Introduction to Differential Geometry

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 600

Subsection 1

The Definition of a Differentiable Manifold

Topological Manifolds

- Recall that a **topological manifold** M of dimension n is a Hausdorff space, with a countable basis of open sets, such that each point has a neighborhood homeomorphic to an open subset of $\mathbb{R}^n$.
- A **coordinate neighborhood** is a pair U, φ, where:
  - U is an open set of M;
  - φ is a homeomorphism of U to an open subset of $\mathbb{R}^n$.
- To $q \in U$ we assign the n coordinates of its image $\varphi(q)$ in $\mathbb{R}^n$,

$$x^1(q), \ldots, x^n(q).$$

- Each $x^i(q)$ is a real-valued function on U, the ***i*th coordinate function**.

Topological Manifolds (Cont'd)

- Suppose q lies also in a second coordinate neighborhood V, ψ.
- Then it has coordinates $y^1(q), \ldots, y^n(q)$ in this neighborhood.
- Since φ and ψ are homeomorphisms, this defines a homeomorphism

$$\psi \circ \varphi^{-1} : \varphi(U \cap V) \to \psi(U \cap V).$$

- The domain and range are the two open subsets of $\mathbb{R}^n$ which correspond to the points of $U \cap V$ by the two coordinate maps φ, ψ, respectively.
- In coordinates, $\psi \circ \varphi^{-1}$ is given by continuous functions

$$y^i = h^i(x^1, \ldots, x^n), \quad i = 1, \ldots, n.$$

- The h^i's give the y-coordinates of each $q \in U \cap V$ in terms of its x-coordinates.

Topological Manifolds (Cont'd)

- Similarly $\varphi \circ \psi^{-1}$ gives the inverse mapping.
- It expresses the x-coordinates as functions of the y-coordinates,

$$x^i = g^i(y^1, \ldots, y^n), \quad i = 1, \ldots, n.$$

- The fact that $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ are homeomorphisms and are inverse to each other is equivalent to the following conditions:
  - The continuity of $h^i(x)$ and $g^j(y)$, $i, j = 1, \ldots, n$;
  - The identities

$$\begin{array}{rcl} h^i(g^1(y), \ldots, g^n(y)) & \equiv & y^i, \quad i = 1, \ldots, n, \\ g^i(h^1(x), \ldots, h^n(x)) & \equiv & x^j, \quad j = 1, \ldots, n. \end{array}$$

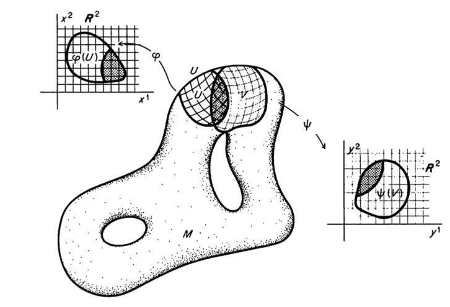
- Thus, every point of a topological manifold M lies in a very large collection of coordinate neighborhoods.
- Whenever two such neighborhoods overlap, we have the formulas just given for change of coordinates.

C^∞-Compatibility

- The basic idea that leads to differentiable manifolds is to try to select a family or subcollection of neighborhoods so that the change of coordinates is always given by differentiable functions.

Definition

We shall say that U, φ and V, ψ are C^∞-**compatible** if $U \cap V$ nonempty implies that the functions $h^i(x)$ and $g^j(y)$ giving the change of coordinates are C^∞. This is equivalent to requiring $\varphi \circ \psi^{-1}$ and $\psi \circ \varphi^{-1}$ to be *diffeomorphisms* of the open subsets $\varphi(U \cap V)$ and $\psi(U \cap V)$ of $\mathbb{R}^n$.



Differentiable Structure and C^∞-Manifolds

Definition

A **differentiable** or C^∞ (or **smooth**) **structure** on a topological manifold M is a family $\mathcal{U} = \{U_\alpha, \varphi_\alpha\}$ of coordinate neighborhoods such that:

(1) The U_α cover M;

(2) For any α, β, the neighborhoods U_α, φ_α and U_β, φ_β are C^∞-compatible;

(3) Any coordinate neighborhood V, ψ C^∞-compatible with every $U_\alpha, \varphi_\alpha \in \mathcal{U}$ is itself in $\mathcal{U}$.

A C^∞ **manifold** is a topological manifold together with a C^∞-differentiable structure.

Existence and Uniqueness

- It is, of course, conceivable that for some topological manifold no such family of compatible coordinate neighborhoods can be singled out.
- It is also conceivable that, on the contrary, families can be chosen in a multiplicity of inequivalent ways so that two inequivalent C^∞ manifolds have the same underlying topological manifold.
- These are basic but very difficult questions.
- What is important, from our point of view, is that we will be able to find an abundance of topological manifolds with at least one differentiable structure.
- So there exists an abundance of C^∞ manifolds.

Terminology and Conventions

- Since there is no danger of confusion, we will often say simply "manifold" for C^∞ manifold;
- We may also sometimes say **differentiable** or **smooth manifold**.
- "Coordinate neighborhood" will refer exclusively to the *coordinate neighborhoods belonging to the differentiable structure*.
- To consider a manifold without differentiable structure, we will say *topological* manifold and *topological* coordinate neighborhood.
- By requiring only that the change of coordinates be given by C^r functions, for $r < \infty$, we could define C^r**-compatible coordinate neighborhoods** and C^r **manifolds**, C^0 denoting a topological manifold.
- One can also require that the change of coordinates be C^ω, that is, real analytic.
- We shall restrict ourselves almost exclusively to the C^∞ case.

Sufficient Conditions for Existence of Structure

- The following proposition shows that Conditions (1) and (2) of the definition are the essential properties defining a C^∞ structure.
- Thus, in examples we need only check the compatibility of a covering by neighborhoods.

Theorem

Let M be a Hausdorff space with a countable basis of open sets. Suppose $V = \{V_\beta, \psi_\beta\}$ is a covering of M by C^∞-compatible coordinate neighborhoods. Then there is a unique C^∞ structure on M containing these coordinate neighborhoods.

- We shall define the differentiable structure to be the collection $\mathcal{U}$ of all topological coordinate neighborhoods U, φ which are C^∞-compatible with each and every one of those of the given collection $\{V_\beta, \psi_\beta\}$.

  This new collection naturally includes the V_β, ψ_β.

  So Property (1) of the definition is automatically satisfied.

Proof (Cont'd)

- Now we turn to Property (2).

  Let U, φ and U', φ', $U \cap U' \neq \emptyset$, be in $\mathcal{U}$.

  We must show that they are C^∞-compatible.

  U, φ and U', φ' are (topological) coordinate neighborhoods.

  So the functions

  $$\varphi' \circ \varphi^{-1} \quad \text{and} \quad \varphi \circ \varphi'^{-1},$$

  giving the change of coordinates, are well-defined homeomorphisms on open subsets of $\mathbb{R}^n$.

  So we need only make sure that they are C^∞.

Proof (Cont'd)

- Let $x = \varphi(p)$ be an arbitrary point of $\varphi(U \cap U')$.

  Then $p \in V_\beta$ for one of the coordinate neighborhoods V_β, ψ_β.

  It follows that:
  - $W = V_\beta \cap U \cap U'$ is an open set containing p;
  - $\varphi(W)$ is an open set containing x.

  On $\varphi(W)$, we have

$$\varphi' \circ \varphi^{-1} = \varphi' \circ \psi_\beta^{-1} \circ \psi_\beta \circ \varphi^{-1}.$$

  But U, φ and U', φ' are both C^∞-compatible with V_β, ψ_β.

  So $\varphi' \circ \psi_\beta^{-1}$ and $\psi_\beta \circ \varphi^{-1}$ are C^∞.

  It follows that their composition $\varphi' \circ \varphi^{-1}$ is C^∞ on $\varphi(W)$.

  Also, $\varphi' \circ \varphi^{-1}$ is C^∞ on a neighborhood of any point of its domain.

  So it is C^∞.

Proof (Cont'd)

- This proves everything except Property (3), which is automatic.

  Suppose U, φ is compatible with all of the coordinate neighborhoods in our collection.

  Then it certainly has this property with respect to the subcollection $\{V_\beta, \psi_\beta\}$.

  So it is in the differentiable structure.

Remark

- A coordinate neighborhood U, φ depends on both the neighborhood U and the map φ of U to $\mathbb{R}^n$.
- If we change either, then we have a different coordinate neighborhood.
- For example, suppose $V \subseteq U$ is an open subset.
- Then $V, \varphi|_V$ is a new coordinate neighborhood, although the coordinates of $p \in V$ are the same as its coordinates in the original neighborhood.
- If $p \in U$, we may choose V so that $\varphi(V)$ is an open ball $B_\epsilon^n(a)$, or cube $C_\epsilon^n(a)$, in $\mathbb{R}^n$ with $\varphi(p) = a$ as center.
- Or we might alter φ by composing it with a map $\theta : \mathbb{R}^n \to \mathbb{R}^n$.
- E.g., by composing with a translation, we can send some $p \in U$ to coordinates $(0, 0, \ldots, 0)$.
- Of course, this gives a new coordinate system on U.
- Thus, we get a new coordinate neighborhood $U, \theta \circ \varphi$.

Example (The Euclidean Plane)

- Once a unit of length is chosen, the Euclidean plane $\boldsymbol{E}^2$ becomes a metric space.
- It is Hausdorff and has a countable basis of open sets.
- The choice of an origin and mutually perpendicular coordinate axes establishes a homeomorphism (even an isometry) $\psi : \boldsymbol{E}^2 \to \mathbb{R}^2$.
- Thus we cover $\boldsymbol{E}^2$ with a single coordinate neighborhood V, ψ, with $V = \boldsymbol{E}^2$ and $\psi(V) = \mathbb{R}^2$.
- It follows that $\boldsymbol{E}^2$ is a topological manifold.
- By the theorem, V, ψ determines a differentiable structure.
- Thus, $\boldsymbol{E}^2$ is a C^∞ manifold.

Example (The Euclidean Plane Cont'd)

- There are many other coordinate neighborhoods on $\boldsymbol{E}^2$ which are C^∞-compatible with V, ψ.
- These also belong to the differentiable structure determined by V, ψ.
- For example, we may choose another rectangular Cartesian coordinate system V', ψ'.
- Then it is shown in analytic geometry that the change of coordinates is given by linear, hence C^∞ (even analytic) functions

$$\begin{aligned} y^1 &= x^1 \cos\theta - x^2 \sin\theta + h, \\ y^2 &= x^1 \sin\theta + x^2 \cos\theta + k. \end{aligned}$$

- Note that $V = V'$, but the coordinate neighborhoods are not the same since $\psi' \neq \psi$.
- That is, the coordinates of each point are different for the two mappings.

Example

- It is also possible to choose as U the plane minus a ray extending from a point 0.
- We use as coordinate functions on U:
  - The angle $\theta(q)$ measured from this ray to $\overline{0q}$;
  - The distance $r(q)$ of q from 0.
- We define a homeomorphism

$$\begin{array}{rrcl} \varphi : & U & \to & \{(r,\theta) \mid r > 0, 0 < \theta < 2\pi\} \subseteq \mathbb{R}^2; \\ & q & \mapsto & (r(q), \theta(q)). \end{array}$$

- The equations for change of coordinates to those above, assuming that 0 is the origin and that the ray is the positive x-axis, are

$$x^1 = r\cos\theta, \quad x^2 = r\sin\theta.$$

- These are analytic and, thus, C^∞.

Example (Cont'd)

- If the origin and axes are not chosen in this special way, then we must compose this mapping on $\mathbb{R}^2$ with a rotation and translation of the previous type to obtain the functions giving the change of coordinates.
- The various coordinate neighborhoods just enumerated are C^∞-compatible with our original V, ψ.
- So they are in the differentiable structure on $\boldsymbol{E}^2$ determined by V, ψ.

Euclidean Space Revisited

- In the same manner, Euclidean space of arbitrary dimension n gives an example of a C^∞ manifold, covered by a single coordinate system.
- Again, this may be done in a variety of ways.
- As we have noted, it is customary to identify $\boldsymbol{E}^n$ and $\mathbb{R}^n$ since the former is difficult to axiomatize.
- This is equivalent to choosing a fixed rectangular Cartesian coordinate system covering all of $\boldsymbol{E}^n$.
- Though many examples, it will become clear that manifolds, in general, cannot be covered by a single coordinate system nor are there preferred coordinates.
- Thus, it is often better in thinking of Euclidean space as a manifold to visualize the model $\boldsymbol{E}^n$ of classical geometry, without coordinates, rather than $\mathbb{R}^n$, Euclidean space with coordinates.

Finite-Dimensional Vector Spaces

- A finite-dimensional vector space $\boldsymbol{V}$ over $\mathbb{R}$ can be identified with $\mathbb{R}^n$, $n = \dim \boldsymbol{V}$, once a basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is chosen.
- Vector $\boldsymbol{v} = x^1 \boldsymbol{e}_1 + \cdots + x^n \boldsymbol{e}_n$ is identified with $(x^1, \ldots, x^n)$ in $\mathbb{R}^n$;
- Similarly, the space of $m \times n$ matrices (a_{ij}) can be identified with $\mathbb{R}^{mn}$.
- The matrix $A = (a_{ij})$ corresponds to $(a_{11}, \ldots, a_{1n}; \ldots; a_{m1}, \ldots, a_{mn})$.
- Using these identifications, we may define a natural topology and C^∞ structure on $\boldsymbol{V}$ and on the set $\mathcal{M}_{mn}(\mathbb{R})$ of $m \times n$ matrices over $\mathbb{R}$.
- We suppose them to be:
  - Homeomorphic to Cartesian or Euclidean space of dimension n in the case of $\boldsymbol{V}$, and mn in the case of $\mathcal{M}_{mn}(\mathbb{R})$;
  - Covered by a single coordinate neighborhood, the identification map above being the coordinate map.

Open Submanifolds

- Let M be a C^∞ manifold.
- Consider an open subset U of M.
- U is itself a C^∞ manifold with differentiable structure consisting of the coordinate neighborhoods V', ψ' obtained by restriction of ψ on those coordinate neighborhoods V, ψ, which intersect U, to the open set $V' = V \cap U$, that is,

$$\psi' = \psi \mid_{V \cap U} .$$

- This gives a covering of U by topological coordinate neighborhoods which are C^∞-compatible.
- Hence, it defines a C^∞ structure on U.
- U, with this structure, is said to be an **open submanifold** of M.

Example ($Gl(n, \mathbb{R})$)

- Consider the subset $U = Gl(n, \mathbb{R})$ of $M = \mathcal{M}_n(\mathbb{R})$, $n \times n$ matrices over $\mathbb{R}$, which consists of all nonsingular $n \times n$ matrices.
- Recall that an $n \times n$ matrix A is nonsingular if and only if its determinant $\det A$ is not zero.
- So we have
$$U = \{A \in \mathcal{M}_n(\mathbb{R}) : \det A \neq 0\}.$$
- This is the usual definition of the group $Gl(n, \mathbb{R})$.
- Now $\det A$ is a polynomial function of its entries a_{ij}.
- Hence, it is a continuous function of its entries.
- So it is a continuous function of A in the topology of identification with $\mathbb{R}^{n^2}$.
- Thus, $U = Gl(n, \mathbb{R})$ is an open set, the complement of the closed set of those A such that $\det A = 0$.
- So $Gl(n, \mathbb{R})$ is an open submanifold of $\mathcal{M}_n(\mathbb{R})$.

Product Manifold

- We state without proof a result on the manifold structure that can be constructed on the product of two manifolds.

Theorem

Let M and N be C^∞ manifolds of dimensions m and n.
Then $M \times N$ is a C^∞ manifold of dimension $m+n$.
Its C^∞ structure is determined by coordinate neighborhoods of the form

$$\{U \times V, \varphi \times \psi\},$$

where:

- U, φ is a coordinate neighborhood on M;
- V, ψ is a coordinate neighborhood on N;
- Homeomorphisms $\varphi \times \psi : U \times V \to \mathbb{R}^{m+n} = \mathbb{R}^m \times \mathbb{R}^n$, defined by

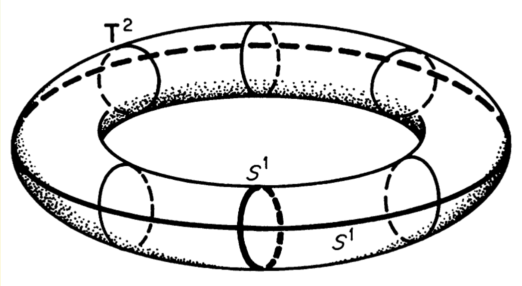
$$\varphi \times \psi(p, q) = (\varphi(p), \psi(q)).$$

Example: The Torus

- An important example is the torus

$$T^2 = S^1 \times S^1,$$

the product of two circles.



- More generally, $T^n = S^1 \times \cdots \times S^1$, the n-fold product of circles, is a C^∞ manifold obtained as a Cartesian product.

Example: The Sphere

- We give a fairly detailed proof, using the theorem, that the unit 2-sphere

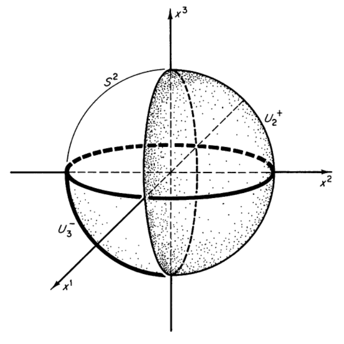
$$S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$$

  is a C^∞ manifold.

- The idea extends in an obvious way to

$$S^{n-1} = \{x \in \mathbb{R}^n : \|x\| = 1\},$$

  the unit $n-1$ sphere in $\mathbb{R}^n$.



Example: The Sphere (Cont'd)

- We take S^2 with its topology as a subspace of $\mathbb{R}^3$.
- That is, U is open in S^2 if $U = \widetilde{U} \cap S^2$, for some open set $\widetilde{U} \subseteq \mathbb{R}^3$.
- This implies that S^2 is Hausdorff with a countable basis.
- We shall show that it is locally Euclidean.
- For $i = 1, 2$, or 3, let

$$\begin{aligned} \widetilde{U}_i^+ &= \{(x^1, x^2, x^3) : x^i > 0\}; \\ \widetilde{U}_i^- &= \{(x^1, x^2, x^3) : x^i < 0\}. \end{aligned}$$

- These $\widetilde{U}_i^\pm$ are two open sets into which the coordinate hyperplane $x^i = 0$ divides $\mathbb{R}^3$.

Example: The Sphere (Cont'd)

- The relatively open sets

$$U_i^{\pm} = \widetilde{U}_i^{\pm} \cap S^2, \quad i = 1, 2, 3,$$

  cover S^2.
- We define $\varphi_i^{\pm} : U_i^{\pm} \to \mathbb{R}^2$ by projection.

$$\begin{aligned} \varphi_1^{\pm}(x^1, x^2, x^3) &= (x^2, x^3); \\ \varphi_2^{\pm}(x^1, x^2, x^3) &= (x^1, x^3); \\ \varphi_3^{\pm}(x^1, x^2, x^3) &= (x^1, x^2). \end{aligned}$$

- It can be checked that these are homeomorphisms to the open set

$$W = \{x \in \mathbb{R}^2 : \|x\| < 1\}.$$

- Thus, S^2 is locally Euclidean and a topological manifold.

Example: The Sphere (Cont'd)

- The formulas for the change of coordinates are C^∞.
- Thus, these coordinate neighborhoods are C^∞-compatible.
- For example, $\varphi_1^+ \circ (\varphi_2^-)^{-1}$ is given on $U_1^+ \cap U_2^-$ by composing $(\varphi_2^-)^{-1}$ and φ_1^+.

$$(x^1, x^3) \overset{(\varphi_2^-)^{-1}}{\to} (x^1, -(1-(x^1)^2-(x^3)^2)^{1/2}, x^3);$$

$$(x^1, -(1-(x^1)^2-(x^3)^2)^{1/2}, x^3) \overset{\varphi_1^+}{\to} (-(1-(x^1)^2-(x^3)^2)^{1/2}, x^3).$$

- Then, by change of notation, using (u^1, u^2) as U_2^--coordinates and (v^1, v^2) as U_1^+-coordinates instead of (x^1, x^3) and (x^2, x^3), we have

$$v^1 = -(1-(u^1)^2-(u^2)^2)^{1/2}, \quad v^2 = u^2.$$

Example: The Sphere (Cont'd)

- The square root term is never zero on
$$\{(u^1, u^2) : (u^1)^2 + (u^2)^2 < 1\}.$$
- So the v^i are C^∞ functions of the u^i.
- By similar computations, $\varphi_2^{-1} \circ (\varphi_1^+)^{-1}$ is C^∞ on
$$\{(v^1, v^2) : (v^1)^2 + (v^2)^2 < 1\}.$$
- Thus the coordinate neighborhoods U_1^+, φ_1^+ and U_2^-, φ_2^- are C^∞-compatible.
- Parallel arguments apply to the other cases.
- This naturally defined covering of S^2 by eight coordinate neighborhoods determines a unique C^∞ structure.

Surfaces and Curves

- Thus S^2 is an example of a manifold which is a subset of another manifold, namely $\mathbb{R}^3$, and which satisfies certain other conditions by virtue of which it is a manifold.
- A two-dimensional submanifold of $\boldsymbol{E}^3$ or $\mathbb{R}^3$ is often called a **surface** in Euclidean space.
- A one-dimensional submanifold is called a **curve**.
- Planes and spheres, circles and lines are the simplest examples.
- Manifolds frequently arise in other ways than as submanifolds.
- So it is natural to ask whether every manifold can be represented as a submanifold of some simple manifold, especially of Euclidean space.

Subsection 2

Further Examples

Quotient Space

- Let X be a topological space.
- Let $\sim$ be an equivalence relation on X.
- Denote by $[x]$ the equivalence class of x,

$$[x] = \{y \in X : y \sim x\}.$$

- For a subset $A \subseteq X$, denote by $[A]$ the set

$$[A] = \bigcup_{a \in A} [a],$$

  that is, all x equivalent to some element of A.
- We let $X/\sim$ stand for the set of equivalence classes.
- Denote by $\pi : X \to X/\sim$ the natural mapping (projection) taking each $x \in X$ to its equivalence class,

$$\pi(x) = [x].$$

Quotient Space (Cont'd)

- Let the **quotient topology** on $X/\sim$ be defined by stipulating that

$$U \subseteq X/\sim \text{ is an open subset if } \pi^{-1}(U) \text{ is open.}$$

- The projection $\pi : X \to X/\sim$ is then continuous.

Definition

$X/\sim$ is called the **quotient space** of X relative to the relation $\sim$.

Example

- Let $X = \mathbb{R}$ be the real numbers and $\mathbb{Z}$ be the integers.
- Define
$$x \sim y \quad \text{if} \quad x - y \in \mathbb{Z}.$$
- Denote by $\mathbb{R}/\sim$ the quotient space.
- This quotient space may be naturally identified with
$$S^1 = \{z \in \mathbb{C} : |z| = 1\},$$
the unit circle in the complex plane.
- The projection $\pi : \mathbb{R} \to \mathbb{R}/\sim$ is then identified with the map
$$\pi(t) = \exp(2\pi t\sqrt{-1}).$$
- Note that $X/\sim$ is a space of cosets of a group relative to a subgroup.
- This situation occurs frequently.

Open Equivalence Relations

Definition

An equivalence relation $\sim$ on a space X is called **open** if, whenever a subset $A \subseteq X$ is open, then $[A]$ is also open.

Lemma

An equivalence relation $\sim$ on X is open if and only if π is an open mapping. When $\sim$ is open and X has a countable basis of open sets, then $X/\sim$ has a countable basis also.

- Suppose, first, that $\sim$ is open.
  Let $A \subseteq X$ be an open subset.
  By hypothesis, $[A]$ is open.
  Note that $[A] = \pi^{-1}(\pi(A))$.
  Thus, by definition, $\pi(A)$ is open in $X/\sim$.
  So π is an open mapping.

Open Equivalence Relations (Cont'd)

- Suppose, conversely, that π is open.

  Let $A \subseteq X$ be an open subset.

  By hypothesis, $\pi(A)$ is open in $X/\sim$.

  Since $[A] = \pi^{-1}(\pi(A))$, $[A]$ is open in X.

  It follows that $\sim$ is open.

  Suppose $\sim$ is open and X has a countable basis $\{U_i\}$ of open sets.

  Let W be an open subset of $X/\sim$.

  Then $\pi^{-1}(W) = \bigcup_{j \in J} U_j$, for some subfamily of $\{U_i\}$.

  Hence, $W = \pi(\pi^{-1}(W)) = \bigcup_{j \in J} \pi(U_j)$.

  It follows that $\{\pi(U_i)\}$ is a basis of open sets for $X/\sim$.

Utility of the Lemma

- Recall that a manifold must be a Hausdorff space with a countable basis of open sets.
- So the lemma is clearly useful in determining those equivalence relations on a manifold M whose quotient space is again a manifold.
- Unfortunately, there is no simple condition which will assure that the quotient space is Hausdorff.
- In fact, a quotient space $X/\sim$ may be locally Euclidean with a countable basis of open sets and still fail to be Hausdorff.
- Nevertheless we obtain important examples by this method.
- The following lemma is sometimes helpful.

Characterization of Hausdorff Quotients

Lemma

Let $\sim$ be an open equivalence relation on a topological space X. Then $R = \{(x, y) : x \sim y\}$ is a closed subset of the space $X \times X$ if and only if the quotient space $X/\sim$ is Hausdorff.

- Assume $X/\sim$ is Hausdorff.
  Suppose $(x, y) \notin R$, that is, $x \nsim y$.
  Then there are disjoint neighborhoods U of $\pi(x)$ and V of $\pi(y)$.
  We denote by $\widetilde{U}$ and $\widetilde{V}$ the open sets $\pi^{-1}(U)$ and $\pi^{-1}(V)$.
  These contain x and y, respectively.
  Suppose the open set $\widetilde{U} \times \widetilde{V}$ containing (x, y) intersects R.
  Then it must contain a point (x', y') for which $x' \sim y'$.
  But then $\pi(x') = \pi(y')$ contrary to the assumption that $U \cap V = \emptyset$.
  This contradiction shows that $\widetilde{U} \times \widetilde{V}$ does not intersect R.
  Therefore, R is closed.

Characterization of Hausdorff Quotients (Converse)

- Conversely, suppose that R is closed.

  Let $\pi(x), \pi(y)$ in $X/\sim$ be a distinct pair of points.

  Then, there is an open set of the form $\widetilde{U} \times \widetilde{V}$ containing (x, y) and having no point in R.

  It follows that $U = \pi(\widetilde{U})$ and $V = \pi(\widetilde{V})$ are disjoint.

  By the preceding lemma and the hypothesis, U and V are open.

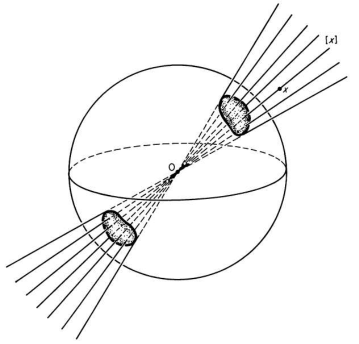
  Thus $X/\sim$ is Hausdorff.

Example: Real Projective Space

- We let $X = \mathbb{R}^{n+1} - \{0\}$, all $(n+1)$-tuples of real numbers $x = (x^1, \ldots, x^{n+1})$ except $0 = (0, ..., 0)$.
- Define $x \sim y$ if, there is a real number $t \neq 0$, such that $y = tx$, that is,

$$(y^1, \ldots, y^{n+1}) = (tx^1, \ldots, tx^{n+1}).$$

- The equivalence classes $[x]$ may be visualized as lines through the origin.
- We denote the quotient space by $P^n(\mathbb{R})$.
- It is called **real projective space**.



Example: Real Projective Space (Cont'd)

- We show that $P^n(\mathbb{R})$ is a differentiable manifold of dimension n.

  First note that $\pi : X \to P^n(\mathbb{R})$ is an open mapping.

  Let $t \neq 0$ be a real number.

  Let $\varphi_t : X \to X$ be the mapping defined by

  $$\varphi_t(x) = tx.$$

  It is clearly a homeomorphism, with $\varphi_t^{-1} = \varphi_{1/t}$.

  Let $U \subseteq X$ be an open set.

  Then $[U] = \bigcup \varphi_t(U)$, the union being over all real $t \neq 0$.

  Each $\varphi_t(U)$ is open.

  So $[U]$ is open.

  By a previous lemma, π is open.

Example: Real Projective Space (Cont'd)

- Next we apply the preceding lemma to prove that $P^n(\mathbb{R})$ is Hausdorff.

  On the open submanifold $X \times X \subseteq \mathbb{R}^{n+1} \times \mathbb{R}^{n+1}$ we define a real-valued function $f(x, y)$ by

  $$f(x^1, \ldots, x^{n+1}; y^1, \ldots, y^{n+1}) = \sum_{i \neq j} (x^i y^j - x^j y^i)^2.$$

  Then $f(x, y)$ is continuous.

  f vanishes if and only if $y = tx$, for some real number $t \neq 0$.

  That is, if and only if $x \sim y$.

  Thus

  $$R = \{(x, y) : x \sim y\} = f^{-1}(0)$$

  is a closed subset of $X \times X$.

  Therefore, $P^n(\mathbb{R})$ is Hausdorff.

Example: Real Projective Space (Cont'd)

- We define $n+1$ coordinate neighborhoods

$$U_i, \varphi_i, \quad i = 1, \ldots, n+1.$$

Let $\widetilde{U}_i = \{x \in X : x^i \neq 0\}$.
Let $U_i = \pi(\widetilde{U}_i)$.
Then $\varphi_i : U_i \to \mathbb{R}^n$ is defined by:
  - Choosing any $x = (x^1, \ldots, x^{n+1})$ representing $[x] \in U_i$;
  - Setting

$$\varphi_i(x) = \left(\frac{x^1}{x^i}, \ldots, \frac{x^{i-1}}{x^i}, \frac{x^{i+1}}{x^i}, \ldots, \frac{x^{n+1}}{x^i} \right).$$

It is seen that if $x \sim y$, then $\varphi_i(x) = \varphi_i(y)$.
Moreover, $\varphi_i(x) = \varphi_i(y)$ implies $x \sim y$.
Thus, $\varphi_i : U_i \to \mathbb{R}^n$ is properly defined, continuous, one-to-one, and even onto.

Example: Real Projective Space (Cont'd)

- $\varphi_i^{-1} : \mathbb{R}^n \to \mathbb{R}^n$ is given by composing a C^∞ map of $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ with π.

  For $z \in \mathbb{R}^n$, we have

  $$\varphi_i^{-1}(z^1, \ldots, z^n) = \pi(z^1, \ldots, z^{i-1}, +1, z^i, \ldots, z^n).$$

  Therefore φ_i^{-1} is continuous.

  Thus, $P^n(\mathbb{R})$ is a (topological) manifold.

  It is C^∞ if the coordinate neighborhoods are C^∞-compatible.

  That is, if $\varphi_i \circ \varphi_j^{-1}$ is C^∞ (where defined), for $1 \leq i, j \leq n+1$.

  This can be verified explicitly.

Example: Grassman Manifolds $G(k,n)$

- The Grassman manifold $G(k,n)$ is the set of all k-planes through the origin of $\mathbb{R}^n$, or k-dimensional subspaces of $\boldsymbol{V}^n = \mathbb{R}^n$ (as a vector space), endowed with a suitable topology and differentiable structure.
- We will realize $G(k,n)$ as a quotient space arising from an equivalence relation on the manifold $F(k,n)$ of k-frames in $\mathbb{R}^n$.
- We define a k-**frame** in $\mathbb{R}^n$ to be a linearly independent set $\boldsymbol{x}$ of k elements of $\mathbb{R}^n$,

$$\begin{array}{rcl} x_1 & = & (x_1^1,\ldots,x_1^n), \\ & \vdots & \\ x_k & = & (x_k^1,\ldots,x_k^n). \end{array}$$

k-Frames

- A k-frame in $\mathbb{R}^n$ may be identified with the $k \times n$ matrix, which we also denote by $\boldsymbol{x}$, whose rows are $x_1, \ldots, x_k$.
- We use the fact that the set $\mathcal{M}_{kn}(\mathbb{R})$ of all $k \times n$ real matrices is a differentiable manifold by virtue of its identification with $\mathbb{R}^{kn}$.
- The matrices which correspond to k-frames, that is, those of rank k, form an open subset.
- Hence, $F(k, n)$ is a differentiable manifold.
- This is because of the fact that "$\boldsymbol{x}$ is of rank k" means that the following two equivalent statements hold:
  (i) The row vectors form a linearly independent set;
  (ii) Not all $k \times k$ minor determinants are zero simultaneously.
- Statement (ii) shows that the rank is less than k at the simultaneous zeros of a set of continuous functions on $\mathcal{M}_{kn}(\mathbb{R})$.
- So the rank is less than k on a closed subset.
- It follows that $F(k, n)$ is open.

Equivalence Relation on $F(k,n)$

- Each frame $\boldsymbol{x}$ determines a k-plane or point of $G(k,n)$, namely, the subspace spanned by $x_1,\ldots,x_k$.
- So we have a natural map of $F(k,n)$ onto $G(k,n)$.
- Moreover $\boldsymbol{x}=(x_1,\ldots,x_k)$ and $\boldsymbol{y}=(y_1,\ldots,y_k)$ determine the same k-plane if and only if
$$y_i=\sum_{j=1}^{k}\alpha_{ij}x_j,$$
where $a=(\alpha_{ij})$ is a nonsingular $k\times k$ matrix.
- Equivalently, if and only if $\boldsymbol{y}=a\boldsymbol{x}$, the product of the matrices a and $\boldsymbol{x}$.
- It is natural to define $\sim$ by
$$\boldsymbol{y}\sim\boldsymbol{x}\quad\text{if}\quad\boldsymbol{y}=a\boldsymbol{x},\quad a\in Gl(k,\mathbb{R}).$$

Construction of $G(k,n)$

- We now identify:
  - $G(k,n)$ with $F(k,n)/\sim$, the set of equivalence classes;
  - The above mentioned natural map with π.
- We sketch a proof that $G(k,n)$ with the quotient space topology has the structure of a differentiable manifold of dimension $k(n-k)$.
- Note that if $k=1$, then $a \in Gl(1,\mathbb{R}) = \mathbb{R}^*$.
- So $G(k,n)$ becomes $P^{n-1}(\mathbb{R})$.
- The proof that π is an open mapping is analogous to the preceding example.
- The proof that $G(k,n)$ is Hausdorff is trickier and is omitted.

Open Subsets

- We describe a covering by coordinate neighborhoods with C^∞-compatible coordinate maps.
- Then a previous theorem may be applied to complete the proof.
- We use the $k \times k$ submatrices of $\boldsymbol{x} \in \mathcal{M}_{kn}(\mathbb{R})$.
- Let $J = (j_1, \ldots, j_k)$ be an ordered subset of $(1, \ldots, n)$.
- Let J' be the complementary subset.
- By $\boldsymbol{x}_J$ we denote the $k \times k$ submatrix

  $$(x_i^{j_\ell}), \quad 1 \le i, \ell \le k,$$

  of the $k \times n$ matrix $\boldsymbol{x}$.
- Denote by $\boldsymbol{x}_{J'}$ the complementary $k \times (n-k)$ submatrix obtained by striking out the columns $j_1, \ldots, j_k$ of $\boldsymbol{x}$.
- Let $\widetilde{U}_J$ be the open set in $F(k, n)$, consisting of matrices for which $\boldsymbol{x}_J$ is nonsingular.
- Let $U_J = \pi(\widetilde{U}_J)$ be the corresponding open set in $G(k, n)$.

Open Subsets (Cont'd)

- Each $\boldsymbol{y} \in \widetilde{U}_J$ is equivalent to exactly one $k \times n$ matrix $\boldsymbol{x}$ in which the submatrix $\boldsymbol{x}_J$ is the $k \times k$ identity matrix.
- For example, if $J = (1, 2, \ldots, k)$, then $\boldsymbol{x}$ is of the form

$$\boldsymbol{x} = \begin{pmatrix} 1 & \cdots & 0 & x_{1,k+1} & \cdots & x_{1n} \\ 0 & \cdots & 0 & & & \\ \vdots & & \vdots & & & \vdots \\ 0 & \cdots & 1 & x_{k,k+1} & \cdots & x_{kn} \end{pmatrix}.$$

- In fact the $\boldsymbol{x}$ equivalent to a matrix $\boldsymbol{y}$, for which $\boldsymbol{y}_J$ is nonsingular, is given by the matrix formula

$$\boldsymbol{x} = \boldsymbol{y}_J^{-1}\boldsymbol{y}.$$

Coordinate Mappings

- We define

$$\varphi_J : U_J \to \mathcal{M}_{k(n-k)}(\mathbb{R}),$$

identified with $\mathbb{R}^{k(n-k)}$, by deleting the k columns corresponding to J in this representative $\boldsymbol{x}$ of $\boldsymbol{y}$,

$$\varphi_J([\boldsymbol{y}]) = \boldsymbol{x}_{J'}.$$

- It can be shown that:
  - φ_J is properly defined;
  - φ_J maps U_J onto $\mathbb{R}^{k(n-k)}$ homeomorphically;
  - The $U_J, \varphi_{J'}$, for all subsets J of k distinct elements of $(1, 2, \ldots, n)$, form a covering of $G(k, n)$ by C^∞-compatible coordinate neighborhoods.
- A verification of this for $G(2, 4)$, the 2-planes through the origin of $\mathbb{R}^4$, is sufficient to show how to proceed in general.
- A different proof of these facts will be provided later.

Subsection 3

Differentiable Functions and Mappings

Functions in Local Coordinates

- Let
$$f : W_f \to \mathbb{R}$$
be a real-valued function defined on an open set W_f of a C^∞ manifold M, possibly all of M.
- Let U, φ is be a coordinate neighborhood such that $W_f \cap U \neq \emptyset$.
- Let $x^1, \ldots, x^n$ denote the local coordinates.
- Then f corresponds to a function $\widehat{f}(x^1, \ldots, x^n)$ on $\varphi(W_f \cap U)$ defined by
$$\widehat{f} = f \circ \varphi^{-1}.$$
- That is, we have, for all $p \in W_f \cap U$,
$$f(p) = \widehat{f}(x^1(p), \ldots, x^n(p)) = \widehat{f}(\varphi(p)).$$

Using Multiple Local Coordinates

- We will customarily omit the hat and use the same letter "f" for:
  - f as defined on W_f;
  - $\widehat{f}$, its expression in local coordinates.
- Ordinarily this will result in no confusion.
- Suppose two coordinate neighborhoods U, φ and V, ψ are involved.
- Then we will use different letters for the coordinates, say

$$x^1, \ldots, x^n \quad \text{and} \quad y^1, \ldots, y^n.$$

- Thus, for $p \in W_f \cap U \cap V$, we have, omitting hats,

$$f(p) = f(x^1(p), \ldots, x^n(p)) = f(y^1(p), \ldots, y^n(p)),$$

  the latter two f's denoting $\widehat{f}$'s, or $f \circ \varphi^{-1}$ and $f \circ \psi^{-1}$, respectively, the expressions in local coordinates.

C^∞ Functions

Definition

Using the notation above, $f : W_f \to \mathbb{R}$ is a C^∞ **function** if each $p \in W_f$ lies in a coordinate neighborhood U, φ such that

$$f \circ \varphi^{-1}(x^1, \ldots, x^n) = \widehat{f}(x^1, \ldots, x^n)$$

is C^∞ on $\varphi(W_f \cap U)$.

- Clearly, a C^∞ function is continuous.

Coordinate Functions

- Among the C^∞ functions on M are the n-coordinate functions $(x^1(q), \ldots, x^n(q))$ of a coordinate neighborhood U, φ.
- More precisely, suppose $\pi^i : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$\pi^i(x^1, \ldots, x^n) = x^i.$$

- These functions are defined by $x^i(q) = \pi^i \circ \varphi(q)$.
- Their expression in local coordinates, on $\varphi(U)$, is given by

$$\widehat{x}^i(x^1, \ldots, x^n) = x^i(\varphi^{-1}(x^1, \ldots, x^n)) = \pi^i(x^1, \ldots, x^n) = x^i.$$

- Since the hat is usually omitted, we have the statement

$$x^i(x^1, \ldots, x^n) = x^i, \quad i = 1, \ldots, n.$$

- This is somewhat confusing since the same letter is used for a function and its values.

Properties

- It is a consequence of the definition that if f is C^∞ on W and $V \subseteq W$ is an open set, then $f|_V$ is C^∞ on V.
- Moreover, if W is a union of open sets on each of which a real-valued function f is C^∞, then f is C^∞ on W.
- Using the C^∞ compatibility of coordinate neighborhoods, it can be verified that, if f is C^∞ on W and V, ψ is any coordinate neighborhood intersecting W, then $f \circ \psi^{-1}$ is C^∞ on the open set $\psi(V \cap W)$ in $\mathbb{R}^n$.

C^∞ Mappings

- Let M and N be C^∞ manifolds.
- Let $W \subseteq M$ be an open subset.
- Let $F : W \to N$ be a mapping.

Definition

F is a C^∞ **mapping** of W into N if, for every $p \in M$, there exist coordinate neighborhoods U, φ of p and V, ψ of $F(p)$, with

$$F(U) \subseteq V,$$

such that

$$\psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V)$$

is C^∞.

C^∞ Mappings (Cont'd)

- Let $x^1, \ldots, x^n$ be local coordinates for $\phi(U)$.
- Let $y^1, \ldots, y^m$ be local coordinates for $\psi(V)$.
- Then F being a C^∞ mapping means that $F|_U : U \to V$ may be written in these local coordinates as a mapping from $\varphi(U)$ into $\psi(V)$ by
$$\widehat{F}(x^1, \ldots, x^n) = (f^1(x^1, \ldots, x^n), \ldots, f^m(x^1, \ldots, x^n))$$
(or simply $y^i = f^i(x)$, $i = 1, \ldots, m$) and each $f^i(x)$ is C^∞ on $\varphi(U)$.
- Note that C^∞ mapping is a more general notion than C^∞ function, the latter being a mapping to $N = \mathbb{R}$.

Remarks

- C^∞ mappings are continuous.
- Their restrictions to open subsets are C^∞.
- Any mapping from an open subset $W \subseteq N$ into M, whose restriction to each of a collection of open sets (which cover W) is C^∞, is necessarily C^∞ on W.
- As is the case with C^∞ functions, the C^∞ compatibility of local coordinate neighborhoods, closure under composition of C^r mappings and the remarks above show that the property does not depend on any particular choice of coordinates.
- Similarly it follows from closure under composition of C^r mappings that composition of C^∞ mappings is again a C^∞ mapping.

Terminology

- C^∞ manifolds, functions and mappings are also called **smooth**.
- From now on we shall refer to **differentiable** manifold, function and mapping.
- Recall, however, that we previously used this term in a much weaker sense than C^∞.
- One reason that C^∞ is a desirable differentiability class to use is that, when we later take derivatives of C^∞ functions on manifolds, we obtain C^∞ functions.
- In contrast, in the C^r case, we would obtain C^{r-1} functions.
- Thus, assuming infinite differentiability relieves us of many irritating concerns about order of differentiability.

Disjoint Compact and Closed Sets

Theorem

Let F be a closed subset and K a compact subset of a C^∞ manifold M, with $F \cap K = \emptyset$. Then there is a C^∞ function f defined on M which has the value $+1$ on K and 0 on F.

Corollary

Let U be an open subset of a manifold M. Suppose $p \in U$. Let f be a C^∞ function on U. Then there is a neighborhood V of p in U and a C^∞ function f^* on M, such that:

- $f^* = f$ on V;
- $f^* = 0$ outside of U.

- The proofs of these results follow along the lines of the proofs of corresponding results already established for C^∞ mappings from open subsets of $\mathbb{R}^n$ to $\mathbb{R}^n$.

Diffeomorphisms

Definition

A C^∞ mapping $F : M \to N$ between C^∞ manifolds is a **diffeomorphism** if it is a homeomorphism and F^{-1} is C^∞.
M and N are **diffeomorphic** if there exists a diffeomorphism $F : M \to N$.

- This extends the concept of diffeomorphism, previously defined for open subsets of $\mathbb{R}^n$ only, to arbitrary C^∞ manifolds.
- Diffeomorphism of manifolds is an equivalence relation.
  - Reflexivity and symmetry are obvious from the definition.
  - Transitivity is a consequence of the following facts:
    - Composition of C^∞ maps is C^∞;
    - Composition of homeomorphisms is a homeomorphism.
- It is important that F^{-1}, as well as F, be C^∞.

Example

- Let $F : \mathbb{R} \to \mathbb{R}$ be defined by

$$F(t) = t^3.$$

- Then F is C^∞ and a homeomorphism.
- It is not a diffeomorphism since $F^{-1}(t) = t^{1/3}$, and this is not even of class C^1, let alone C^∞, at $t = 0$.
- This example shows how it is possible to define two distinct C^∞ structures on $\mathbb{R}$.
- The first is the usual one defined by:
  - $U = \mathbb{R}$;
  - $\varphi : U \to \mathbb{R}$ be the identity map.
- This determines a C^∞ structure on $\mathbb{R}$ by a previous theorem.

Example (Cont'd)

- We may also consider the structure defined by the coordinate neighborhood V, ψ, where:
  - $V = \mathbb{R}$;
  - $\psi : V \to \mathbb{R}$ is defined by $\psi(t) = t^3$.
- Then $\varphi \circ \psi^{-1} = t^{1/3}$.
- So U, φ and V, ψ are not C^∞-compatible.
- Hence they are not in the same differentiable structure.
- However, $\mathbb{R}$ with its first structure is diffeomorphic to $\widetilde{\mathbb{R}}$, denoting $\mathbb{R}$ with its second structure.
- The diffeomorphism $F : \mathbb{R} \to \widetilde{\mathbb{R}}$ being defined by
$$F(t) = t^{1/3}.$$
- So in local coordinates it is given by $\psi \circ F \circ \varphi^{-1} = t$.

Remarks

- The preceding examples shows that two C^∞ manifolds with the same underlying topological manifold but incompatible C^∞ structures can still be diffeomorphic.
- A fundamental question is:

  Can the same manifold M or homeomorphic manifolds have C^∞ structures which are not diffeomorphic?

- This was an unsolved problem for many years.
- It was finally settled by Milnor, who proved the existence of two C^∞ structures on S^7 which were not diffeomorphic.

Remark: Characterization of Coordinate Neighborhoods

- We conclude with a remark which is occasionally useful.
- A necessary and sufficient condition that an open set U of M, together with a mapping $\varphi : U \to \mathbb{R}^n$, be a coordinate neighborhood is that φ be a diffeomorphism of U onto an open subset W of $\mathbb{R}^n$.
- Conversely, if W is an open subset of $\mathbb{R}^n$ and $\psi : W \to M$ is a diffeomorphism onto an open subset U, then U, ψ^{-1} is a coordinate neighborhood.
- We sometimes call W, ψ a **parametrization**, especially in the case $\dim M = 1$.

Subsection 4

Rank of a Mapping. Immersions

Rank of a Differentiable Mapping

- Let N and M be C^∞ manifolds.
- Let $F : N \to M$ be a differentiable mapping.
- Let $p \in N$.
- Suppose U, φ and V, ψ are coordinate neighborhoods of p and $F(p)$, respectively, such that $F(U) \subseteq V$.
- Then we have a corresponding expression for F in local coordinates,

$$\widehat{F} = \psi \circ F \circ \varphi^{-1} : \varphi(U) \to \psi(V).$$

Definition

The **rank of** F **at** p is defined to be the rank of $\widehat{F}$ at $\varphi(p)$.

Rank of a Differentiable Mapping (Cont'd)

- Thus, the rank at p is the rank at $a = \varphi(p)$ of the Jacobian matrix

$$\begin{pmatrix} \frac{\partial f^1}{\partial x^1} & \cdots & \frac{\partial f^1}{\partial x^n} \\ \vdots & & \vdots \\ \frac{\partial f^m}{\partial x^1} & \cdots & \frac{\partial f^m}{\partial x^n} \end{pmatrix}_a$$

of the mapping

$$\widehat{F}(x^1,\ldots,x^n) = (f^1(x^1,\ldots,x^n),\ldots,f^m(x^1,\ldots,x^n))$$

expressing F in the local coordinates.

- This definition must be validated by showing that the rank is independent of the choice of coordinates.
- Another definition which is clearly independent of this choice is given in the next chapter.

The Case of Constant Rank

- The important case for us will be that in which the rank is constant.
- The theorem on rank of the previous chapter, and its corollary, can be restated as follows:

  Let N and M be C^∞ manifolds.
  Let $F : N \to M$ be a differentiable mapping.
  Suppose $\dim N = n$, $\dim M = m$ and $\operatorname{rank} F = k$ at every point of N.
  If $p \in N$, then, there exist coordinate neighborhoods U, φ and V, ψ of p and $F(p)$, respectively, with $F(U) \subseteq V$, such that:
  - $\varphi(p) = (0, \ldots, 0)$;
  - $\psi(F(p)) = (0, \ldots, 0)$;
  - $\widehat{F} = \psi \circ F \circ \varphi^{-1}$ is given by

  $$\widehat{F}(x^1, \ldots, x^n) = (x^1, \ldots, x^k, 0, \ldots, 0).$$

  Moreover, we may assume $\varphi(U) = C^n_\varepsilon(0)$ and $\psi(V) = C^m_\varepsilon(0)$ with the same $\varepsilon > 0$.

Necessary Condition for Diffeomorphism

- An obvious corollary to this remark is:

  A necessary condition for $F : N \to M$ to be a diffeomorphism is that

$$\dim M = \dim N = \operatorname{rank} F.$$

- Otherwise k would be either less than n or less than m.

  In that case the expression in local coordinates implies that it is not possible for both F and F^{-1} to be one-to-one, even locally.

  For example, suppose $k < n$ in the expression above.

  Then all points in U with coordinates of the form

$$(0, \ldots, 0, x^{k+1}, \ldots, x^n)$$

  are mapped onto the same point of V.

Immersions

Definition

Let N and M be C^∞ manifolds and $F : N \to M$ be a differentiable mapping.
Suppose, using the notation above, that $n < m$.
We say that F is an **immersion** of N in M if

$$\text{rank} F = n, \quad \text{at every point.}$$

If an immersion $F : N \to M$ is univalent (injective), then we say that the image $\widetilde{N} = F(N)$, endowed with the topology and C^∞ structure which makes $F : N \to \widetilde{N}$ a diffeomorphism, is a **submanifold** (or an **immersed submanifold**).

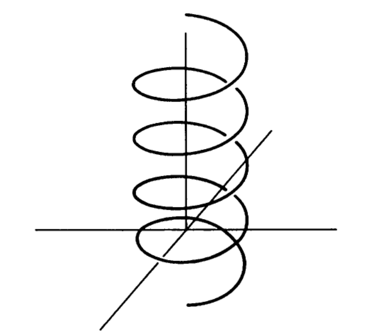
Remarks

- In every case that follows:
  - $N = \mathbb{R}$ or an open interval of $\mathbb{R}$;
  - $M = \mathbb{R}^2$, except in the first example where $M = \mathbb{R}^3$.
- We use the natural coordinates (given by the identity map).
- To verify that F is an immersion it is necessary to check that the Jacobian has rank 1 at every point.
- Equivalently, one of the derivatives with respect to t differs from zero, for every value of t for which the mapping F is defined.
- The demonstration of this is usually omitted.

Example

- $F : \mathbb{R} \to \mathbb{R}^3$ is given by

$$F(t) = (\cos 2\pi t, \sin 2\pi t, t).$$

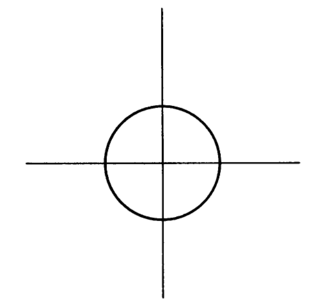


- The image $F(\mathbb{R})$ is a helix lying on a unit cylinder whose axis is the x^3-axis in $\mathbb{R}^3$.

Example

- $F : \mathbb{R} \to \mathbb{R}^2$ is given by

$$F(t) = (\cos 2\pi t, \sin 2\pi t).$$



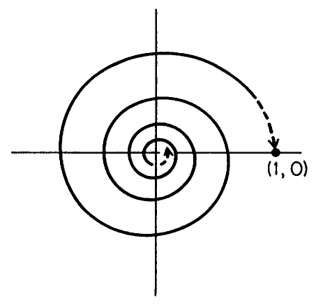
- The image $F(\mathbb{R})$ is the unit circle in $\mathbb{R}^2$,

$$S^1 = \{(x^1, x^2) : (x^1)^2 + (x^2)^2 = 1\}.$$

Example

- $F : (1, \infty) \to \mathbb{R}^2$ is given by

$$F(t) = \left(\frac{1}{t} \cos 2\pi t, \frac{1}{t} \sin 2\pi t \right).$$

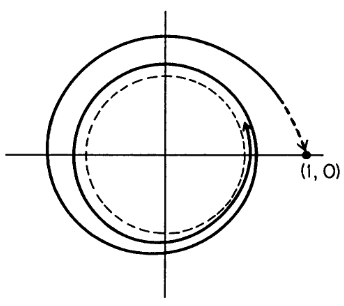


- The image is a curve with the following properties:
  - It spirals to $(0, 0)$ as $t \to \infty$;
  - It tends to $(1, 0)$ as $t \to 1$.

Example

- In this example, $F : (1, \infty) \to \mathbb{R}^2$ is also a spiral.
- However, F is modified so that the image $F(\mathbb{R})$ spirals toward the circle with center at $(0, 0)$ and radius $\frac{1}{2}$ as $t \to \infty$.
- The mapping is given by

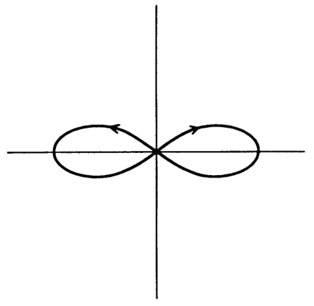
$$F(t) = \left(\frac{t+1}{2t} \cos 2\pi t, \frac{t+1}{2t} \sin 2\pi t \right).$$



- It is not difficult to check that the Jacobian could have rank 0, i.e., both derivatives $\frac{dx^1}{dt}$ and $\frac{dx^2}{dt}$ could vanish simultaneously on $1 < t < \infty$, if and only if $\cot 2\pi t = -\tan 2\pi t$.
- This, however, is impossible.

Example

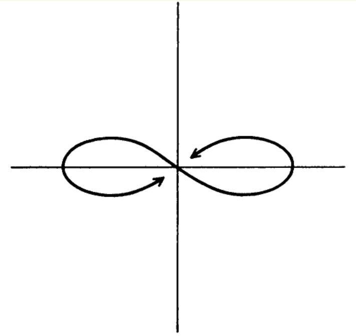
- $F : \mathbb{R} \to \mathbb{R}^2$ is given by
$$F(t) = \left(2\cos\left(t - \frac{1}{2}\pi\right), \sin 2\left(t - \frac{1}{2}\pi\right)\right).$$



- The image is a "figure eight" traversed in the sense shown.
- The image point makes a complete circuit starting at the origin as t goes from 0 to 2π.

Example

- $G : \mathbb{R} \to \mathbb{R}^2$ again and the image is the "figure eight" as in the previous example, but with an important difference.



- We pass through $(0, 0)$ only once, when $t = \frac{1}{2}$.
- For $t \to -\infty$ and $t \to +\infty$ we only approach $(0, 0)$ as limit.

Example (Cont'd)

- The immersion is given by changing parameter in the previous example.
- Let $g(t)$ be a monotone increasing C^∞ function on $-\infty < t < \infty$, such that:
  - $g(0) = \pi$;
  - $\lim_{t \to -\infty} g(t) = 0$;
  - $\lim_{t \to +\infty} g(t) = 2\pi$.
- For example, we may use

$$g(t) = \pi + 2\tan^{-1} t.$$

- Then $G(t)$ is given by composition of $g(t)$ with $F(t)$ from the previous example:

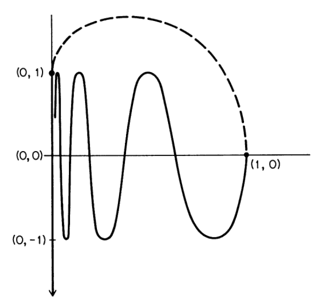
$$G(t) = F(g(t)) = \left(2\cos\left(g(t) - \frac{\pi}{2}\right), \sin 2\left(g(t) - \frac{\pi}{2}\right)\right).$$

Example

- Again $F : \mathbb{R} \to \mathbb{R}^2$ so that

$$F(t) = \begin{cases} \left(\frac{1}{t}, \sin \pi t\right), & \text{if } 1 \leq t < \infty, \\ (0, t+2), & \text{if } -\infty < t \leq -1. \end{cases}$$

- This gives a curve with a gap as shown.
- For $-1 \leq t \leq +1$ we connect the two pieces together smoothly, as shown by the dotted line.
- This gives a C^∞ immersion of all of $\mathbb{R}$ in $\mathbb{R}^2$ whose image is as shown.



Injectivity

- We may draw some conclusions from these examples about the nature of immersions.
- First we note that an immersion need not be univalent, that is, one-to-one into (injective), at large, even though it is one-to-one locally.
- The unit circle ($F(t) = (\cos 2\pi t, \sin 2\pi t)$) and the figure eight ($F(t) = \left(2\cos\left(t - \frac{1}{2}\pi\right), \sin 2\left(t - \frac{1}{2}\pi\right)\right)$) show this.
- For example, in both cases

$$t = 0, +2\pi, +4\pi, \ldots$$

  all have the same image point:
  - $(0, 1)$ in the case of the circle;
  - $(0, 0)$ for the figure eight.

Immersions versus Homeomorphisms

- The second conclusion we can draw is that even when it is one-to-one, an immersion is not necessarily a homeomorphism onto its image.
- That is, $F : N \to M$ a one-to-one immersion does not imply that F is a homeomorphism of N onto $\widetilde{N} = F(N)$ considered as a subspace of M.
- The second figure eight and the last example show this:
  - In the case of the former, $\widetilde{N}$ is the figure eight whereas N is the real line $\mathbb{R}$, two spaces which are not homeomorphic.
  - In the case of the latter, N is again the real line and $\widetilde{N} = F(N)$ as a subspace of $\mathbb{R}^2$ is not locally connected at all of its points.
    There are points on the x^2-axis, such as $(0,1)$, which do not have arbitrarily small connected neighborhoods.
    Hence, $\widetilde{N}$ and $N = \mathbb{R}$ are not homeomorphic.
- In any case, $F : N \to M$ is continuous, since it is differentiable.

Imbeddings

Definition

An **imbedding** is a one-to-one immersion $F : N \to M$ which is a homeomorphism of N into M. That is, F is a homeomorphism of N onto its image, $\widetilde{N} = F(N)$, with its topology as a subspace of M.
The image of an imbedding is called an **imbedded submanifold**.

- The examples of the helix and the two spirals above are imbeddings.

Immersions and Imbeddings

- The following theorem, essentially a restatement of the theorem on rank and its corollary, shows that the distinction between immersions and imbeddings is a global one.

Theorem

Let $F : N \to M$ be an immersion. Then each $p \in N$ has a neighborhood U such that $F|_U$ is an imbedding of U in M.

- According to a previous remark, we may choose cubical coordinate neighborhoods U, φ and V, ψ of $p \in N$ and $F(p) \in M$, respectively, such that:
  - $\varphi(p) = (0, \ldots, 0)$ in $\mathbb{R}^n$;
  - $\psi(F(p)) = (0, \ldots, 0)$ in $\mathbb{R}^m$;
  - $\varphi(U) = C_\varepsilon^n(0)$ and $\psi(V) = C_\varepsilon^m(0)$ (cubes of the same breadth ε);
  - $\widehat{F} = \psi \circ F \circ \varphi^{-1}$, the expression of F in these local coordinates, is given by
$$\widehat{F}(x^1, \ldots, x^n) = (x^1, \ldots, x^n, 0, \ldots, 0).$$

Proof of the Theorem

- We want to show that $F|_U$ is a homeomorphism of U onto $F(U)$ with the relative topology.

  It is enough to show that $\widehat{F}$ is a homeomorphism of $C_\varepsilon^n(0)$ onto its image in $C_\varepsilon^m(0)$.

  First, note that $F(U) \subseteq V$, an open subset of M.

  So the topology of $F(U)$ as a subspace of M is the same as its topology as a subspace of V.

  Now $\varphi : U \to C_\varepsilon^n(0)$ and $\psi : V \to C_\varepsilon^m(0)$ are homeomorphisms.

  So $\widehat{F}$ is a homeomorphism of $C_\varepsilon^n(0)$ onto its image in $C_\varepsilon^m(0)$.

  But it is clear that $\widetilde{F}$ is a homeomorphism of $C_\varepsilon^n(0)$ onto the subset $x^{n+1} = \cdots = x^m = 0$ of $C_\varepsilon^m(0)$.

  Hence, the theorem holds.

Slices

- We call a subset S of a cube $C_\varepsilon^m(a)$ in $\mathbb{R}^m$ a **slice** if it consists of all points for which certain of the coordinates are held constant.

  Example: The set

$$S = \{x \in C_\varepsilon^m(0) : x^{n+1} = \cdots = x^m = 0\}$$

  is a slice through the center $0 = (0, \ldots, 0)$ of $C_\varepsilon^m(0)$.

- Suppose V, ψ is a cubical coordinate neighborhood on a manifold M.

  Let S' is a subset of V, such that $\psi(S')$ is a slice S of the cube $\psi(V)$.

  Then S' is called a **slice** of V.

- Note that, in the proof of the theorem, $S' = F(U)$ is a slice of V.
- In general this slice is not equal to the set $V \cap F(N)$ but only contained in it, even if F is univalent and U is chosen very small.

Subsection 5

Submanifolds

Submanifolds

- A **submanifold** N is the image in M of a one-to-one immersion $F : N' \to M$, $N = F(N')$, of a manifold N' into M, together with the topology and C^∞ structure which makes $F : N' \to N$ a diffeomorphism.
- We frequently refer to N in this case as an **immersed submanifold**.
- As shown by the second figure eight and the last example above, the C^∞ structure of N has an obscure and complicated relation to that of M.
- A more natural notion is that of a *regular* submanifold.
- As its name implies, it will be a special case of the one above.
- It is more natural since its topology and differentiable structure are derived directly from that of M.

The Submanifold Property

Definition

A subset N of a C^∞ manifold M is said to have the ***n*-submanifold property** if each $p \in N$ has a coordinate neighborhood U, φ on M with local coordinates $x^1, \ldots, x^m$ such that:

(i) $\varphi(p) = (0, \ldots, 0)$;

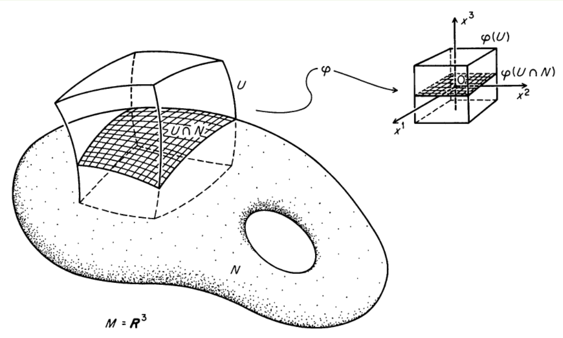
(ii) $\varphi(U) = C_\varepsilon^m(0)$;

(iii) $\varphi(U \cap N) = \{x \in C_\varepsilon^m(0) : x^{n+1} = \cdots = x^m = 0\}$.

If N has this property, coordinate neighborhoods of this type are called **preferred coordinates (relative to N)**.

Illustration

- The figure shows such a subset N in $M = \mathbb{R}^3$ ($n = 2$ and $m = 3$).



- Note that immersed submanifolds do not always have this property.
- For example, take $p = (0, 0)$ in the second figure eight and the last example above.

Consequence of Submanifold Property

- Denote by

$$\pi : \mathbb{R}^m \to \mathbb{R}^n, \quad n \leq m,$$

  the projection to the first n coordinates.
- Then we may state the following lemma, using the notation above.

Lemma

Let $N \subseteq M$ have the n-submanifold property. Then N, with the relative topology, is a topological n manifold and each preferred coordinate system U, φ of M (relative to N) defines a local coordinate neighborhood $V, \widetilde{\varphi}$ on N by

$$V = U \cap N \quad \text{and} \quad \widetilde{\varphi} = \pi \circ \varphi|_V.$$

These local coordinates on N are C^∞-compatible wherever they overlap. Moreover, they determine a C^∞ structure on N relative to which the inclusion $i : N \to M$ is an imbedding.

Proof

- Assume N has the subspace topology relative to M.

  Now $V = U \cap N$ is an open set in the relative topology.

  Also, $\widetilde{\varphi}$ is a homeomorphism onto $C_\varepsilon^n(0) = \pi(C_\varepsilon^m(0))$ in $\mathbb{R}^n$.

  Thus, $V, \widetilde{\varphi}$ are topological coordinate neighborhoods covering N.

  Suppose that for two preferred neighborhoods, U, φ and U', φ', $V = U \cap N$ and $V' = U' \cap N$ have nonempty intersection.

  $V, \widetilde{\varphi}$ and $V', \widetilde{\varphi}'$ are topological coordinate neighborhoods.

  So the change of coordinates is given by homeomorphisms

  $$\widetilde{\varphi}' \circ \widetilde{\varphi}^{-1} \quad \text{and} \quad \widetilde{\varphi} \circ (\widetilde{\varphi}')^{-1}.$$

  It suffices to show that these are C^∞.

Proof (Cont'd)

- Let $\theta : \mathbb{R}^n \to \mathbb{R}^m$ be given by

$$\theta(x^1, \dots, x^n) = (x^1, \dots, x^n, 0, \dots, 0).$$

So we have that $\pi \circ \theta$ is the identity on $\mathbb{R}^n$.

This map θ is C^∞ as is its restriction to $C_\varepsilon^n(0)$, an open subset of $\mathbb{R}^n$.

Thus, $\widetilde{\varphi}^{-1} = \varphi^{-1} \circ \theta$ is C^∞ since it is a composition of C^∞ maps.

On the other hand, $\widetilde{\varphi}' = \pi \circ \varphi'$.

φ' is a C^∞ map of U' and its open subset $U' \cap U$ to $\mathbb{R}^m$.

So $\widetilde{\varphi}'$ is C^∞ on $V \cap V'$.

Thus $\widetilde{\varphi}' \circ \widetilde{\varphi}^{-1}$ is C^∞ on its domain, $\widetilde{\varphi}(V \cap V')$.

Proof (Cont'd)

- We can see this if we write the expressions in local coordinates.

  Suppose
  $$y^i = f^i(x^1, \ldots, x^m), \quad i = 1, \ldots, m,$$
  are the functions giving $\varphi' \circ \varphi^{-1}$, which we know to be C^∞.

  It can be checked that $\widetilde{\varphi}' \circ \widetilde{\varphi}^{-1}$ is given by
  $$y^i = f^i(x^1, \ldots, x^n, 0, \ldots, 0), \quad i = 1, \ldots, n.$$
  Therefore, $\widetilde{\varphi}' \circ \widetilde{\varphi}^{-1}$ is C^∞ by a previous definition.

Proof (Cont'd)

- By a previous theorem, the totality of these neighborhoods defines a unique differentiable structure on N.

  In preferred local coordinates $V, \widetilde{\varphi}$, $i : N \to M$ is given on V by

  $$(x^1, \ldots, x^n) \to (x^1, \ldots, x^n, 0, \ldots, 0).$$

  So it is obviously an immersion.

  But we have taken the relative topology on N.

  So $i : N \to M$ is, by definition, a homeomorphism to its image $i(N) = N$, with the subspace topology.

  So, i is an imbedding.

Regular Submanifolds

Definition

A **regular submanifold** of a C^∞ manifold M is any subspace N with the submanifold property and with the C^∞ structure that the corresponding preferred coordinate neighborhoods determine on it.

Example: We see that $S^2 = \{x \in \mathbb{R}^3 : \|x\| = 1\}$ is really a submanifold, as was indicated previously.

Let $q = (a^1, a^2, a^3)$ be an arbitrary point on S^2.

q cannot lie on more than one coordinate axis.

For convenience we suppose that it does not lie on the x^3-axis.

We introduce the usual spherical coordinates (r, θ, φ).

They are defined on $\mathbb{R}^3 - \{x^3\text{-axis}\}$.

Suppose $(1, \theta_0, \varphi_0)$ are the coordinates of q.

Regular Submanifolds (Cont'd)

- We may change the coordinate map slightly so that:
  - r is replaced by $\widetilde{r} = r - 1$;
  - θ is replaced by $\widetilde{\theta} = \theta - \theta_0$
  - φ is replaced by $\widetilde{\varphi} = \varphi - \varphi_0$.

  Consider the neighborhood V, ψ with coordinate function

  $$\psi : p \to (\widetilde{r}(p), \widetilde{\theta}(p), \widetilde{\varphi}(p)),$$

  defined for p, such that $|\widetilde{r}| < \varepsilon$, $|\widetilde{\theta}| < \varepsilon$, and $|\widetilde{\varphi}| < \varepsilon$.
  For sufficiently small θ, V, ψ defines a coordinate neighborhood of q, with:
  - q having coordinates $(0, 0, 0)$;
  - $V \cap S^2$ the open subset of S^2 corresponding to $\widetilde{r} = 0$.

  The fact that these neighborhoods are compatible with the ones previously defined for S^2 can be proved by writing down the standard equations giving rectangular Cartesian coordinates as functions of the spherical coordinates.

Remark

- At this point we have defined three classes of submanifolds in a manifold M, **immersed**, **imbedded** and **regular**.
- The first of these, usually called simply a submanifold, was defined as the image $N = F(N')$ of a C^∞ univalent immersion F of N' into M.
- Since $F : N' \to N \subseteq M$ is one-to-one and onto, we may and do (as part of the definition) carry over to N the topology and differentiable structure of N'.
  - Open sets of N are the images of open sets of N';
  - Coordinate neighborhoods U, φ of N are of the form:
    - $U = F(U')$, where U' is a coordinate neighborhood of N';
    - $\varphi = \varphi' \circ F^{-1}$.
- The continuity of F implies that the topology of N, thus obtained, is in general finer than its relative topology as a subspace of M.
- That is, if V is open in M, then $V \cap N$ is open in N, but there may be open sets of N which are not of this form.

Remark (Cont'd)

- An imbedding is a particular type of univalent immersion, one in which U' is open in N' if and only if $F(U') = U \cap N$, for some open set U of M.
- So the topology of the submanifold $N = F(N')$ is exactly its relative topology as a subspace of M.
- An imbedded submanifold is thus a special type of (immersed) submanifold.

  Note: Although submanifold and immersed submanifold are the same thing by definition, nevertheless we will frequently use the latter term both for emphasis and to avoid potential confusion.
- Finally, if $N \subseteq M$ is a regular submanifold, then it is also an imbedded submanifold, since the inclusion $i : N \to M$ is an imbedding.

Imbedded and Regular Submanifolds

Theorem

Let $F : N' \to M$ be an imbedding of a C^∞ manifold N' of dimension n in a C^∞ manifold M of dimension m. Then $N = F(N')$ has the n-submanifold property and, thus, N is a regular submanifold. As such it is diffeomorphic to N' with respect to the mapping $F : N' \to N$.

- Let $q = F(p)$ be any point of N. According to a previous theorem (and its proof), there are cubical coordinate neighborhoods U, φ of p and V, ψ of q such that:
  (i) $\varphi(p) = (0, \ldots, 0) \in C^n_\varepsilon(0) = \varphi(U)$;
  (ii) $\psi(q) = C^m_\varepsilon(0) = \psi(V)$;
  (iii) The mapping $F|_U$ is given in local coordinates by

$$\widehat{F} : (x^1, \ldots, x^n) \mapsto (x^1, \ldots, x^n, 0, \ldots, 0).$$

  If $F(U) = V \cap N$, then the neighborhood V, ψ would be a preferred coordinate neighborhood relative to N.

Imbedded and Regular Submanifolds (Cont'd)

- To achieve this situation, we use the fact that F is an imbedding.

  This implies at least that $F(V)$ is a relatively open set of N.

  That is, $F(V) = W \cap N$, where W is open in M.

  Since $V \supseteq F(U)$, it is no loss of generality to suppose $W \subseteq V$.

  Thus:
  - $\psi(W)$ is an open subset of $C_\varepsilon^m(0)$ containing the origin;
  - $\psi(W) \supseteq \psi(F(U))$, which is a slice S of $C_\varepsilon^m(0)$,

  $$S = \{x \in C_\varepsilon^m(0) : x^{n+1} = \cdots = x^m = 0\}.$$

  Therefore, we may choose a (smaller) open cube $C_\delta^m(0) \subseteq \psi(W)$ and let

  $$V' = \psi^{-1}(C_\delta^m(0)), \quad \psi' = \psi|_V'.$$

  This is a cubical coordinate neighborhood of q, with $F(U) \cap V' = V' \cap N$.

Imbedded and Regular Submanifolds (Cont'd)

- Take $U' = \varphi^{-1}(C_\delta^n(0)) = F^{-1}(V')$.

  We see that U', φ', with $\varphi' = \varphi|_{U'}$, is a coordinate neighborhood of p.

  Moreover, the pair U', φ', and V', ψ' have exactly the properties needed, namely, Properties (i), (ii), (iii) and $F(U') = V' \cap N$.

  This proves simultaneously that:
  - N has the n-submanifold property;
  - F is a diffeomorphism.

  The latter is true since the inverse of $F : N' \to N$ is given in the local preferred coordinates $V', \pi \circ \psi'$ and U', φ' by

$$\widehat{F}^{-1}(x^1, \ldots, x^n) = (x^1, \ldots, x^n),$$

  which is clearly C^∞.

Remark

- Suppose that $N \subseteq M$ is an (immersed) submanifold.
- Let $q \in N$.
- Then there is a cubical neighborhood V, ψ of q with

$$\psi(q) = (0, \ldots, 0) \in C_\varepsilon^m(0) = \psi(V),$$

such that the slice $S' \subseteq V$ consisting of all points of V whose last $m - n$ coordinates vanish is:
  - An open set;
  - A cubical coordinate neighborhood of the submanifold structure of N, with coordinate map

$$\psi'(r) = \pi \circ \psi(r) = (x^1(r), \ldots, x^n(r)).$$

One-One Immersion From Compact Domain

- It is usually easier to determine that a map from one C^∞ manifold into another is an immersion than to see that it is an imbedding.
- So the following theorem is useful.

Theorem

Suppose $F : N \to M$ is a one-to-one immersion and N is compact. Then F is an imbedding and $\widetilde{N} = F(N)$ is a regular submanifold.

- We know that F is continuous.

  Also both N and $\widetilde{N}$, with the subspace topology, are Hausdorff.

  So we have a continuous (one-to-one) mapping from a compact space to a Hausdorff space.

One-One Immersion From Compact Domain (Cont'd)

- A closed subset K of N is compact.

  So $F(K)$ is compact.

  Therefore, $F(K)$ is closed.

  Thus, F takes closed subsets of N to closed subsets of $\widetilde{N}$.

  Since it is one-to-one onto, it takes open subsets to open subsets also.

  It follows that F^{-1} is continuous.

  So $F : N \to \widetilde{N}$ is a homeomorphism and, therefore, an imbedding.

  The rest of the statement follows from the preceding remarks.

Submanifolds via Maps of Constant Rank

Theorem

Let N be a C^∞ manifold of dimension n and M be a C^∞ manifold of dimension m. Let $F : N \to M$ be a C^∞ mapping. Suppose that F has constant rank k on N. Let $q \in F(N)$. Then $F^{-1}(q)$ is a closed, regular submanifold of N of dimension $n - k$.

- Let A denote $F^{-1}(q)$.

  $\{q\}$ is a closed subset of M.

  A is the inverse image of $\{q\}$ under a continuous map.

  So A is a closed subset.

Submanifolds via Maps of Constant Rank (Cont'd)

- We show A has the submanifold property for the dimension $n-k$.

  Let $p \in A$.

  F has constant rank k on a neighborhood of p.

  By the theorem on rank we may find coordinate neighborhoods U, φ and V, ψ of p and q, respectively, such that:
  - $\varphi(p)$ and $\psi(q)$ are the origins in $\mathbb{R}^n$ and $\mathbb{R}^m$;
  - $\varphi(U) = C_\varepsilon^n(0)$, $\psi(V) = C_\varepsilon^m(0)$;
  - In local coordinate $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^m)$, $F|_U$ is given by

$$\psi \circ F \circ \varphi^{-1}(x) = \widehat{F}(x^1, \ldots, x^n) = (x^1, \ldots, x^k, 0, \ldots, 0).$$

Submanifolds via Maps of Constant Rank (Cont'd)

- This means that the only points of U mapping onto q are those whose first k coordinates are zero.

  That is,

  $$\begin{aligned} A \cap U &= \varphi^{-1}(\varphi \circ F^{-1} \circ \psi^{-1}(0)) \\ &= \varphi^{-1}(\widehat{F}^{-1}(0)) \\ &= \varphi^{-1}(\{x \in C_\varepsilon^n(0) : x^1 = \cdots = x^k = 0\}). \end{aligned}$$

  Hence, A has the submanifold property.

  So it is a regular manifold of dimension $n - k$.

Corollary

Corollary

Let $F: N \to M$ be a C^∞ mapping of manifolds. Suppose

$$\dim M = m < n = \dim N,$$

and $\text{rank} F = m$ at every point of $A = F^{-1}(a)$. Then A is a closed, regular submanifold of N.

- At $p \in A$, F has the maximum rank possible, namely m.

  By a previous section and the independence of rank on local coordinates, in some neighborhood of p in N, F has this rank also.

  Thus, the rank of F is m on an open subset of N containing A.

  But such a subset is itself a (open) manifold of dimension n.

  Now we may apply the preceding theorem to that subset.

Example

- Consider the map $F : \mathbb{R}^n \to \mathbb{R}$ defined by

$$F(x^1, \ldots, x^n) = \sum_{i=1}^{n} (x^i)^2.$$

- It has rank 1 on $\mathbb{R}^n - \{0\}$.
- Moreover, $\mathbb{R}^n - \{0\}$ contains $F^{-1}(+1) = S^{n-1}$.
- Thus, by the corollary, S^{n-1} is an $(n-1)$-dimensional submanifold of $\mathbb{R}^n$.

Example

- Consider the map $F : \mathbb{R}^3 \to \mathbb{R}$ given by

$$F(x^1, x^2, x^3) = \left(a - ((x^1)^2 + (x^2)^2)^{1/2}\right)^2 + (x^3)^2.$$

- Its has rank 1 at each point of $F^{-1}(b^2)$, $a > b > 0$.
- The locus $F^{-1}(b^2)$ is the torus in $\mathbb{R}^3$.
- So, by the corollary, the torus in $\mathbb{R}^3$ is a submanifold.

Subsection 6

Lie Groups

Introducing Lie Groups

- The space $\mathbb{R}^n$ is:
  - A C^∞ manifold;
  - An Abelian group with group operation given by componentwise addition.
- Moreover the algebraic and differentiable structures are related.
- The mapping

$$(x, y) \mapsto x + y$$

  is a C^∞ mapping of the product manifold $\mathbb{R}^n \times \mathbb{R}^n$ onto $\mathbb{R}^n$, that is, the group operation is differentiable.
- We also see that the mapping of $\mathbb{R}^n$ onto $\mathbb{R}^n$ given by

$$x \mapsto -x \quad \text{(its inverse)}$$

  is differentiable.

Lie Groups

- Let G be a group which is at the same time a differentiable manifold.
- For $x, y \in G$ let:
  - xy denote their product;
  - x^{-1} the inverse of x.

Definition

G is a **Lie group** provided that the following are both C^∞ mappings:

- The mapping of $G \times G \to G$ defined by
$$(x, y) \to xy;$$
- The mapping of $G \to G$ defined by
$$x \to x^{-1}.$$

Example

- Consider $Gl(n, \mathbb{R})$, the set of nonsingular $n \times n$ matrices.
- We have seen that it is an open submanifold of $\mathcal{M}_n(\mathbb{R})$, the set of $n \times n$ real matrices identified with $\mathbb{R}^{n^2}$.
- Moreover $Gl(n, \mathbb{R})$ is a group with respect to matrix multiplication.
- In fact, an $n \times n$ matrix A is nonsingular if and only if $\det A \neq 0$.

  But we also have
  $$\det(AB) = (\det A)(\det B).$$
  So if A and B are nonsingular, AB is also.
- An $n \times n$ matrix A is nonsingular, that is, $\det A \neq 0$, if and only if A has a multiplicative inverse.
- Thus $Gl(n, \mathbb{R})$ is a group.

Example (Cont'd)

- Both the maps $(A, B) \to AB$ and $A \to A^{-1}$ are C^ω.
- The product has entries which are polynomials in those of A and B.

  These entries are exactly the expressions in local coordinates of the product map.

  So the product map is C^ω and, hence, C^∞.

Example (Cont'd)

- The inverse of $A = (a_{ij})$ may be written as

$$A^{-1} = \frac{1}{\det A}(\widetilde{a}_{ij}),$$

where:
  - The $\widetilde{a}_{ij}$ are the cofactors of A (thus polynomials in the entries of A);
  - $\det A$ is a polynomial in these entries which does not vanish on $Gl(n, \mathbb{R})$.

It follows that the entries of A^{-1} are rational functions on $Gl(n, \mathbb{R})$ with non-vanishing denominators.

Hence they are C^ω (and C^∞).

- Therefore $Gl(n, \mathbb{R})$ is a Lie group.
- A special case is $Gl(1, \mathbb{R}) = \mathbb{R}^*$, the multiplicative group of nonzero real numbers.

Example

- Let $\mathbb{C}^*$ be the set of nonzero complex numbers.
- Then $\mathbb{C}^*$ is a group with respect to multiplication of complex numbers, the inverse being

$$z^{-1} = \frac{1}{z}.$$

- Moreover, $\mathbb{C}^*$ is a one-dimensional C^∞ manifold covered by a single coordinate neighborhood $U = \mathbb{C}^*$, with coordinate map $z \to \varphi(z)$ given by

$$\varphi(x + iy) = (x, y), \quad \text{for } z = x + iy.$$

Example (Cont'd)

- Using these coordinates:
  - The product $w = zz'$, $z = x + iy$, and $z' = x' + iy'$, is given by
$$((x,y)(x',y')) \to (xx' - yy', xy' + yx');$$
  - The mapping $z \to z^{-1}$ by
$$(x,y) \to \left(\frac{x}{x^2+y^2}, \frac{-y}{x^2+y^2}\right).$$
- This means that the two mappings are C^∞.
- Therefore $\mathbb{C}^*$ is a Lie group.

Products of Lie Groups

Theorem

Let G_1 and G_2 be Lie groups. Then the direct product $G_1 \times G_2$ of these groups with the C^∞ structure of the Cartesian product of manifolds is a Lie group.

Example (**Toral Groups**):

The circle S^1 may be identified with the complex numbers of absolute value $+1$.

We have

$$|z_1 z_2| = |z_1||z_2|.$$

So it is a group with respect to multiplication of complex numbers.

It is actually a subgroup of $\mathbb{C}^*$.

Products of Lie Groups (Cont'd)

- It is a Lie group as can be checked directly or proved as a consequence of the previous example and the next theorem.

  Combining this with the preceding theorem, we see that

  $$T^n = \underbrace{S^1 \times \cdots \times S^1}_{n}.$$

  the *n*-fold Cartesian product, is a Lie group.

  It is called the **toral group**.

  Since S^1 is Abelian, T^n is Abelian also.

Subgroups as Lie Groups

Theorem

Let G be a Lie group. Let H a subgroup which is also a regular submanifold. Then, with its differentiable structure as a submanifold, H is a Lie group.

- It can be shown that $H \times H$ is a regular submanifold of $G \times G$. Thus, the inclusion map

$$F_1 : H \times H \to G \times G$$

is a C^∞ imbedding.
Let $F_2 : G \times G \to G$ be the C^∞ mapping

$$(g, g') \to gg'.$$

Let

$$F = F_2 \circ F_1$$

be the composition.

Subgroups as Lie Groups (Cont'd)

- Then F is a C^∞ mapping from $H \times H \to G$ with image in H.

  Let F denote this map considered as a map into H.

  It is not the same mapping as F, since we have changed the range.

  We must show that F is C^∞.

  Similarly, we must show that the map $H \to G$, given by taking

  $$h \to h^{-1},$$

  is C^∞ as a map onto H.

  These facts both follow from the next lemma, which completes the proof.

Changing the Codomain

Lemma

Let A and M be C^∞ manifolds. Let $F : A \to M$ be a C^∞ mapping. Suppose $F(A) \subseteq N$, N being a regular submanifold of M. Then F is C^∞ as a mapping into N.

- By hypothesis, N is a regular submanifold of M.

  So each point is contained in a preferred coordinate neighborhood.

  Let $p \in A$ and let $q = F(p)$ be its image.

  Let U, φ be a neighborhood of p which maps into a preferred coordinate neighborhood V, ψ of q.

  For $m = \dim M$ and $n = \dim N$, we have:
  - $\psi(V) = C_\varepsilon^m(0)$;
  - $\psi(q) = (0, \ldots, 0)$, the origin of $\mathbb{R}^m$;
  - $V \cap N$ consists of those points of V whose last $m - n$ coordinates are zero.

Changing the Codomain (Cont'd)

- Let $(x^1, \ldots, x^p)$ be the local coordinates in U, φ on A.

  Then the expression in local coordinates for F is

  $$\widehat{F}(x^1, \ldots, x^p) = (f^1(x), \ldots, f^n(x), 0, \ldots, 0).$$

  That is, $f^{n+1}(x) = \cdots = f^m(x) = 0$ since $F(A) \subseteq N$.

  But $V \cap N$, $\pi \circ \psi|_{V \cap N}$ is a coordinate neighborhood of q on N.

  So F, considered as a mapping into N, is given in local coordinates by

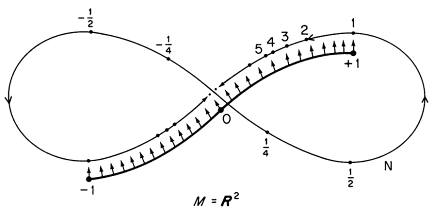
  $$(x^1, \ldots, x^n) \to (f^1(x), \ldots, f^n(x)).$$

  This is $\widehat{F}$, followed by projection to the first n coordinates (projection of $\mathbb{R}^m$ to $\mathbb{R}^n$).

  Being a composition of C^∞ maps, it is, therefore, C^∞.

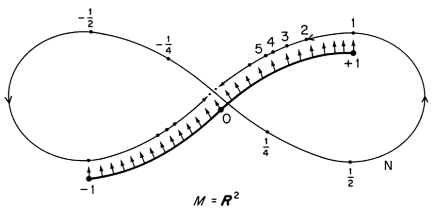
Remark

- The lemma does not hold for immersed submanifolds.
- Consider the second figure eight example.



- Suppose we map the open interval $(-1, 1)$ by a mapping G into $N = F(\mathbb{R})$, the figure eight, so that it crosses the origin as shown in the figure.
- Then G is C^∞ as a mapping into $\mathbb{R}^2$.
- But it is not even continuous as a mapping to N.

Remark (Cont'd)



- Thus, N is diffeomorphic to the real line by $F : \mathbb{R} \to N$.
- Identifying N and $\mathbb{R}$, we may think of G as taking:
  - Part of the open interval $(-1, 1)$, say $(0, 1)$, onto the real numbers $t > 1$;
  - 0 onto 0;
  - $(-1, 0)$, the remaining part, onto the real numbers $t < 1$.
- The image is not even connected.
- So G is not continuous.

Diffeomorphisms of a Lie Group

- We make use of the following naturally defined maps of a Lie group G onto itself.
  - (i) $x \to x^{-1}$;
  - (ii) Left and right translations by a fixed element a of G:
    - $L_a : G \to G$, defined by $L_a(x) = ax$;
    - $R_a : G \to G$, defined by $R_a(x) = xa$.
- These maps are C^∞, by the definition of Lie group.
- Moreover, they have inverses which are C^∞.
- So they are, in fact, diffeomorphisms.
- The mapping $x \to x^{-1}$ is its own inverse.
- Additionally, we have

$$(L_a)^{-1} = L_{a^{-1}} \quad \text{and} \quad (R_a)^{-1} = R_{a^{-1}}.$$

Example

- Consider
$$Sl(n,\mathbb{R}) = \{X \in Gl(n,\mathbb{R}) : \det X = +1\}.$$
- It is a subgroup and regular submanifold of $Gl(n,\mathbb{R})$.
- Hence, it is a Lie group.
- To prove this, we consider the mapping $F : Gl(n,\mathbb{R}) \to \mathbb{R}^*$,
$$F(X) = \det X.$$
- According to the product rule,
$$\det(XY) = (\det X)(\det Y).$$
- Thus F is a homomorphism onto $\mathbb{R}^* = Gl(1,\mathbb{R})$;
- It is also C^∞ since it is given by polynomials in the entries.

Example (Cont'd)

- Finally, its rank is constant.
- Let $A \in Gl(n, \mathbb{R})$, with $a = \det A$.
- Let L_X, $L_{\boldsymbol{x}}$ denote left translations in $Gl(n, \mathbb{R})$ and $Gl(1, \mathbb{R}) = \mathbb{R}^*$.
- Then we have $a \cdot \det(A^{-1}X) = \det X$.
- Therefore,
$$F(X) = L_a \circ F \circ L_{A^{-1}}(X).$$
- Now we get, for all $A \in Gl(n, \mathbb{R})$,
$$\begin{aligned} \operatorname{rank} DF(X) &= \operatorname{rank}[aDF(A^{-1}X)DL_{A^{-1}}(X)] \quad \text{(chain rule)} \\ &= \operatorname{rank} DF(A^{-1}X). \\ &\quad (DL_a = a \neq 0 \text{ and } L_{A^{-1}} \text{ diffeomorphism}) \end{aligned}$$

  In particular,
$$\operatorname{rank} DF(X) = \operatorname{rank} DF(X^{-1}X) = \operatorname{rank} DF(I).$$

  Thus, we see that the rank is constant as claimed.

Example (Cont'd)

- By a previous theorem, it follows that $Sl(n, \mathbb{R}) = F^{-1}(+1)$ is a closed, regular submanifold.
- It is also a subgroup - in fact the kernel of a homomorphism - by virtue of the product rule for determinants.
- Therefore it is a Lie group.

Example

- Consider
$$O(n) = \{X \in Gl(n, \mathbb{R}) : X^t X = I\},$$
the subgroup of orthogonal $n \times n$ matrices.
- It is a regular submanifold and, thus, a Lie group.
- Consider the mapping F from $Gl(n, \mathbb{R})$ to $Gl(n, \mathbb{R})$,
$$F(X) = X^t X, \quad X^t = \text{transpose of } X.$$
- For $A \in Gl(n, \mathbb{R})$, we will show that
$$\text{rank} DF(X) = \text{rank} DF(XA^{-1}).$$
- But any $Y \in Gl(n, \mathbb{R})$ can be written in the form $Y = XA^{-1}$.
- It follows that rank DF is constant on $Gl(n, \mathbb{R})$.
- To prove this equality we note that
$$F(XA^{-1}) = L_{(A^t)^{-1}} \circ R_{A^{-1}} \circ F(X).$$

Example (Cont'd)

- Therefore

$$DF(XA^{-1}) = DL_{(A^t)^{-1}} \circ DR_{A^{-1}} \circ DF(X),$$

  where:
  - $DR_{A^{-1}}$ is evaluated at $F(X)$;
  - $DL_{(A^t)^{-1}}$ is evaluated at $R_{A^{-1}}(F(X))$.
- Then the equality of rank$DF(XA^{-1})$ and rank$DF(X)$ follows as above from the fact that $DL_{(A^t)^{-1}}$ and $DR_{A^{-1}}$ are everywhere nonsingular.
- Now $O(n) = F^{-1}(I)$, where I is the identity matrix.
- So the statement follows from a previous theorem.

Homomorphisms of Lie Groups

Definition

Let $F : G_1 \to G_2$ be an algebraic homomorphism of Lie groups G_1 and G_2. We shall call F a **homomorphism (of Lie groups)** if F is also a C^∞ mapping.

Example: Let $G_1 = Gl(n, \mathbb{R})$ and $G_2 = \mathbb{R}^*$ $[= Gl(1, \mathbb{R})]$.

Consider the map F given by

$$F(X) = \det X.$$

$F : G_1 \to G_2$ is a homomorphism.

Example

- Let $G_1 = \mathbb{R}$, the additive group of real numbers.
- Let $G_2 = S^1$, identified with the multiplicative group of real numbers of absolute value 1.
- Consider the mapping
$$F(t) = e^{2\pi i t}.$$
- F is a homomorphism.

Example

- Similarly, let $G_1 = \mathbb{R}^n$ be a Lie group with componentwise addition.
- Let $G_2 = T^n = S^1 \times \cdots \times S^1$.
- Consider the mapping $F : \mathbb{R}^n \to T^n$ given by

$$F(t_1, \ldots, t_n) = (e^{2\pi i t_1}, \ldots, e^{2\pi i t_n}).$$

- F is a homomorphism.
- Its kernel is the discrete additive group $\mathbb{Z}^n$ consisting of all n-tuples of integers.
- It is called the **integral lattice** of $\mathbb{R}^n$.

Rank of Homomorphisms of Lie Groups

Theorem

Let $F : G_1 \to G_2$ be a homomorphism of Lie groups. Then:

- The rank of F is constant
- The kernel is a closed regular submanifold and, thus, a Lie group;
- $\dim \ker F = \dim G_1 - \operatorname{rank} F$.

- Let $a \in G_1$ be arbitrarily chosen.
  Let $b = F(a)$ be its image in G_2.
  Denote by e_1, e_2 the unit elements of G_1, G_2, respectively.
  Then we may write
  $$F(x) = F(aa^{-1}x) = F(a)F(a^{-1}x) = L_b \circ F \circ L_{a^{-1}}(x).$$
  So for all $a \in G_1$,
  $$DF(a) = DL_b(e_2) \cdot DL_{a^{-1}}(a).$$

Rank of Homomorphisms of Lie Groups (Cont'd)

- For all $a \in G_1$,
$$DF(a) = DL_b(e_2) \cdot DL_{a^{-1}}(a).$$
Now $L_{a^{-1}}$ and L_b are diffeomorphisms.
Thus, they have nonsingular Jacobian matrices at each point.
The rank of F at a and at e_1 is the same.
By a previous theorem, $\ker F = F^{-1}(e_1)$ is a closed regular submanifold whose dimension is $\dim G_1 - \operatorname{rank} F$.
By another theorem, $\ker F$ is a Lie group since it is a regular submanifold (and a group).

Example

- A very useful example of a submanifold which is not regular but is a subgroup of a Lie group is obtained as follows.
- Let $T^2 = S^1 \times S^1$.
- Let $F : \mathbb{R}^2 \to T^2$ be given by

$$F(x^1, x^2) = (e^{2\pi i x^1}, e^{2\pi i x^2}).$$

- Then F is a C^∞ map of rank 2 everywhere.
- Moreover, it is a homomorphism of Lie groups.
- The rank may be easily computed at $(0, 0)$.
- It is constant by the theorem.

Example

- Let α be an irrational number.
- Define $G : \mathbb{R} \to \mathbb{R}^2$ by

$$G(t) = (t, \alpha t).$$

- G is obviously an imbedding.
- Its image is the line through the origin of slope α.
- Let $F : \mathbb{R}^2 \to T^2$ be the map of the preceding slide.
- Let

$$H = F \circ G : \mathbb{R} \to T^2.$$

- $DH = DF \cdot DG$ has rank 1, for all $t \in \mathbb{R}$.
- It follows that H is an immersion of $\mathbb{R}$ into T^2

Example (Cont'd)

- Note that H is one-to-one.

  Suppose $H(t_1) = H(t_2)$.

  Then $e^{2\pi i t_1} = e^{2\pi i t_2}$ and $e^{2\pi i \alpha t_1} = e^{2\pi i \alpha t_2}$.

  However, $e^{2\pi i u} = e^{2\pi i v}$ if and only if $u - v$ is an integer.

  Clearly $t_1 - t_2$ and $\alpha(t_1 - t_2)$ are both integers only if $t_1 = t_2$.
- Thus $H : \mathbb{R} \to T^2$ is a one-to-one immersion.
- So $H(\mathbb{R})$ is an immersed submanifold.
- However, the interesting fact is that $H(\mathbb{R})$ is a dense subset of T^2.
- So it is about as far from being a regular submanifold.
- For example, as a subspace it is not locally connected at any point.

Example (Cont'd)

- We shall prove that $H(\mathbb{R})$ is dense in T^2.
- F is continuous and onto.
- Thus, a dense subset D of $\mathbb{R}^2$ is mapped to a dense subset of T^2.
- We will show that $D = F^{-1}(H(\mathbb{R}))$ is dense.
- D consists not only of the line of slope α through the origin but of all lines which can be obtained from it by translation by an integral vector in either direction.
- Let $(x^1 + m, x^2 + n)$ be a point, with:
  - m, n integers;
  - $x^1 = t$, $x^2 = \alpha t$.
- We have $F(x^1, x^2) = F(x^1 + m, x^2 + n)$.
- So $(x^1 + m, x^2 + n)$ must also be in D.
- These lines are all parallel to the given one $H(\mathbb{R})$.
- In fact D consists of the union of all lines $t \to (t + m, \alpha t + n)$.
- That is, all lines with equation $x^2 = \alpha x^1 + (n - \alpha m)$, $n, m \in \mathbb{N}$.

Example (Cont'd)

- Obviously, D is dense on the plane if the y-intercepts $(n - \alpha m)$ form a dense subset of the y-axis.
- Thus, we must show that given α, any real number b, and any $\varepsilon > 0$, there is a pair of integers n, m with $|b - (n - \alpha m)| < \varepsilon$.
- Assume that there exist integers n', m' such that $0 \leq n' - \alpha m' < \varepsilon$;
- Since $n' - \alpha m'$ is irrational, it must then in fact be positive.
- It follows that for some integer k,

$$k(n' - \alpha m') \leq b \leq (k+1)(n' - \alpha m').$$

- This implies

$$0 < b - k(n' - \alpha m') < n' - \alpha m' < \varepsilon.$$

- Now $n - \alpha m = kn' - \alpha km'$ is a y-intercept of a line of D.
- So, since either $n' - \alpha m'$ or $(-n') - \alpha(-m')$ is nonnegative, the following fact from number theory completes the proof.

Example (Cont'd)

- If $\alpha > 0$ is any irrational number, then there exist arbitrarily large integers n', m' such that

$$\left|\frac{n'}{m'} - \alpha\right| < \frac{1}{m'^2}.$$

- This is asserted by the Kronecker Approximation Theorem.
- We remark that $H : \mathbb{R} \to \mathbb{R}^2$ in addition to being a one-to-one immersion is a homomorphism of Lie groups.
- So that $\widetilde{R} = H(\mathbb{R})$ is:
  - A subgroup algebraically;
  - An immersed submanifold.
- It is clearly a Lie group with the manifold structure of $\mathbb{R}$.
- However, it is not a regular submanifold nor is it a closed subset.

Lie Subgroups

Definition

A (**Lie**) **subgroup** H of a Lie group G is any algebraic subgroup which is a submanifold and is a Lie group with its C^∞ structure as an (immersed) submanifold.

Theorem

If H is a regular submanifold and subgroup of a Lie group G, then H is closed as a subset of G.

- It is enough to show that whenever a sequence $\{h_n\}$ of elements of H has a limit $g \in G$, then g is in H.

  Let U, φ be a preferred coordinate neighborhood of the identity e relative to the regular submanifold H.

Lie Subgroups (Cont'd)

- Then the following hold:
  - $\varphi(U) = C_\varepsilon^m(0)$ is a cube with $\varphi(e) = 0$;
  - $V = H \cap U$ consists exactly of those points whose last $m - n$ coordinates are zero;
  - $\varphi' = \varphi|_V$ maps V homeomorphically onto this slice of the cube.

  Let $\{\widetilde{h}_n\}$ is a sequence in $V = H \cap U$.

  Suppose $\lim \widetilde{h}_n = \widetilde{g}$, with $\widetilde{g} \in V$.

  Then the last $m - n$ coordinates of $\widetilde{g}$ are also zero.

  So $\widetilde{g} \in H \cap U \subseteq H$.

Lie Subgroups (Cont'd)

- Let $\{h_n\}$ be any sequence of H with $\lim h_n = g$.

  Let W be a neighborhood of e small enough so that $W^{-1}W \subseteq V$, where
  $$W^{-1}W = \{x^{-1}y \in G : x, y \in W\}.$$
  Such W exist by continuity of the group operations.

  There exists N, such that, for $n \geq N$, $h_n \in gW$.

  In particular, $h_N \in gW$.

  Using group operations, we may verify that:
  (i) $\widetilde{g} = g^{-1}h_N \in W$;
  (ii) $\lim \widetilde{h}_n = \widetilde{g}$, where $\widetilde{h}_n := h_n^{-1}h_N$.

  But for $n \geq N$, $\widetilde{h}_n = h_n^{-1}h_N$ lies in $(gW)^{-1}gW = W^{-1}W \subseteq V$.

  Thus, by preceding remarks, $\widetilde{g} \in H$.

  Hence, $g = h_N\widetilde{g}^{-1} \in H$.

Closed Subgroups

- A converse statement is also true:

  A Lie subgroup H of a Lie group G that is closed as a subset is necessarily a regular submanifold.
- In fact it is even true that an algebraic subgroup (not assumed to be an immersed submanifold), which is closed as a subset, is a regular submanifold.
- The proof is complicated and we omit it.
- However, it validates the following terminology.
- A subgroup H of a Lie group G, which is a regular submanifold, will be called a **closed subgroup** of G.

Subsection 7

The Action of a Lie Group on a Manifold

Actions of a Group

Definition

Let G be a group and X a set.
Then G is said to **act on** X (on the left) if there is a mapping

$$\theta : G \times X \to X$$

satisfying two conditions:

(i) If e is the identity element of G, then $\theta(e, x) = x$, for all $x \in X$;

(ii) If $g_1, g_2 \in G$, then $\theta(g_1, \theta(g_2, x)) = \theta(g_1 g_2, x)$, for all $x \in X$.

When G is a topological group, X is a topological space, and θ is continuous, then the action is called **continuous**.
When G is a Lie group, X is a C^∞ manifold, and θ is a C^∞ mapping, we speak of a C^∞ **action**.

- Note that a C^∞ action is a fortiori continuous.

Notation

- As a matter of notation we shall often write gx for $\theta(g,x)$.
- So Condition (ii) reads

$$(g_1 g_2)x = g_1(g_2 x).$$

- For g fixed, we let $\theta_g(x)$ denote the mapping $\theta_g : X \to X$ defined by

$$\theta_g(x) = \theta(g,x).$$

- So Condition (ii) may also be written

$$\theta_{g_1 g_2} = \theta_{g_1} \circ \theta_{g_2}.$$

- When we define right action, Conditions (i) and (ii) become:
  (i) $\theta(x,e) = x$;
  (ii) $\theta(\theta(x,g_1),g_2) = \theta(x,g_1 g_2)$.
- Usually we are concerned with left action, but in both cases we usually say G **acts on** X, and leave the rest to be determined by the context.

Actions and Permutations

- We have $\theta_{g^{-1}} = (\theta_g)^{-1}$.

$$\theta_{g^{-1}} \circ \theta_g = \theta_{g^{-1}g} = \theta_e = i_X.$$

- So each mapping θ_g is one-to-one onto.
- This and Condition (ii) show that the following statement holds.
- If G acts on a set X, then the map

$$g \to \theta_g$$

  is a homomorphism of G into $S(X)$, the group of all permutations on X.
- Conversely, any such homomorphism determines an action with

$$\theta(g, x) = \theta_g(x).$$

Special Kinds of Actions

- We note that the homomorphism is injective if and only if

$$\theta_g = i_X \quad \text{implies} \quad g = e.$$

- If this is so, we shall call the action **effective**.
- When the action is effective, G may be identified with a subgroup of $S(X)$ by the map $g \to \theta_g$.
- The preceding considerations all refer only to the set-theoretic aspects, since $S(X)$ has not been topologized.
- We also note that if X is a topological space (C^∞ manifold), G a topological group (Lie group), and the action is continuous (C^∞), then each θ_g is a homeomorphism (diffeomorphism).

Actions via Homomorphisms

- Let H, G be groups.
- Let $\psi : H \to G$ be a homomorphism.
- Then $\theta : H \times G \to G$ defined by

$$\theta(h, x) = \psi(h)x$$

  is a left action.
- Indeed, we have:
  (i) $\theta(e_H, x) = \psi(e_H)x = e_G x = x$;
  (ii) Moreover,

$$\begin{array}{rcl}\theta(h_1, \theta(h_2, x)) & = & \theta(h_1, \psi(h_2)x) \\ & = & \psi(h_1)(\psi(h_2)x) \\ & = & (\psi(h_1)\psi(h_2))x \\ & = & \psi(h_1 h_2)x \\ & = & \theta(h_1 h_2, x).\end{array}$$

Actions by Left Translations

- Suppose H and G are Lie groups.
- Suppose $\psi : H \to G$ is a homomorphism of Lie groups.
- Then the action $\theta : H \times G \to G$ defined by

$$\theta(h, x) = \psi(h)x$$

  is C^∞.
- This may be applied to the case where H is a Lie subgroup of G (or even if $H = G$).
- In this case ψ is the identity (inclusion) mapping of H into G.
- We say that H **acts on** G **by left translations**.

Natural Action of $Gl(n, \mathbb{R})$ on $\mathbb{R}^n$

- Let $G = Gl(n, \mathbb{R})$ and $X = \mathbb{R}^n$.
- Define $\theta : G \times \mathbb{R}^n \to \mathbb{R}^n$ by

$$\theta(A, x) = Ax,$$

  i.e., multiplication of the $n \times n$ matrix A by the $n \times 1$ column vector obtained by writing $x \in \mathbb{R}^n$ vertically.
- This satisfies Conditions (i) and (ii) rather trivially.
- Condition (ii) is associativity (of matrix products):

$$(AB)x = A(Bx).$$

- Since $\theta : G \times \mathbb{R}^n \to \mathbb{R}^n$ is given by polynomials in the entries of $A \in Gl(n, \mathbb{R})$ and $x \in \mathbb{R}^n$, it is a C^∞-map:

$$\theta\left((a_{ij})\begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}\right) = \left(\sum_{j=1}^{n} a_{ij} x^j\right).$$

Natural Action of $Gl(n, \mathbb{R})$ on $\mathbb{R}^n$ (Cont'd)

- Let $H \subseteq Gl(n, \mathbb{R})$ be a subgroup in the sense of Lie groups.
- That is, H has its own Lie group structure such that the inclusion map $i : H \to Gl(n, \mathbb{R})$ is an immersion, or, if H is a closed subgroup, an imbedding.
- Then θ restricted to H defines a C^∞ action

$$\theta_H : H \times \mathbb{R}^n \to \mathbb{R}^n.$$

- This is because:
  - $\theta_H = \theta \circ i$, $i : H \to G$ the inclusion map;
  - Both θ and i are C^∞.

Example

- Let $H \subseteq Gl(2, \mathbb{R})$ be the subgroup of all matrices of the form

$$\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \quad a > 0.$$

- Then H is seen to be a two-dimensional submanifold of $Gl(2, \mathbb{R})$.
- Therefore, it is a closed subgroup.
- The restriction to H of the natural action of $Gl(2, \mathbb{R})$ on $\mathbb{R}^2$ is

$$\theta_H\left(\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}, \begin{pmatrix} x^1 \\ x^2 \end{pmatrix}\right) = \begin{pmatrix} ax^1 + bx^2 \\ ax^2 \end{pmatrix}.$$

- θ_H is obviously C^∞, as expected.

Example

- Identify $\boldsymbol{E}^n$ with $\mathbb{R}^n$.
- Let d be the usual metric,

$$d(x,y)=\sqrt{\sum_{i=1}^{n}(x^i-y^i)^2}.$$

- Consider the group G of all rigid motions.
- These are diffeomorphisms $T:\mathbb{R}^n\to\mathbb{R}^n$ such that

$$d(Tx,Ty)=d(x,y).$$

- They are transformations T of the form

$$T(x)=Ax+b,$$

  where:
  - $A\in O(n)$, a rotation of $\mathbb{R}^n$ about the origin;
  - $b\in\mathbb{R}^n$, inducing a translation taking the origin to b.

Example (Cont'd)

- The group operation is composition of rigid motions.
- The group of rigid motions is a Lie group.
- It is in one-to-one correspondence with $O(n) \times \mathbb{R}^n$.
- It takes its manifold structure from this correspondence.
- The correspondence is given by assigning to each rigid motion, as above, the pair $(A, b) \in O(n) \times \mathbb{R}^n$.
- However, G is not a direct product in the group theoretic sense.
- Now $\theta : G \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\theta((A, b), x) = Ax + b.$$

- So θ is a C^∞ mapping.

Orbits

Definition

Let a group G act on a set M. Suppose that $A \subseteq M$ is a subset. Then GA denotes the set

$$GA = \{ga : g \in G \text{ and } a \in A\}.$$

The **orbit of** $x \in M$ is the set Gx.
If $Gx = x$, then x is a **fixed point** of G.
If $Gx = M$, for some x, then G said to be **transitive** on M.
In this case, $Gx = M$, for all x.

Example

- Consider the natural action of $Gl(n,\mathbb{R})$ on $M = \mathbb{R}^n$.
- The origin 0 is a fixed point of $Gl(n,\mathbb{R})$
- $Gl(n,\mathbb{R})$ is transitive on $\mathbb{R}^n - \{0\}$.
- To see this, let $x = (x^1,\ldots,x^n) \neq 0$.
- There is a basis $\boldsymbol{f}_1,\ldots,\boldsymbol{f}_n$ with $x = \boldsymbol{f}_1$.
- Express these basis elements in terms of the canonical basis

$$\boldsymbol{f}_i = \sum_{j=1}^{n} a_{ij}\boldsymbol{e}_j, \quad i = 1,\ldots,n.$$

- Then we see that

$$x = A \cdot \boldsymbol{e}_1, \quad A = (a_{ij}) \in Gl(n,\mathbb{R}).$$

- From this it follows that every $x \neq 0$ is in the orbit of $\boldsymbol{e}_1$.

Example (Cont'd)

- This action is not very interesting from the point of view of its orbits.
- However, if we consider this action restricted to various subgroups $G \subseteq Gl(n, \mathbb{R})$, then the orbits can be quite complicated.
- A relatively simple case of this type is obtained by letting $G = O(n)$, the subgroup of $n \times n$ orthogonal matrices in $Gl(n, \mathbb{R})$.
- This is a closed subgroup as we have seen.
- Moreover, by a previous example, the natural action of $Gl(n, \mathbb{R})$ restricted to $O(n)$ is a C^∞ action.
- The orbits are the concentric spheres.
- The origin is a fixed point (sphere of radius zero).

Space of Frames

- The same facts from linear algebra that we used above also show that $Gl(n, \mathbb{R})$ is transitive on the collection $\boldsymbol{B}$ of all bases of $\mathbb{R}^n$.
- Given any basis $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n\}$, there exists $A \in Gl(n, \mathbb{R})$, such that
$$A \cdot \boldsymbol{e}_i = \boldsymbol{f}_i.$$
- In fact, there is exactly one such A.
- Let $\boldsymbol{f} = \{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n\}$ and $\boldsymbol{e} = \{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ be elements of $\boldsymbol{B}$.
- By the preceding, we may define a left action of $Gl(n, \mathbb{R})$ on $\boldsymbol{B}$, that is, a mapping $\theta : Gl(n, \mathbb{R}) \times \boldsymbol{B} \to \boldsymbol{B}$ by
$$\theta(A, \boldsymbol{e}) = A \cdot \boldsymbol{e} = \boldsymbol{f} = \{A\boldsymbol{e}_1, \ldots, A\boldsymbol{e}_n\}.$$

Space of Frames (Cont'd)

- This action is transitive as mentioned.
- Moreover, the uniqueness of A (such that $A \cdot \boldsymbol{e} = \boldsymbol{f}$) implies that it is *simply transitive*.
- That is, given bases $\boldsymbol{f}, \widetilde{\boldsymbol{f}}$, there is exactly one $A \in Gl(n, \mathbb{R})$, such that
$$A \cdot \boldsymbol{f} = \widetilde{\boldsymbol{f}}.$$
- This means that $Gl(n, \mathbb{R})$ is in one-to-one correspondence with $\boldsymbol{B}$.
- $A \in Gl(n, \mathbb{R})$ corresponds to $A \cdot \boldsymbol{e}$, where $\boldsymbol{e}$ is the canonical basis.
- We may use this correspondence to give $\boldsymbol{B}$ the topology and C^∞ structure which makes it diffeomorphic to $Gl(n, \mathbb{R})$.
- As a C^∞ manifold it is called the **space of frames** of $\mathbb{R}^n$.

Equivalence Induced By Action

- Let G denote a Lie group and M a C^∞ manifold.
- Assume a C^∞ action $\theta : G \times M \to M$.
- We define a relation $\sim$ on M by

$$p \sim q \quad \text{iff} \quad \text{for some } g \in G, q = \theta_g(p) = gp.$$

- We can show that $\sim$ is an equivalence relation.
  - $p \sim p$, since $p = ep$. So $\sim$ is reflexive.
  - $p \sim q$ means $q = gp$. This implies $p = g^{-1}q$. Hence, $q \sim p$. So $\sim$ is symmetric.
  - $p \sim q$ and $q \sim r$ imply $q = gp$ and $r = hq$. So $r = (hg)p$. Hence, $p \sim r$. So $\sim$ is transitive.
- Moreover, the equivalence classes coincide with the orbits of G.

  Obviously, $p \sim q$ implies that p and q are on the same orbit.

  So the equivalence class $[p] \subseteq Gp$.

  Conversely, if $q \in Gp$, then $p \sim q$. So $Gp \subseteq [p]$.

Orbit Space of an Action

- We denote by M/G the set of equivalence classes.
- It will always be taken with the quotient topology.
- It is often called the **orbit space** of the action.
- With this topology the projection $\pi : M \to M/G$ (taking each $x \in M$ to its orbit) is continuous.
- Since the action θ is continuous, π is also open.

  Let $U \subseteq M$ be an open set.

  Then so is $\theta_g(U)$ for every $g \in G$.

  Now $GU = [U] = \bigcup_{g \in G} \theta_g(U)$.

  Hence GU, being a union of open sets, is open.

Orbit Space of an Action (Cont'd)

- The orbit space need not be Hausdorff.
- But, if it is, then the orbits must be closed subsets of M.

  Note that each orbit is the inverse image by π of a point of G/H.

  Points are closed in a Hausdorff space.
- We shall be particularly interested in discovering examples in which:
  - M/G is a C^∞ manifold;
  - $\pi : M \to M/G$ a C^∞ mapping.

Example

- When $M = \mathbb{R}^n$ and $G = O(n)$ acting naturally as a subgroup of $Gl(n, \mathbb{R})$, then the orbits correspond to concentric spheres.
- Thus, they are in one-to-one correspondence with the real numbers $r \geq 0$ by the mapping which assigns to each sphere its radius.
- This is a homeomorphism of $\mathbb{R}^n / O(n)$ and the ray $0 \leq r < \infty$.
- This is not a manifold, but it is almost one.

Example

- Let G be a Lie group and H a subgroup (in the algebraic sense).
- Then H acts on G on the right by right translations.
- If H is a Lie subgroup, then, according to a previous example, this is a C^∞ action.
- The set G/H of left cosets coincides with the orbits of this action.
- It is, thus, a space with the quotient topology.
- The following theorem concerns G/H (with this topology).

The Set of Left Cosets of a Lie Group

Theorem

The natural map $\pi : G \to G/H$, taking each element of G to its orbit, that is, to its left coset, is not only continuous but open.
G/H is Hausdorff if and only if H is closed.

- Note that the space G/H, usually called the **(left) coset space**, coincides with the orbit space of H acting on G.
  So π is continuous and open.
  For the last statement, use the C^∞ mapping $F : G \times G \to G$, with
  $$F(x,y) = y^{-1}x.$$
  F is continuous and $F^{-1}(H)$ is the subset
  $$R = \{(x,y) : x \sim y\} \subseteq G \times G.$$
  By a previous lemma, R is closed.
  G/H is Hausdorff if and only if H is a closed subset of G.

Stability Group and Free Action

Definition

Let G be a group acting on a set X and let $x \in X$.
The **stability** or **isotropy group of** x, denoted by G_x, is the subgroup of all elements of G leaving x fixed,

$$G_x = \{g \in G : gx = x\}.$$

Definition

Let G be a group acting on a set X.
Then G is said to **act freely** on X if

$$gx = x \quad \text{implies} \quad g = e.$$

That is, the identity is the only element of G having a fixed point.

Subsection 8

The Action of a Discrete Group on a Manifold

The Action of a Discrete Group on a Manifold

- By a **discrete group** Γ we shall mean a group with a countable number of elements and the discrete topology (every point is an open set).
- The countability means that Γ falls within our definition of a manifold.
  - It has a countable basis of open sets;
  - Each is homeomorphic to a zero-dimensional Euclidean space, i.e., a point.
- Thus Γ is a zero-dimensional Lie group.
- In this case to verify that an action $\theta : \Gamma \times \widetilde{M} \to \widetilde{M}$ is C^∞, we must show that, for each $h \in \Gamma$, $\theta_h : \widetilde{M} \to \widetilde{M}$ is a diffeomorphism.
- For convenience of notation, we will let h denote θ_h, writing hx for $\theta_h(x)$, and so on.

Set of Orbits and Topology

- Suppose that a C^∞ action is given.
- Consider the set of orbits

$$M = \widetilde{M}/\Gamma,$$

with the quotient topology.
- $U \subseteq M$ is open if and only if $\pi^{-1}(U)$ is open in M, where

$$\pi : \widetilde{M} \to M$$

denotes the natural map taking each x to its orbit Γx.
- We have seen that π is then continuous and open.

Discontinuous Group Actions

- If M is Hausdorff in the topology, then points are closed sets and the inverse image of any $p \in M$, that is, the orbit $\pi^{-1}(p)$, must be closed.
- Thus, an obvious necessary condition for M to possess some kind of reasonable topology and manifold structure is that, for each $x \in \widetilde{M}$, the orbit Γx is closed.
- However, this condition is not sufficient.
- A stronger requirement is the following:
  - Given any point $x \in \widetilde{M}$ and any sequence $\{h_n\}$ of distinct elements of Γ, then $\{h_n x\}$ does not converge to any point of $\widetilde{M}$.
- A group action with this property is called **discontinuous**.
- Discontinuity is equivalent to the requirement that each orbit be a closed, discrete subset of $\widetilde{M}$.

Properly Discontinuous Group Actions

- In the presence of other conditions, discontinuity is sometimes enough to ensure that $\widetilde{M}/\Gamma$ is Hausdorff.
- In general we need the following condition, which is even stronger.

Definition

A discrete group Γ is said to act **properly discontinuously** on a manifold $\widetilde{M}$ if the action is C^∞ and satisfies the following two conditions:

(i) Each $x \in \widetilde{M}$ has a neighborhood U, such that the following is finite

$$\{h \in \Gamma : hU \cap U \neq \emptyset\};$$

(ii) If $x, y \in \widetilde{M}$ are not in the same orbit, then there are neighborhoods U, V of x, y, such that $U \cap \Gamma V = \emptyset$.

Consequences of Proper Discontinuity

- Condition (ii) implies at once that $M = \widetilde{M}/\Gamma$ is Hausdorff.
- In fact, Condition (ii) is equivalent to the statement that

$$R = \{(x, y) : x \sim y\} \subseteq M \times M \text{ is closed.}$$

- A consequence of proper discontinuity is the following statement.

(i') The isotropy group Γ_x of each $x \in \widetilde{M}$ is finite, and each x has a neighborhood U, such that

$$\begin{cases} hU \cap U = \emptyset, & \text{if } h \notin \Gamma_x, \\ hU = U, & \text{if } h \in \Gamma_x. \end{cases}$$

- This condition is denoted Condition (i') because it could be used to replace Condition (i) in the definition.

Example

- Let $M = S^{n-1}$, the set
$$\{x \in \mathbb{R}^n : \|x\| = 1\}.$$
- Let $\Gamma = \mathbb{Z}_2$, the cyclic group of order 2 with generator h.
- Γ consists of h and $h^2 = e$, the identity.
- Define an action $\theta : \mathbb{Z}_2 \times S^{n-1} \to S^{n-1}$ by setting
$$h(x) = -x \quad \text{and} \quad e(x) = x.$$
- It can be shown that $\theta : \mathbb{Z}_2 \times S^{n-1} \to S^{n-1}$ is free and properly discontinuous.
- The quotient space $S^{n-1}/\mathbb{Z}_2$ is none other than real projective $n-1$ space $P^{n-1}(\mathbb{R})$.

Free and Properly Discontinuous Action

Theorem

Let Γ be a discrete group which acts freely and properly discontinuously on a manifold $\widetilde{M}$. Then there is a unique C^∞ structure of differentiable manifold on $M = \widetilde{M}/\Gamma$ (with the quotient topology), such that each $p \in M$ has a connected neighborhood U with the property:

$\pi^{-1}(U) = \bigcup \widetilde{U}_\alpha$ is a decomposition of $\pi^{-1}(U)$ into its (open) connected components and $\pi|_{\widetilde{U}_\alpha}$ is a diffeomorphism onto U for each component $\widetilde{U}_\alpha$.

- The manifold M is Hausdorff since Γ acts properly discontinuously.
  By a previous lemma it has a countable basis of open sets.
  Using both Condition (i') and the assumption that the action is free, we may find, for each $x \in \widetilde{M}$, a neighborhood $\widetilde{U}$ such that $h\widetilde{U} \cap \widetilde{U} = \emptyset$ except when $h = e$.
  This implies that $\pi_{\widetilde{U}}$ $(= \pi|_{\widetilde{U}})$ is one-to-one onto its image U.

Free and Properly Discontinuous Action (Cont'd)

- We know the mapping π is both continuous and open.

  Therefore, $\pi_{\widetilde{U}} : \widetilde{U} \to U$ is a homeomorphism of $\widetilde{U}$ to the open set U.

  We may assume, without loss of generality, that $\widetilde{U}$ is a connected coordinate neighborhood $\widetilde{U}, \widetilde{\varphi}$.

  Let $\varphi = \widetilde{\varphi} \circ \pi_{\widetilde{U}}^{-1}$.

  Then $\varphi : U \to \widetilde{\varphi}(\widetilde{U}) \subseteq \mathbb{R}^n$ is a homeomorphism.

  But every $p \in M$ is the image of some $x \in \widetilde{M}$.

  So we see that M is locally Euclidean.

  Thus, M is a topological manifold.

Free and Properly Discontinuous Action (Cont'd)

- The coordinate neighborhoods U, φ will be called **admissible**.

  The differentiable structure is determined by the admissible coordinate neighborhoods.

  Note that

$$\pi^{-1}(U) = \bigcup_{h \in \Gamma} h\widetilde{U},$$

  a disjoint union of connected open sets each diffeomorphic to $\widetilde{U}$.

  Now $\pi : h\widetilde{U} \to U$ is the same map as $\pi \circ h^{-1} : h\widetilde{U} \to U$.

  So that $\pi|_{h\widetilde{U}}$ is a diffeomorphism will follow trivially from the fact that h^{-1} and $\pi|_{\widetilde{U}} : \widetilde{U} \to U$ are diffeomorphisms.

  But, we must first establish that any overlapping admissible neighborhoods U, φ and V, ψ are C^∞-compatible, so that they define a C^∞ structure.

Free and Properly Discontinuous Action (Cont'd)

- To prove this let $U = \pi(\widetilde{U})$ and $V = \pi(\widetilde{V})$, where $\widetilde{U}, \widetilde{\varphi}$ and $\widetilde{V}, \widetilde{\psi}$ are the corresponding coordinate neighborhoods on $\widetilde{M}$.

  If $p \in U \cap V$, then there are points $x \in \widetilde{U}$ and $y \in \widetilde{V}$ (possibly not distinct), with $\pi(x) = p = \pi(y)$.

  This implies that $x = h(y)$, for some $h \in \Gamma$.

  Since h is a diffeomorphism, $\widetilde{V}_1 = h(\widetilde{V})$, with $\widetilde{\psi}_1 = \widetilde{\psi} \circ h^{-1}$, is a coordinate neighborhood and

$$\psi = \widetilde{\psi} \circ \pi_{\widetilde{V}}^{-1} = \widetilde{\psi}_1 \circ h \circ \pi_{\widetilde{V}}^{-1} = \psi_1 \circ \pi_{\widetilde{V}_1}^{-1}.$$

  However, $\widetilde{U}, \widetilde{\varphi}$ and $\widetilde{V}_1, \widetilde{\varphi}_1$ are C^∞-compatible.

  Thus U, φ and V, ψ are also compatible.

  Because of the requirement that $\pi(\widetilde{U})$ be a diffeomorphism, no other C^∞ structure is possible.

Discrete Subgroups

Lemma

Let G be a Lie group. Let Γ be a subgroup which has the property that, there exists a neighborhood U of e, such that $U \cap \Gamma = \{e\}$. Then Γ is a countable, closed subset of G and is discrete as a subspace.

- We first show that:
  - Γ is closed as a subset of G;
  - Γ is discrete in the relative topology.

  Let V be a neighborhood of e, such that $VV^{-1} \subseteq U$.

  Such V exists, since the map

  $$(g_1, g_2) \to g_1 g_2^{-1}$$

  is continuous and takes $(e, e) \to e$.

Discrete Subgroups (Cont'd)

- Suppose $\{h_n\} \subseteq \Gamma$ is a sequence, such that $\lim h_n = g$.

  Now Vg is a neighborhood of g.

  So there exists $N > 0$, such that, for $n > N$, $h_n \in Vg$.

  Suppose $v_n, v_m \in V$ so chosen that $h_n = v_n g$ and $h_m = v_m g$.

  Then $h_n h_m^{-1} = v_n v_m^{-1} \in U$.

  From $U \cap \Gamma = \{e\}$ it follows that $h_n h_m^{-1} = e$.

  So $h_n = h_m$, for all $n, m > N$.

  Thus $g = h_N \in \Gamma$.

  So Γ is closed.

  Moreover, for U of the hypothesis and $h \in \Gamma$, hU is a neighborhood of h whose intersection with Γ is just h.

  This proves the discreteness.

Discrete Subgroups (Cont'd)

- Finally Γ must be countable, since $\{hV : h \in \Gamma\}$ form a nonintersecting family of disjoint open sets indexed by Γ.

  In fact, suppose $h_1 V \cap h_2 V \neq \emptyset$.

  Then $h_1 v_1 = h_2 v_2$ for $v_1, v_2 \in V$.

  This implies $h_2 h_1^{-1} = v_2 v_1^{-1} \in VV^{-1} \subseteq U$.

  So $h_1 = h_2$.

  Were Γ not countable, this would mean we could not have a countable basis of open sets.
- We remark that a Γ with this property is a closed zero-dimensional Lie subgroup of G.
- Such subgroups are often called simply **discrete subgroups**.

Properties of Discrete Subgroups

Theorem

Any discrete subgroup Γ of a Lie group G acts freely and properly discontinuously on G by left translations.

- No other translation than the identity has a fixed point so the action is free.

  To see that it is properly discontinuous we must check Properties (i) and (ii) of the definition.

  Choose U, V neighborhoods of e, as in the proof of the preceding lemma so that $VV^{-1} \subseteq U$ and $U \cap \Gamma = \{e\}$.

  Then the only $h \in \Gamma$ such that $hV \cap V \neq \emptyset$ is $h = e$.

  This proves Condition (i).

Properties of Discrete Subgroups (Cont'd)

- To prove Condition (ii) we argue as follows.

  Suppose Γx and Γy are distinct orbits.

  Then $x \notin \Gamma y$.

  Now Γy is closed.

  By the regularity of G, there is a neighborhood U of x, such that $U \cap \Gamma y = \emptyset$.

  Let V be a neighborhood of e such that $xVV^{-1} \subseteq U$.

  Assume the open sets ΓxV and ΓyV intersect.

  Then some element of xVV^{-1} must be in Γy.

  This is an immediate contradiction.

Corollary

If Γ is a discrete subgroup of a Lie group G, then the space of right (or left) cosets G/Γ is a C^∞ manifold and $\pi : G \to G/\Gamma$ is a C^∞ mapping.

Example

- Let $G = \boldsymbol{V}^n$, that is, $\mathbb{R}^n$ considered as a vector space.
- Let $\Gamma = \mathbb{Z}^n$, the n-tuples of integers, called the **integral lattice**.
- More generally one could take for Γ the integral linear combinations of any basis $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n$ of $\boldsymbol{V}^n$.
- Γ is a discrete subgroup.
- The neighborhood $C_\varepsilon^n(0)$ of the origin with $\varepsilon < 1$ does not contain any element of Γ other than $(0, \ldots, 0)$.
- $\boldsymbol{V}^n/\Gamma = \boldsymbol{V}^n/\mathbb{Z}^n$ is diffeomorphic to $T^n = S^1 \times \cdots \times S^1$, the n-dimensional torus.
- Additionally, π is a Lie group homomorphism of $\boldsymbol{V}^n$ onto T^n.
- Its kernel is Γ.

Example

- Any finite subgroup Γ of a Lie group G is a discrete subgroup.
- When G is compact, a discrete subgroup must be finite.
- But even in this case there are interesting examples.
- Consider the case of $SO(3)$ the group of 3×3 orthogonal matrices of determinant $+1$.
- The subgroups of symmetries of the five regular solids give examples among which is the famous icosahedral group, which contains 60 elements.

Example

- In the case of groups which are not compact we have many variations of the following theme.
- Let $G_0 = Gl(n, \mathbb{R})$ and $\Gamma_0 = Sl(n, \mathbb{Z})$, the $n \times n$ matrices with integer coefficients and determinant $+1$.
- The topology of G_0 is obtained by viewing it as an open subset of $\mathbb{R}^{n^2}$.
- So it is clear that Γ_0 corresponds to the intersection of G_0 with the integral lattice $\mathbb{Z}^{n^2}$.
- Hence Γ_0 is discrete.
- Suppose G is a Lie subgroup of G_0.
- Let $\Gamma = \Gamma_0 \cap G$.
- Then Γ is discrete in G.
- For an illustration, let:
  - G be the set of all matrices in $Gl(n, \mathbb{R})$ with $+1$ on the main diagonal and zero below;
  - Γ be its intersection with $Sl(n, \mathbb{Z})$.

On Compactness

- An interesting question about which one can speculate is the following:

  In which, if any, of these cases is G/Γ compact?
- Note that it is compact when $G = \boldsymbol{V}^n$ and $\Gamma = \mathbb{Z}^n$.
- A necessary and sufficient condition for compactness is the existence of a compact subset $K \subseteq G$ whose Γ-orbit covers G, $\Gamma K = G$.
- In the first example above, any cube K of side one or greater has this property.

Tiling

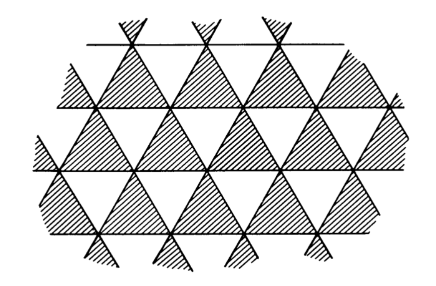
- Note that reflection in a line is a rigid motion of the plane.
- In fact, any rigid motion is a product of reflections.
- So reflections generate the group of motions of the plane.
- For example, the group Γ generated by reflections in the four lines $x = 0$, $x = \frac{1}{2}$, $y = 0$, $y = 1$ relative to a fixed Cartesian coordinate system contains the group of translations $(x, y) \to (x + m, y + n)$, m, n integers.
- This latter group may be identified with the subgroup $\mathbb{Z}^2$ of $\boldsymbol{V}^2$ discussed above.
- The action of Γ leaves unchanged the figure consisting of lines

$$x = \frac{k}{2}, \quad y = \frac{\ell}{2}, \quad k, \ell \text{ integers},$$

that is, a collection of squares which "tile" the plane.

Tiling (Cont'd)

- Similarly, suppose we tile the plane with other polygons as shown.



- We see that the group Γ of reflections in all lines forming edges of these polygons leaves the whole configuration or tiling unchanged.
- We may verify geometrically that the group Γ in these illustrations acts properly discontinuously.
- Is the action free?
- This is an important method of obtaining such group actions.

Subsection 9

Covering Manifolds

Covering Manifolds

- Let $\widetilde{M}$ and M be two C^∞ manifolds of the same dimension.
- Let $\pi : \widetilde{M} \to M$ be a C^∞ mapping.

Definition

$\widetilde{M}$ is said to be a **covering** (**manifold**) of M, with covering mapping π, if it is connected and if each $p \in M$ has a connected neighborhood U, such that

$$\pi^{-1}(U) = \bigcup U_\alpha,$$

a union of open components U_α, with the property that π_{U_α}, the restriction of π to U_α, is a diffeomorphism onto U.
The U are called **admissible neighborhoods** and π is called the **projection** or **covering mapping**.

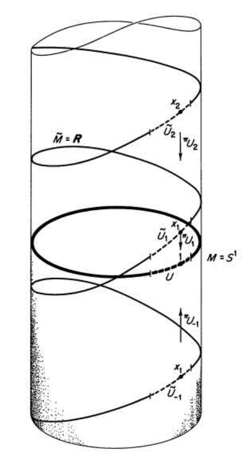
Examples

- $\widetilde{M} = \mathbb{R}$ covers $M = S^1$ realized as complex numbers of absolute value $+1$, with

$$\pi(t) = e^{2\pi i t}.$$

- More generally $\widetilde{M} = \mathbb{R}^n$ covers T^n.
- S^{n-1} covers $P^{n-1}(\mathbb{R})$.
- In a very general way the main theorem of the preceding section tells us that, if Γ acts freely and properly discontinuously on $\widetilde{M}$, then $\widetilde{M}$ covers $M = \widetilde{M}/\Gamma$.

  The map π is the obvious one, taking each $x \in \widetilde{M}$, to its orbit Γx which is a point of M.



Covering or Deck Transformations

- Let us assume that $\pi : \widetilde{M} \to M$ is any covering of a manifold M by a connected manifold $\widetilde{M}$.
- We indicate how this may give rise to a group Γ acting freely and properly discontinuously on $\widetilde{M}$.

Definition

A diffeomorphism $h : \widetilde{M} \to \widetilde{M}$ is said to be a **covering transformation**, or **deck transformation**, if $\pi \circ h = \pi$.

- Note that this is equivalent to the requirement that each set $\pi^{-1}(p)$ is carried into itself.
- If the covering is one arising from a free, properly discontinuous action of a group Γ on $\widetilde{M}$, then each $h \in \Gamma$ is a covering transformation of the covering $\pi : \widetilde{M} \to \widetilde{M}/\Gamma$.

Group Property of Covering Transformations

- We verify at once that the set $\widetilde{\Gamma}$ of all covering transformations is a group acting on $\widetilde{M}$.

  It contains at least the identity.

  So it is not empty.

  Let $x \in \widetilde{M}$ and $p = \pi(x)$.

  Let U be an admissible neighborhood of p so

  $$\pi^{-1}(U) = \bigcup \widetilde{U}_\alpha, \quad \alpha = 1, 2, \ldots,$$

  (the collection of mutually disjoint neighborhoods $\{\widetilde{U}_\alpha\}$ must be countable).

Group Property of Covering Transformations (Cont'd)

- Let $x_\alpha = \pi^{-1}(p) \cap \widetilde{U}_\alpha$.

  Then x is one of the x_α's, say x_1.

  The set of x_α's is exactly $\pi^{-1}(p)$ and $h : \pi^{-1}(p) \to \pi^{-1}(p)$ is a permutation of this set.

  It follows that $h(x_\alpha) = x_{\alpha'}$ and $h : \widetilde{U}_\alpha \to \widetilde{U}_{\alpha'}$ is a diffeomorphism.

  In fact

$$h|_{\widetilde{U}_\alpha} = \pi^{-1}_{\widetilde{U}_{\alpha'}} \circ \pi_{\widetilde{U}_\alpha}.$$

Group Property of Covering Transformations (Cont'd)

- We can conclude that the points left fixed by h form an open set.

  By continuity of h they also form a closed set.

  M being connected, this set is empty or h is the identity.

  In particular, two covering transformations with the same value on a point x must be identical.

  Thus covering transformations are completely determined by the permutation $\alpha \to \alpha'$ they induce on the set of points $\{x_\alpha\} = \pi^{-1}(p)$ for an arbitrary (but fixed) point $p \in M$.

  In particular, the action of $\widetilde{\Gamma}$ on $\widetilde{M}$ is free.

  If $x_1 \in \pi^{-1}(p)$, then $h \to hx_1$ maps $\widetilde{\Gamma}$ into $\pi^{-1}(p)$.

  This mapping is an injection so $\widetilde{\Gamma}$ must be countable.

  Also, as a discrete group of diffeomorphisms of $\widetilde{M}$, it acts differentiably on $\widetilde{M}$.

  This proves, in part, the following theorem.

Properties of the Action

Theorem

Let $\widetilde{M}$ be a covering manifold of M, with covering mapping π.
Let $\widetilde{\Gamma}$ be the set of all covering transformations on $\widetilde{M}$.
Then $\widetilde{\Gamma}$ acts freely and properly discontinuously on $\widetilde{M}$.
If $p \in M$ and $\widetilde{\Gamma}$ is transitive on $\pi^{-1}(p)$, then $\widetilde{M}/\widetilde{\Gamma}$ is naturally diffeomorphic to M. Relative to this diffeomorphism the covering map $\pi : \widetilde{M} \to M$ corresponds to the projection of each $x \in \widetilde{M}$ to its orbit $\widetilde{\Gamma}_x$.

- We have already seen that $\widetilde{\Gamma}$ acts on $\widetilde{M}$ freely since only the identity has a fixed point.

  We must check (using admissible neighborhoods) that the action is properly discontinuous.

Properties of the Action (Cont'd)

- Suppose $x \in \widetilde{M}$ and $p = \pi(x)$.

  Then $x \in \{x_\alpha\} = \pi^{-1}(p)$, say $x = x_1$.

  Moreover, if $h \neq e$, then $h(x_1) = x_\beta \neq x_1$.

  So $h(\widetilde{U}_1) = \widetilde{U}_\beta$, with $\widetilde{U}_\beta \cap \widetilde{U}_1 = \emptyset$.

  Thus the first part of proper discontinuity is proved.

Properties of the Action (Cont'd)

- Next, we prove the second part.

  Take $x, y \in \widetilde{M}$ not in the same orbit of $\widetilde{\Gamma}$.

  Consider two cases, depending on whether $\pi(x) = \pi(y)$ or not.
  - Suppose $\pi(x) = \pi(y)$ and let $p = \pi(x) = \pi(y)$.
    Note that, in permuting $\{x_\alpha\} = \pi^{-1}(p)$, no $h \in \widetilde{\Gamma}$ takes $x = x_\alpha$ to $y = x_\beta$, for $\alpha \neq \beta$.
    Thus, $\widetilde{U}_\alpha$ is not carried to $\widetilde{U}_\beta$ by any $h \in \widetilde{\Gamma}$.
    This establishes Condition (ii) of the definition in this case.
  - Suppose $\pi(x) = p$ and $\pi(y) = q$ are distinct.
    Let U, V be disjoint admissible neighborhoods of p, q, respectively.
    Then the open sets $\pi^{-1}(U)$ and $\pi^{-1}(V)$ are disjoint and carried into themselves by every $h \in \widetilde{\Gamma}$.
    So Condition (ii) is satisfied in this case also.

  We conclude that the action is properly discontinuous.

Properties of the Action (Cont'd)

- We now define a map
$$\pi_1 : \widetilde{M}/\widetilde{\Gamma} \to M.$$
For $[y]$ a point of $\widetilde{M}/\widetilde{\Gamma}$, i.e., an orbit $\widetilde{\Gamma}y$ of $\widetilde{\Gamma}$,
$$\pi_1([y]) = \pi(y).$$
This is well-defined, since $\pi(hy) = \pi(y)$.
Since $\widetilde{M}$ is connected, $\widetilde{M}/\widetilde{\Gamma}$ is connected.
The mapping π_1 is onto, since $\pi : \widetilde{M} \to M$ is onto.
Further π_1 is a covering map.
To see this one checks the definition of $\widetilde{M}/\widetilde{\Gamma}$ from the main theorem of the preceding section.

Properties of the Action (Cont'd)

- Suppose, further, that $\widetilde{\Gamma}$ is transitive on $\pi^{-1}(p)$, for some $p \in M$.

  Then $\pi_1^{-1}(p)$ consists of a single point.

  This reduces the proof of the last part of the theorem to the following lemma.

Lemma

Let $\pi : \widetilde{M} \to M$ be a covering and suppose that for some $p \in M$,

$$\pi^{-1}(p) \text{ is a single point.}$$

Then π is a diffeomorphism.

- We omit the proof.