p \in M$, which varies continuously with $M$.
- This is possible for $S^2$ and $T^2$.
- It is not for the Mobius band (which is actually a manifold with boundary) or the Klein bottle.
- We say that a manifold or manifold with boundary is **orientable** if such a choice of $N_p$ is possible.

# Fundamental Theorem of 2-Manifolds

## Theorem

Every compact, connected, orientable 2-manifold is homeomorphic to a sphere with handles added. Two such manifolds with the same number of handles are homeomorphic and conversely, so that the number of handles is the only topological invariant.

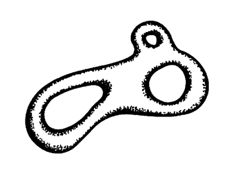
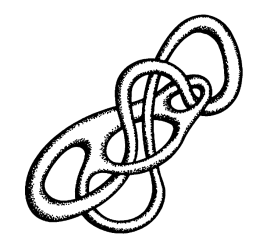
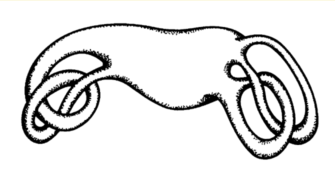
- Nonorientable, as well as noncompact 2-manifolds, can be described equally completely, although the noncompact case is more involved.
- Also, every connected, one-dimensional manifold is homeomorphic to $S^1$ or to $\mathbb{R}$, depending on whether it is compact or not.
- However, beginning with $n = 3$ everything is far more complicated and no such classification is known, even in the compact case.

Subsection 5

## Abstract Manifolds. Some Examples

# Manifolds Pictured in $\boldsymbol{E}^3$

- The manifolds of dimensions 1 and 2 considered above are pictured as subspaces of $\boldsymbol{E}^3$ except in the case of the Klein bottle.
- This is the way in which manifolds are first and most easily visualized.
- However, the definition makes no such requirement.
- Such visualization makes equivalent (homeomorphic) manifolds look different just because they are differently placed in Euclidean space.
- In spite of appearances, the following are homeomorphic manifolds.

# "Abstractly" Defined Manifolds

- As we might expect from the definition, it is possible to give examples of manifolds which we do not think of as lying in Euclidean space.
- Indeed, it is not clear that they can be realized at all as a subspace of Euclidean space.
- This can already be guessed from the construction of manifolds by pasting, which does not really use $\boldsymbol{E}^3$ at all.
- The simplest, as well as one of the most important examples of manifolds defined "abstractly", that is, not as a subspace of Euclidean space, is real projective space $P^n(\mathbb{R})$, the space of (real) projective geometry.

# Real Projective Space $P^n(\mathbb{R})$

- Let an equivalence relation $\sim$ be defined on $\mathbb{R}^{n+1} - \{0\}$ by

$$(x^1, \ldots, x^{n+1}) \sim (y^1, \ldots, y^{n+1})$$

  if there is a real number $t$, such that $y^i = tx^i$, $i = 1, \ldots, n+1$, i.e., $y = tx$.
- We denote by $[x]$ the equivalence class of $x$.
- Let $P^n(\mathbb{R})$ be the set of equivalence classes.
- There is a natural map $\pi : \mathbb{R}^{n+1} - \{0\} \to P^n(\mathbb{R})$ given by

$$\pi(x) = [x].$$

- We topologize $P^n(\mathbb{R})$ by saying that $U \subseteq P^n(\mathbb{R})$ is open if and only if $\pi^{-1}(U)$ is open in $\mathbb{R}^{n+1}$.
- This gives $P^n(\mathbb{R})$ the structure of an $n$-manifold.

# Alternative Description of $P^n(\mathbb{R})$

- We note that there is an alternative description of $P^n(\mathbb{R})$ as the space of all lines through the origin 0 of $\mathbb{R}^{n+1}$.
- $\pi$ takes each $x \neq 0$ to the line through 0 which contains it.
- Then we define the topology as follows.
- A collection $\widetilde{U}$ of lines is open if it is the set of all lines through 0 which meet a given open set $U$.

# Generalization

- Let $M$ be the set of all $r$-planes through the origin in $\mathbb{R}^n$, where $n$ and $r$ are fixed.
- E.g., the set of all planes through the origin in $\mathbb{R}^3$ or the set of all three-dimensional planes through the origin of $\mathbb{R}^5$, and so on.
- This set has a natural topology which makes it a manifold.
- Intuitively it consists of defining a neighborhood of a given plane $p$ to be all planes $q$ which are "close" to it in a relatively obvious sense.
- There exist corresponding bases of both planes $p$ and $q$ (considered as $r$-dimensional subspaces of $\mathbb{R}^n$, viewed as a vector space), such that corresponding basis vectors are close, say, for example, that their differences have norm less than some $\varepsilon > 0$.

# Tangent Bundle of $S^2$

- Consider $S^2$, the unit sphere in $\mathbb{R}^3$.
- We denote by $T(S^2)$ the collection of all tangent vectors to points of $S^2$, including the zero vector at each point.
- Thus,

$$T(S^2) = \bigcup_{p \in S^2} T_p(S^2).$$

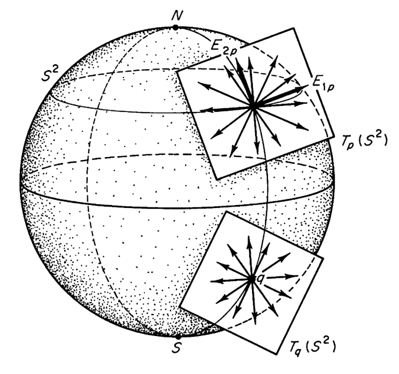
- This set has a natural topology.
- Two tangent vectors $X_p$ and $Y_q$ are "close" if their initial points $p$ and $q$ and their terminal points are close.

# Tangent Bundle of $M$

- Let $M$ be any 2-manifold, lying "smoothly" in $\boldsymbol{E}^3$, so as to have a tangent plane at each point which turns continuously as we move about on $M$.
- Then $T(M) = \bigcup_{p \in M} T_p(M)$ is a manifold.
- It is called the **tangent bundle of** $M$.
- The dimension of $T(M)$ is 4 since, roughly speaking, $X_p$ depends locally on four parameters:
  - Two being the local coordinates of $p$ relative to some coordinate neighborhood $U$;
  - Two more being the components which determine $X_p$ relative to some basis $\{E_{1p}, E_{2p}\}$ of $T_p(M)$, a basis which varies continuously over the neighborhood $U$.

# Tangent Bundle of $M$ (Cont'd)

- We later make these statements quite precise.
- At the same time, we exhibit the locally Euclidean character of $T(M)$.
- For now, we note that $E_1$ and $E_2$ can be visualized as vectors tangent to the coordinate curves $x^1 =$ constant and $x^2 =$ constant in $U$.



# Remark on Tangent Bundles

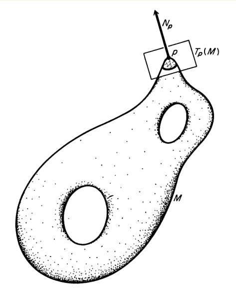
- We should note that these manifolds are not subspaces of $\boldsymbol{E}^3$, even though $M$ is and although the geometry of $\boldsymbol{E}^3$ is used here to describe them.
- One of our major tasks is to describe $T_p(M)$ and $T(M)$ independently of any way of placing $M$ in Euclidean space.
- In other words, we wish to give a description valid for an abstract manifold.

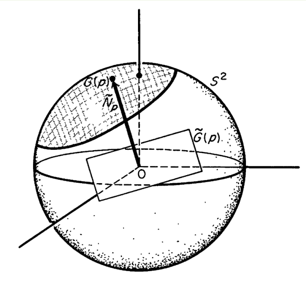
# The Gauss Mapping

- Let $M$ be such an orientable surface in $\boldsymbol{E}^3$.
- Let $N_p$ be a unit normal vector at each $p \in M$, such that $N_p$ varies continuously with $p$ on $M$.
- Translate $N_p$ to $\widetilde{N}_p$ from a fixed origin 0.
- Let $G(p)$ be the endpoint of $\widetilde{N}_p$ on $S^2$, the unit sphere at 0.
- The mapping taking $p$ to $G(p)$ is known as the **Gauss mapping**.
- The **Gaussian curvature** is a measure of the distortion of areas under this mapping.
- If $M$ is sharply curved near $p$, then the area of a small region around $p$ would be greatly magnified in mapping to $S^2$.

# The Gauss Mapping (Cont'd)

- Even if $M$ is not orientable, we still have a tangent plane $T_p(M)$ at each $p$ parallel to a uniquely determined plane $\widetilde{G}(p)$ through 0.






- Thus a slight variant of the previous definition defines a mapping of $M$ to the manifold of 2-planes through 0, introduced above.
- Using normal lines instead of tangent planes, we can obtain a mapping from $M$ to the manifold of lines through 0.
- This, as we have remarked, is equivalent to $P^2(\mathbb{R})$.