Introduction to Differential Geometry

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Subsection 1

The Tangent Space at a Point on a Manifold

Algebra of Functions

- Let M denote a C^∞ manifold of dimension n.
- We have defined for M the concepts of:
  - C^∞ function on an open subset U;
  - C^∞ mapping to another manifold.
- This allows us to consider $C^\infty(U)$, the collection of all C^∞ functions on the open subset U (including the special case $U = M$).
- We can verify, as we did for $U \in \mathbb{R}^n$, that $C^\infty(U)$ is a commutative algebra over the real numbers $\mathbb{R}$.
- As before, $\mathbb{R}$ may be identified in a natural way with the constant functions and the constant 1 with the unit.

Germs

- Let M denote a C^∞ manifold of dimension n.
- Let $p \in M$ be a given point.
- We define $C^\infty(p)$ as the algebra of C^∞ functions whose domain of definition includes some open neighborhood of p.
- In $C^\infty(p)$, functions are identified if they agree on any neighborhood of p.
- The objects so obtained are called "**germs**" of C^∞ functions.

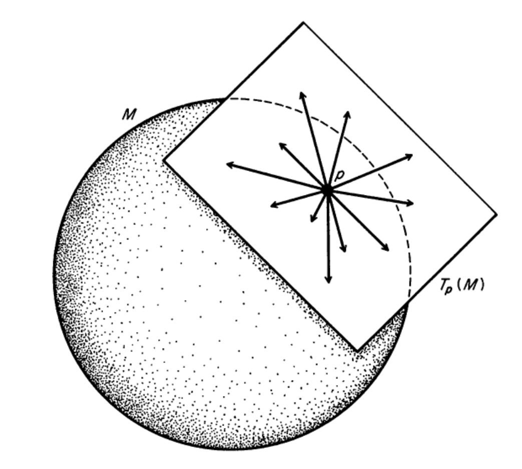
Equivalence of Algebras of Germs

- Choose an arbitrary coordinate neighborhood U, φ of p.
- Consider the mapping $\varphi^* : C^\infty(\varphi(p)) \to C^\infty(p)$ given by

$$\varphi^*(f) = f \circ \varphi.$$

- It can be verified that φ^* is an isomorphism of the algebra of "germs" of C^∞ functions at $\varphi(p) \in \mathbb{R}^n$ onto the algebra $C^\infty(p)$.
- This is to be expected since locally M is C^∞-equivalent to $\mathbb{R}^n$ by the diffeomorphism φ.
- Our main purpose is to attach to each $p \in M$ a tangent vector space $T_p(M)$, as was done for $\mathbb{R}^n$ and $\boldsymbol{E}^n$.
- Our first definitions in the latter case giving $T_p(\mathbb{R}^n)$ as directed line segments do not generalize.
- But the identification of $T_p(\mathbb{R}^n)$ with directional derivatives generalizes.

Geometric Idea of $T_p(M)$



Tangent Space

Definition

We define the **tangent space** $T_p(M)$ **to** M **at** p to be the set of all mappings $X_p : C^\infty(p) \to \mathbb{R}$ satisfying for all $\alpha, \beta \in \mathbb{R}$ and $f, g \in C^\infty(p)$ the two conditions:

(i) $X_p(\alpha f + \beta g) = \alpha(X_p f) + \beta(X_p g)$ (**linearity**);

(ii) $X_p(fg) = (X_p f)g(p) + f(p)(X_p g)$ (**Leibniz Rule**),

with the vector space operations in $T_p(M)$ defined by

$$\begin{array}{rcl}(X_p + Y_p)f & = & X_p f + Y_p f; \\ (\alpha X_p)f & = & \alpha(X_p f).\end{array}$$

A **tangent vector to** M **at** p is any $X_p \in T_p(M)$.

- One can check that this defines a vector space $T_p(M)$ at each $p \in M$.

Remark

- The definition of $T_p(M)$ uses only $C^\infty(p)$, not all of M.
- Thus, if U is any open set of M containing p, then $T_p(U)$ and $T_p(M)$ are naturally identified.
- The proof that $T_p(M)$ is a vector space includes the case of $\mathbb{R}^n$.
- The difference is that we no longer have the alternative "geometric" way of defining $T_p(M)$ as pairs of points $\overrightarrow{px}$ as we did in $\mathbb{R}^n$.
- That method used special features of $\mathbb{R}^n$, namely the existence of a natural one-to-one correspondence with the vector space $\boldsymbol{V}^n$.
- For manifolds in general, any such correspondence entails a choice of a coordinate neighborhood and depends on the particular choice.
- So, for manifolds, it is not natural in the preceding sense.
- However, for each choice of coordinate neighborhood U, φ containing $p \in M$ we obtain an isomorphism to $\boldsymbol{V}^n$, as we shall see.
- Using this method, we can establish that $\dim T_p(M) = \dim M$.

Tangent Space Homomorphisms

Theorem

Let $F : M \to N$ be a C^∞ map of manifolds. Then, for $p \in M$, the map $F^* : C^\infty(F(p)) \to C^\infty(p)$ defined by

$$F^*(f) = f \circ F$$

is a homomorphism of algebras. Moreover, it induces a dual vector space homomorphism $F_* : T_p(M) \to T_{F(p)}(N)$, defined by

$$F_*(X_p)f = X_p(F^* f),$$

which gives $F_*(X_p)$ as a map of $C^\infty(F(p))$ to $\mathbb{R}$.
When $F : M \to M$ is the identity, both F^* and F_* are the identity isomorphism.

Tangent Space Homomorphisms (Cont'd)

Theorem (Cont'd)

Finally, if $H = G \circ F$ is a composition of C^∞ maps, then

$$H^* = F^* \circ G^* \quad \text{and} \quad H_* = G_* \circ F_*.$$

- The proof consists of checking the statements against definitions.
  We omit the verification that F^* is a homomorphism.
  We only consider F_* only.
  Let $X_p \in T_p(M)$ and $f, g \in C^\infty(F(p))$.
  We must prove that the map

  $$F_*(X_p) : C^\infty(F(p)) \to \mathbb{R}$$

  is a vector at $F(p)$.
  That is, we must show it is a linear map satisfying the Leibniz rule.

Tangent Space Homomorphisms (Cont'd)

- We have

$$\begin{array}{rcl} F_*(X_p)(fg) & = & X_pF^*(fg) \\ & = & X_p[(f\circ F)(g\circ F)] \\ & = & X_p(f\circ F)g(F(p))+f(F(p))X_p(g\circ F). \end{array}$$

So we obtain

$$F_*(X_p)(fg)=(F_*(X_p)f)g(F(p))+f(F(p))F_*(X_p)g.$$

Linearity is even simpler.

Thus, $F_*:T_p(M)\to T_{F(p)}(N)$.

Further, F_* is a homomorphism:

$$\begin{array}{rcl} F_*(\alpha X_p+\beta Y_p)f & = & (\alpha X_p+\beta Y_p)(F\circ f) \\ & = & \alpha X_p(F\circ f)+\beta Y_p(F\circ f) \\ & = & \alpha F_*(X_p)f+\beta F_*(Y_p)f \\ & = & [\alpha F_*(X_p)+\beta F_*(Y_p)]f. \end{array}$$

Remark

- The homomorphism $F_* : T_p(M) \to T_{F(p)}(M)$ is often called the **differential** of F.
- One frequently sees other notations for F_*.
- Other notations include dF, DF, F', and so on.
- The $_*$ is a subscript since the mapping is in the same "direction" as F, that is, from M to N.
- In contrast, $F^* : C^\infty(F(p)) \to C^\infty(p)$ goes opposite to the direction of F.
- This notational convention can be quite important and reflects a similar situation in linear algebra related to linear mappings of vector spaces and their duals.

The Case of a Diffeomorphism

Corollary

Let $F : M \to N$ be a diffeomorphism of M onto an open set $U \subseteq N$. For $p \in M$, $F_* : T_p(M) \to T_{F(p)}(N)$ is an isomorphism onto.

- This follows at once from:
  - The last statement of the theorem;
  - The remark after the definition of tangent space.

  Taking G to be the inverse to F, we get that each of

$$G_* \circ F_* : T_p(M) \to T_p(M),$$
$$F_* \circ G_* : T_{F(p)}(N) \to T_{F(p)}(N)$$

  is the identity isomorphism on the corresponding vector space.

The Coordinate Frames

- Recall that any open subset of a manifold M is a (sub)manifold of the same dimension.
- Let U, φ be a coordinate neighborhood on M.
- Then the coordinate map φ induces an isomorphism

$$\varphi_* : T_p(M) \to T_{\varphi(p)}(\mathbb{R}^n)$$

of the tangent space at each point $p \in U$ onto $T_a(\mathbb{R}^n)$, $a = \varphi(p)$.
- Similarly, the map φ^{-1} maps $T_a(\mathbb{R}^n)$ isomorphically onto $T_p(M)$.
- Consider, now, the natural basis $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$ at each $a \in \varphi(U) \subseteq \mathbb{R}^n$.
- The images

$$E_{ip} = \varphi_*^{-1}\left(\frac{\partial}{\partial x^i}\right), \quad i = 1, \ldots, n,$$

determine at $p = \varphi^{-1}(a) \in M$ a basis $E_{1p}, \ldots, E_{np}$ of $T_p(M)$.
- We call these bases the **coordinate frames**.

Natural Basis

Corollary

To each coordinate neighborhood U on M there corresponds a natural basis $E_{1p}, \ldots, E_{np}$ of $T_p(M)$, for every $p \in U$. In particular,

$$\dim T_p(M) = \dim M.$$

Let f be a C^∞ function defined in a neighborhood of p, and

$$\widehat{f} = f \circ \varphi^{-1}$$

its expression in local coordinates relative to U, φ. Then

$$E_{ip} f = \left(\frac{\partial \widehat{f}}{\partial x^i} \right)_{\varphi(p)}.$$

Natural Basis (Cont'd)

Corollary (Cont'd)

In particular, if $x^i(q)$ is the ith coordinate function, $X_p x^i$ is the ith component of X_p in this basis, that is,

$$X_p = \sum_{i=1}^{n} (X_p x^i) E_{ip}.$$

- The last statement of the corollary is a restatement of the definition of $_*$ for $E_{ip} = \varphi_*^{-1}\left(\frac{\partial}{\partial x^i}\right)$.

  Namely,

$$E_{ip} f = \left(\varphi_*^{-1}\left(\frac{\partial}{\partial x^i}\right)\right) f = \frac{\partial}{\partial x^i}(f \circ \varphi^{-1})|_{x=\varphi(p)}.$$

Matrix of F_* in Local Coordinates

- Take f to be the ith coordinate function, $f(q) = x^i(q)$.
- Moreover, let
$$X_p = \sum \alpha^j E_{jp}.$$
- Then
$$X_p x^i = \sum_j \alpha^j (E_{jp} x^i) = \sum_j \alpha^j \left(\frac{\partial x^i}{\partial x^j}\right)_{\varphi(p)} = \alpha^i.$$
- We may use this to derive a standard formula which gives the matrix of the linear map F_* relative to local coordinate systems.

Matrix of F_* in Local Coordinates (Cont'd)

- Let $F : M \to N$ be a smooth map.
- Let U, φ and V, ψ be coordinate neighborhoods on M and N, with
$$F(U) \subseteq V.$$
- Suppose that, in these local coordinates, F is given by
$$y^i = f^i(x^1, \ldots, x^n), \quad i = 1, \ldots, m.$$
- Let p is a point with coordinates $a = (a^1, \ldots, a^n)$.
- Then $F(p)$ has y coordinates determined by these functions.
- Further let $\frac{\partial y^j}{\partial x^i}$ denote $\frac{\partial f^j}{\partial x^i}$.

The Coordinate Theorem

Theorem

Let $E_{ip} = \varphi_*^{-1}(\frac{\partial}{\partial x^i})$ and $\widetilde{E}_{jF(p)} = \psi_*^{-1}(\frac{\partial}{\partial y^j})$, $i = 1, \ldots, n$ and $j = 1, \ldots, m$, be the basis of $T_p(M)$ and $T_{F(p)}(N)$, respectively, determined by the given coordinate neighborhoods. Then

$$F_*(E_{ip}) = \sum_{j=1}^{m} \left(\frac{\partial y^j}{\partial x^i}\right)_a \widetilde{E}_{jF(p)}, \quad i = 1, \ldots, n.$$

In terms of components, if $X = \sum \alpha^i E_{ip}$ maps to $F_*(X_p) = \sum \beta^j Y_{jF(p)}$, then we have

$$\beta^j = \sum_{i=1}^{n} \alpha^i \left(\frac{\partial y^j}{\partial x^i}\right)_a, \quad j = 1, \ldots, m.$$

The partial derivatives in these formulas are evaluated at the coordinates of p: $a = (a^1, \ldots, a^n) = \varphi(p)$.

The Coordinate Theorem (Proof)

- We have $F_*(E_{ip}) = F_* \circ \varphi_*^{-1} \left(\frac{\partial}{\partial x^i}\right)_{\varphi(p)}$.

  According to the preceding corollary, to compute its components relative to $\widetilde{E}_{jF(p)}$, we must apply this vector as an operator on $C^\infty(F(p))$ to the coordinate functions y_j,

$$F_*(E_{ip})y_j = \left(F_* \circ \varphi_*^{-1}\left(\frac{\partial}{\partial x^i}\right)\right)y_j = \frac{\partial}{\partial x^i}y_j(F \circ \varphi^{-1})(x) = \frac{\partial f^i}{\partial x^i},$$

  the derivatives being evaluated at the coordinates of p, i.e., at $\varphi(p)$.

  They could also be written $\left(\frac{\partial y^j}{\partial x^i}\right)_{\varphi(p)}$.

The Rank of a Mapping

- In the following F, M, and N are as in the preceding theorem.

Corollary

The rank of F at p is exactly the dimension of the image of $F_*(T_p(M))$. F_* is an isomorphism into if and only if this rank is the dimension of M. It is onto if and only if the rank equals $\dim N$.

- Note that $(\frac{\partial y^i}{\partial x^j})$ is exactly the Jacobian of $\psi \circ F \circ \varphi^{-1}$.

  This matrix was used to define the rank.

  It is also the matrix of the linear transformation $F_* : T_p(M) \to T_p(N)$ in the given bases.

  So we obtain the conclusion from linear algebra.

- This corollary gives a characterization of the rank which is independent of any coordinate systems.

Change of Basis Formulas

- We apply the theorem to the maps

$$F = \widetilde{\varphi} \circ \varphi^{-1} \quad \text{and} \quad F^{-1} = \varphi \circ \widetilde{\varphi}^{-1}.$$

- These maps give the change of coordinates between U, φ and $\widetilde{U}, \widetilde{\varphi}$ in $U \cap \widetilde{U}$ on M.
- We obtain formulas for:
  - Change of basis in $T_p(M)$;
  - Corresponding change of components relative to these bases.

Change of Basis Formulas (Cont'd)

Corollary

Let $p \in U \cap \widetilde{U}$ and let $E_{ip} = \varphi_*^{-1}\left(\frac{\partial}{\partial x^i}\right)$ and $\widetilde{E}_{ip} = \widetilde{\varphi}_*^{-1}\left(\frac{\partial}{\partial x^i}\right)$ be the bases of $T_p(M)$ corresponding to the two coordinate systems. Then with indices running from 1 to n, we have

$$E_{ip} = \sum_k \left(\frac{\partial x^k}{\partial x^i}\right)_{\varphi(p)} \widetilde{E}_{kp} \quad \text{and} \quad \widetilde{E}_{jp} = \sum_\ell \left(\frac{\partial x^\ell}{\partial \widetilde{x}^j}\right)_{\widetilde{\varphi}(p)} E_{\ell p}.$$

If $X_p = \sum \alpha^i E_{ip} = \sum \beta^j \widetilde{E}_{jp}$, then

$$\alpha^i = \sum_j \beta^j \frac{\partial x^i}{\partial \widetilde{x}^j} \quad \text{and} \quad \beta^j = \sum_i \alpha^i \frac{\partial \widetilde{x}^j}{\partial x^i}.$$

Tangent Vector

- The second set of formulas in the preceding corollary is often used to define tangent vector at a point p of a manifold.
- A **tangent vector** X_p is an equivalence class of the collection of all n-tuples

$$\{(\alpha^1, \ldots, \alpha^n)_{(U,\varphi)} : \alpha^i \in \mathbb{R}, U, \varphi \text{ a coordinate neighborhood of } p\}.$$

- Two such n-tuples

$$(\alpha^1, \ldots, \alpha^n)_{U,\varphi} \quad \text{and} \quad (\beta^1, \ldots, \beta^n)_{\widetilde{U},\widetilde{\varphi}}$$

are equivalent if they are related as in the last formula of the corollary.

The Case of Submanifolds

- Let M be a submanifold of N.
- Let $F : M \to N$ be the immersion or inclusion map of M into N.
- In either case, the mapping F from M (with its C^∞ manifold structure) into N (with its C^∞ structure) is a C^∞ mapping, and

$$\text{rank} F = \dim M.$$

- This means that $F_* : T_p(M) \to T_p(N)$ is an injective isomorphism.
- So $T_p(M)$ can be identified with a subspace of $T_p(N)$.
- Under this identification, we can think of $T_p(M)$, the tangent space to M, as a subspace in $T_p(N)$ for each $p \in M$.
- Applying this principle to our examples of submanifolds of $\mathbb{R}^n$, especially when $n = 2$ or 3, will enable us to recapture some of the intuitive meaning of tangent vector which was lost in the transition from Euclidean space to general manifolds.

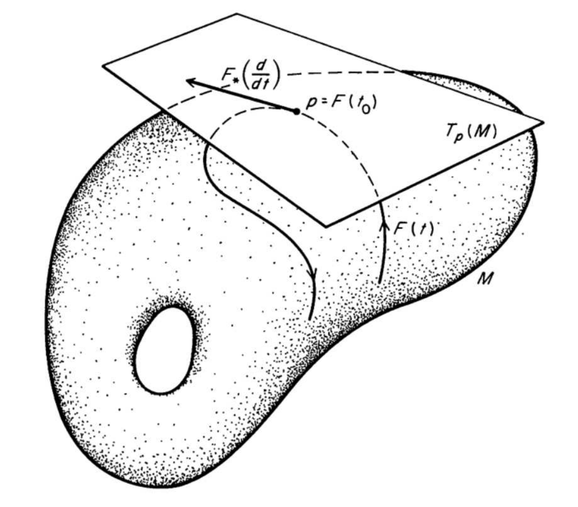
Example

- Let $M = (a, b)$ be an open interval of $\mathbb{R}$.
- Consider the case of a C^∞ curve $F : M \to N$ in a manifold.
- For the moment we drop the requirement that F is an immersion.
- Given $t_0 \in M$, $a < t_0 < b$, then $\frac{d}{dt}$ taken at t_0 is a basis for $T_{t_0}(M)$.
- Suppose $p = F(t_0)$ and $f \in C^\infty(p)$.
- Then $F_*(\frac{d}{dt})$ is determined by its value on all such f:

$$F_* \left(\frac{d}{dt} \right) f = \left(\frac{d}{dt}(f \circ F) \right)_{t_0}.$$

- We call this vector the (**tangent**) **velocity vector** to the curve at p.
- In this interpretation we use the parameter $t \in \mathbb{R}$ as time, and we think of $F(t)$ as a point moving in N.

Example (Illustration)



Example (Cont'd)

- Let U, φ be coordinates around p.
- Then, in the local coordinates, F is given by

$$\widehat{F}(t) = \varphi \circ F(t) = (x^1(t), \ldots, x^n(t)).$$

- The ith coordinate x^i is a function on U.
- Using somewhat sloppy notation, we write $x^i(t) = (x^i \circ F)(t)$;
- Thus, $F_*(\frac{d}{dt})x^i = (\frac{dx^i}{dt})_{t_0}$, which we denote

$$\dot{x}^i(t_0), \quad i = 1, \ldots, n.$$

- So by the theorem (with $E_{ip} = \frac{d}{dt}$ and E's replacing $\widetilde{E}$'s),

$$F_*\left(\frac{d}{dt}\right) = \sum_{i=1}^{n} \dot{x}^i(t_0) E_{ip}.$$

Example (Special Case)

- As a special case let $N = \mathbb{R}^n$.
- Take the usual (canonical) coordinates of $\mathbb{R}^n$
- The formula means that the image of $\frac{d}{dt}$ is just the velocity vector at the point $p = (x^1(t_0), \ldots, x^n(t_0))$ of the curve.
- Its components relative to the natural basis at the point p are $\dot{x}^1(t_0), \ldots, \dot{x}^n(t_0)$;
- It is the vector of $T_p(\mathbb{R}^n)$ whose:
  - Initial point is $p = x(t_0)$;
  - Terminal point is $(x^1(t_0) + \dot{x}^1(t_0), \ldots, x^n(t_0) + \dot{x}^n(t_0))$.
- If the rank of F at t_0 is 1, then F_* is an isomorphism.
- Then, we may identify the tangent space to the image curve at p with the subspace of $T_p(\mathbb{R}^n)$ spanned by this vector.
- Thus, we obtain the usual tangent line at the point p of the curve.
- If the rank of F at t_0 is 0, then $F_*(\frac{d}{dt}) = 0$.

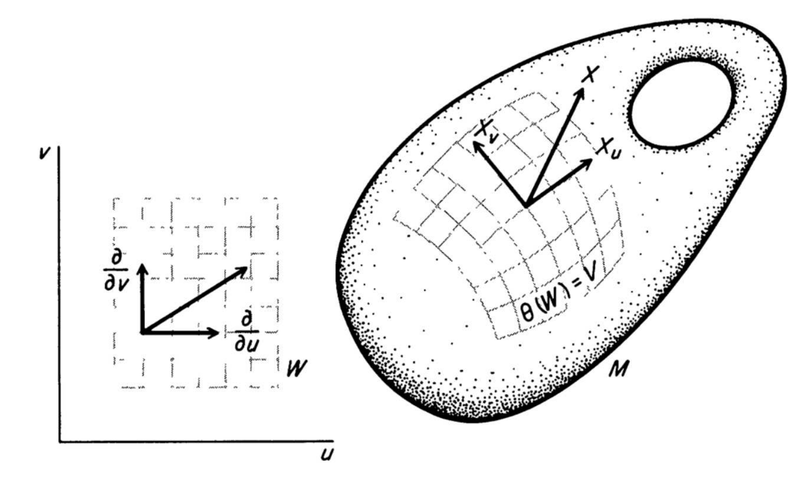
Example

- Let M be a two-dimensional submanifold of $\mathbb{R}^3$, that is, a surface.
- Let W be an open subset, say a rectangle in the (u, v)-plane $\mathbb{R}^2$.
- Let $\theta : W \to \mathbb{R}^3$ be a parametrization of a portion of M, that is, θ is an imbedding whose image is an open subset V of M.
- V, θ^{-1} is a coordinate neighborhood on M.
- Suppose $\theta(u_0, v_0) = (x_0, y_0, z_0)$, where we now use (x, y, z) as the natural coordinates in $\mathbb{R}^3$.
- We may assume that θ is given by coordinate functions

$$x = f(u, v), \quad y = g(u, v), \quad z = h(u, v).$$

- Since θ is an imbedding, the Jacobian matrix $\frac{\partial(f,g,h)}{\partial(u,v)}$ has rank 2 at each point of W.
- We consider the image of the basis vectors $\frac{\partial}{\partial u}$ and $\frac{\partial}{\partial v}$ at (u_0, v_0).
- We denote these by $(X_u)_0$ and $(X_v)_0$.

Example (Illustration)



Example (Cont'd)

- According to the first formula of the theorem,

$$\begin{aligned}(X_u)_0 &= \theta_*(\frac{\partial}{\partial u}) = \frac{\partial x}{\partial u}\frac{\partial}{\partial x} + \frac{\partial y}{\partial u}\frac{\partial}{\partial y} + \frac{\partial z}{\partial u}\frac{\partial}{\partial z},\\ (X_v)_0 &= \theta_*(\frac{\partial}{\partial v}) = \frac{\partial x}{\partial v}\frac{\partial}{\partial x} + \frac{\partial y}{\partial v}\frac{\partial}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial}{\partial z},\end{aligned}$$

  where we have written $\frac{\partial x}{\partial u}, \frac{\partial x}{\partial v}$ for $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$, and so on, these derivatives being evaluated at u_0, v_0.
- Since θ_* has rank 2, these are linearly independent vectors.
- So they span a two-dimensional subspace of $T_{(x_0,y_0,z_0)}(\mathbb{R}^3)$.
- This subspace is what we have, by our identification, agreed to call the **tangent space of M at the point** (x_0, y_0, z_0).

Example (Cont'd)

- The tangent space $T_{(x_0,y_0,z_0)}(\mathbb{R}^3)$ of M at the point (x_0, y_0, z_0) consists of all the vectors of the form

$$\alpha\theta_*\left(\frac{\partial}{\partial u}\right) + \beta\theta_*\left(\frac{\partial}{\partial v}\right) = \alpha(X_u)_0 + \beta(X_v)_0, \quad \alpha, \beta \in \mathbb{R}.$$

- Their initial point, of course, is always at (x_0, y_0, z_0).
- It can be seen that this subspace is the usual tangent plane to a surface, as we would naturally expect it to be.
- We next use one of the standard descriptions of the tangent plane at a point p of a surface M in $\mathbb{R}^3$ as the collection of all tangent vectors at p to curves through p which lie on M.

Example (Cont'd)

- Let I be an open interval about $t = t_0$.
- Let us consider a curve on N through (x_0, y_0, z_0).
- It is no loss of generality to suppose the curve is given by $F : I \to W$ composed with $\theta : W \to \mathbb{R}^3$.
- Thus, u, v are functions of t with:
  - $u(t_0) = u_0, \ v(t_0) = v_0$;
  - $\theta(F(t)) = (x(u(t), v(t)), y(u(t), v(t)), z(u(t), v(t)))$.
- The tangent to the curve at (x_0, y_0, z_0) is given by

$$(\theta \circ F)_* \left(\frac{d}{dt}\right) = \dot{x}(t_0)\frac{\partial}{\partial x} + \dot{y}(t_0)\frac{\partial}{\partial y} + \dot{z}(t_0)\frac{\partial}{\partial z},$$

where

$$\dot{x}(t_0) = \left(\frac{dx}{dt}\right)_{t_0} = \frac{\partial x}{\partial u}\frac{du}{dt} + \frac{\partial x}{\partial v}\frac{dv}{dt},$$

evaluated at (x_0, y_0, z_0) and $t = t_0$.

Example (Cont'd)

- Substituting and collecting terms, we have

$$\begin{aligned}(\theta \circ F)_*(\tfrac{d}{dt}) &= \tfrac{du}{dt}(\tfrac{\partial x}{\partial u}\tfrac{\partial}{\partial x} + \tfrac{\partial y}{\partial u}\tfrac{\partial}{\partial y} + \tfrac{\partial z}{\partial u}\tfrac{\partial}{\partial z}) \\ &\quad + \tfrac{dv}{dt}(\tfrac{\partial x}{\partial v}\tfrac{\partial}{\partial x} + \tfrac{\partial y}{\partial v}\tfrac{\partial}{\partial y} + \tfrac{\partial z}{\partial v}\tfrac{\partial}{\partial z}) \\ &= \tfrac{du}{dt}\theta_*(\tfrac{\partial}{\partial u}) + \tfrac{du}{dt}\theta_*(\tfrac{\partial}{\partial v}) \\ &= \dot{u}(t_0)(X_u)_0 + \dot{v}(t_0)(X_v)_0.\end{aligned}$$

- If we let $u = t$, $v = v_0$, we obtain just $(X_u)_0 = \theta_*(\frac{\partial}{\partial u})$.
- Similarly, $(X_v)_0$ is tangent to the parameter curve $u = u_0$, $v = t$.
- The coordinate frame vectors are tangent to the coordinate curves.
- This means that the (tangent) velocity to every curve in M through $p = (x_0, y_0, z_0)$ lies in the subspace $T_p(M) \subseteq T_p(\mathbb{R}^3)$ spanned by $(X_u)_0$ and $(X_v)_0$.
- Conversely, by suitable choice of the curve, every vector of $T_p(M)$ may be so represented.

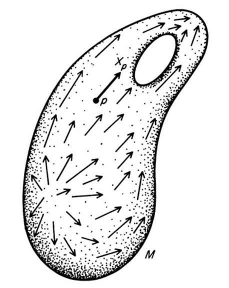
Subsection 2

Vector Fields

Introducing Vector Fields

- We defined the notion of a tangent vector to a manifold at a point $p \in M$, that is, of an element X_p of $T_p(M)$.
- Now we define and give examples of a C^r-vector field on M, $r \geq 0$.

- A vector field X on M is a "function" assigning to each point p of M an element X_p of $T_p(M)$.
- We place the word "function" in quotation marks since we have not really defined its range, only its domain M.
- The range is, in fact, the set $T(M)$ consisting of all tangent vectors at all points of M,



$$T(M) = \bigcup_{p \in M} T_p(M).$$

Partition Property of Vector Fields

- The set $T(M)$ is partitioned into disjoint subsets $\{T_p(M)\}$ which are indexed by the points of M.
- That is, to $p \in M$ corresponds its tangent space $T_p(M)$.
- It follows that there is a natural projection

$$\begin{array}{rrcl} \pi : & T(M) & \to & M; \\ & X_p & \mapsto & p. \end{array}$$

- The vector field X as a function $X : M \to T(M)$, must satisfy the condition

$$\pi \circ X = i_M,$$

the identity on M.

Regularity of Vector Fields

- A vector field X is also required to satisfy some condition of regularity, that is, of continuity or differentiability.
- For $p \in M$, let U, φ be any coordinate neighborhood of p.
- Let $E_{1p}, \ldots, E_{np}$ be the corresponding basis (coordinate frames) of $T_p(M)$.
- Then X_p, the value of X at p, may be written uniquely as

$$X_p = \sum_{i=1}^{n} \alpha^i E_{ip}.$$

Regularity of Vector Fields (Cont'd)

- Suppose p varies in U.
- Then the components $\alpha^1, \ldots, \alpha^n$ are well-defined functions of p.
- They must, then, be given by functions of the local coordinates (denoted by the same letters)

$$\alpha^1 = \alpha^i(x^1, \ldots, x^n), \quad i = 1, \ldots, n, \text{ on } \varphi(U) \subseteq \mathbb{R}^n.$$

- We say that X *is of class* C^r, $r \geq 0$, if these functions are of class C^r on U, for every local coordinate system U, φ.
- The expressions, given in a previous corollary, for changing coordinate systems are linear, with C^∞ coefficients.
- We see that this definition is independent of the coordinates used.

Vector Fields

Definition

A **vector field X of class C^r on M** is a function assigning, to each point p of M, a vector

$$X_p \in T_p(M)$$

whose components in the frames of any local coordinates U, φ are functions of class C^r on the domain U of the coordinates.

Unless otherwise noted, we will use **vector field** to mean C^∞-vector field.

Alternative Definitions of Vector Fields

- One way to avoid reliance on local coordinates is to define X to be C^r if, for every C^∞ function f whose domain W_f is an open subset of U, the function Xf, defined by

$$(Xf)(p) = X_p f,$$

is of class C^r.
- Another very elegant approach is to:
  - Give $T(M)$ the structure of a C^∞ manifold;
  - Then X becomes a mapping

$$X : M \to T(M)$$

of one C^∞ manifold to another.

In this case we have already defined the meaning of C^r.

Example

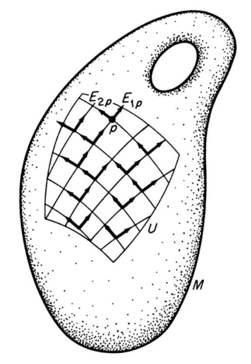
- Let $M = \mathbb{R}^3 - \{0\}$.
- Consider the gravitational field of an object of unit mass at 0.
- It is a C^∞-vector field.
- Consider the basis
$$\frac{\partial}{\partial x^1} = E_1, \quad \frac{\partial}{\partial x^2} = E_2, \quad \frac{\partial}{\partial x^3} = E_3.$$
- The components $\alpha^1, \alpha^2, \alpha^3$ relative to this basis are
$$\alpha^i = \frac{x^i}{r^3}, \ i = 1, 2, 3,$$
where
$$r = \sqrt{(x^1)^2 + (x^2)^2 + (x^3)^2}.$$

Example

- Let M be a manifold.
- Let U, φ be a coordinate neighborhood on M.
- Then U is an open set of M.
- So it is itself a manifold of the same dimension, say n.
- Consider the vector fields

$$E_i = \varphi_*^{-1}\left(\frac{\partial}{\partial x^i}\right), \quad i = 1, \ldots, n.$$

- They have components $\alpha^j = \delta_i^j$.
- These are constants.
- Hence, they are C^∞ functions on U.
- So each E_i is a C^∞-vector field on U.
- The set $E_1, \ldots, E_n$ is a basis of $T_p(M)$ at each $p \in U$, the **coordinate frames**.



Field of k-Frames

- Consider a manifold M, with $\dim M = n$.
- A set of k vector fields on M which is linearly independent at each point is called a **field of k-frames on** M.
- If $k = n$, then the frames form a basis at each point.
- It would be convenient if on a manifold one could always find such a field of n-frames.
- Then the components of any vector field would be globally defined.
- That is, they would be functions whose domain is all of M.

Field of k-Frames (Cont'd)

- This would relieve us of the necessity of using local coordinate neighborhoods and the associated frames $E_1, \ldots, E_n$.
- However, it is known that this is not possible in general.
- For example, on the sphere S^2 it is not possible to define even one continuous vector field X which is linearly independent (nonzero) at each point of S^2.
- This a classical theorem of algebraic topology discovered by Brouwer that will be proved later.

Vector Fields on Regular Submanifolds

Lemma

Let M be a manifold. Let N be a regular submanifold of M. Let X be a C^∞-vector field on M, such that, for each $p \in N$, $X_p \in T_p(N)$. Then X restricted to N is a C^∞-vector field on N.

- By hypothesis, X assigns to each $p \in N$ the tangent vector X_p in the subspace $T_p(N)$ of $T_p(M)$.

  We must prove that X restricted to N is of class C^∞.

  Let U, φ be a preferred coordinate neighborhood in M relative to N.

  So $V = U \cap N, \psi = \varphi|_V$ is a coordinate neighborhood on N, such that $p \in V$ if and only if its last $m - n$ coordinates are zero,

$$x^{n+1}(p) = \cdots = x^m(p) = 0, \quad \dim N = n, \ \dim M = m.$$

Vector Fields on Regular Submanifolds (Cont'd)

- Suppose on U we have $X = \sum_{i=1}^{m} \alpha^i E_i$.

  By a previous corollary, $E_{1p}, \ldots, E_{mp}$ span $T_p(N)$ for $p \in V$.

  So, on $V = U \cap N$, we must have

  $$\alpha^{n+1} = \cdots = \alpha^m = 0.$$

  The α^i are the same functions as in the case of U but with the last $m - n$ variables equated to zero when we restrict to V.

  Thus, X restricted to N has C^∞-components relative to the frames $E_1, \ldots, E_n$ of preferred coordinate systems.

  However, by a previous corollary, it is clearly sufficient to check that X is C^∞ for a covering by coordinate neighborhoods.

  It must then be C^∞ relative to any coordinates.

Example

- On the 2-sphere S^2, there do not exist any nonvanishing continuous vector fields.
- However, there are three mutually perpendicular unit vector fields on $S^3 \subseteq \mathbb{R}^4$, that is, a frame field.
- Let
$$S^3 = \left\{ (x_1, x_2, x_3, x_4) : \sum_{i=1}^{4} (x^i)^2 = 1 \right\}.$$
- Let the vector fields be given by
$$\begin{aligned} X &= -x^2 \frac{\partial}{\partial x^1} + x^1 \frac{\partial}{\partial x^2} + x^4 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^4}, \\ Y &= -x^3 \frac{\partial}{\partial x^1} - x^4 \frac{\partial}{\partial x^2} + x^1 \frac{\partial}{\partial x^3} + x^2 \frac{\partial}{\partial x^4}, \\ Z &= -x^4 \frac{\partial}{\partial x^1} + x^3 \frac{\partial}{\partial x^2} - x^2 \frac{\partial}{\partial x^3} + x^1 \frac{\partial}{\partial x^4}, \end{aligned}$$

  at the point $x = (x^l, x^2, x^3, x^4)$ of S^3.

Example (Cont'd)

- At each point these are mutually orthogonal unit vectors in $\mathbb{R}^4$.
- So they are independent.
- It can be seen that they are orthogonal to the radius vector from the origin 0 to the point x of S^3.
- This shows that they are tangent to S^3.
- Finally, by the preceding lemma with $N = S^3$ and $M = \mathbb{R}^4$, they are C^∞-vector fields.

Parallelizable Manifolds

- It is possible to show that all odd-dimensional spheres have at least one nonvanishing C^∞-vector field.
- Moreover, like S^2, no even-dimensional sphere has any continuous nonvanishing field of tangent vectors.
- It has been proved that only the spheres S^1, S^3, S^7 have a C^∞ field of bases, as we have just seen to be the case for S^3.
- Manifolds with this very special property are called **parallelizable**.
- As already mentioned, coordinate neighborhoods are parallelizable.

Situation with Mappings

- We have established the concept of *vector field on a manifold*.
- We must now consider what happens when we map a manifold N on which a vector field is defined into another manifold M.
- We saw that if $F : N \to M$ is a C^∞ map, then to each point $p \in M$ there is associated a homomorphism

$$F_* : T_p(N) \to T_{F(p)}(M).$$

- If X is a vector field on N, then $F_*(X_p)$ is a vector at $F(p)$.
- But this process does not, in general, induce a vector field on M:
  - $F(N)$ may not be all of M, that is, given $q \in M$ it may well happen that for no $p \in N$ is $F(p) = q$.
  - Even if $F^{-1}(q)$ is not empty, it may contain more than one element, say p_1, p_2, with $p_1 \neq p_2$. Then it may happen that $F_*(X_{p_1}) \neq F_*(X_{p_2})$. So that there would be no uniquely determined vector Y_q at q which is the image of vectors of the field X on N.

Example

- It is easy to construct examples of these mishaps.
- Let N be the half-space $x^1 > 0$ in $\mathbb{R}^3$.
- Let $F : N \to M$ be projection to the coordinate plane $x^3 = 0$.
- Let X be the gravitational field restricted to N.
- The image vectors do not determine a vector field on M.

Related Vector Fields

Definition

Let N and M be manifolds.
Let $F : N \to M$ be a C^∞ map.
Let X be a vector field on N.
Suppose we have a vector field Y on M, such that, for each $q \in M$ and $p \in F^{-1}(q) \subseteq N$,

$$F_*(X_p) = Y_q.$$

Then we say that the vector fields X and Y are F-**related** and we write, briefly,

$$Y = F_*(X).$$

We do not require F to be onto. If $F^{-1}(q)$ is empty, then the condition is vacuously satisfied.

Diffeomorphisms and Related Vector Fields

Theorem

If $F : N \to M$ is a diffeomorphism, then each vector field X on N is F-related to a uniquely determined vector field Y on M.

- Since F is a diffeomorphism, it has an inverse $G : M \to N$.

  Moreover, at each point p we have

  $$F_* : T_p(N) \to T_{F(p)}(M)$$

  is an isomorphism onto, with G_* as inverse.

  Let X be a C^∞-vector field on N.

  Then, at each point q of M, the vector

  $$Y_q = F_*(X_{G(q)})$$

  is uniquely determined.

F-Related Vector Fields (Cont'd)

- It then remains to check that Y is a C^∞-vector field.

  This is immediate if we:
  - Introduce local coordinates;
  - Apply a previous theorem to the component functions.

  Remark: Under the hypotheses of the lemma we have a second example of F-related vector fields.

  Let $F : N \to M$ be the inclusion map.

  Let X' be X restricted to N.

  Then X' and X are F-related by the lemma.

Invariance With Respect to a Diffeomorphism

Definition

Let $F : M \to M$ be a diffeomorphism. Let X be a C^∞ vector field on M, such that

$$F_*(X) = X,$$

that is, X is F-related to itself. Then X is said to be **invariant with respect to** F or F-**invariant**.

Lie Groups and Invariance Under Translations

Theorem

Let G be a Lie group and $T_e(G)$ the tangent space at the identity. Then each $X_e \in T_e(G)$ determines uniquely a C^∞-vector field X on G which is invariant under left translations. In particular, G is parallelizable.

- Let $g \in G$.
  Consider the unique left translation L_g taking e to g.
  Therefore, if it exists, X is uniquely determined by the formula
  $$X_g = L_{g*}(X_e).$$
  Except for differentiability, this formula does define a left invariant vector field, since for $a \in G$, we have
  $$L_{a*}(X_g) = L_{a*} \circ L_{g*}(X_e) = L_{ag*}(X_e) = X_{ag}.$$

Lie Groups and Invariance Under Translations (Cont'd)

- We must show that X, so determined, is C^∞.
  Let U, φ be a coordinate neighborhood of e, such that
  $$\varphi(e) = (0, \ldots, 0).$$
  Let V be a neighborhood of e satisfying $VV \subseteq U$.
  Let $g, h \in V$ have coordinates
  $$x = (x^1, \ldots, x^n) \quad \text{and} \quad y = (y^1, \ldots, y^n).$$
  Let the coordinates of the product gh be
  $$z = (z^1, \ldots, z^n).$$
  Then
  $$z^i = f^i(x, y), \quad i = 1, \ldots, n,$$
  are C^∞ functions on $\varphi(V) \times \varphi(V)$.

Lie Groups and Invariance Under Translations (Cont'd)

- Write
$$X_e = \sum_{i=1}^{n} \gamma^i E_{ie}, \quad \gamma^1, \ldots, \gamma^n \text{ real numbers.}$$
In local coordinates L_g is given by
$$z^i = f^i(x, y), \quad i = 1, \ldots, n,$$
with the coordinates x of g fixed.
So, by a previous theorem, the formula above for X_g becomes
$$X_g = L_{g*}(X_e) = \sum \gamma^j \left(\frac{\partial f^i}{\partial y^j}\right)_{(x,0)} E_{ig}.$$
It follows that, on V, the components of X_g in the coordinate frames are C^∞ functions of the local coordinates.

Lie Groups and Invariance Under Translations (Cont'd)

- However, for any $a \in G$, the open set aV is the diffeomorphic image by L_a of V.

  Moreover, X, as noted above, is L_a-invariant.

  So, for every $g = ah \in aV$, we have

  $$X_g = L_{a*}(X_h).$$

  It follows that X on aV is L_a-related to X on V.

  Therefore, X is C^∞ on aV by the previous theorem.

  But X is C^∞ in a neighborhood of each element of G.

  So X is C^∞ on G.

Lie Group Homomorphisms and Invariant Vector Fields

Corollary

Let G_1 and G_2 be Lie groups and $F : G_1 \to G_2$ a homomorphism. Then to each left-invariant vector field X on G_1, there is a uniquely determined left-invariant vector field Y on G_2 which is F-related to X.

- By the theorem, X is determined by X_{e_1}, where e_1 is the identity of G_1.
  Let $e_2 = F(e_1)$ be the identity of G_2.
  Let Y be the uniquely determined left-invariant vector field on G_2, such that
  $$Y_{e_2} = F_*(X_{e_1}).$$
  This is certainly a necessary condition for Y to be F-related to X.
  It remains to see whether Y satisfies
  $$F_*(X_g) = Y_{F(g)}, \quad \text{for every } g \in G_1.$$

Proof (Cont'd)

- We must show that the vector field Y satisfies

$$F_*(X_g) = Y_{F(g)}, \text{ for every } g \in G_1.$$

If so, Y is indeed F-related (and uniquely determined).
We have $F(x) = F(g)F(g^{-1}x)$.
Using this, we write F as a composition

$$F = L_{F(g)} \circ F \circ L_{g^{-1}}.$$

Now both X and Y are left-invariant by assumption.
So this gives

$$\begin{array}{rcl} F_*(X_g) & = & L_{F(g)*} \circ F_* \circ L_{g^{-1}*}(X_g), \\ F_*(X_g) & = & L_{F(g)*} \circ F_*(X_e) = L_{F(g)*} Y_{e_2}, \\ F_*(X_g) & = & Y_{F(g)}. \end{array}$$

Therefore, Y meets all conditions and the corollary is true.

Subsection 3

One-Parameter Groups Acting on a Manifold

Introduction

- We study the case of a connected Lie group of dimension 1 acting on a manifold M.
- When we looked at the case of a Lie group of dimension 0 we focused in the space of orbits.
- Here we are mainly concerned with the relation to vector fields on M.
- For this reason we shall limit ourselves to the action of R, by which we denote the additive (Lie) group of real numbers $\mathbb{R}$, acting on M.
- This will illustrate all the relevant facts.
- We note that R and S^1 are the only connected Lie groups of dimension 1.
- These two cases, discrete Lie groups and the one-dimensional Lie group R acting on M, will give some idea of the depth and diversity of the whole subject of group action on manifolds.

The Action

- Consider the general definition of action specialized to an action θ of R on M.
- Let
$$\theta : R \times M \to M$$
be a C^∞ mapping which satisfies the two conditions:
  (i) $\theta_0(p) = p$, for all $p \in M$;
  (ii) $\theta_t \circ \theta_s(p) = \theta_{t+s}(p) = \theta_s \circ \theta_t(p)$, for all $p \in M$ and $s, t \in R$.
- We will often write $\theta(t, p)$ as $\theta_t(p)$ or $\theta_p(t)$, depending on which variable is to be emphasized.

Example: Translations

- Suppose that $M = \mathbb{R}^3$.
- Let $a = (a^1, a^2, a^3)$ be fixed and different from 0.
- Consider the mapping

$$\theta_t(x) = (x^1 + a^1 t, x^2 + a^2 t, x^3 + a^3 t).$$

- It defines a C^∞ action of R on M.
- To each $t \in R$, it assigns the translation $\theta_t : \mathbb{R}^3 \to \mathbb{R}^3$, taking the point x to the point $x + ta$.
- This is a free action.
- The orbits consist of straight lines parallel to the vector a.
- A particularly simple special case is given by $a = (1, 0, 0)$.
- Then

$$\theta_t(x) = (x^1 + t, x^2, x^3).$$

Infinitesimal Generator of an Action

- Suppose that $\theta : R \times M \to M$ is any such C^∞ action.
- It defines on M a C^∞-vector field X, which we shall call the **infinitesimal generator** of θ, according to the following prescription.
- For each $p \in M$, $X_p : C^\infty(p) \to \mathbb{R}$ is given by

$$X_p f = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[f(\theta_{\Delta t}(p)) - f(p)].$$

- We may check directly that X_p is a vector at p.
- We may then verify that $p \to X_p$ defines a vector field.

Infinitesimal Generator of an Action (Cont'd)

- Alternatively, we may proceed as follows
- Let U, φ be a coordinate neighborhood of $p \in M$.
- Let $I_\delta \times V$ be an open subset of $(0, p)$ in $R \times M$, where:
  - $I = \{t \in R : -\delta < t < \delta\}$;
  - V and $\delta > 0$ are so chosen that

$$\theta(I_\delta \times V) \subseteq U.$$

- In particular, $V = \theta_0(V)$ is contained in U and contains p.

Infinitesimal Generator of an Action (Cont'd)

- Restricted to the open set $I_\delta \times V$, we may write θ in local coordinates

$$y^1 = h^1(t, x^1, \dots, x^n), \dots, y^n = h^n(t, x^1, \dots, x^n)$$

  or $y = h(t, x)$, where:
  - $x = (x^1, \dots, x^n)$ are the coordinates of $q \in V$;
  - $y = (y^1, \dots, y^n)$ are the coordinates of $\theta_t(q)$, its image.
- The h_i are defined and C^∞ on $I_\delta \times \varphi(V)$.
- The range of $h(t, x)$ is in $\varphi(U)$.
- The fact that θ_0 is the identity and $\theta_{t_1+t_2} = \theta_{t_1} \circ \theta_{t_2}$ is reflected in having, for all $i = 1, \dots, n$,
  - $h^i(0, x) = x^i$;
  - $h^i(t_1 + t_2, x) = h^i(t_1, h(t_2, x))$.

Infinitesimal Generator of an Action (Cont'd)

- Let $\widehat{f}(x^1, \ldots, x^n)$ be the local expression for $f \in C^\infty(p)$
- Then
$$\frac{1}{\Delta t}[f(\theta_{\Delta t}(p)) - f(p)] = \frac{1}{\Delta t}[\widehat{f}(h(\Delta t, x)) - \widehat{f}(x)].$$
- Let dot indicate differentiation with respect to t.
- Then, we also have
$$X_p f = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[\widehat{f}(h(\Delta t, x)) - \widehat{f}(x)] = \sum_{i=1}^{n} \dot{h}^i(0, x) \left(\frac{\partial \widehat{f}}{\partial x^i}\right)_{\varphi(p)}.$$
- This formula is valid for every $p \in V$.

Infinitesimal Generator of an Action (Cont'd)

- We obtained, for all $p \in V$,

$$X_p f = \sum_{i=1}^{n} \dot{h}^i(0, x) \left(\frac{\partial \widehat{f}}{\partial x^i} \right)_{\varphi(p)}.$$

- The formula implies that on V;

$$X_p = \sum \dot{h}^i(0, x) E_{ip},$$

  where:
  - $E_i = \varphi_*^{-1}(\frac{\partial}{\partial x^i})$;
  - $x = \varphi(p)$.
- This shows that X is a C^∞-vector field over V.
- But every point of M lies in such a neighborhood.
- So X is C^∞ on M.
- Definition of X at $p \in M$ involves only the values of θ on $I_\delta \times V$.
- That is, like derivatives in general, it is defined locally and involves only values of t near $t = 0$.

Invariance

Definition

Let $\theta : G \times M \to M$ be the action of a group G on a manifold M.
Let X be a vector field on M.
X is said to be **invariant under the action of** G or G-**invariant** if X is invariant under each of the diffeomorphisms

$$\theta_g : M \to M.$$

In brief if

$$\theta_{g*}(X) = X.$$

Invariance of the Infinitesimal Generator

Theorem

Let $\theta : R \times M \to M$ be a C^∞ action of R on M.
Then the infinitesimal generator X is invariant under this action, that is,

$$\theta_{t*}(X_p) = X_{\theta_t}(p), \quad \text{for all } t \in R.$$

- Let $f \in C^\infty(\theta_t(p))$, for some $(t, p) \in R \times M$.
  Compute $\theta_{t*}(X_p)f$,

$$\theta_{t*}(X_p)f = X_p(f \circ \theta_t) = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[f \circ \theta_t(\theta_{\Delta t}(p)) - f \circ \theta_t(p)].$$

  But R is Abelian and we have $\theta_t \circ \theta_{\Delta t} = \theta_{t+\Delta t} = \theta_{\Delta t} \circ \theta_t$.
  So

$$\theta_{t*}(X_p)f = \lim_{\Delta t \to 0} [(f \circ \theta_{\Delta t})(\theta_t(p)) - f(\theta_t(p))] = X_{\theta_t(p)}f.$$

  Since this holds for all f, the result follows.

Vanishing and Orbits

Corollary

If $X_p = 0$, then for each q in the orbit of p we have $X_q = 0$.
That is, at the points of an orbit the associated vector field vanishes identically or is never zero.

- The orbit of p consists of all q such that $q = \theta_t(p)$ for some $t \in R$.
  Thus, by the theorem,
  $$X_q = \theta_{t*} X_p.$$
  Now θ_t is a diffeomorphism.
  So θ_{t*} is an isomorphism of $T_p(M)$ onto $T_q(M)$.
  So $X_q = 0$ if and only if $X_p = 0$.

Orbits as Immersions

Theorem

The orbit of p is either a single point or an immersion of R in M by the map $t \to \theta_t(p)$, depending on whether or not $X_p = 0$.

- The orbit of p is the image of R under the C^∞ map

$$\begin{array}{rrcl} F: & R & \to & M; \\ & t & \mapsto & \theta_t(p). \end{array}$$

Let $t_0 \in R$ and $\frac{d}{dt}$ denote the standard basis of $T_{t_0}(R)$.
F is an immersion if and only if

$$F_* \left(\frac{d}{dt} \right) \neq 0, \quad \text{for every } t_0 \in R.$$

Orbits as Immersions (Cont'd)

- Let $f \in C^\infty(F(t_0)) = C^\infty(\theta_{t_0}(p))$.

  Observe that

$$\begin{aligned} F_*\left(\frac{d}{dt}\right) f &= \frac{d}{dt}(f \circ F)_{t_0} \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[f \circ F(t_0 + \Delta t) - f \circ F(t_0)] \\ &= \lim_{\Delta t \to 0} \frac{1}{\Delta t}[f(\theta_{t_0 + \Delta t}(p)) - f(\theta_{t_0}(p))] \\ &= X_{\theta_{t_0}(p)} f. \end{aligned}$$

  This formula and the preceding corollary show that either $X_p \neq 0$ and F is an immersion or else $X_{F(t)} = F_*(\frac{d}{dt}) \equiv 0$.

  In the latter case F is a constant map with $F(R) = p$.

Remarks and Notation

- Consider again the formula just obtained,

$$F_*\left(\frac{d}{dt}\right) = X_{\theta_{t_0}(p)} = X_{F(t_0)}.$$

- It shows that, at each point $p \in M$, the vector X_p is tangent to its orbit.
- It is, in fact, the (tangent) velocity vector of the curve $t \to F(t)$ in M, in the sense in which we have previously defined the velocity vector to a parameterized curve.
- Recall that, for a differentiable map of an open interval J of R into M, this was defined by $F_*(\frac{d}{dt})$.

Remarks and Notation (Cont'd)

- The notation $F_*(\frac{d}{dt})$ does not indicate that:
  - $\frac{d}{dt} \in T_{t_0}(R)$;
  - F_* is a homomorphism of $T_{t_0}(R)$ into $T_{F(t_0)}(M)$.
- For this reason we often write either

$$\dot{F}(t_0) \quad \text{or} \quad \left(\frac{dF}{dt}\right)_{t_0}$$

  to denote the velocity vector.
- Sometimes we use $t \to p(t)$ to denote the mapping rather than F.
- Then its velocity vector is written

$$\frac{dp}{dt} \quad \text{or} \quad \dot{p}(t).$$

- In the notation of the theorem, the formula above can be written

$$\dot{\theta}(t,p) = X_{\theta(t,p)}.$$

The Chain Rule

- Suppose we change parameter by a function $t = f(s)$.
- Then $s \to G(s) = F(f(s))$ represents the curve.
- So for $t_0 = f(s_0)$,

$$\left(\frac{dG}{ds}\right)_{s_0} = G_*\left(\frac{d}{ds}\right) = F_* \circ f_*\left(\frac{d}{ds}\right) = F_*\left(\frac{dt}{ds}\frac{d}{dt}\right).$$

- These give the formula

$$\left(\frac{dG}{ds}\right)_{s_0} = \left(\frac{dt}{ds}\right)_{s_0} F_*\left(\frac{d}{dt}\right)_{t_0}.$$

- Thus the velocity vector with respect to s is a scalar multiple by $(\frac{dt}{ds})_{s_0}$ of the velocity vector with respect to t.
- This may be conveniently written

$$\dot{G} = \left(\frac{dt}{ds}\right)\dot{F}(f(s)) \quad \text{or} \quad \frac{dp}{ds} = \frac{dp}{dt}\frac{dt}{ds}.$$

- This vector equation is, of course, just a special case of the chain rule.

Integral Curves

Definition

Let M be a manifold. Let X be a vector field on M.
We say that a curve

$$t \to F(t)$$

defined on an open interval J of $\mathbb{R}$ is an **integral curve of** X if

$$\frac{dF}{dt} = X_{F(t)} \quad \text{on } J.$$

- We have just shown that each orbit of the action θ is an integral curve of the infinitesimal generator X of θ.
- That is, for each fixed $p \in M$,

$$\dot{\theta}(t, p) = X_{\theta(t,p)}.$$

Questions

- Some natural questions arise concerning vector fields and one-parameter group actions.
  - Is every C^∞-vector field the infinitesimal generator of some group action?
  - Can two different actions of R on M give rise to the same vector field X as infinitesimal generator?
- These questions will be answered next.
- First we use a simple, but instructive, example to:
  - Illustrate the difficulties involved;
  - Show the necessity for a less restrictive concept of one-parameter group action.

Example

- Let $M = \mathbb{R}^2$ and let $\theta : R \times M \to M$ be defined by
$$\theta(t, (x, y)) = (x + t, y).$$
- Then the infinitesimal generator is
$$X = \frac{\partial}{\partial x}.$$
- This action is given by translation of each point (x, y) to a point t units to the right.
- Suppose now that we remove the origin $(0, 0)$ from $\mathbb{R}^2$ and let
$$M_0 = \mathbb{R}^2 - \{(0, 0)\}.$$
- For most points θ_t is defined as before.

Example (Cont'd)

- However, we cannot obtain an action of R on M_0 by restriction of θ to $R \times M_0$.
- This is because points of the closed set
$$F = \{(t,(x,0)) : t + x = 0\} = \theta^{-1}(0,0)$$
of $R \times M$ are mapped by θ to the origin.
- On the other hand, let $W \subseteq R \times M_0$ be the open set defined by
$$W = R \times M_0 - F \cap (R \times M_0).$$
- Then $\overline{\theta} = \theta|_W$ maps W onto M_0.
- Moreover, it preserves many of the features of θ which we have used.

Example (Cont'd)

- For example, let $p = (x, y) \in M_0$.
- Then, if all terms are defined we get:
  (i) $(0, p) \in W$ and $\theta_0(p) = p$;
  (ii) $\theta_s \circ \theta_t(p) = \theta_{s+t}(p) = \theta_t \circ \theta_s(p)$.
- The infinitesimal generator X is defined, as before, by

$$X_p = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[f(\theta_{\Delta t}(p)) - f(p)]$$

- It is again $X = \frac{\partial}{\partial x}$.
- Finally we have orbits $t \to \theta_t(p)$, which are:
  - The lines $y =$ constant when $p = (x, y)$, $y \neq 0$;
  - The portion of the x-axis minus the origin which contains p, for $p = (x, 0)$.
- This curve is not defined for all values of t in the case of the orbit of a point on the x-axis.

Local One Parameter Group Actions

- Let M be a C^∞ manifold.
- Let $W \subseteq R \times M$ be an open set which satisfies:

  For every $p \in M$, there exist real numbers $\alpha(p) < 0 < \beta(p)$, such that
$$W \cap (R \times \{p\}) = \{(t, p) : \alpha(p) < t < \beta(p)\}.$$

- Denote by $I(p)$ the interval $\alpha(p) < t < \beta(p)$.
- Denote by I_δ the interval defined by $|t| < \delta$.
- The displayed condition simply states that
$$W = \bigcup_{p \in M} I(p) \times \{p\}.$$

Local One Parameter Group Actions (Cont'd)

- We use the preceding notation and consider W as above.

Definition

A **local one-parameter group action** or **flow** on a manifold M is a C^∞ map

$$\theta : W \to M$$

which satisfies the following two conditions:

(i) $\theta_0(p) = p$, for all $p \in M$;

(ii) If $(s, p) \in W$, then

$$\alpha(\theta_s(p)) = \alpha(p) - s \quad \text{and} \quad \beta(\theta_s(p)) = \beta(p) - s.$$

Moreover, for any t, such that $\alpha(p) - s < t < \beta(p) - s$, $\theta_{t+s}(p)$ is defined and

$$\theta_t \circ \theta_s(p) = \theta_{t+s}(p).$$

On the Preceding Example

- It is easy to check that the preceding example has these properties.
- This example also shows that, to obtain a correspondence between one-parameter group actions and vector fields, we must abandon the requirement that W is all of $R \times M$.
- Such actions are called **global actions**.
- The set W is open and contains $(0, p)$, for each $p \in M$.
- So it also contains $I_\delta \times U$, U a neighborhood of p, for sufficiently small $\delta > 0$.
- Therefore, the definition of the vector field X (infinitesimal generator) associated with θ is valid in the case of local action also.
- Moreover, it associates a C^∞-vector field to each flow θ.

Local One-Parameter Actions as Diffeomorphisms

- Let R act on M, as in the case of any group acting on M.
- For each t, $\theta_t : M \to M$ is a diffeomorphism, with $\theta_t^{-1} = \theta_{-t}$.
- Something like this is also true for the local case.
- The difference is that θ_t is not defined on all of M in general.
- Let $V_t \subseteq M$ be the domain of definition of θ_t,

$$V_t = \{p \in M : (t, p) \in W\}.$$

Theorem

V_t is an open set for every $t \in R$ and $\theta_t : V_t \to V_{-t}$ is a diffeomorphism with $\theta_t^{-1} = \theta_{-t}$.

- Let $p_0 \in V_{t_0}$ so that $(t_0, p_0) \in W$.
  Since W is open, there is a $\delta > 0$ and a neighborhood V of p_0, such that

$$\{t : |t - t_0| < \delta\} \times V \subseteq W.$$

Local One-Parameter Actions as Diffeomorphisms (Cont'd)

- In particular, $\{t_0\} \times V \subseteq W$.
  So $V \subseteq V_{t_0}$.
  Next, note that if $p \in V_t$, then $\alpha(p) < t < \beta(p)$.
  By definition $t + (-t)$ lies in the same interval.
  It follows that $\theta_t(p) \in V_{-t}$ and
  $$\theta_{-t} \circ \theta_t(p) = p.$$
  Similarly, $\theta_{-t}(V_{-t}) \subseteq V_t$ and
  $$\theta_t \circ \theta_{-t}(q) = q, \quad \text{for any } q \in V_{-t}.$$
  Combining these statements with the fact that θ_t, θ_{-t} are C^∞ on any open subsets of M on which they are defined completes the proof.

Remarks

- For local one-parameter actions we may show as in the global case that:

$$\theta_{t*}(X_p) = X_{\theta_t(p)}, \text{ if } p \in V_t.$$

- As before,

$$F(t) = \theta_t(p), \quad \alpha(p) < t < \beta(p)$$

  is a C^∞-integral curve of X.
- It is an immersion of $I(p)$ in M, provided that $X_p \neq 0$.
- It is a single point if $X_p = 0$.
- We shall continue to refer to these curves as **orbits** of the local one-parameter group, just as in the global case.
- It is a consequence of our definitions that these curves (and points) partition M into a union of mutually disjoint sets.
- The proofs are the same, essentially, as in the global case.

Flows in Local Coordinates

Theorem

Let $\theta : W \to M$ be as in the definition of local one-parameter group actions. Let X be the associated infinitesimal generator.
Suppose $p \in M$ such that $X_p \neq 0$. Then there exist:

- A coordinate neighborhood V, ψ around p;
- A $\nu > 0$;
- A corresponding neighborhood V' of p, $V' \subseteq V$,

such that, in local coordinates, θ restricted to $I_\nu \times V'$ is given by

$$(t, y^1, \ldots, y^n) \to (y^1 + t, y^2, \ldots, y^n).$$

In these coordinates

$$X = \psi_*^{-1}\left(\frac{\partial}{\partial y^1}\right) \quad \text{at every point of } V'.$$

Flows in Local Coordinates (Cont'd)

- In W introduce coordinates U, φ around p.
  Express θ in the local coordinates by
  $$x \to h(t; x),$$
  where $x = (x^1, \ldots, x^n)$ and $h(t; x)$ stands for an n-tuple of functions satisfying:
  (i) $h(0; x) = x$;
  (ii) $h(t; h(t'; x)) = h(t + t'; x)$.
  We will assume coordinates so chosen that:
  - $\varphi(p) = (0, \ldots, 0)$;
  - $\varphi(U) = C^n_\varepsilon(0)$;
  - $X_p = \varphi_*^{-1}(\frac{\partial}{\partial x^1}) = E_{ip}$.

  Recall the expression for X_p, $X_p = \sum \dot{h}^i(0; 0, \ldots, 0) E_{ip}$.
  It implies that
  $$\dot{h}^i(0; 0, \ldots, 0) = \begin{cases} 1, & \text{for } i = 1, \\ 0, & \text{for } i > 1. \end{cases}$$

Flows in Local Coordinates (Cont'd)

- Choose $\delta > 0$ small enough so that:
  - $V'' = \varphi^{-1}(C_\delta^n(0)) \subseteq U$;
  - $\theta(I_\delta \times V'') \subseteq U$.

  Then map the cube $C_\delta^n(0) \subseteq I_\delta \times \mathbb{R}^{n-1}$ into $C_\delta^n(0) \subseteq \varphi(U)$ by a map F, given in local coordinates by

  $$F : (y^1, \ldots, y^n) \to (h^1(y^1; 0, y^2, \ldots, y^n), \ldots, h^n(y^1; 0, y^2, \ldots, y^n)).$$

  From the expression for X_p, we see that $(\frac{\partial h^i}{\partial y^1})_0 = \delta_1^i$.

  From $y^i = h^i(0; 0, y^2, \ldots, y^n)$, we see that $(\frac{\partial h^i}{\partial y^j})_0 = \delta_j^i$, for $j > 1$.

  Thus, the Jacobian of F at $y = (0, \ldots, 0)$ is the identity matrix.

  Hence, there is a $\mu > 0$, with $\mu \leq \delta$, such that F is a diffeomorphism of $C_\mu^n(0)$ onto an open set of $C_\varepsilon^n(0) = \varphi(U)$.

  Let $V = \varphi^{-1} \circ F(C_\mu^n(0))$ and $\psi = F^{-1} \circ \varphi$.

  They form a coordinate neighborhood of p with $V \subseteq U$.

Flows in Local Coordinates (Cont'd)

- The relations satisfied by $h^i(t,x)$, $i = 1, \ldots, n$, give:
  (i) $\psi(p) = F^{-1}(\varphi(p)) = F^{-1}(0, \ldots, 0)$.
  For $(y^1, \ldots, y^n) \in C_\nu(0)$ and $|t| < \nu$ with $\nu = \frac{\mu}{2}$, they give:
  (ii) $h^i(t + y^1; 0, y^2, \ldots, y^n) = h^i(t, h(y^1; 0, y^2, \ldots, y^n))$, $i = 1, \ldots, n$.

  Formula (ii) may be interpreted as follows.

  In the coordinate system (V, ψ), if $\psi(q) = (y^1, \ldots, y^n)$, then

$$\psi(\theta_t(q)) = (t + y^1, \ldots, y^n),$$

  provided only that $|t| < \nu$ and $q \in \psi^{-1}(C_\nu^n(0))$, so that all functions are defined.

Flows in Local Coordinates (Cont'd)

- In other words, in the y-coordinates of V, ψ, the mapping θ_t is expressed by functions $\widetilde{h}^i(t, y)$, defined on $I_\nu \times C_\nu^n(0)$ by

$$\begin{aligned} \widetilde{h}^1(t, y^1, \ldots, y^n) &= t + y^1, \\ \widetilde{h}^i(t, y^1, \ldots, y^n) &= y^i, \text{ for } i > 1. \end{aligned}$$

We also have

$$\psi_*(X_q) = \sum \dot{h}^i(0, y) \frac{\partial}{\partial y^i} = \frac{\partial}{\partial y^1}.$$

From these formulas, we get that, on $V' = \psi^{-1}(C_\nu^n(0))$,

$$X_q = \psi_*^{-1}\left(\frac{\partial}{\partial y^1}\right).$$

Subsection 4

The Existence Theorem for Ordinary Differential Equations

Existence Theorem for Ordinary Differential Equations

Theorem (Existence Theorem for Ordinary Differential Equations)

Let $U \subseteq \mathbb{R}^n$ be an open set. For $\varepsilon > 0$, let $I_\varepsilon = (-\varepsilon, \varepsilon)$. Let

$$f^i(t, x^1, \ldots, x^n), \quad i = 1, \ldots, n,$$

be functions of class C^r, $r \geq 1$, on $I_\varepsilon \times U$.
Then, for each $x \in U$, there exists $\delta > 0$ and a neighborhood V of x, $V \subseteq U$, such that:

(I) For each $a = (a^1, \ldots, a^n) \in V$ there exists an n-tuple of C^r functions $x(t) = (x^1(t), \ldots, x^n(t))$, defined on I_δ and mapping I_δ into U, which satisfy the system of first-order differential equations

$$\frac{dx^i}{dt} = f^i(t, x), \quad i = 1, \ldots, n,$$

and the initial conditions $x^i(0) = a^i$, $i = 1, \ldots, n$.

Existence Theorem (Cont'd)

Theorem (Existence Theorem Cont'd)

For each a, the functions $x(t) = (x^1(t), \ldots, x^n(t))$ are uniquely determined, in the sense that any other functions $\overline{x}^1(t), \ldots, \overline{x}^n(t)$ satisfying the same condition must agree with $x(t)$ on their common domain, which includes I_δ.

(II) These functions being uniquely determined by $a = (a^1, \ldots, a^n)$ for every $a \in V$, we write them

$$x^i(t, a^1, \ldots, a^n), \quad i = 1, \ldots, n.$$

They are of class C^r in all variables and, thus, determine a C^r map of $I_\delta \times V \to U$.

Autonomous Systems

- If $f^i(t,x)$, $i = 1, \ldots, n$, is independent of t, then the system of differential equations is called **autonomous**.
- Throughout the remainder of this chapter we shall deal only with autonomous systems.
- In this case it is possible to restate the hypotheses and conclusions of the fundamental existence theorem in coordinate-free form using the concepts of vector field and integral curve.
- This will allow us to derive various global theorems useful in both geometry and analysis from a purely local existence theorem about open subsets of $\mathbb{R}^n$.

The Autonomous Case

- In the autonomous case, the f^i depend on $x = (x^1, \ldots, x^n)$ alone.
- For simplicity we shall also assume hereafter that all data are C^∞.
- Define on $U \subseteq \mathbb{R}^n$ a C^∞-vector field X by

$$X = f^1(x)\frac{\partial}{\partial x^1} + \cdots + f^n(x)\frac{\partial}{\partial x^n}.$$

- Recall that an **integral curve** of X is a C^∞ mapping F of an open interval (α, β) of $\mathbb{R}$ into U such that

$$\dot{F}(t) = X_{F(t)}, \quad \alpha < t < \beta.$$

The Autonomous Case (Cont'd)

- Write F in terms of its coordinate functions

$$F(t) = (x^1(t), \ldots, x^n(t)).$$

- Then the vector equation $\dot{F}(t) = X_{F(t)}$ is satisfied if and only if

$$\frac{dx^i}{dt} = f^i(x^1(t), \ldots, x^n(t)), \quad i = 1, \ldots, n.$$

- This states precisely that the functions

$$x(t) = (x^1(t), \ldots, x^n(t))$$

are a solution of the system of the theorem.

The Autonomous Case (Cont'd)

- Given $x \in U$, Part (I) states that, for each a in a neighborhood V of x, there is a unique integral curve $F(t)$ satisfying $F(0) = a$.
- $F(t)$ is defined at least for $-\delta < t < \delta$, with same $\delta > 0$, for every $a \in V$.

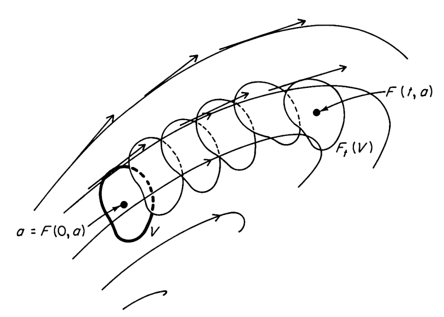
- Use a notation for these integral curves through points of V, indicating dependence on a, say $F(t, a) = (x^1(t, a), \ldots, x^n(t, a))$.
- Use an overdot for differentiation with respect to t.
- Then these equations become

$$\dot{x}^i(t, a) = f^i(x(t, a)), \quad x^i(0, a) = a^i, \quad i = 1, \ldots, n.$$

- Part (II) states that these functions $x^i(t, a)$ are C^∞ - in all variables - on $I_\delta \times V$, an open subset of $\mathbb{R} \times U$.

Interpreting the Solution as a Flow

- As an aid to intuition we may interpret the mapping $F : I_\delta \times V \to U$ as a "flow", that is, a motion within U of the points of V so that the point at position a at time $t = 0$ moves to $F(t, a)$ at time t.



- The path of a moving point is the integral curve.
- Moreover, its velocity at any of its positions is given by the vector X assigned to that point of U.

The Case of Vector Fields on Manifolds

Theorem

Let X be a C^∞-vector field on a manifold M.
Then, for each $p \in M$, there exists a neighborhood V and real number $\delta > 0$, such that there corresponds a C^∞ mapping $\theta^V : I_\delta \times V \to M$, with

$$\dot{\theta}^V(t,q) = X_{\theta^V(t,q)}$$

and

$$\theta^V(0,q) = q, \quad \text{for all } q \in V.$$

If $F(t)$ is an integral curve of X, with $F(0) = q \in V$, then

$$F(t) = \theta^V(t,q), \quad \text{for } |t| < \delta.$$

In particular, this mapping is unique in the sense that if V_1, δ_1 is another such pair for $p \in M$, then $\theta^V = \theta^{V_1}$ on the common part of their domains.

Proof of the Theorem

- This is basically a restatement of the existence theorem as follows.
  For $p \in M$, we choose:
  - A coordinate neighborhood U, φ;
  - A map X to the φ-related vector field $\widetilde{X} = \varphi_*(X)$ on $\widetilde{U} = \varphi(U) \subseteq \mathbb{R}^n$.

  Apply the local existence theorem to obtain $F : I_\delta \times \widetilde{V} \to \widetilde{U}$ defined by
  $$F(t,a) = (x^1(t,a), \ldots, x^n(t,a))$$
  on a neighborhood $\widetilde{V} \subseteq \widetilde{U}$ of $\varphi(p)$.
  Set $V = \varphi^{-1}(\widetilde{V})$ and define $\theta^V : I_\delta \times V \to U$ by
  $$\theta^V(t,q) = \varphi^{-1}(F(t,\varphi(q))).$$
  Now φ and φ^{-1} are diffeomorphisms.
  So θ^V satisfies the required conditions.
  The final assertion is a consequence of the uniqueness of solutions.

Vector Fields and Integral Curves

Theorem

Let X be a C^∞-vector field on a manifold M and suppose $p \in M$. Then there is a uniquely determined open interval of R,

$$I(p) = \{\alpha(p) < t < \beta(p)\},$$

containing $t = 0$, and having the properties:

(1) There exists a C^∞-integral curve $F(t)$ defined on $I(p)$ and such that $F(0) = p$;

(2) Given any other integral curve $G(t)$ with $G(0) = p$, then the interval of definition of G is contained in $I(p)$ and $F(t) = G(t)$ on this interval.

Vector Fields and Integral Curves (Cont'd)

- Let $F(t)$ and $G(t)$ be two integral curves such that $F(0) = p = G(0)$.
  Suppose I_F, I_G to be the open intervals on which they are defined.
  Let I^* the set on which they agree.
  I^* is not empty since it contains $t = 0$.
  $F(t)$ and $G(t)$ are C^∞ mappings (hence continuous).
  So I^* is closed.
  Suppose $s \in I^*$.
  Now $s \in I_F \cap I_G$, an open set.
  So there is some interval $-\delta < t < \delta$ on which
  $$\widetilde{F}(t) = F(t+s) \quad \text{and} \quad \widetilde{G}(t) = G(t+s)$$
  are both defined.
  They are both integral curves, satisfying the same initial condition,
  $$\widetilde{F}(0) = F(s) = G(s) = \widetilde{G}(0).$$

Vector Fields and Integral Curves (Cont'd)

- From the existence theorem they agree on some open interval $|t| < \delta$ around $t = 0$.

  Thus, $F(t) = G(t)$ on an open set around s and I^* is open.

  It follows that $I^* = I_F \cap I_G$.

  Therefore $I(p)$ is defined.

  It is the union of the domains of all integral curves which pass through p at $t = 0$.

  The asserted properties are immediate.

  Note that it is possible for $\alpha(p) = -\infty$ and/or $\beta(p) = +\infty$.

  If both occur, then $I(p) = R$.
- We shall use the notation $F(t) = \theta(t, p)$ for the unique integral curve $F(t)$ such that $F(0) = p$.
- When we wish to emphasize dependence on t, we may write $\theta_p(t)$ for $\theta(t, p)$.

Notation and Summary

- Let the subset $W \subseteq R \times M$ be defined by

$$W = \{(t, p) \in R \times M : \alpha(p) < t < \beta(p)\}.$$

- According to what has been shown thus far:
  - Both W and θ are uniquely determined by X;
  - W is the domain of $\theta : W \to M$.
- Moreover we have the following properties of θ and W:
  (i) $\{0\} \times M \in W$ and $\theta(0, p) = p$ for all $p \in M$.
  (ii) For each (fixed) $p \in M$, let $\theta_p(t) = \theta(t, p)$.
  Then $\theta_p : I(p) \to M$ is a C^∞-integral curve, that is,

$$\dot{\theta}_p(t) = X_{\theta_p(t)}.$$

  (iii) For each $p \in M$, there is a neighborhood V and a $\delta > 0$, such that $I_\delta \times V \subseteq W$ and θ is C^∞ on $I_\delta \times V$.

Relation Between $I(p)$ and $I(q)$

Corollary

Let $s \in I(p)$ and $q = \theta_p(s) = \theta(s, p)$ be the corresponding point of the integral curve determined by p. Then

$$\alpha(q) = \alpha(p) - s \quad \text{and} \quad \beta(q) = \beta(p) - s$$

so that

$$I(q) = I(\theta_p(s)) = \{\alpha(p) - s < t < \beta(p) - s\}.$$

Thus $t \in I(q)$ if and only if $t + s \in I(p)$, and then we have

$$\theta(t, \theta(s, p)) = \theta(t + s, p).$$

Relation Between $I(p)$ and $I(q)$ (Cont'd)

- Suppose that $s \in I(p)$ and let

$$F(t) = \theta_p(s+t).$$

Then $F(t)$ is defined on the open interval $\alpha(p) < s+t < \beta(p)$ and

$$F(0) = \theta_p(s) = q.$$

By hypothesis, $F(t)$ is an integral curve.

So, by uniqueness, we have

$$F(t) = \theta(t, \theta_p(s)) = \theta(t, q).$$

So its domain must be $I(q) = \{\alpha(q) < t < \beta(q)\}$.

On W and θ

Theorem

Consider a C^∞-vector field X.

- The domain W of $\theta(t, p)$ is open in $R \times M$.
- θ is a C^∞ map onto M.

- Let $(t', p_0) \in W$.
  We must show that there is a neighborhood V of p_0 and $\delta > 0$, such that:
  - The open set $(t' - \delta, t' + \delta) \times V$ is in W;
  - θ is C^∞ on it.

  This is already known to be the case for $(0, p_0)$.

  Suppose, to the contrary, that the theorem fails.

  There exists $(t_0, p_0) \in W$, such that, for each $0 \leq t' < t_0$ there exists $(t' - \delta, t' + \delta) \times V$ with the above properties, but not for (t_0, p_0).

  We have assumed, without loss of generality, that $t_0 > 0$.

On W and θ (Cont'd)

- We shall show by contradiction that there can be no (t_0, p_0).

  Using a previous theorem, we find $\delta_0 > 0$ and a neighborhood V_0 of $q_0 = \theta(t_0, p_0)$, such that:
  - $I_{\delta_0} \times V_0 \subseteq W$;
  - θ is C^∞ on it.

  By continuity of $\theta(t, p_0)$ in t we may find $t_1 < t_0$, with:
  - $|t_0 - t_1| < \frac{1}{3}\delta_0$;
  - $\theta(t_1, p_0) \in V_0$.

  Since $t_1 < t_0$, by our assumption on (t_0, p_0), there is a $\delta_1 > 0$ and a neighborhood V_1 of p_0 such that:
  - $(t_1 - \delta_1, t_1 + \delta_1) \times V_1 \subseteq W$;
  - θ is C^∞ on this open set.

  In particular, $\theta(t_1, p_0)$ is in V_0 and $\theta_{t_1} : V_1 \to M$ is defined and C^∞.

On W and θ (Cont'd)

- We may suppose by continuity (restricting V_1 if necessary) that

$$\theta_{t_1}(V_1) \subseteq V_0.$$

We now have:

- $\theta(s + t_1, q)$ defined and C^∞ on the open set $|s| < \delta_1$ and $q \in V_1$;
- $\theta(s + t_1, q)$'s values for $s = 0$ are in V_0.

By a previous corollary, for $\alpha(\theta(t_1, q)) < s < \beta(\theta(t_1, q))$,

$$\theta(s + t_1, q) = \theta(s, \theta(t_1, q)).$$

Since $\theta(t_1, q)$ is in V_0, by the definition of δ_0 and V_0, the interval $I(\theta(t_1, q))$ contains all s for which $|s| < \delta_0$.

Thus, $\theta(s + t_1, q)$ is defined and C^∞ for $|s| < \delta_0$ and any $q \in V_1$.

This is an open set and, since $|t_0 - t_1| < \frac{1}{3}\delta_0$, it contains (t_0, p_0).

This shows that our assumption on (t_0, p_0) leads to a contradiction.

Equality of Flows

- Recall that a (local) one-parameter group θ acting on M was defined in terms of a C^∞ mapping θ of an open set $W \subseteq R \times M$ into M.
- Suppose
$$\theta_i, W_i, \quad i = 1, 2,$$
are two such local group actions.
- We say that $\theta_1 = \theta_2$ if they are equal (as mappings) on $W_1 \cap W_2$.
- Recall, also, the expression
$$X_p f = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[f(\theta_{\Delta t}(p)) - f(p)].$$
- So, if $\theta_1 = \theta_2$, then they have the same infinitesimal generator X.
- We note once again that if $W = R \times M$, then θ defines an action of R on M, that is, a global one-parameter group action.

Summary of Results

- Collecting the preceding results, we have the following

Theorem

- To each local one-parameter group action θ on M is associated a unique maximal domain of definition W.
- If θ_1, W_1 is equal to θ, W, then
$$W_1 \subseteq W \quad \text{and} \quad \theta_1 = \theta|_{W_1}.$$
- Two local one-parameter groups are equal if and only if they have the same infinitesimal generator X.
- Each vector field X on M determines a local one-parameter group θ, W of which it is the infinitesimal generator.

Summary of Results (Cont'd)

- This theorem summarizes the results of the last two sections.
- We saw those for the autonomous case, in which the vector field X does not depend on t (time), but only on the point of the manifold.
- It follows from the Existence Theorem.
- But, conversely, it implies the Existence Theorem as a special case when M is assumed to be an open set of $\mathbb{R}^n$.

First Generalization

- A general nth order ordinary differential equation in the independent variable t and dependent variable x and its derivatives is given by a relation
$$F\left(t, x, \frac{dx}{dt}, \ldots, \frac{d^n x}{dt^n}\right) = 0.$$
- We suppose that this is a function of class C^r defined on some neighborhood in $\mathbb{R}^{n+2}$ of the point $(0, a_0, a_1, a_2, \ldots, a_n)$.
- Also, in a neighborhood U of this point we can write it in the form
$$\frac{d^n x}{dt^n} = G\left(t, x, \frac{dx}{dt}, \ldots, \frac{d^{n-1} x}{dt^{n-1}}\right).$$
- This can be done if the derivative of F with respect to its last variable is not zero at the point.

First Generalization (Cont'd)

- Let
$$x = x^1,\ \frac{dx}{dt} = x^2, \ldots, \frac{d^{n-1}x}{dt^{n-1}} = x^n.$$
- Consider the first-order system of ordinary differential equations
$$\begin{aligned} \frac{dx^1}{dt} &= x^2 \\ \frac{dx^2}{dt} &= x^3 \\ &\vdots \\ \frac{dx^n}{dt} &= G(t, x^1, x^2, \ldots, x^{n-1}) \end{aligned}$$
with initial conditions
$$x^i(0) = a^i, \quad i = 1, \ldots, n.$$

First Generalization (Cont'd)

- The original nth order equation has a solution $x(t)$ satisfying initial conditions (at $t = 0$):

$$x(0) = a^1, \ \left(\frac{dx}{dt}\right)_0 = a^2, \ldots, \left(\frac{d^{n-1}x}{dt^{n-1}}\right)_0 = a^n$$

  if and only if the first-order system above has a solution satisfying the indicated initial conditions.
- Hence, the existence theorem gives the existence and uniqueness of solutions of ordinary differential equations of nth order.
- This can be extended also to systems of ordinary differential equations of higher order than one.

Second Generalization

- Suppose the functions $\frac{dx^i}{dt} = f^i$ depend on parameters $z^1, \ldots, z^m$.
- So the system becomes

$$\frac{dx^i}{dt} = f^i(t, x^1, \ldots, x^n, z^1, \ldots, z^m), \quad i = 1, \ldots, n.$$

- Assume that the functions f^i are of class C^r in the z's also, on some open set $V' \subseteq \mathbb{R}^m$.
- That is, f^i is a function of class C^r on

$$I_\varepsilon \times U \times U' \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m.$$

Second Generalization (Cont'd)

- Then the solutions will depend on the z's as well as on the initial conditions,
$$x^i = x^i(t, a^1, \ldots, a^n, z^1, \ldots, z^m).$$
- It is a further consequence of the theorem that these functions are of class C^r in all variables on an open set
$$I_\varepsilon \times V \times V' \subseteq \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m.$$
- This is very easily proved by introducing new equations of the form
$$\frac{dz^j}{dt} = 0, \quad j = 1, \ldots, m.$$
- In this way, we are dealing with a system of $n + m$ ordinary equations to which we apply the theorem.

Application

- Choose a basis
$$E_1, \ldots, E_n$$
of the tangent space at the identity e of a Lie group G.
- Consider the uniquely determined left-invariant vector field X whose value X_e at e has components $z^1, \ldots, z^n$,
$$X_e = \sum_{i=1}^{n} z^i E_i.$$
- Let $X_g(z^1, \ldots, z^n)$ denote the value at $g \in G$ of X.

Application (Cont'd)

- With the choice of basis fixed, the left-invariant vector fields on G are, thus, parameterized by $\mathbb{R}^n$.
- The dependence on g and on the parameters is C^∞.
- So the solutions of the system of equations corresponding to each of the vector fields $X(z^1, \ldots, z^n)$ is C^∞ in all variables.
- Thus, we have $\theta(t; g; z^1, \ldots, z^n)$, which gives a C^∞ mapping

$$\theta : R \times G \times \mathbb{R}^n \to G.$$

- For g, z fixed, θ determines the integral curve through g.

Subsection 5

Examples of One-Parameter Groups Acting on a Manifold

The Setup

- We consider a local one-parameter group θ with (maximal) domain W and infinitesimal generator X acting on a manifold M.
- For $p \in M$, we denote by $I(p)$ the set $\alpha(p) < t < \beta(p)$ of all real numbers t such that (t, p) is in W.
- The integral curve of X through p is given by

$$\theta_p : I(p) \to M, \quad \theta_p(t) = \theta(t, p).$$

- If $X_p = 0$, the curve is a single point p.
- Otherwise θ_p is an immersion, as was shown earlier.
- In the latter case, we consider the nature of the integral curves on M.

Sequence Converging to Endpoints

Lemma

Suppose that $\beta(p) < \infty$ and that $\{t_n\} \subseteq I(p)$ is an increasing sequence converging to $\beta(p)$. Then $\{\theta(t_n, p)\}$ cannot lie in any compact set.
In particular, the sequence $\{\theta(t_n, p)\}$ cannot approach a limit on M.
A similar statement holds for a decreasing sequence approaching $\alpha(p)$ if $\alpha(p)$ is finite.

- Let K be a compact subset of M.

  Let X be a C^∞-vector field on M.

  By the Existence Theorem, to each $q \in M$ corresponds a $\delta > 0$ and a neighborhood V of q, such that θ is defined on $I_\delta \times V$.

  A finite number of such neighborhoods cover K.

  We let δ_0 be the minimum δ for these neighborhoods.

  Then for each $q \in K$, $\theta(t, q)$ is defined for $|t| < \delta_0$.

Sequence Converging to Endpoints (Cont'd)

- Suppose $\{\theta(t_n, p)\} \subseteq K$.
  Take N is so large that $\beta(p) - t_N < \frac{1}{3}\delta_0$.
  Then we see that

$$\theta(t_N + t, p) = \theta(t, \theta(t_N, p)).$$

  The right side is defined for all t with $|t| < \delta_0$, since $\theta(t_N, p) \in K$.
  The left side is also defined for such t, e.g., for $t_N + \frac{2}{3}\delta_0 > \beta(p)$.
  This contradicts a previous corollary and proves the first statement.
  For the second, suppose $\lim_{n\to\infty} \theta(t_n, p) = q$.
  Then there is a neighborhood of q whose closure K is compact and contains all but a finite number of terms of the sequence $\{\theta(t_n, p)\}$.
  We discard the terms not in K and obtain the same contradiction.
  Obviously the same arguments apply to decreasing sequences approaching $\alpha(p)$, if $\alpha(p)$ is finite.

Bounded Interval and Constant Infinitesimal Generator

Corollary

If $I(p)$ is a bounded interval, then the integral curve is a closed subset of M.

Corollary

If $X_p = 0$, then $I(p) = \mathbb{R}$.

- We skip the proofs.

Singular and Regular Points of a Vector Field

- A point p of M at which $X_p = 0$ is called a **singular point** of the vector field.
- Any other point is referred to as **regular**.
- We have seen that in the neighborhood of a regular point the integral curves are - to within diffeomorphism - the family of parallel lines

$$x^2 = c^2, \ldots, x^n = c^n$$

  in $\mathbb{R}^n$.
- On the other hand the pattern of integral curves at an isolated singularity can take many forms, even in the two-dimensional case.
- These patterns have been extensively studied.
- At least in the two-dimensional case singularities can be visualized in terms of the integral curves of the vector field X near p.

Complete Vector Fields

Definition

A vector field X on a manifold M is said to be **complete** if it generates a (global) action of R on M, that is, if $W = R \times M$.

- This is clearly the most desirable case.
- So it is very convenient to have sufficient conditions for completeness.

Corollary

If M is a compact manifold, then every vector field X on M is complete.

- To see that this is so, we take $K = M$ in the lemma.

  Note that, in this case, $\alpha(p) = -\infty$ and $\beta(p) = +\infty$.

  That is $I(p) = R$, for every $p \in M$.

Diffeomorphisms and Invariance

Theorem

Let X be a C^∞-vector field on a manifold M.
Let $F : M \to M$ be a diffeomorphism.
Let $\theta(t, p)$ denote the C^∞ map $\theta : W \to M$ defined by X.
Then X is invariant under F if and only if

$$F(\theta(t, p)) = \theta(t, F(p)),$$

whenever both sides are defined.

- Suppose that X is invariant under F.
  Let $\theta_p : I(p) \to M$ be the integral curve of X with $\theta_p(0) = p$.
  F takes it to an integral curve $F(\theta_p(t))$ of the vector field $F_*(X)$.
  Now $F_*(X) = X$ and $F(\theta_p(0)) = F(p)$.
  By uniqueness of integral curves, we get $F(\theta_p(t)) = \theta(t, F(p))$.
  This proves the "only if" part of the theorem.

Diffeomorphisms and Invariance (Cont'd)

- Suppose, conversely, that $F(\theta(t,p)) = \theta(t, F(p))$.

  We must show that $F_*(X_p) = X_{F(p)}$.

  This could be done directly from the expression for the infinitesimal generator X, but we proceed in a slightly different way.

  Let
  $$\theta_p(t) = \theta(t,p).$$
  Let $\frac{d}{dt}$ be the natural basis of $T_0(R)$, the tangent space to R at $t=0$.

  Then, by definition,
  $$X_p = \dot{\theta}_p(0) = \theta_{p*}\left(\frac{d}{dt}\right).$$

Diffeomorphisms and Invariance (Cont'd)

- Applying the isomorphism $F_* : T_p(M) \to T_{F(p)}(M)$ to this definition,

$$\begin{array}{rcl}
F_*(X_p) & = & F_* \circ \theta_{p*}\left(\frac{d}{dt}\right) \\
& = & (F \circ \theta_p)_*\left(\frac{d}{dt}\right) \\
& & \text{(chain rule applied to } \theta_R \text{ and } F) \\
& = & \theta_{F(p)*}\left(\frac{d}{dt}\right) \\
& & \text{(by hypothesis, } F \circ \theta_p(t) = \theta_{F(P)}(t)) \\
& = & X_{F(p)}.
\end{array}$$

Completeness of Left-Invariant Vector Field

Corollary

A left-invariant vector field on a Lie group G is complete.

- Let X be such a vector field. Then, there is a neighborhood V of e and a $\delta > 0$ such that $\theta(t, g)$ is defined on $I_\delta \times V$.

  For $h \in G$, let L_h denote the left translation by h.

  By the theorem, with $F = L_h$, we get $\theta(t, L_h g) = L_h \theta(t, g)$.

  So θ is defined on $I_\delta \times L_h(V)$, a neighborhood of $(0, h)$ in $R \times G$.

  Thus, for every $h \in G$, there is a neighborhood $U = L_h(V)$, such that $I_\delta \times U \subseteq W$, the domain of θ with the same $\delta > 0$ as obtained for V.

  Hence, δ is fixed and independent of h.

  As in the compact case, we obtain a contradiction if we assume for any $g \in M$ that either $\alpha(g)$ or $\beta(g)$ is finite.

  Therefore, $W = R \times M$ and X is complete.

One-Parameter Subgroups of Lie Groups

Definition

Let R be the additive group of real numbers, considered as a Lie group.
Let G be an arbitrary Lie group.
A **one-parameter subgroup** H of G is the homomorphic image

$$H = F(R)$$

of a homomorphism $F : R \to G$.

Comments

- Let G be a Lie group which acts on a manifold M by
$$\theta : G \times M \to M.$$
- Let $F : R \to G$ be a homomorphism.
- Then $\theta : R \times M \to M$ defined by
$$\theta(t, p) = \theta(F(t), p)$$
defines an action of R on M.
- Applying our theory, we have an associated infinitesimal generator X, integral curves as orbits of the action, and so on.
- The same G may act on different manifolds, or in different ways on the same manifold.
- Consequently, a fixed one-parameter subgroup of G will give many examples of a one-parameter group of transformations of a manifold.

Example

- Let G be the group $Gl(3,\mathbb{R})$.
- We consider two one-parameter subgroups, that is, two homomorphisms F_1, F_2 of $\mathbb{R}$ into G, defined as follows ($a, b, c \in \mathbb{R}$ are constants):

$$F_1(t) = \begin{pmatrix} e^{at} & 0 & 0 \\ 0 & e^{at} & 0 \\ 0 & 0 & e^{at} \end{pmatrix}, \quad F_2(t) = \begin{pmatrix} 1 & at & bt + \frac{1}{2}act^2 \\ 0 & 1 & ct \\ 0 & 0 & 1 \end{pmatrix}.$$

- We can check that these are indeed homomorphisms.
- Now $Gl(3,\mathbb{R})$ acts naturally on $\mathbb{R}^3$.
- Hence each F_i defines an action on $\mathbb{R}^3$.

Example (Cont'd)

- In the case of F_1, we have

$$\theta(t, x^1, x^2, x^3) = (e^{at}x^1, e^{at}x^2, e^{at}x^3).$$

- Therefore the infinitesimal generator X is given at $x \in \mathbb{R}^3$ by

$$X_x = \dot{\theta}(0, x) = ax^1 \frac{\partial}{\partial x^1} + ax^2 \frac{\partial}{\partial x^2} + ax^3 \frac{\partial}{\partial x^3}.$$

- The integral curves, or orbits, are the lines through the origin.
- The group $Gl(n, \mathbb{R})$ also acts on $P^{n-1}(\mathbb{R})$, since it preserves the equivalence relation (proportionality) of n-tuples which defines it.
- Therefore $Gl(3, \mathbb{R})$ acts on two-dimensional projective space $P^2(\mathbb{R})$.
- In this case F_1 defines a trivial action $\theta(t, p) \equiv p$.

Example

- Let G be the Lie group $SO(3)$ of orthogonal matrices with determinant $+1$.
- Define $F : R \to SO(3)$ and, thus, a one-parameter subgroup by

$$F(t) = \begin{pmatrix} \cos at & \sin at & 0 \\ -\sin at & \cos at & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

- It can be checked that this is in fact a homomorphism.
- Thus, $SO(3)$ acts on the unit sphere S^2 in a standard manner.

Example (Cont'd)

- The action is just the usual rotation of the sphere.
- F defines a one-parameter group of rotations holding the x^3 axis fixed:

$$\theta(t, x^1, x^2, x^3) = (x^1 \cos at + x^2 \sin at, -x^1 \sin at + x^2 \cos at, x^3).$$

- The orbits are the lines of latitude.
- The generator X is tangent to them and orthogonal to the x^3-axis.
- $X = 0$ at the north and south poles $(0, 0, \pm 1)$.

Example

- We recall also that a Lie group G acts on itself (on the right) by right translations.
- Thus if we are given a homomorphism $F : R \to G$, we may define an action θ of R on $M = G$ by

$$\theta(t, g) = R_{F(t)}(g) = gF(t).$$

- We have used R_a to denote right translation: $R_a(g) = ga$.
- As previously noted, this is a composition of the C^∞ maps F and right translation.
- It is an action, since F is a homomorphism and multiplication is associative:
  - (i) $\theta(0, g) = gF(0) = g$;
  - (ii) $\theta(t + s, g) = gF(t + s) = g(F(t)F(s)) = (gF(t))F(s) = \theta(t, \theta(s, g))$.

Characterization of One-Parameter Subgroups

- Recall that a left-invariant vector field on G is uniquely determined by its value at the identity e.

Theorem

Let $F : R \to G$ be a one-parameter subgroup of the Lie group G.
Let X be the left-invariant vector field on G defined by $X_e = \dot{F}(0)$.
Then

$$\theta(t, g) = R_{F(t)}(g)$$

defines an action $\theta : R \times G \to G$ of R on G (as a manifold) having X as infinitesimal generator.
Conversely, let X be a left-invariant vector field.
Let $\theta : R \times G \to G$ the corresponding action.
Then $F(t) = \theta(t, e)$ is a one- parameter subgroup of G and

$$\theta(t, g) = R_{F(t)}(g).$$

Characterization (Cont'd)

- Consider the C^∞ homomorphism $F : R \to G$.
  $\theta : R \times G \to G$, defined by
  $$\theta(t, g) = R_{F(t)}(g) = gF(t)$$
  is an action of R on G.
  If $a \in G$, then
  $$L_a\theta(t, g) = a(gF(t)) = (ag)F(t) = \theta(t, L_a(g)).$$
  By a previous theorem, the generator X of θ is L_a-invariant, for any $a \in G$.
  However, $\theta(t, e) = F(t)$.
  So
  $$X_e = \dot{\theta}(0, e) = \dot{F}(0).$$

Characterization (Cont'd)

- For the converse X, being left-invariant, is both C^∞ and complete and it generates an action θ of R on G.
  By a previous theorem, for any left translation L_h,
  $$L_h\theta(t,g) = \theta(t, L_h(g)).$$
  Equivalently, $h\theta(t,g) = \theta(t,hg)$.
  Let $F(t) = \theta(t,e)$ and $h = F(s)$.
  Then this relation implies
  $$F(s)F(t) = F(s)\theta(t,e) = \theta(t,\theta(s,e)) = \theta(t+s,e) = F(s+t).$$
  Thus, $t \to F(t)$ is a C^∞ homomorphism.
  But $\dot{F}(0) = \dot{\theta}(0,e) = X_e$. Moreover, X is left-invariant.
  So, by uniqueness of the action generated by X,
  $$\theta(t,g) = R_{F(t)}(g),$$
  the action defined just previously.

Tangent Vectors and One-Parameter Subgroups

Corollary

There is a one-to-one correspondence between the elements of $T_e(G)$ and one-parameter subgroups of G. For $Z \in T_e(G)$, let

$$t \to F(t, Z)$$

denote the (unique) corresponding one-parameter subgroup.
Then $F : R \times T_e(G) \to G$ is C^∞ and satisfies

$$F(t, sZ) = F(st, Z).$$

Tangent Vectors and One-Parameter Subgroups

- According to the theorem, each $Z \in T_e(G)$ determines a unique homomorphism $t \to F(t, Z)$ of R into G, such that
$$\dot{F}(0, Z) = Z.$$
Identify $T_e(G)$ with $\mathbb{R}^n$ via some choice of basis.
By our extension of the Existence Theorem, F is C^∞ simultaneously in t and Z.
Using the rule for change of parameter in a curve on a manifold,
$$\left[\frac{d}{dt}F(ts, Z)\right]_{t=0} = s\left[\frac{d}{dt}F(t, Z)\right]_{t=0} = sZ.$$
On the other hand, $t \to F(ts, Z)$ is a homomorphism.
Therefore, by uniqueness,
$$F(st, Z) = F(t, sZ).$$

Subsection 6

One-Parameter Subgroups of Lie Groups

One-Parameter Subgroups of $Gl(n, \mathbb{R})$

- We have seen that one-parameter subgroups of a Lie group G are in one-to-one correspondence with the elements of $T_e(G)$.
- We shall use this to help determine all one-parameter subgroups of various matrix groups.
- We first consider $G = Gl(n, \mathbb{R})$.
- The matrix entries x_{ij}, $1 \leq i, j \leq n$, for any $X = (x_{ij}) \in Gl(n, \mathbb{R})$ are coordinates on a single neighborhood covering the group, which is an open subset of $\mathcal{M}_n(\mathbb{R})$, the $n \times n$ matrices over $\mathbb{R}$.
- Therefore $\frac{\partial}{\partial x_{ij}}$, $1 \leq i, j \leq n$, is a field of frames on G.
- Relative to these frames as a basis at $e = I$ (the identity $n \times n$ matrix), there an isomorphism of $\mathcal{M}_n(\mathbb{R})$ as a vector space onto $T_e(G)$ given by
$$A = (a_{ij}) \to \sum_{i,j} a_{ij} \left(\frac{\partial}{\partial x_{ij}} \right)_e .$$

Exponential of a Matrix

Definition

The **exponential** e^X **of a matrix** $X \in \mathcal{M}_n(\mathbb{R})$ is defined to be the matrix given by

$$e^X = I + X + \frac{1}{2!}X^2 + \frac{1}{3!}X^3 + \cdots$$

if the series converges.

Properties of the Exponential of a Matrix

Theorem

The exponential series converges absolutely, for all $X \in \mathcal{M}_n(\mathbb{R})$, and uniformly on compact subsets.

The mapping $\mathcal{M}_n(\mathbb{R}) \to \mathcal{M}_n(\mathbb{R})$ defined by

$$X \to e^X$$

is C^∞.

Its has nonsingular Jacobian at $X = 0$.

Its image lies in $Gl(n, \mathbb{R})$.

If $A, B \in \mathcal{M}_n(\mathbb{R})$ such that $AB = BA$, then

$$e^{A+B} = e^A e^B.$$

Proof of the Theorem

- Denote by $x_{ij}^{(k)}$ the entries of the matrix X^k, with

$$X^1 = X = (x_{ij}) \quad \text{and} \quad X^0 = I = (\delta_{ij}).$$

Let

$$\rho = \sup_{1 \leq i,j \leq n} |x_{ij}|.$$

By induction on k, we have the inequality

$$|x_{ij}^{(k)}| \leq (n\rho)^k.$$

This is true for $k = 0$. Suppose it holds for k.

Then

$$|x_{ij}^{(k+1)}| = \left| \sum_{\ell} x_{i\ell}^{(k)} x_{\ell j} \right| \leq n(n\rho)^k \rho = (n\rho)^{k+1}.$$

So the sequence e^X converges absolutely for every X.

Proof (Cont'd)

- It also converges uniformly on every compact subset of $\mathcal{M}_n(\mathbb{R})$.

  Indeed each compact set is contained in a set $K_\rho = \{X : |x_{ij}| \leq \rho\}$.

  Consider the mapping $X \to e^X$.

  The entries of the partial sums are polynomials in x_{ij}.

  So, by uniformity of convergence, the mapping is C^∞ (even analytic) in the x_{ij}.

  Denote by $f_{ij}(X)$ the coordinate functions of the mapping.

  Then the terms of degree less than 2 in the variables x_{ij} are

$$f_{ij}(X) = \delta_{ij} + x_{ij}, \quad 1 \leq i, j \leq n.$$

  Hence the Jacobian at $X = 0$ reduces to the $n^2 \times n^2$ identity matrix.

Proof (Cont'd)

- We know the convergence is absolute.

  So we may rearrange terms.

  Moreover, an analog of Cauchy's Theorem for multiplication of series also holds for matrices.

  So, when $AB = BA$, we obtain the equality

$$\begin{aligned}(\textstyle\sum_{k=0}^{\infty}\frac{1}{k!}A^k)(\sum_{\ell=0}^{\infty}B^{\ell}) &= \textstyle\sum_{m=0}^{\infty}\sum_{p=0}^{m}\frac{1}{(m-p)!}A^{m-p}\frac{1}{p!}B^p \\ &= \textstyle\sum_m \frac{1}{m!}(A+B)^m.\end{aligned}$$

  From this we may deduce $e^A e^B = e^{A+B}$.

  In particular, this implies $e^A e^{-A} = e^0 = I$.

  Hence, e^A is nonsingular.

  It follows that $e^A \in Gl(n, \mathbb{R})$, for any $A \in \mathcal{M}_n(\mathbb{R})$.

Exponential One-Parameter Subgroup

Corollary

$t \to e^{tA}$ is the one-parameter subgroup of $Gl(n, \mathbb{R})$ whose corresponding left-invariant vector field has the value $\sum_{i,j} a_{ij}(\frac{\partial}{\partial x_{ij}})_e$. Moreover, all one-parameter subgroups of $Gl(n, \mathbb{R})$ are of this form.

- For every $t \in \mathbb{R}$, $t_1 A$ and $t_2 A$ commute. Thus

$$e^{(t_1+t_2)A} = e^{t_1 A} e^{t_2 A}.$$

  So $t \to e^{tA}$ is a group homomorphism.

  It is C^∞ since it is a restriction of a C^∞-map on $\mathcal{M}_n(\mathbb{R})$ to the submanifold $\{tA : t \in \mathbb{R}\}$.

  Write $x_{ij}(t)$ for the ij-th entry of e^{tA}.

  Letting $A = (a_{ij})$, we have $x_{ij}(t) = \delta_{ij} + ta_{ij} + O(t^2)$.

  So $\dot{x}_{ij}(0) = a_{ij}$, $1 \leq i, j \leq n$. Equivalently $(\frac{de^{tA}}{dt})_{t=0} = A$.

Example

- Consider
$$A = \begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix} \in \mathcal{M}_3(\mathbb{R}).$$
- We have
$$e^{tA} = I + tA + \frac{1}{2}t^2A^2 + \cdots .$$
- However, $A^k = 0$ if $k > 2$.
- So we obtain once again
$$e^{tA} = \begin{pmatrix} 1 & ta & tb + \frac{1}{2}t^2ac \\ 0 & 1 & tc \\ 0 & 0 & 1 \end{pmatrix}.$$

Characterization of One-Parameter Subgroups

Theorem

Let H be a Lie subgroup of G. Then the one-parameter subgroups of H are exactly those one-parameter subgroups $t \to F(t)$ of G, such that

$$\dot{F}(0) \in T_e(H),$$

considered as a subspace of $T_e(G)$.

- Let $F : R \to H$ be any one-parameter subgroup of H.

  The inclusion $H \subseteq G$ is an immersion, and so C^∞.

  So F, followed by inclusion, is a one-parameter subgroup of G.

  Its tangent vector at any point is tangent to H.

  In particular, $\dot{F}(0) \in T_e(H)$ a subspace of $T_e(G)$.

Characterization of One-Parameter Subgroups (Cont'd)

- Conversely, let $F : R \to G$ is a one-parameter subgroup, such that

$$\dot{F}(0) \in T_e(H).$$

Then $\dot{F}(0)$ determines a one-parameter subgroup of H, $F_1 : R \to H$, with

$$\dot{F}_1(0) = \dot{F}(0).$$

As just seen, F_1 can be considered a one-parameter subgroup of G.

But F and F_1 have the same tangent vector at e.

So they must agree.

Therefore, the correspondence is one-to-one, as claimed.

One-Parameter Subgroups of $Gl(n, \mathbb{R})$

- Suppose that $G = Gl(n, \mathbb{R})$ in the preceding discussion.

  Then we have the following application.

Corollary

The one-parameter subgroups of a subgroup $H \subseteq Gl(n, \mathbb{R})$ are all of the form $t \to e^{tA}$, where $A = (a_{ij})$ are the components of a vector

$$\sum_{i,j} \left(\frac{\partial}{\partial x_{ij}}\right)_e \in T_e(G),$$

which is tangent to H at e, that is, is in $T_e(H) \subseteq T_e(G)$.

- This is an immediate consequence of the theorem and the fact that every one-parameter subgroup of $G = Gl(n, \mathbb{R})$ is of the form $F(t) = e^{tA}$.

Example

- Let $H = O(n)$ and $G = Gl(n, \mathbb{R})$.
- We determine the one-parameter subgroups of H.
- Suppose $e^{tA} \in H$ for all t.
- Then $(e^{tA})(e^{tA})' = I$, where the prime indicates the transpose.
- By the definition, $(e^{tA})' = e^{tA'}$.
- By a previous theorem, $(e^{tA})^{-1} = e^{-tA}$.
- We conclude that $e^{tA} \in H$ implies $e^{tA'} = e^{-tA}$.
- Now, $X \to e^X$ maps the (linear) submanifold of $\mathcal{M}_n(\mathbb{R})$ of skew symmetric matrices to the submanifold $O(n)$ of G.
- Both manifolds have the same dimension.
- The Jacobian is nonsingular at $X = 0$, by a previous theorem.
- Hence, on some neighborhood of the 0 matrix, $X \to e^X$ is a diffeomorphism.
- Therefore, there is a δ such that if $|t| < \delta$, then $tA' = -tA$.
- It follows that A is skew symmetric.

Example (Cont'd)

- Conversely, suppose $A' = -A$.
- Then
$$e^{tA}(e^{tA})' = e^{tA}e^{tA'} = e^{tA}e^{-tA} = I.$$
- This means that e^{tA} is an orthogonal matrix.
- We have, therefore, proven the following:

  The homomorphism $t \to e^{tA}$ is a one-parameter subgroup of $O(n)$ if and only if
$$A' = -A.$$
  This is the necessary and sufficient condition on $A = (a_{ij})$ in order that the tangent vector
$$\sum_{i,j} \left(\frac{\partial}{\partial x_{ij}}\right)_e$$
  to $Gl(n, \mathbb{R})$ at the identity be tangent to the subgroup $O(n)$.

Generalization of the Exponential Mapping

- Recall that, if G is a Lie group and $Z \in T_e(G)$, then Z determines uniquely a one-parameter subgroup, denoted earlier by $F(t, Z)$.
- We now define an exponential mapping on an arbitrary Lie group.

Definition

The **exponential mapping** $\exp : T_e(G) \to G$ is defined by the formula

$$\exp Z = F(1, Z).$$

Generalization of the Exponential Mapping (Cont'd)

- According to a previous theorem, we have the following properties.

Theorem

For any Lie group G the mapping $\exp : T_e(G) \to G$ is C^∞ and

$$F(t) = \exp tZ$$

is the unique one-parameter subgroup such that $\dot{F}(0) = Z$.
The Jacobian matrix at 0 of exp is the identity.
That is, at e, $\exp_*$ is the identity.
Finally, if G is a subgroup of $Gl(n, \mathbb{R})$, then for each $Z \in T_e(G)$, there is an $A = (a_{ij}) \in \mathcal{M}_n(\mathbb{R})$ such that

$$Z = \sum a_{ij} \left(\frac{\partial}{\partial x_{ij}} \right)_e .$$

For this Z, $\exp tZ = e^{tA}$.

Subsection 7

The Lie Algebra of Vector Field on a Manifold

Space of Vector Fields on a Manifold

- We denote by $\mathfrak{X}(M)$ the set of all C^∞-vector fields defined on the C^∞ manifold M.
- Suppose X and Y are C^∞-vector fields on M.
- Then so is any linear combination of them with constant coefficients.
- So $\mathfrak{X}(M)$ is itself a vector space over $\mathbb{R}$.
- More generally, any linear combination with coefficients which are C^∞ functions on M is again a C^∞-vector field.
- Let $X, Y \in \mathfrak{X}(M)$ and $f, g \in C^\infty(M)$.
- The vector field $Z = fX + gY$, with the obvious definition
$$Z_p = f(p)X_p + g(p)Y_p, \quad \text{for each } p \in M,$$
is a C^∞-vector field.
- Hence, $\mathfrak{X}(M)$ is a vector space over $\mathbb{R}$ and a module over $C^\infty(M)$.
- As a vector space $\mathfrak{X}(M)$ is not finite-dimensional over $\mathbb{R}$.

Lie Algebras

Definition

We shall say that a vector space $\mathcal{L}$ over $\mathbb{R}$ is a (**real**) **Lie algebra** if, in addition to its vector space structure, it possesses a product, that is, a map $\mathcal{L} \times \mathcal{L} \to \mathcal{L}$, taking the pair (X, Y) to the element $[X, Y]$ of $\mathcal{L}$, which has the following properties:

(1) It is *bilinear over* $\mathbb{R}$:

$$\begin{aligned} [\alpha_1 X_1 + \alpha_2 X_2, Y] &= \alpha_1 [X_1, Y] + \alpha_2 [X_2, Y], \\ [X, \alpha_1 Y_1 + \alpha_2 Y_2] &= \alpha_1 [X, Y_1] + \alpha_2 [X, Y_2]; \end{aligned}$$

(2) It is *skew commutative*: $[Y, X] = -[X, Y]$;

(3) It satisfies the **Jacobi identity**:

$$[X, [Y, Z]] + [Y, [Z, X]] + [Z, [X, Y]] = 0.$$

Examples

- A vector space $\boldsymbol{V}^3$, of dimension 3 over $\mathbb{R}$, with the usual vector product of vector calculus, is a Lie algebra.
- Let $\mathcal{M}_n(\mathbb{R})$ denote the algebra of $n \times n$ matrices over $\mathbb{R}$, with XY denoting the usual matrix product of X and Y.

  Let the product $[X, Y]$ be defined as the "commutator" of X and Y,

  $$[X, Y] = XY - YX.$$

  This defines a Lie algebra structure on $\mathcal{M}_n(\mathbb{R})$.

A Product on $\mathfrak{X}(M)$

- Now suppose that X and Y denote C^∞-vector fields on a manifold M, that is, $X, Y \in \mathfrak{X}(M)$.
- Let f be a C^∞ function on a neighborhood of p.
- Let $f \to X_p(Yf)$ be the operator defined on $C^\infty(p)$.
- In general, $f \to X_p(Yf)$ does not define a vector at p.
- Thus XY, considered as an operator on C^∞ functions on M, does not in general determine a C^∞-vector field.
- However, $XY - YX$ defines a vector field $Z \in \mathfrak{X}(M)$ according to the prescription

$$Z_p f = (XY - YX)_p f = X_p(Yf) - Y_p(Xf), \quad f \in C^\infty(p).$$

- If $f \in C^\infty(p)$, then Xf and Yf are C^∞ on a neighborhood of p.
- So the formula determines a linear map $Z_p : C^\infty(p) \to \mathbb{R}$.

The Leibniz Rule for Z_p

- It follows that, if the Leibniz rule holds for Z_p, then Z_p is an element of $T_p(M)$ at each $p \in M$.

  Consider $f, g \in C^\infty(p)$.

  Then $f, g \in C^\infty(U)$, for some open set U containing p.

  We have the relations

$$\begin{aligned}
(XY - YX)_p(fg) &= X_p(Yfg) - Y_p(Xfg) \\
&= X_p(fYg - gYf) - Y_p(fXg - gXf) \\
&= (X_pf)(Yg)_p + f(p)X_p(Yg) - (X_pg)(Yf)_p \\
&\quad - g(p)X_p(Yf) - (Y_pf)(Xg)_p - f(p)Y_p(Xg) \\
&\quad + (Y_pg)(Xf)_p + g(p)Y_p(Xf).
\end{aligned}$$

The Leibniz Rule for Z_p (Cont'd)

- So we get

$$\begin{aligned} Z_p(fg) &= (XY - YX)_p(fg) \\ &= f(p)X_p(Yg) - f(p)Y_p(Xg) \\ &\quad - g(p)X_p(Yf) + g(p)Y_p(Xf) \\ &= f(p)(XY - YX)_p g - g(p)(XY - YX)_p f \\ &= f(p)Z_p g + g(p)Z_p f. \end{aligned}$$

Finally, if f is C^∞ on any open set $U \subseteq M$, then so is $(XY - YX)f$. Therefore, Z is a C^∞-vector field on M as claimed.

Lie Algebra Structure on $\mathfrak{X}(M)$

- We may now define on $\mathfrak{X}(M)$ the product of X and Y by

$$[X, Y] = XY - YX.$$

Theorem

$\mathfrak{X}(M)$ with the product $[X, Y]$ is a Lie algebra.

- Let $\alpha, \beta \in \mathbb{R}$ and X_1, X_2, Y be C^∞-vector fields.

  Then we can verify that

$$[\alpha X_1 + \beta X_1, Y]f = \alpha[X_1, Y]f + \beta[X_2, Y]f.$$

  Thus, $[X, Y]$ is linear in the first variable.

  Skew commutativity is immediate from the definition.

  So linearity in the first variable implies linearity in the second.

  Therefore, $[X, Y]$ is bilinear and skew-commutative.

Lie Algebra Structure on $\mathfrak{X}(M)$ (The Jacobi Identity)

- There remains the Jacobi identity which follows immediately if we evaluate $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]$ applied to a C^∞-function f.

  Using the definition, we obtain

$$\begin{array}{rcl}[X,[Y,Z]]f & = & X(([Y,Z])f)-[Y,Z](Xf) \\ & = & X(Y(Zf))-X(Z(Yf)) \\ & & \quad -Y(Z(Xf))+Z(Y(Xf)).\end{array}$$

  Permuting cyclically, we get

$$\begin{array}{rcl}[Y,[Z,X]]f & = & Y(Z(Xf))-Y(X(Zf)) \\ & & \quad -Z(X(Yf))+X(Z(Yf)); \\ [Z,[X,Y]]f & = & Z(X(Yf))-Z(Y(Xf)) \\ & & \quad -X(Y(Zf))+Y(X(Zf)).\end{array}$$

  Adding these establishes the identity.

Rate of Change of Y in the Direction of X

- Let X be a vector field on M.
- There is an associated one-parameter group $\theta : W \to M$ generated by X.
- By a previous theorem, for each $t \in R$, $\theta_t : V_t \to V_{-t}$ is a diffeomorphism (with inverse θ_{-t}) of the open set V_t, provided V_t is not empty.
- In particular, for each $p \in M$, there is a neighborhood V and a $\delta > 0$, such that
$$V \subseteq V_t, \quad \text{for } |t| < \delta.$$
- The isomorphism
$$\theta_{t*} : T_p(M) \to T_{\theta_t(p)}(M)$$
and its inverse allow us to compare the values of vector fields at these two points.

Rate of Change of Y in the Direction of X (Cont'd)

- Indeed, suppose Y is a second C^∞-vector field on M.
- We may use this idea to compute for each p the rate of change of Y in the direction of X.
- This is the rate of change of Y along the integral curve of the vector field X passing through p.
- We denote this rate of change by $L_X Y$.
- It is itself a C^∞-vector field.

The Lie Derivative

Definition

The vector field $L_X Y$, called the **Lie derivative of X with respect to Y** is defined at each $p \in M$ by either of the following limits:

$$\begin{aligned}(L_X Y)_p &= \lim_{t\to 0}\frac{1}{t}[\theta_{-t*}(Y_{\theta(t,p)}) - Y_p] \\ &= \lim_{t\to 0}\frac{1}{t}[Y_p - \theta_{t*}(Y_{\theta(-t,p)})].\end{aligned}$$

- The second definition is obtained from the first by replacing t by $-t$.

The Lie Derivative (Cont'd)

- We interpret the first expression as follows.
- Apply to $Y_{\theta(t,p)} \in T_{\theta(t,p)}(M)$ the isomorphism θ_{-t*}, taking $T_{\theta(t,p)}(M)$ to $T_p(M)$.
- Then in $T_p(M)$:
  - Take the difference of this vector and Y_p;
  - Multiply by the scalar $\frac{1}{t}$.
  - Pass to the limit as $t \to 0$.
- This limit is a vector $(L_X Y)_p \in T_p(M)$, if it exists at all.
- The existence and the fact that the vector field so defined is C^∞ may be verified by writing the formula above in local coordinates.

A Technical Lemma

Lemma

Let X be a C^∞-vector field on M.
Let θ be the corresponding map of $W \subseteq R \times M$ onto M.
Let $p \in M$ and $f \in C^\infty(U)$, where U an open set containing p.
Choose $\delta > 0$ and a neighborhood V of p in U, such that $\theta(I_\delta \times V) \subseteq U$.
Then there is a C^∞ function $g(q, t)$ defined on $V \times I_\delta$, such that, for $q \in V$ and $t \in I_\delta$, we have

$$f(\theta_t(q)) = f(q) + tg(q, t) \quad \text{and} \quad X_q f = g(q, 0).$$

- By a previous theorem, there is a neighborhood V of p and a $\delta > 0$, such that:
  - $\theta_t(p) = \theta(t, p)$ is defined and C^∞ on $I_\delta \times V$;
  - θ maps $I_\delta \times V$ into U.

A Technical Lemma (Cont'd)

- The function

$$r(t,q) = f(\theta_t(q)) - f(q)$$

is C^∞ on $I_\delta \times V$ and $r(0,q) = 0$.

We denote by $\dot{r}(t,q)$ its derivative with respect to t.

We define $g(q,t)$ - for each fixed q - by the formula

$$g(q,t) = \int_0^1 \dot{r}(ts,q)ds.$$

This function is also C^∞ on $I_\delta \times V$.

This can be verified by use of local coordinates and properties of the integral.

A Technical Lemma (Cont'd)

By the Fundamental Theorem of Calculus,

$$tg(q,t) = \int_0^1 \dot{r}(ts,q)tds = r(t,q) - r(0,q) = r(t,q).$$

Using the definition of r, this becomes

$$f(\theta_t(q)) = f(q) + tg(q,t).$$

Now, by the definition of the infinitesimal generator of θ,

$$\begin{aligned} g(q,0) &= \lim_{t\to 0} g(q,t) \\ &= \lim_{t\to 0} \tfrac{1}{t} r(t,q) \\ &= \lim_{t\to 0} \tfrac{1}{t}[f(\theta_t(q)) - f(q)] \\ &= X_q f. \end{aligned}$$

Characterization of $L_X Y$

Theorem

If X and Y are C^∞-vector fields on M, then $L_X Y = [X, Y]$.

- By definition

$$(L_X Y)_p f = \left(\lim_{t\to 0} \frac{1}{t}[Y_p - \theta_{t*}(Y_{\theta_{-t}(p)})]\right) f.$$

This differential quotient and that of the following expression, whose limit is the derivative of a C^∞ function of t, are equal for all $0 < |t| < \delta$, and, hence, are equal in the limit

$$(L_X Y)_p f = \lim_{t\to 0} \frac{1}{t}[Y_p f - Y_{\theta_{-t}(p)}(f \circ \theta_t)].$$

Using the lemma and denoting $g(q, t)$ by g_t we have

$$(L_X Y)_p f = \lim_{t\to 0} \frac{1}{t}[Y_p f - Y_{\theta_{-t}(p)}(f + t g_t)].$$

Characterization of $L_X Y$ (Cont'd)

- Then, replacing t by $-t$ and rearranging terms, we get

$$(L_X Y)_p = \lim_{t\to 0} \frac{1}{t}[(Yf)(\theta_t(p)) - (Yf)(p)] - \lim_{t\to 0} Y_{\theta_t(p)} g_t.$$

Now, using both the formula of the definition of the infinitesimal generator, with f replaced by Yf and Δt by t, and the fact that $g_0 = g(q,0) = Xf(q)$, we obtain in the limit

$$(L_X Y)_p f = X_p(Yf) - Y_p(Xf) = [X, Y]_p f.$$

This completes the proof of the theorem.

It also shows that $L_X Y$ is C^∞.

F_* and Lie Derivatives

Theorem

Let $F : N \to M$ be a C^∞ mapping and suppose that X_1, X_2 and Y_1, Y_2 are vector fields on N, M, respectively, which are F-related, that is,

$$F_*(X_i) = Y_i, \quad i = 1, 2.$$

Then $[X_1, X_2]$ and $[Y_1, Y_2]$ are F-related, i.e.,

$$F_*[X_1, X_2] = [F_*(X_1), F_*(X_2)].$$

F_* and Lie Derivatives (Cont'd)

- Before proving the theorem we note the following necessary and sufficient condition for X on N and Y on M to be F-related.
  For any g which is C^∞ on some open set $V \subseteq M$,
  $$(Yg) \circ F = X(g \circ F)$$
  on $F^{-1}(V)$.
  We show this is a restatement of the definition of F-related.
  Suppose $q \in F^{-1}(V)$. Then, we have:
  - On the one hand,
    $$F_*(X_q)g = X_q(g \circ F) = X(g \circ F)(q);$$
  - On the other, $Y_{F(q)}g$ is the value of the C^∞ function Yg at $F(q)$. Therefore, $Y_{F(q)}g = ((Yg) \circ F)(q)$.

  Thus, the condition holds if and only if
  $$F_*(X_q) = Y_{F(q)}, \quad \text{for all } q \in M.$$

F_* and Lie Derivatives (Cont'd)

- Returning to the proof, consider $f \in C^\infty(V)$, $V \subseteq M$, so that $Y_1 f$ and $Y_2 f \in C^\infty(V)$ also.

  Apply the formula above, first with $g = Y_2 f$ and then with $g = f$.

  We get

  $$[Y_1(Y_2 f)] \circ F = X_1((Y_2 f) \circ F) = X_1[X_2(f \circ F)].$$

  Interchange the roles of Y_1, Y_2 and X_1, X_2 to get

  $$[Y_2(Y_1 f)] \circ F = X_2[X_1(f \circ F)].$$

  Subtract to obtain

  $$([Y_1, Y_2]f) \circ F = [X_1, X_2](f \circ F).$$

  By the formula above, $[X_1, X_2]$ and $[Y_1, Y_2]$ are F-related.

Lie Algebra of a Lie Group

Corollary

If G is a Lie group, then the left-invariant vector fields on G form a Lie algebra $\mathfrak{g}$ with the product $[X, Y]$ and $\dim \mathfrak{g} = \dim G$. If $F : G_1 \to G_2$ is a homomorphism of Lie groups, $F_* : \mathfrak{g}_1 \to \mathfrak{g}_2$ is a homomorphism of Lie algebras.

- Let $a \in G$, and let X and Y be left-invariant vector fields.
  L_a (left translation) is a diffeomorphism and $L_{a*}X = X$, $L_{a*}Y = Y$.
  Therefore, by the theorem, $L_{a*}[X, Y] = [X, Y]$.
  So $[X, Y]$ is L_a-invariant, for any a.
  Hence, the subspace $\mathfrak{g}$ is closed with respect to $[X, Y]$.
  Now each $X \in \mathfrak{g}$ is uniquely determined by X_e.
  So $X \to X_e$ is an isomorphism of $\mathfrak{g}$ and $T_e(G)$ as vector spaces.
  The last statement follows from a previous corollary and the preceding theorem.

Lie Algebra of a Lie Subgroup

- Let $H \subseteq G$ be a Lie subgroup.
- Then, by the corollary, $i_*(\mathfrak{h})$ is a subalgebra of $\mathfrak{g}$.
- It consists of the elements of $\mathfrak{g}$ tangent to H and its cosets gH.

Commutativity of Actions

Theorem

Let X and Y be complete C^∞-vector fields on a manifold M.
Let θ, σ denote the corresponding actions of R on M.
Then

$$\theta_t \circ \sigma_s = \sigma_s \circ \theta_t, \text{ for all } s, t \in R, \quad \text{if and only if} \quad [X, Y] = 0.$$

- We first suppose that $\theta_t \circ \sigma_s = \sigma_s \circ \theta_t$, for all $s, t \in R$.
  Applying a previous theorem to the diffeomorphism $\theta_t : M \to M$, we see that Y is θ_t-invariant.
  In particular, $\theta_{t*} Y = Y$.
  This implies that

$$[X, Y] = L_X Y = \lim_{\Delta t \to 0} [Y - \theta_{-t*} Y] = 0.$$

Commutativity of Actions (Converse)

- Next assume $[X, Y] = 0$.

  By the previous theorem,

  $$0 = \theta_{t*}[X, Y] = [\theta_{t*}X, \theta_{t*}Y] = [X, \theta_{t*}Y].$$

  So, for any $p \in M$ and any $f \in C^\infty(p)$, we have

  $$0 = (L_X(\theta_{t*}Y))_p f = \lim_{\Delta t \to 0} \frac{1}{\Delta t}[(\theta_{t*}Y)_p f - (\theta_{t-\Delta t*}Y)_p f].$$

  So

  $$\frac{d(\theta_{t*}Y)_p f}{dt} = 0, \quad \text{for every } t.$$

Commutativity of Actions (Converse)

- We got, for every t,
$$\frac{d(\theta_{t*}Y)_p f}{dt} = 0.$$
That is, $(\theta_{t*}Y)_p f$ is constant.
When $t = 0$, this constant function has the value $Y_p f$.
Therefore
$$(\theta_{t*}Y)_p f = Y_p f.$$
Since p and $f \in C^\infty(p)$ were arbitrary, it follows that
$$\theta_{t*}Y = Y.$$
By a previous theorem, we conclude that, for each $t \in R$,
$$\theta_t \circ \sigma_s = \sigma_s \circ \theta_t.$$

Subsection 8

Frobenius' Theorem

Example

- Let
$$F_\alpha(x^1,x^2,x^3;y^1,y^2,p_\ell^k)=0,\quad \alpha=1,\ldots,6,$$
be a system of six partial differential equations involving:
  - Two unknown functions y^1 and y^2 of three variables x^1,x^2,x^3;
  - Their first derivatives $p_\ell^k=\frac{\partial y^k}{\partial x^\ell}$, $k=1,2$, $\ell=1,2,3$.
- To simplify matters, we assume that these equations can be solved for p_ℓ^k and written equivalently
$$\frac{\partial y^k}{\partial x^\ell}=G_\ell^k(x;y),\quad k=1,2,\ \ell=1,2,3,$$
in some neighborhood U of a point $(a;b)=(a^1,a^2,a^3;b^1,b^2)$.

Example (Cont'd)

- Consider again

$$\frac{\partial y^k}{\partial x^\ell} = G^k_\ell(x;y), \quad k=1,2,\ \ell=1,2,3,$$

in some neighborhood U of a point $(a;b)=(a^1,a^2,a^3;b^1,b^2)$.

- A **solution** of the system consists of functions

$$y^k = f^k(x^1,x^2,x^3), \quad k=1,2,$$

such that they satisfy:

  - The system of equations

$$\frac{\partial f^k}{\partial x^\ell} \equiv G^k_\ell(x;f^1(x),f^2(x))$$

in a neighborhood of $x=a$;

  - $f(a)=b$, this last being "initial conditions".

Example (Solutions)

- This is equivalent to defining a three-dimensional submanifold of $\mathbb{R}^5 = \mathbb{R}^3 \times \mathbb{R}^2$ given by

$$(x^1, x^2, x^3) \to (x^1, x^2, x^3; f^1(x), f^2(x)).$$

- The tangent plane at the point $(x; y)$ is spanned by three vectors X_1, X_2, X_3, with components given by

$$\begin{aligned} X_1 &= (1, 0, 0, G_1^1(x, y), G_1^2(x, y)), \\ X_2 &= (0, 1, 0, G_2^1(x, y), G_2^2(x, y)), \\ X_3 &= (0, 0, 1, G_3^1(x, y), G_3^2(x, y)). \end{aligned}$$

Example (Solutions)

- Any such surface gives a solution.
- The initial conditions add the requirement that it pass through $(a; b)$.
- Such solutions may not exist.
- The equations must satisfy certain necessary conditions on the functions G_ℓ^k.
- They reflect the fact that if there is a solution, then one can interchange the order of differentiation of f^1 and f^2.
- These conditions can be written as relations among X_i and $[X_i, X_j]$, $i, j = 1, 2, 3$.

Example (The Vector Fields)

- The vector fields X_1, X_2, X_3 are determined by the system.
- They define, at each point q of U, a three-dimensional subspace $\Delta_q \subset T_q(\mathbb{R}^5)$, at least if they are linearly independent, which we will assume.
- Thus, such a system of equations determines in some domain of $\mathbb{R}^5$ three linearly independent vector fields X_1, X_2, X_3 at each point.
- A solution is a three-dimensional submanifold whose tangent space at each of its points q is spanned by X_1, X_2, X_3.

Example (Equivalent Systems)

- Two systems of differential equations will be **equivalent** if they determine, at each q of this domain, the same three-dimensional subspace Δ_q of $T_q(\mathbb{R}^5)$.
- In that case, they would - if some sort of uniqueness prevailed -have the same solutions.

Example (Complete Integrability)

- A system of equations is **completely integrable**, roughly speaking, if there is a single such solution manifold through each point of some domain of $\mathbb{R}^5$.
- That is, if the domain, up to diffeomorphism, is like an open subspace of $\mathbb{R}^5$, presented as a union of disjoint "surfaces", like the surfaces obtained by holding two coordinates fixed and letting the other three vary.

Distributions and Local Bases

- Let M be a manifold of dimension $m = n + k$.
- Suppose that to each $p \in M$ is assigned an n-dimensional subspace Δ_p of $T_p(M)$.
- Suppose, also, that, in a neighborhood U of each $p \in M$, there are n linearly independent C^∞-vector fields $X_1, \ldots, X_n$ which form a basis of Δ_q, for every $q \in U$.
- In this case we shall say that:
  - Δ is a C^∞ **distribution** of dimension n on M;
  - $X_1, \ldots, X_n$ is a **local basis** of Δ.

Involutive Distributions

- We shall say that the distribution Δ is **involutive** if, there exists a local basis $X_1, \ldots, X_n$ in a neighborhood of each point, such that

$$[X_i, X_j] = \sum_{k=1}^{n} c_{ij}^k X_k, \quad 1 \leq i, j \leq n.$$

- The c_{ij}^k will not in general be constants, but will be C^∞ functions on the neighborhood.

Integral Manifolds

- Suppose Δ is a C^∞ distribution on M.
- Let N be a connected C^∞ manifold.
- Suppose $F : N \to M$ is a one-to-one immersion, such that, for each $q \in N$, we have
$$F_*(T_q(N)) \subseteq \Delta_{F(q)}.$$
- Then we say that the immersed submanifold is an **integral manifold** of Δ.
- Note that an integral manifold may be of lower dimension than Δ.

Example: Involutive Distribution

- Consider $M = \mathbb{R}^n \times \mathbb{R}^k$.
- Suppose
$$X_i = \frac{\partial}{\partial x^i}, \quad i = 1, \ldots, n.$$
- Then the distribution is the subspace of dimension n consisting of all those vectors parallel to $\mathbb{R}^n$ at each point q of M.
- We will see that this apparently rather special example is actually typical, locally, of involutive distributions.

Completely Integrable Distributions

- Let M be a manifold of dimension $m = n + k$.
- Let Δ be a C^∞ distribution on M of dimension n.
- We shall say that Δ is **completely integrable** if each point $p \in M$ has a coordinate neighborhood U, φ, such that if $x^1, \ldots, x^m$ denote the local coordinates, then the n vectors

$$E_i = \varphi_*^{-1}\left(\frac{\partial}{\partial x^i}\right), \quad i = 1, \ldots, n,$$

are a local basis on U for Δ.

Completely Integrable Distributions (Cont'd)

- Note that, if Δ is completely integral, there is an n-dimensional integral manifold N through each point q of U, such that

$$T_q(N) = \Delta_q.$$

- That is, the tangent space to N is exactly Δ.
- In fact, let $(a^1, \ldots, a^m)$ denote the coordinates of q.
- Then an integral manifold through q is the set of all points whose coordinates satisfy

$$x^{n+1} = a^{n+1}, \ldots, x^m = a^m.$$

- In other words, N is the **slice** of U

$$N = \varphi^{-1}\{x \in \varphi(U) : x^j = a^j, j = n+1, \ldots, m\}.$$

Complete Integrability and Involutivity

- In the completely integrable case the distribution is involutive, since

$$[E_i, E_j] = \varphi_*^{-1}\left[\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right] = 0, \quad 1 \leq i, j \leq n.$$

- Thus any completely integrable distribution is involutive.
- However, most distributions are not involutive.
- For example, on $\mathbb{R}^3$ the distribution

$$X_1 = x^3 \frac{\partial}{\partial x^1} + \frac{\partial}{\partial x^3}, \quad X_2 = \frac{\partial}{\partial x^2} + \frac{\partial}{\partial x^3}$$

  is not involutive since $[X_1, X_2] = -\frac{\partial}{\partial x^1}$, which is not a linear combination of X_1 and X_2.
- This means, in particular, that X_1, X_2 could not be tangent vectors to a surface $x^3 = f(x^1, x^2)$.

Involutive Distributions and Lie Algebras

- An important and instructive example of an involutive distribution is furnished by the Lie algebra $\mathfrak{h}$ of a subgroup H of a Lie group G.
- $\mathfrak{h}$ consists of left-invariant vector fields on G which are tangent to H at the identity.
- We saw that this determines a subalgebra, the image of the Lie algebra of H under the inclusion mapping.
- These give a (left-invariant) distribution Δ on G, such that

$$\Delta_h = T_h(H), \quad \text{for every } h \in H.$$

- The cosets gH are the integral manifolds of this distribution.
- Δ is evidently involutive since $\mathfrak{h}$ is a subalgebra of $\mathfrak{g}$.

One-Dimensional Distributions

- A distribution Δ of dimension 1 is just a field of line elements.
- That is, Δ consists of one-dimensional subspaces.
- A local basis is given by any nonvanishing vector field X which belongs to Δ at each point.
- An integral curve of X is an integral manifold.
- We know from the existence theorem that such integral manifolds, passing through any given point, exist and are unique.
- In fact, a previous theorem says precisely that any such distribution is completely integrable.
- It is also involutive since $[X, X] = 0$.

Frobenius' Generalized Existence Theorem

- The following theorem may be considered a generalization of the existence theorem to certain types of partial differential equations.
- In the general case, however, there is a necessary condition which is not automatic, as it is in the case of a one-dimensional distribution.
- This condition is the involutivity of Δ.

Theorem (Frobenius)

A distribution Δ on a manifold M is completely integrable if and only if it is involutive.

Frobenius' Theorem (Cont'd)

- We showed that a completely integrable distribution is involutive.

  This is an easy consequence of the definitions.

  We shall prove that involutive distributions are completely integrable by induction on their dimensions, which we denote by n.

  We let $m = \dim M$.

  When $n = 1$, we have seen that we may introduce local coordinates V, ψ, such that $\widetilde{E}_1 = \psi_*^{-1}(\frac{\partial}{\partial y^1})$ is a local basis for Δ.

  This establishes complete integrability when $n = 1$.

  Suppose that the theorem is true for involutive distributions of dimensions $1, 2, \ldots, n-1$.

  Let Δ be of dimension n and in involution.

Frobenius' Theorem (Cont'd)

- Around any $p \in M$, we may find local coordinates V, ψ and a local basis $X_1, \ldots, X_n$ of Δ on V, such that $X_1 = \widetilde{E}_1$.

  By assumption,

  $$[X_i, X_j] = \sum_{\ell=1}^{n} c_{ij}^{\ell} X_{\ell}.$$

  Let $y^1, \ldots, y^m$ denote the local coordinates.

  We may suppose that $\psi(p) = 0$.

  We know that the components of X_j relative to the coordinate frames $\widetilde{E}_1, \ldots, \widetilde{E}_m$ are $X_j y^1, \ldots, X_j y^m$, which are C^∞ functions on V.

Frobenius' Theorem (Cont'd)

- Define a new basis of Δ on V by

$$\begin{array}{rcl} Y_1 & = & X_1 \ (= \widetilde{E}_1), \\ Y_2 & = & X_2 - (X_2 y^1) X_1, \\ & \vdots & \\ Y_n & = & X_n - (X_n y^1) X_1. \end{array}$$

By involutivity $[Y_i, Y_j] = \sum_{i=1}^n d_{ij}^\ell Y_\ell$.

But we have arranged that $Y_2, \ldots, Y_n$ are linear combinations of $\widetilde{E}_2, \ldots, \widetilde{E}_m$ at each point and do not involve $\widetilde{E}_1$ $(= Y_1)$.

Therefore, they are tangent to the manifolds $y^1 =$ constant.

So $[Y_i, Y_j]$, $2 \leq i, j \leq n$, must be tangent to the submanifolds $y^1 =$ constant.

Hence, $d_{ij}^1 = 0$, $2 \leq i, j \leq n$.

Frobenius' Theorem (Cont'd)

- The distribution on V defined by $Y_2, \ldots, Y_n$ is in involution on V.

  Moreover, on each submanifold $y^1 =$ constant of V, including $N_0 \subseteq U$, it is defined by $y^1 = 0$.

  The functions $(y^2, \ldots, y^m)$, restricted to N_0, give coordinates on $V \cap N_0$.

  By the induction hypothesis, we may change coordinates on N_0 in a neighborhood of p by, say, functions

$$y^i = f^i(x^2, \ldots, x^m), \quad i = 2, \ldots, m,$$

  defined on a neighborhood of the origin of $\mathbb{R}^{m-1}$, so that:
  - The image on N_0 of $\frac{\partial}{\partial x^2}, \ldots, \frac{\partial}{\partial x^n}$ is a basis at each point of the subspace spanned by $Y_2, \ldots, Y_n$;
  - We have $f^i(0, \ldots, 0) = 0$, $i = 2, \ldots, m$.

Frobenius' Theorem (Cont'd)

- We extend this to a change of coordinates in a neighborhood $U \subseteq V$ of p by adding the function $f^1(x) = x^1$, giving

$$y^1 = x^1, \quad y^i = f^i(x^2, \dots, x^m), \quad i = 2, \dots, m.$$

  Note that the Jacobian matrix is nonsingular at the origin.

  So this is a valid change of coordinates.

  We may suppose, with no loss of generality, that the image of U in the $(x^1, \dots, x^m)$ space is the cube $C_\varepsilon^m(0)$.

  Let φ denote the coordinate map.

Frobenius' Theorem (Cont'd)

- We have $\varphi = \psi \circ F^{-1}$ with
$$F(x^1, \ldots, x^m) = (f^1(x), \ldots, f^m(x)).$$
Also $\varphi(p) = (0, \ldots, 0)$.
Moreover, in terms of the new coordinates, we have the following three facts:
  (i) $Y_1 = \varphi_*^{-1}(\frac{\partial}{\partial x^1})$;
  (ii) $N_0 \cap U$ consists of those points for which $x^1 = 0$, so $(x^2, \ldots, x^m)$ are coordinates on this submanifold;
  (iii) at each point of $N_0 \cap U$, $Y_2, \ldots, Y_n$ are linear combinations of
$$E_2 = \varphi_* \left(\frac{\partial}{\partial x^2} \right), \ldots, E_n = \varphi_* \left(\frac{\partial}{\partial x^n} \right).$$
Equivalently, when $x^1 = 0$,
$$Y_2 x^\ell = \cdots = Y_n x^\ell = 0, \quad \text{for } \ell = n+1, \ldots, m,$$
that is, the last $m - n$ components vanish.

Frobenius' Theorem (Cont'd)

- We now prove that (iii) holds throughout U, without restriction on x^1.

  We consider $Y_1(Y_j x^\ell)$, for $j = 2, \ldots, n$ and each $\ell > n$.

  Using the definition of brackets, we get

$$Y_1(Y_j x^\ell) = Y_j(Y_1 x^\ell) + [Y_1, Y_j]x^\ell.$$

  We have

$$Y_1 x^\ell = \frac{\partial x^\ell}{\partial x^1} = 0.$$

  Moreover,

$$[Y_1, Y_j] = \sum_{s=1}^{n} d_{1j}^s Y_s.$$

  So

$$Y_1(Y_j x^\ell) = \sum_{s=2}^{n} d_{1j}^s (Y_s x^\ell).$$

Frobenius' Theorem (Cont'd)

- We found
$$Y_1(Y_j x^\ell) = \sum_{s=2}^{n} d_{1j}^s (Y_s x^\ell).$$
Now write d_{1j}^s and $Y_j x^i$ as functions of $(x^1, \ldots, x^m)$, passing from functions on U to the corresponding functions in local coordinates.
Then we see that $Y_2 x^1, \ldots, Y_n x^\ell$, for fixed $\ell > n$ and fixed $x^2, \ldots, x^m$, are solutions of the system of ordinary differential equations
$$\frac{dz_j}{dx^1} = \sum_{s=2}^{n} d_{1j}^s z_s, \quad j = 2, \ldots, n,$$
satisfying initial conditions
$$z_j = 0,\ j = 2, \ldots, n, \quad \text{when } x^1 = 0.$$

Frobenius' Theorem (Cont'd)

- However, the functions $z_j = 0$ also satisfy the system and these same initial conditions.

  So, by the uniqueness of solutions, whenever $\ell > n$,

  $$Y_2 x^\ell = \cdots = Y_k x^\ell = 0, \text{ for all values of } x^1.$$

  This shows that the vectors $Y_2, \ldots, Y_n$ are linear combinations of the vectors $E_2, \ldots, E_n$ (of the coordinate frames) throughout U.

  We also have $E_1 = Y_1$.

  It follows that

  $$E_i = \varphi_*^{-1} \left(\frac{\partial}{\partial x^i} \right), \quad i = 1, \ldots, n,$$

  is a local basis for Δ.

Local Uniqueness

Corollary

Let U, φ be a cubical coordinate neighborhood of $p \in M$, relative to the involutive distribution Δ, whose slices - corresponding to $x^{n+1}, \ldots, x^m$ fixed - are integral manifolds in U. Then any connected integral manifold $V \subseteq U$ lies on such a slice. That is, there are constants $a^{n+1}, \ldots, a^m$ such that

$$V \subseteq \{q \in U : x^{n+1}(q) = a^{n+1}, \ldots, x^m(q) = a^m\}.$$

- By hypothesis, V is an integral manifold.

  So its tangent space at each point lies in the space spanned by the first n vectors $E_1, \ldots, E_n$ of the coordinate frames.

Local Uniqueness (Cont'd)

- Let x^j be a coordinate function of U, with $j > n$.

  Let p be any point of V.

  Let X_p be any vector at p tangent to V.

  Then

  $$X_p = \sum_{i=1}^{n} \alpha_i E_{ip}.$$

  So we get

  $$X_p x^j = \sum_{i=1}^{n} \alpha^i E_{ip} x^j = \sum_{i=1}^{n} \alpha^j \left(\frac{\partial x^j}{\partial x^i} \right)_{\varphi(p)} = 0.$$

  But x^j is defined on all of V and V is connected.

  It follows that $x^j = a^j$, a constant, on V.

Restriction of a Mapping

Theorem

Let $N \subseteq M$ be an integral manifold of an involutive distribution Δ, with

$$\dim N = \dim \Delta.$$

Suppose $F : A \to M$ is a C^∞ mapping of a manifold A into M. If $F(A) \subseteq N$, then F is C^∞ as a mapping into N.

- Let $p \in A$ and let $q = F(p)$ be its image.
  Choose a cubical coordinate neighborhood U, φ of q, such that:
  - $\varphi(q) = (0, \ldots, 0)$;
  - $\varphi(U) = C_\varepsilon^m(0)$;
  - Its slices $x^{n+1} = a^{n+1}, \ldots, x^m = a^m$ are integral manifolds, where $n = \dim \Delta$ and $m = \dim M$.

  Now the inclusion $i : N \to M$ is an immersion.
  So $i^{-1}(U) = N \cap U$ is an open set in N.
  Therefore, it is an open submanifold.

Restriction of a Mapping (Cont'd)

- Manifolds are locally connected.

  So the components of $N \cap U$ are open sets of N and countable in number.

  Each component is itself a (connected) integral manifold.

  Thus, by the preceding corollary, it lies on a slice.

  It follows that, if x^j, $j > n$, is a coordinate function on U, it can have only a countable number of values on $N \cap U$.

  The function x^j maps any connected set $C \subseteq N \cap U$ continuously into this countable subset of $\mathbb{R}$.

  Hence, it must be constant on C [the only connected, countable subset of $\mathbb{R}$ is a single point].

Restriction of a Mapping (Cont'd)

- Using the continuity of $F : A \to M$, choose a connected coordinate neighborhood W, ψ of p such that $F(W) \subseteq U$.

  $F(W)$ is a connected subset of U and lies in $N \cap U$.

  Therefore, it lies on a single slice.

  Because $q \in F(W)$, this is the slice $x^{n+1} = \cdots = x^m = 0$.

  Let $\widetilde{U}$ be the subset of N which lies on this slice.

  We know it must be a union of components of $i^{-1}(U) = N \cap U$.

  So it is an open subset of N in the topology of N.

Restriction of a Mapping (Cont'd)

- The coordinate functions $x^1, \ldots, x^n$ restricted to $\widetilde{U}$ are coordinates on $\widetilde{U}$. That is, they define a mapping

$$\widetilde{\varphi} : \widetilde{U} \to \mathbb{R}^n,$$

such that $\widetilde{U}, \widetilde{\varphi}$ is a coordinate neighborhood of q on N.
Let $y^1, \ldots, y^r$ be the local coordinates on W, ψ.
Suppose $F : A \to M$ is given on W by C^∞ functions

$$x^j = f^j(y^1, \ldots, y^r), \quad j = 1, \ldots, m.$$

Then

$$f^j(y) = 0, \quad j = n+1, \ldots, m.$$

Moreover, as a mapping into N, F is given (in local coordinates) on W by the same functions $f^j(y)$, $1 \leq j \leq m$.
So it must be C^∞, as claimed.

Maximal Integral Manifolds

Definition

A **maximal integral manifold** N of an involutive distribution Δ is a connected integral manifold which contains every connected integral manifold which has a point in common with it.

- It is immediate from the preceding corollary that a maximal integral manifold has the same dimension as Δ.
- It is also clear that at most one maximal integral manifold can pass through a point p of M.
- It is true, but more difficult to prove, that there is a maximal integral manifold through every point of M.
  - The idea is to piece together local slices using the corollary and build up an immersed submanifold.
  - The difficulty is in showing that there are not too many such slices, that is, in proving that we have a countable basis of open sets.

Application to Lie Groups

Theorem

Let G be a Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{h}$ a subalgebra of $\mathfrak{g}$. Then there is a connected subgroup H of G whose Lie algebra is $\mathfrak{h}$.

- Let the left-invariant vector fields $X_1, \ldots, X_n$ on $\mathfrak{g}$ be a basis of $\mathfrak{h}$.

  They define a distribution Δ which is invariant under left translations.

  Hence, each integral manifold N is carried by any left translation L_g diffeomorphically to an integral manifold $L_g(N)$.

  Let H be the *maximal integral manifold through the identity* e.

  If $h \in H$, then $L_{h^{-1}}(h) = e$.

  So both H and $L_{h^{-1}}(H)$ have e in common.

  Since H is maximal, $L_{h^{-1}}(H) = H$.

  It follows that, if $h_1, h_2 \in H$, then $h_1^{-1} h_2 \in H$.

  Thus, H is a subgroup as well as an immersed submanifold.

Application to Lie Groups (Cont'd)

- The product mapping

$$H \times H \to H$$

is a composition of:

  - Inclusion $i : H \times H \to G \times G$;
  - The product mapping $\theta : G \times G \to G$.

Both are C^∞.

So $\theta \circ i$ is C^∞ as a mapping into G.

Its image is in H because H is a subgroup.

By the preceding theorem, we see that the product mapping $H \times H \to H$ is also C^∞.

A similar argument shows that the mapping taking each $h \in H$ to its inverse h^{-1} is also C^∞.

This completes the proof, subject to the unproved assertion concerning integral manifolds.

Subsection 9

Homogeneous Spaces

Transitive Actions and Homogeneous Spaces

- Suppose G is a Lie group and M a manifold.
- Let $\theta : G \times M \to M$ be an action of G on M.
- We recall that θ is **transitive** if, for every pair $p, q \in M$, there is a $g \in G$, such that
$$\theta_g(p) = q.$$
- This means that, as far as properties preserved by G are concerned, any two points of the manifold are alike.

Definition

A manifold M is said to be a **homogeneous space of the Lie group** G if there is a transitive C^∞ action of G on M.

Examples

- Many examples of group actions have this property.
  - $O(n)$ acts transitively on S^{n-1};
  - $Gl(n, \mathbb{R})$ acts transitively on $\mathbb{R}^n - \{0\}$;

    $\vdots$

Action on Cosets

- Let G be a group.
- Let H be a subgroup of G.
- Consider the set G/H of left cosets of H in G.
- We define a left action $\lambda : G \times G/H \to G/H$ by

$$\lambda(g, xH) = gxH.$$

- This defines a left action, since:
  (i) $\lambda(e, xH) = xH$;
  (ii) $\lambda(g_1, \lambda(g_2, xH)) = \lambda(g_1, g_2xH) = (g_1g_2)xH = \lambda(g_1g_2, xH)$.

Properties of the Action on Cosets

- Suppose $\pi : G \to G/H$ is the natural mapping of each $g \in G$ to the coset which contains it,
$$\pi(g) = gH.$$
- Let $L_g : G \to G$ denote left translation.
- Then, for all $g \in G$,
$$\pi \circ L_g = \lambda_g \circ \pi.$$
- The transitivity is apparent, since, for all $x, y \in G$,
$$\lambda_{yx^{-1}}(xH) = yH.$$

Outline of Work

- We would like to assert that:
  - When G is a Lie group, then G/H is a manifold;
  - The mappings λ and π, defined by G and H as above, are C^∞.
- We saw that, if H is closed in G (a Lie group), then the quotient topology on G/H makes it a Hausdorff space and π an open - as well as continuous - mapping.
- Moreover, we asserted that, with this topology on G/H, λ is a continuous group action.
- In this section we show that G/H is a manifold and λ is a C^∞ action.
- This will give us many new examples of manifolds and group action.
- More importantly, the manifolds G/H, with G acting by left translation, form a universal model for all transitive actions, that is, for all homogeneous spaces.

Set-Theoretic Action

- Consider universality from the set-theoretic viewpoint, without topology or differentiable structure.
- Let X be a set on which a group G acts transitively by the rule
$$\theta : G \times X \to X.$$
- Choose, arbitrarily, a point $a \in X$.
- Let the isotropy subgroup (or stability group) of a be
$$H = \{g \in G : \theta_g(a) = a\}.$$
- We then define a mapping $\widetilde{F} : G \to X$ by
$$\widetilde{F}(g) = \theta_g(a).$$
- Since θ is transitive, $\widetilde{F}$ is onto.
- Moreover for any $x \in X$, $\widetilde{F}^{-1}(x) = gH$, where g is any element of G such that $\widetilde{F}(g) = x$.

Set-Theoretic Action (Cont'd)

- It is then easily verified that $\widetilde{F}$ induces a one-to-one onto mapping $F : G/H \to X$ by
$$F(gh) = \widetilde{F}(g).$$
- For these mappings we have the relation
$$F \circ \pi = \widetilde{F}.$$
- Finally F carries the natural action of G on G/H, which we denoted by λ, to the action θ, that is,
$$F \circ \lambda_g = \theta_g \circ F, \quad g \in G.$$
- Thus, from the set-theoretic and abstract group viewpoint, $\lambda : G \times G/H \to G/H$ is equivalent as an action to $\theta : G \times X \to X$.

Sections on a Quotient

- Recall that a **Lie subgroup** H of a Lie group G is an immersed submanifold which is a Lie group with respect to the group operations of G.
- We intend to use the quotient topology on G/H.
- Moreover, we wish G/H to be a Hausdorff space.
- So we must restrict our attention to those Lie subgroups that are closed subsets.
- Therefore, H will be assumed to be a closed Lie subgroup.
- We show later that this implies that H is a submanifold of G.
- A **section** V, σ on G/H is a continuous mapping σ of an open subset V of G/H into G, $\sigma : V \to G$, satisfying

$$\pi \circ \sigma = i_V, \text{ the identity on } V.$$

Manifold Structure on Quotient

Theorem

Let G be a Lie group and H a closed Lie subgroup.
Then there exists a unique C^∞-manifold structure on the space G/H, satisfying the following properties:

(i) π is C^∞;

(ii) Each $g \in G$ is in the image $\sigma(V)$ of a C^∞ section V, σ on G/H.

The natural action

$$\lambda : G \times G/H \to G/H,$$

described above, is a C^∞ action of G on G/H with this structure.
Moreover, we have

$$\dim(G/H) = \dim G - \dim H.$$

- The proof will be given shortly.

Diffeomorphism Between Quotient and Manifold

- Now suppose that a Lie group G acts transitively on a manifold M, the action being given by the C^∞-mapping
$$\theta : G \times M \to M.$$
- We use the notation above, with X replaced by M.
- Suppose $a \in M$ is fixed.
- Let H be the isotropy subgroup of a.
- We then have a closely related theorem that completes the picture.

Diffeomorphism Between Quotient and Manifold (Cont'd)

Theorem

The mapping $\widetilde{F} : G \to M$, defined by

$$\widetilde{F}(g) = \theta(g, a),$$

is C^∞ and has rank equal to $\dim M$ everywhere on G. The isotropy group H is a closed Lie subgroup. So G/H is a C^∞ manifold. The mapping $F : G/H \to M$ defined by

$$F(gH) = \widetilde{F}(g)$$

is a diffeomorphism. Moreover, for every $g \in G$,

$$F \circ \lambda_g = \theta_g \circ F.$$

Example

- Consider briefly some of the spaces associated with classical geometries:
  - $\boldsymbol{E}^n$, Euclidean space;
  - $P^n(\mathbb{R})$, the space of real projective geometry;
  - $\boldsymbol{H}^2$, the space of plane non-Euclidean geometry.
- All of these were discovered and studied before Lie groups (or groups of any kind) were invented.
- However, in each case there is an underlying group, the group of automorphisms of the geometry.
- It is the group by which we can bring congruent figures into congruence.
- In fact each geometry studies precisely the objects and properties which are invariant under the transformations expressed by the actions of this group on the space.

Example (Cont'd)

- For $\boldsymbol{E}^n$, or $\mathbb{R}^n$, the group consists of all isometries (rigid motions), that is, translations, rotations and reflections.
- For $P^n(\mathbb{R})$ it consists of the projective transformations.
- For $\boldsymbol{H}^2$ it is the group whose actions leave non-Euclidean distances unchanged ("rigid" motions again!).
- In each case the group is a Lie group and in each case it is transitive.
- This means that the theorems above can be used as a sort of underlying unifying principle of all these geometries.
- Thus the study of any of these classical geometries can be reduced to a study of Lie groups G and their subgroups H.

Example

- Consider the space $\boldsymbol{E}^n$, identified with $\mathbb{R}^n$.
- We have seen that the group of its rigid motions is a group G which is $O(n) \times \boldsymbol{V}^n$ as a manifold.
- However, its group product was defined by

$$(A, v)(B, w) = (AB, Aw + v).$$

- Moreover, the action on $\mathbb{R}^n$ is given by

$$(A, v) \cdot x = Ax + v.$$

Example (Cont'd)

- Another approach is the following.
- We identify G with the $(n+1)\times(n+1)$ matrices of the form

$$g = \left(\begin{array}{ccc|c} a_{11} & \cdots & a_{1n} & v_1 \\ \vdots & & \vdots & \vdots \\ a_{n1} & \cdots & a_{nm} & v_n \\ \hline 0 & \cdots & 0 & 1 \end{array}\right), \quad A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \in O(n).$$

- We identify points $x = (x^1, \ldots, x^n)$ of $\mathbb{R}^n$ with the column vector

$$\widetilde{x} = \begin{pmatrix} x^1 \\ \vdots \\ x^n \\ 1 \end{pmatrix} = (x^1, \ldots, x^n, 1)^T.$$

Example (Cont'd)

- Then the action
$$\theta(g, x) = g\widetilde{x},$$
the product of the matrices g and $\widetilde{x}$.
- The subgroup H leaving the origin $x = (0, \ldots, 0)$ fixed is the set of all of these matrices for which
$$v_1 = \cdots = v_n = 0.$$
- Hence, it is a closed Lie subgroup isomorphic to $O(n)$.

Example

- The group $G = Sl(n+1,\mathbb{R})$ acts transitively on $P^n(\mathbb{R})$ as follows.
- Let $[x] \in P^n(\mathbb{R})$.
- Then $[x]$ is an equivalence class of nonzero elements
$$x = (x^1,\ldots,x^{n+1}) \in \mathbb{R}^{n+1}.$$
- Given any $g \in Sl(n+1,\mathbb{R})$, we define $\theta(g,[x])$ by
$$\theta(g,[x]) = [gx],$$
where gx is the matrix product of g with x, an $(n+1)\times 1$ matrix.
- This is a C^∞ action and is transitive.
- The isotropy subgroup H of $[(1,0,\ldots,0)]$ is the set of elements (a_{ij}) of $Sl(n+1,\mathbb{R})$ with $a_{11} \neq 0$ and all other entries of the first column equal to zero.
- It can be shown that H is a closed Lie subgroup of G.

The Grassman Manifolds Revisited

- These ideas and the preceding theorem give a relatively simple method for establishing that certain sets are C^∞ manifolds in a natural way.
- The best illustrations are the Grassman manifolds $G(k,n)$ of k-planes through the origin in $\mathbb{R}^n$.
- It was proved that these were manifolds, but the proof was quite complicated and only sketched at some points.
- We revisit the same result to illustrate the new approach.

The Grassman Manifolds Revisited (Cont'd)

- The group $Gl(n, \mathbb{R})$ acting in the natural manner on $\mathbb{R}^n$ is transitive on k-planes through the origin.
- Let $\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n\}$ be a linearly independent set of vectors.
- There is a uniquely determined, nonsingular, linear transformation taking it to any second linearly independent set

$$\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_n\}.$$

- Recall that each set of k linearly independent vectors can be completed to a basis.
- So, if $Gl(n, \mathbb{R})$ is transitive on n-frames, it is also transitive on k-frames.
- So $Gl(n, \mathbb{R})$ acts transitively on the set $M = G(k, n)$ of k-planes through 0.

The Grassman Manifolds Revisited (Cont'd)

- Suppose the isotropy subgroup H of some point of M, that is, a k-plane through 0, is a closed Lie subgroup.
- Then, by the theorem, $Gl(n, \mathbb{R})/H$ is a C^∞ manifold.
- Moreover, it is in natural one-to-one correspondence with M.
- Thus, we may take on M the topology and C^∞ structure which makes this correspondence a diffeomorphism.
- So it suffices to show that H is in fact a closed Lie subgroup.

The Grassman Manifolds Revisited (Cont'd)

- Recall H is the isotropy group of some point of $M = G(k,n)$, i.e., a k plane through 0.
- Consider such a k-plane of $\mathbb{R}^n$ spanned by the vectors

$$\boldsymbol{e}_1 = (1,0,\ldots,0),\ldots,\boldsymbol{e}_k = (0,\ldots,1,0,\ldots,0).$$

- It is carried onto itself by the subgroup $H \subseteq Gl(n,\mathbb{R})$ consisting of matrices of the form

$$h = \left(\begin{array}{c|c} A & C \\ \hline 0 & B \end{array}\right),$$

  where:
  - $A \in Gl(k,\mathbb{R})$;
  - $B \in Gl(n-k,\mathbb{R})$;
  - C is an arbitrary $k \times (n-k)$ matrix.
- Therefore, the Grassmann manifold $G(k,n)$ is indeed a C^∞ manifold.

Outline of the Method

- This method is frequently used in practice to show that some rather complicated objects can be endowed with the structure of a differentiable manifold (uniquely, according to the theorem).
- It may be summarized as follows:

  Suppose G is a Lie group and G acts on a set X transitively in such a way that the isotropy subgroup of some point a of X is a closed Lie subgroup. Then there exists a (unique) C^∞ structure on X such that the action is C^∞.
- This principle as well as other results of this section are susceptible to further refinements and weakening of hypotheses.

The First Theorem

Theorem

Let G be a Lie group and H a closed Lie subgroup.
Then there exists a unique C^∞-manifold structure on the space G/H, satisfying the following properties:

(i) π is C^∞;

(ii) Each $g \in G$ is in the image $\sigma(V)$ of a C^∞ section V, σ on G/H.

The natural action

$$\lambda : G \times G/H \to G/H,$$

described above, is a C^∞ action of G on G/H with this structure.
Moreover, we have

$$\dim(G/H) = \dim G - \dim H.$$

- We now give the proof.

Proof of the First Theorem

- The topology on G/H is uniquely determined by the requirement that $\pi : G \to G/H$ be open and continuous.

  We show $\lambda : G \times G/H \to G/H$ is a continuous action.

  Let U be an open set of G/H.

  We show that $\lambda^{-1}(U)$ is open.

  Let W be the subset of $G \times G$, such that every pair $(g_1, g_2) \in W$ has its product $g_1 g_2$ in $\pi^{-1}(U)$, an open subset of G.

  W is the inverse image of $\pi^{-1}(U)$ under the continuous mapping $(g_1, g_2) \to g_1 g_2$.

  So W is open.

  The natural mapping of $G \times G \to G \times G/H$ given by $(g_1, g_2) \to (g_1, \pi(g_2))$ is open.

  So it carries W onto an open set, which is exactly $\lambda^{-1}(U)$.

Proof of the First Theorem (Cont'd)

- We now need to use Frobenius' theorem, which we apply to the left-invariant distribution Δ, determined by $\Delta_e = T_e(H)$.

  Denote by $\mathfrak{h}$ the the Lie algebra of H, viewed as a subalgebra of $\mathfrak{g}$.

  Δ has as a basis any basis of left-invariant vector fields in $\mathfrak{h}$.

  Moreover, the integral manifolds of Δ are exactly the left cosets gH, as remarked previously.

  It follows that there is a cubical neighborhood of e whose intersections with the cosets gH are a union of slices.

  To complete the proof we need a sharper result given by the following lemma.

Lemma Auxiliary to the Proof of the First Theorem

Lemma

If H is a Lie subgroup of G which is closed as a subset, then each coset gH is a submanifold. Moreover, there is a cubical neighborhood U, φ of any $g \in G$, such that, for each coset xH, either $xH \cap U$ is empty or a single (connected) slice.

- That H and each of its cosets is a submanifold is an immediate consequence of the second part of the statement, which asserts, in particular, that H and its cosets have the submanifold property.

  We know each coset is an integral manifold of the distribution Δ.

  So every $g \in G$ has a cubical coordinate neighborhood with $\varphi(g) = C_\varepsilon^m(0)$, $m = \dim G$, whose slices, determined by fixing the last $m - n$ coordinates ($n = \dim H = \dim \Delta$), are integral manifolds, each an open set of a coset xH of H.

Lemma (Cont'd)

- We must now verify that U may be taken sufficiently small that each coset $xH \cap U$ is empty or is a single slice.

  Δ, integral manifolds, and so on, are invariant under left translation by elements of G.

  So it is enough to check this for the special case $g = e$.

  Let U', φ' be a cubical neighborhood of e, whose slices are cosets of H, and such that $U' \cap H$ consists of a single slice.

  It suffices to choose $U \subseteq U'$ small enough that:
  - $U^{-1}U \subseteq U'$;
  - $U, \varphi|_U$ is also a cube.

  Assume $x, y \in U$ are on distinct slices of U but $xH = yH$.

  $L_{y^{-1}}$ is a diffeomorphism and carries slices into slices.

  So $y^{-1}x$ and e are elements of $U' \cap H$ but lie on distinct slices.

  This contradicts our assumption about U', so it cannot happen.

Lemma (Existence of U', φ')

- Let V, ψ be a cubical neighborhood of e, $\psi(V) = C_\varepsilon^m(0)$, whose slices

$$S(a^{n+1}, \ldots, a^m) = \{q \in V : x^j(q) = a^j, j = n+1, \ldots, m\}$$

are integral manifolds.

We saw in the proof of a previous theorem that the collection of distinct slices on H, that is, $V \cap H$, is countable.

Hence, it corresponds to a countable set of points $\{(a^{n+1}, \ldots, a^m)\}$ of the cube $C_\delta^{m-n}(0)$.

Restricting slightly to a closed cube $\overline{V}' = \psi^{-1}(\overline{C_{\delta'}^m(0)})$, $\delta > \delta' > 0$, we may suppose this countable set is closed, for H is closed and $\overline{V}' \cap H$ is closed.

Lemma (Existence of U', φ' Cont'd)

- A closed countable subset of $\mathbb{R}^{m-n}$ must contain an isolated point.

  It follows that $H \cap V'$ contains an isolated slice.

  By translation invariance, we may assume this is the slice through e.

  Then it is possible to choose ε', $\delta' > \varepsilon' > 0$, so that

  $$\psi^{-1}(C^m_{\varepsilon'}(0)) = U', \quad \varphi' = \psi|_{U'}$$

  have exactly the property needed.

  That is, $H \cap U'$ is a single slice and contains the identity e.

  This U', φ', as we have seen, enables us to complete the proof of the lemma.

Proof of First Theorem (Cont'd)

- We restrict our discussion entirely to cubical neighborhoods U, φ of the type described above, with $\varphi(U) = C_\varepsilon^m(0)$.

  We also suppose that, in the local coordinates

  $$x^1, \ldots, x^n, x^{n+1}, \ldots, x^m,$$

  the slices obtained by holding $x^{n+1}, \ldots, x^m$ fixed are the intersections of cosets gH with V.

  Let

  $$A = \{q \in U : x^1(q) = \cdots = x^n(q) = 0\}.$$

  Let $\psi' : A \to C_\varepsilon^{m-n}(0) \subseteq \mathbb{R}^{m-n}$ be defined by

  $$\psi'(q) = (x^{n+1}(q), \ldots, x^m(q)).$$

Proof of First Theorem (Cont'd)

- A is a C^∞ submanifold of G, contained in U.

  In addition, ψ' is a diffeomorphism.

  By our choice of U, φ, we see that A meets each coset of H which intersects U in exactly one point.

  Therefore, π maps A homeomorphically onto an open subset V of G/H.

  We denote the inverse by σ.

  Thus $\sigma : V \to G$ is a continuous section with $\sigma(V) = A$.

Proof of First Theorem (Cont'd)

- Suppose that U, φ and $\widetilde{U}, \widetilde{\varphi}$, as just chosen, are such that $\widetilde{V} = \pi(\widetilde{A})$ and $V = \pi(A)$ have common points.

  The set $V \cap \widetilde{V}$ is open.

  Moreover, it can be verified that the corresponding subsets

  $$W = \sigma(V \cap \widetilde{V}) \quad \text{and} \quad \widetilde{W} = \widetilde{\sigma}(V \cap \widetilde{V})$$

  are diffeomorphic with respect to the natural correspondences

  $$\widetilde{\sigma} \circ \pi : W \to \widetilde{W} \quad \text{and} \quad \sigma \circ \pi : \widetilde{W} \to W.$$

  We consider:
  - The collection of open sets $V = \pi(A)$, over all U, φ of the type above;
  - The homeomorphisms $\psi = \psi' \circ \sigma : V \to C_\varepsilon^{m-n}(0)$.

  It follows that they determine a C^∞ structure of the type required by the conclusions of the theorem.

Proof of First Theorem (Cont'd)

- The uniqueness follows from Requirements (i) and (ii).

  Suppose we have two differentiable structures on G/H.

  We show that the identity is a diffeomorphism.

  Factor it locally into:
  - A section $\sigma : V \to G$ of the first structure;
  - A projection π, which is C^∞, onto the second structure.

  Thus, the identity is a C^∞ mapping of G/H with structure one to G/H with structure two, since this holds on each domain V.

  The converse is also true.

  So the structures agree.

  Finally $\lambda : G \times G/H \to G/H$ is C^∞, since it may be written on the domain V of a section as

$$\lambda(g, xH) = \pi(g\sigma(x)).$$

The Second Theorem

Theorem

The mapping $\widetilde{F} : G \to M$, defined by

$$\widetilde{F}(g) = \theta(g, a),$$

is C^∞ and has rank equal to $\dim M$ everywhere on G.
The isotropy group H is a closed Lie subgroup. So G/H is a C^∞ manifold.
The mapping $F : G/H \to M$ defined by

$$F(gH) = \widetilde{F}(g)$$

is a diffeomorphism. Moreover, for every $g \in G$,

$$F \circ \lambda_g = \theta_g \circ F.$$

Proof of the Second Theorem

- By hypothesis, θ is C^∞.

  By definition, $\widetilde{F}(g) = \theta(g, a)$.

  So $\widetilde{F} : G \to M$ is C^∞.

  Note that
  $$\widetilde{F} \circ L_g(x) = \widetilde{F}(gx) = \theta_g \circ \widetilde{F}(x).$$

  Moreover, both L_g and θ_g are diffeomorphisms.

  By the chain rule, the rank of $\widetilde{F}$ is the same at every $g \in G$.

  By a previous theorem, that $\widetilde{F}^{-1}(a) = H$ is a closed submanifold and satisfies the hypotheses of the preceding theorem.

Proof of the Second Theorem (Cont'd)

- At e we have $\widetilde{F}_* : T_e(G) \to T_a(M)$.
  But each $X_e \in T_e(G)$ is the tangent vector at $t = 0$ to the curve
  $$g(t) = \exp tX.$$
  So the vector $\widetilde{F}_*(X_e)$ is the tangent vector to
  $$\widetilde{F}(\exp tX) = \theta(\exp tX, a)$$
  at a (which corresponds to $t = 0$).
  θ restricted to $g(t) = \exp tX$ is an action of R on M.
  By a previous theorem, $\widetilde{F}_*(X_e)$ is zero if and only if
  $$\theta(\exp tX, a) = a, \text{ for all } t.$$

Proof of the Second Theorem (Cont'd)

- That is, $\widetilde{F}_*(X_e)$ is zero iff $\exp tX \subseteq H$.
  Equivalently, $X \in T_e(H)$, the subspace of $T_e(G)$ corresponding to the subgroup H.
  Hence,
  $$\ker \widetilde{F}_{*e} = T_e(H) = \ker \pi_{*e}.$$
  As noted, $\dim \ker \widetilde{F}_*$ is constant on G, as is $\dim \ker \pi_*$.
  Since $\widetilde{F}$ is onto, it follows from a previous theorem that
  $$\dim M = \dim G - \dim H = \dim G/H.$$
  Now consider $F : G/H \to M$.
  Let $q \in G/H$.
  Let V, σ be a section defined on a neighborhood V of q.
  σ is C^∞ and $F|_V = \widetilde{F} \circ \sigma$
  So F is C^∞ in a neighborhood of every point.

Proof of the Second Theorem (Cont'd)

- Hence F is C^∞ on G/H.

  F is one-to-one and onto from set-theoretic considerations.

  If $\ker F_* = \{0\}$, that is, $\text{rank} F = \dim G/H = \dim M$ everywhere, then F must be a diffeomorphism.

  Let q be any point of G/H and suppose $q = \pi(g)$.

  Using $\widetilde{F} = F \circ \pi$ and the chain rule, we see that $\widetilde{F}_* : T_g(G) \to T_{\widetilde{F}(g)}(M)$ is given also by $F_* \circ \pi_*$.

  But $\dim\ker \widetilde{F}_* = \dim\ker \pi_*$.

  So we must have $\dim\ker F_* = 0$, as we wished to prove.

  The fact that $F \circ \lambda_g = \theta_g \circ F$ was already noted.

  By a previous theorem, λ_g is a diffeomorphism.

  Finally, by hypothesis, θ_g is also a diffeomorphism.

Subsection 10

Appendix: Partial Proof of Existence Theorem

The Existence Theorem Revisited

Theorem (Existence Theorem for Ordinary Differential Equations)

Let $U \subseteq \mathbb{R}^n$ be an open set. For $\varepsilon > 0$, let $I_\varepsilon = (-\varepsilon, \varepsilon)$. Let

$$f^i(t, x^1, \ldots, x^n), \quad i = 1, \ldots, n,$$

be functions of class C^r, $r \geq 1$, on $I_\varepsilon \times U$.

Then, for each $x \in U$, there exists $\delta > 0$ and a neighborhood V of x, $V \subseteq U$, such that:

(I) For each $a = (a^1, \ldots, a^n) \in V$ there exists an n-tuple of C^r functions $x(t) = (x^1(t), \ldots, x^n(t))$, defined on I_δ and mapping I_δ into U, which satisfy the system of first-order differential equations

$$\frac{dx^i}{dt} = f^i(t, x), \quad i = 1, \ldots, n,$$

and the initial conditions $x^i(0) = a^i$, $i = 1, \ldots, n$.

The Existence Theorem (Cont'd)

Theorem (Existence Theorem Cont'd)

For each a, the functions $x(t) = (x^1(t), \ldots, x^n(t))$ are uniquely determined, in the sense that any other functions $\overline{x}^1(t), \ldots, \overline{x}^n(t)$ satisfying the same condition must agree with $x(t)$ on their common domain, which includes I_δ.

(II) These functions being uniquely determined by $a = (a^1, \ldots, a^n)$ for every $a \in V$, we write them

$$x^i(t, a^1, \ldots, a^n), \quad i = 1, \ldots, n.$$

They are of class C^r in all variables and, thus, determine a C^r map of $I_\delta \times V \to U$.

Partial Proof of Part I

- We are given n functions $f^i(t,x)$ defined and of class C^r on an open subset $I_\varepsilon \times U \subseteq \mathbb{R} \times \mathbb{R}^n$, with $I_\varepsilon = \{-\varepsilon < t < \varepsilon, \varepsilon > 0\}$.
- We must show that, for each $x \in U$, there is a neighborhood V and a $\delta > 0$, such that, for each $a \in V$, there exist unique functions $x^i(t)$, $-\delta < t < \delta$, satisfying

$$\frac{dx^i}{dt} = f^i(t, x(t)) \quad \text{and} \quad x^i(0) = a^i, \quad i = 1, \ldots, n.$$

- Suppose $x^i(t)$, $i = 1, \ldots, n$, are continuous functions defined for $|t| < \delta$ and they satisfy

$$x^i(t) = a^i + \int_0^t f^i(\tau, x(\tau)) d\tau.$$

Partial Proof of Part I (Cont'd)

- By the Fundamental Theorem of Calculus, they are of class C^1 at least and satisfy the required conditions.
- By the first condition above, it follows that they must be of class C^{r+1} at least, since their derivatives are of class C^r.
- We may write this set of integral equations for $x^1(t), \ldots, x^n(t)$ as an equation in n-tuples

$$x(t) = a + \int_0^t f(\tau, x(\tau)) d\tau.$$

- For a given $x_0 \in U$, we choose:
  - r, $0 < r < 1$, such that $\overline{B}_{3r}(x_0) \subseteq U$;
  - An ε', satisfying $\varepsilon > \varepsilon' > 0$, so that $\overline{I}_{\varepsilon'} \subseteq I_\varepsilon$.
- Thus, the functions $f^i(t, x)$ are of class C^r, $r \geq 1$, on the compact set $\overline{I}_{\varepsilon'} \times \overline{B}_{3r}(x_0)$.

Partial Proof of Part I (Cont'd)

- Therefore, both the given functions f^i and their derivatives are bounded on $\overline{I}_{\varepsilon'} \times \overline{B}_{3r}(x_0)$.
- It follows that we may choose $M > 1$ such that:
  - $M \geq \sup \|f(t,x)\|$;
  - $M\|x - y\| \geq \|f(t,x) - f(t,y)\|$, for all $t \in \overline{I}_{\varepsilon}$, and $x, y \in \overline{B}_{3r}(x_0)$.
- The last inequality results from the Mean Value Theorem and the continuity of the derivatives.
- Choose a positive δ, such that $\delta < \frac{r}{M^2}$.
- We shall prove the theorem with this δ and with $V = B_r(x_0)$, which we denote by B_r here.

Partial Proof of Part I (Cont'd)

- Let $a \in \overline{B}_r$.
- Let $\mathcal{F}$ be the collection of all continuous maps

$$\varphi(t) = (\varphi^1(t), \ldots, \varphi^n(t))$$

of $\overline{I}_\delta$ into $\overline{B}_{2r}(a)$ satisfying $\varphi(0) = a$.

- By virtue of the preceding comments, it is enough to show that, there is a unique member of this collection satisfying

$$\varphi = L(\varphi) = a + \int_0^t f(\tau, \varphi(\tau)) d\tau.$$

- This will be done by:
  - Proving that $L : \mathcal{F} \to \mathcal{F}$ is a contracting mapping on a complete metric space;
  - Applying the Contracting Mapping Theorem.

Partial Proof of Part I (Cont'd)

(1) $\mathcal{F}$ is a complete metric space with

$$d(\varphi, \psi) = \sup_{t \in \overline{I}_\delta} \|\varphi(t) - \psi(t)\|.$$

Indeed, this is the topology of uniform convergence of continuous functions on a compact space.

(2) If $\varphi \in \mathcal{F}$, then $L(\varphi) \in \mathcal{F}$ so that L maps $\mathcal{F}$ to $\mathcal{F}$.
It is clear that $L(\varphi)$ is continuous. In fact, it is at least C^1.
When $t = 0$, the function $L(\varphi)$ has the value a.
It is only necessary to check that if $|t| \leq \delta$, then $\|L(\varphi)(t) - a\| \leq 2r$.
This results from

$$\begin{array}{rcl} \|L(\varphi)(t) - a\| & = & \| \int_0^t f(\tau, \varphi(\tau)) d\tau \| \\ & \leq & \int_0^t \|f(\tau, \varphi(\tau))\| d\tau \\ & \leq & M\delta < \frac{r}{M} < r. \end{array}$$

Partial Proof of Part I (Cont'd)

(3) Finally, L is contracting.

Let $\varphi, \psi \in \mathcal{F}$:

$$\begin{array}{rcl} \|L(\varphi) - L(\psi)\| & \leq & \int_0^t \|f(\tau, \varphi(\tau)) - f(\tau, \psi(\tau))\| d\tau \\ & \leq & \delta M \sup_{t \in \bar{I}_\delta} \|\varphi(t) - \psi(t)\| \\ & \leq & \delta M d(\varphi, \psi) = \frac{r}{M} d(\varphi, \psi). \end{array}$$

But $r < 1$ and $M > 1$.

So we have

$$\|L(\varphi) - L(\psi)\| \leq k d(\varphi, \psi), \quad \text{where } 0 < k < 1.$$

- By the contracting mapping theorem there is a unique $\varphi(t)$ satisfying the conditions.