Introduction to Differential Geometry

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Subsection 1

Tangent Covectors

Dual Space and Covectors

- We suppose that $\boldsymbol{V}$ is a finite-dimensional vector space over $\mathbb{R}$.
- Let $\boldsymbol{V}^*$ denote its dual space.
- $\boldsymbol{V}^*$ is the space whose elements are linear functions from $\boldsymbol{V}$ to $\mathbb{R}$.
- Linear functions from $\boldsymbol{V}$ to $\mathbb{R}$ are called **covectors**.

Notation

- Suppose $\sigma \in \boldsymbol{V}^*$ so that $\sigma : \boldsymbol{V} \to \mathbb{R}$.
- Then, for $\boldsymbol{v} \in \boldsymbol{V}$, we denote the value of σ on $\boldsymbol{v}$ by

$$\sigma(\boldsymbol{v}) \quad \text{or} \quad \langle \boldsymbol{v}, \sigma \rangle.$$

- Recall that addition and multiplication by scalars in $\boldsymbol{V}^*$ are defined by the equations

$$\begin{aligned} (\sigma_1 + \sigma_2)(\boldsymbol{v}) &= \sigma_1(\boldsymbol{v}) + \sigma_2(\boldsymbol{v}), \\ (\alpha\sigma)(\boldsymbol{v}) &= \alpha(\sigma(\boldsymbol{v})). \end{aligned}$$

- These give the values of $\sigma_1 + \sigma_2$ and $\alpha\sigma$, $\alpha \in \mathbb{R}$, on an arbitrary $\boldsymbol{v} \in \boldsymbol{V}$, the right-hand operations taking place in $\mathbb{R}$.

Linear Algebra Fact (i)

- Let $F_* : \boldsymbol{V} \to \boldsymbol{W}$ be a linear map of vector spaces.
- It uniquely determines a dual linear map $F^* : \boldsymbol{W}^* \to \boldsymbol{V}^*$ by the prescription
$$(F^*\sigma)(\boldsymbol{v}) = \sigma(F_*(\boldsymbol{v})).$$
- This can be written, equivalently,
$$\langle \boldsymbol{v}, F^*(\sigma) \rangle = \langle F_*(\boldsymbol{v}), \sigma \rangle.$$
- When F_* is injective, then F^* is surjective.
- When F_* is surjective, then F^* is injective.

Linear Algebra Fact (ii)

- Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be a basis of $\boldsymbol{V}$.
- There exists a unique **dual basis**

$$\omega^1, \ldots, \omega^n$$

of $\boldsymbol{V}^*$ such that

$$\omega^i(\boldsymbol{v}_j) = \delta^i_j = \begin{cases} 0, & \text{if } i \neq j, \\ 1, & \text{if } i = j. \end{cases}$$

Linear Algebra Fact (ii) (Cont'd)

- If $\boldsymbol{v} \in \boldsymbol{V}$, then $\omega^1(\boldsymbol{v}), \ldots, \omega^n(\boldsymbol{v})$ are exactly the components of $\boldsymbol{v}$ in the basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$,

$$\boldsymbol{v} = \sum_{j=1}^{n} \omega^j(\boldsymbol{v})\boldsymbol{e}_j.$$

- Indeed, if $\boldsymbol{v} = \sum_{i=1}^{n} \alpha_i \boldsymbol{e}_i$,

$$\omega^j(\boldsymbol{v}) = \omega^j\left(\sum_{i=1}^{n} \alpha_i \boldsymbol{e}_i\right) = \sum_{i=1}^{n} \alpha_i \omega^j(\boldsymbol{e}_i) = \alpha_j.$$

Linear Algebra Facts (Cont'd)

- Observe that in Fact (i), the definition of F^* does not require the choice of a basis.
- Therefore F^* is **naturally** or **canonically** determined by F_*.
- According to Fact (ii), the vector spaces $\boldsymbol{V}$, $\boldsymbol{V}^*$ have the same dimension.
- Thus, they must be isomorphic.
- There is no natural isomorphism.
- However, the following Fact (iii) holds.

Linear Algebra Fact (iii)

- There is a natural isomorphism of $\boldsymbol{V}$ onto $(\boldsymbol{V}^*)^*$ given by

$$\boldsymbol{v} \to \langle \boldsymbol{v}, \cdot \rangle.$$

- That is, $\boldsymbol{v}$ is mapped to the linear function on $\boldsymbol{V}^*$ whose value on any $\sigma \in \boldsymbol{V}^*$ is $\langle \boldsymbol{v}, \sigma \rangle$.
- Note that $\langle \boldsymbol{v}, \sigma \rangle$ is linear in each variable separately (with the other fixed).
- This shows that:
  - The dual of $\boldsymbol{V}^*$ is $\boldsymbol{V}$ itself;
  - Accounts for the name "dual" space;
  - Validates the use of the symmetric notation

$$\langle \boldsymbol{v}, \sigma \rangle$$

  in preference to the functional notation $\sigma(\boldsymbol{v})$.

Covectors on Manifolds

- Let M be a C^∞ manifold and assume $p \in M$.
- We denote by $T_p^*(M)$ the dual space to $T_p(M)$.
- Thus, $\sigma_p \in T_p^*(M)$ is a linear mapping $\sigma_p : T_p(M) \to \mathbb{R}$.
- Its value on $X_p \in T_p(M)$ is denoted by $\sigma_p(X_p)$ or $\langle X_p, \sigma_p \rangle$.
- Suppose $E_{1p}, \ldots, E_{np}$ is a basis of $T_p(M)$.
- There is a uniquely determined dual basis $\omega_p^1, \ldots, \omega_p^n$ satisfying, by definition,
$$\omega_p^i(E_{jp}) = \delta_j^i.$$
- The components of σ_p relative to this basis are equal to the values of σ_p on the basis vectors $E_{1p}, \ldots, E_{np}$,
$$\sigma_p = \sum_{i=1}^n \sigma_p(E_{ip})\omega_p^i.$$

Covector Fields on Manifolds

- We have defined a vector field on M.
- Similarly, we may define a **covector field**.
- It is a (regular) function σ, assigning to each $p \in M$ an element σ_p of $T_p^*(M)$.
- We denote such a function by $\sigma, \lambda, \ldots$.
- We denote by $\sigma_p, \lambda_p, \ldots$ its value at p.
- This is the element of $T_p^*(M)$ assigned to p.

Vector and Covector Fields on Manifolds

- Let σ be a covector field on M.
- Let X be a vector field on on an open subset U of M.
- Then $\sigma(X)$ defines a function on U.
- To each $p \in U$ we assign the number

$$\sigma(X)(p) = \sigma_p(X_p).$$

- We often write $\sigma(X_p)$ for $\sigma_p(X_p)$ if σ is a covector field.

Covector Fields

Definition

A C^r-**covector field** σ on M, $r \geq 0$, is a function which assigns to each $p \in M$ a covector $\sigma_p \in T_p^*(M)$ in such a manner that for any coordinate neighborhood U, φ with coordinate frames $E_1, \ldots, E_n$, the functions $\sigma(E_i)$, $i = 1, \ldots, n$, are of class C^r on U.
For convenience, "covector field" will mean C^∞-covector field.

- One may wish to avoid the use of local coordinates.
- In that case, the following (apparently stronger) regularity condition could be used to replace the requirement of the definition.

  Suppose that σ assigns to each $p \in M$ an element σ_p of $T_p^*(M)$.
  σ is of class C^r, iff, for any C^∞-vector field X on an open subset W of M, the function $\sigma(X)$ is of class C^r on W.

Covector Fields (Cont'd)

- We show why the preceding equivalence holds.
- Take a covering of W by coordinate neighborhoods of M (whose domains are in W).
- Let U, φ be such a neighborhood.
- Then, for some α^i, which are C^∞ on U,

$$X = \sum \alpha^i E_i.$$

- Thus, on U,

$$\sigma(X) = \sum \alpha^i \sigma(E_i).$$

- This is C^r if $\sigma(E_1), \ldots, \sigma(E_n)$ are.
- Hence the condition given implies $\sigma(X)$ is of class C^r on a collection of open sets covering W.
- So it is C^r on W itself.
- The converse is obvious.

Field of Coframes

- Let $E_1, \ldots, E_n$ be a field of (C^∞) frames on an open set $U \subseteq M$.
- Consider the dual basis at each point of U.
- These define a field of dual bases $\omega^1, \ldots, \omega^n$ on U satisfying

$$\omega^i(E_j) = \delta^i_j.$$

- We call this a field of **coordinate coframes** if $E_1, \ldots, E_n$ are coordinate frames.
- The $\omega^1, \ldots, \omega^n$ are of class C^∞ by the criterion just stated.
- Covector field σ is of class C^r if and only if, for each coordinate neighborhood U, φ, the components of σ relative to the coordinate coframes are functions of class C^r on U.

Remark

- Let M be a manifold.
- Recall that $\mathfrak{X}(M)$ denotes the collection of all C^∞ vector fields on M.
- It is important to note that a C^r-covector field defines a map of

$$\mathfrak{X}(M) \to C^r(M).$$

- This map is not only $\mathbb{R}$-linear but even $C^r(M)$-linear.
- More precisely, suppose:
  - $f, g \in C^r(M)$;
  - X and Y are vector fields on M.
- Then

$$\sigma(fX + gY) = f\sigma(X) + g\sigma(Y),$$

since these functions are equal at each $p \in M$.

Example: Differential Covector Field

- Let f be a C^∞ function on M.
- f defines a C^∞-covector field, denoted df, by the formula

$$\langle X_p, df_p \rangle = X_p f \quad \text{or} \quad df_p(X_p) = X_p f.$$

- For a vector field X on M, this gives

$$df(X) = Xf,$$

  a C^∞ function on M.
- This covector field df is called the **differential of** f.
- Its value at p, df_p, is called the **differential of** f **at** p.

Example (The Case of $\mathbb{R}^n$)

- In the case of an open set $U \subseteq \mathbb{R}^n$, we verify that it coincides with the usual notion of differential of a function in advanced calculus.
- In fact, it makes the notion of differential more precise.
- In this case, the coordinates x^i of a point of U are functions on U.
- By our definition, dx^i assigns to each vector X at $p \in U$ a number $X_p x^i$, its ith component in the natural basis of $\mathbb{R}^n$.
- In particular,

$$\left\langle \frac{\partial}{\partial x^j}, dx^i \right\rangle = \frac{\partial x^i}{\partial x^j} = \delta^i_j.$$

- So we see that $dx^1, \ldots, dx^n$ is exactly the field of coframes dual to $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$.

Example (Cont'd)

- Suppose f is a C^∞ function on U.
- Then we may express df as a linear combination of $dx^1, \ldots, dx^n$.
- We know that the coefficients in this combination, that is the components of df, are given by $df(\frac{\partial}{\partial x^i}) = \frac{\partial f}{\partial x^i}$.
- Thus we have

$$df = \frac{\partial f}{\partial x^1} dx^1 + \cdots + \frac{\partial f}{\partial x^n} dx^n.$$

- Suppose $a \in U$ and $X_a \in T_a(\mathbb{R}^n)$.
- Then X_a has components, say, $h^1, \ldots, h^n$ and geometrically X_a is the vector from a to $a + h$.
- We have

$$df(X_a) = X_a f = \left(\sum h^i \frac{\partial}{\partial x^i} \right) f = \sum h^i \left(\frac{\partial f}{\partial x^i} \right)_a.$$

Example (Cont'd)

- In particular, $dx^i(X_a) = h^i$.
- That is, dx^i measures the change in the ith coordinate of a point which moves from the initial to the terminal point of X_a.
- The preceding formula may thus be written

$$df(X_a) = \left(\frac{\partial f}{\partial x^1}\right)_a dx^1(X_a) + \cdots + \left(\frac{\partial f}{\partial x^n}\right)_a dx^n(X_a).$$

- This gives us a very good definition of the **differential of a function** f on $U \subseteq \mathbb{R}^n$.
  - df is a field of linear functions which, at each point a of the domain of f, assigns to the vector X_a a number.
  - X_a can be interpreted as the displacement of the n independent variables from a, i.e., it has a as initial and $a + h$ as terminal point.
  - $df(X_a)$ approximates (linearly) the change in f between these points.

Covector Fields and Mappings

- Let $F : M \to N$ be a smooth mapping and suppose $p \in M$.
- Then, as we know, there is induced a linear map

$$F_* : T_p(M) \to T_{F(p)}(N).$$

- We know that F_* determines a linear map $F^* : T^*_{F(p)}(N) \to T^*_p(M)$, given by the formula

$$F^*(\sigma_{F(p)})(X_p) = \sigma_{F(p)}(F_*(X_p)).$$

- In general, F_* does not map vector fields on M to vector fields on N.
- It is surprising, then, that given any C^r-covector field on N, F^* determines (uniquely) a covector field of the same class C^r on M by this formula.

Covector Field Determined by a Mapping

Theorem

Let $F : M \to N$ be C^∞ and let σ be a covector field of class C^r on N. Then

$$F^*(\sigma_{F(p)})(X_p) = \sigma_{F(p)}(F_*(X_p))$$

defines a C^r-covector field on M.

- Let σ be a covector field on N.

  By definition, for any $p \in M$, there is exactly one image point $F(p)$.

  It is, thus, clear that $F^*(\sigma)$ is defined uniquely at each point of M.

  Suppose that, for $p_0 \in M$, we take coordinate neighborhoods U, φ of p_0 and V, ψ of $F(p_0)$, such that $F(U) \subseteq V$.

  Denote the coordinates on U by $(x^1, \ldots, x^m)$.

  Denote the coordinates on V by $(y^1, \ldots, y^n)$.

Covector Field Determined by a Mapping (Cont'd)

- Then we may suppose the mapping F to be given in local coordinates by
$$y^i = f^i(x^1, \ldots, x^m), \quad i = 1, \ldots, n.$$
Let the expression for σ on V, in the local coframes, at $q \in V$ be
$$\sigma_q = \sum_{i=1}^{n} \alpha_i(q) \widetilde{\omega}_q^i,$$
where $\widetilde{\omega}_q^1, \ldots, \widetilde{\omega}_q^n$ is the basis of $T_q^*(N)$ dual to the coordinate frames.
The functions $\alpha^i(q)$ are of class C^r on V, by hypothesis.
Let p be any point on U and $q = F(p)$ its image.
Using the formula defining F^*, we see that
$$(F^*(\sigma))_p(E_{jp}) = \sigma_{F(p)}(F_*(E_{jp})) = \sum \alpha_i(F(p)) \widetilde{\omega}_{F(p)}^i (F_*(E_{jp})).$$

Covector Field Determined by a Mapping (Cont'd)

- We got

$$(F^*(\sigma))_p(E_{jp}) = \sum \alpha_i(F(p))\widetilde{\omega}^i_{F(p)}(F_*(E_{jp})).$$

However, we have previously obtained the formula

$$F_*(E_{jp}) = \sum_{k=1}^{n} \frac{\partial y^k}{\partial x^j} \widetilde{E}_{kF(p)}, \quad j = 1, \ldots, m,$$

the derivatives being evaluated at $\varphi(p) = (x^1(p), \ldots, x^m(p))$.
Using $\widetilde{\omega}^i(\widetilde{E}_j) = \delta^i_j$, we obtain

$$(F^*(\sigma))_p(E_{jp}) = \sum_{i=1}^{n} \alpha_i(F(p)) \left(\frac{\partial y^i}{\partial x^j}\right)_{\varphi(p)}.$$

As p varies over U these expressions give the components of $F^*(\sigma)$ relative to $\omega^1, \ldots, \omega^m$ on U, the coframes dual to $E_1, \ldots, E_m$.
They are clearly of class C^r at least, completing the proof.

Formulas for $F^*(\sigma)$

Corollary

Using the notation above, suppose:

- $\sigma = \sum_{i=1}^{n} \alpha_i \widetilde{\omega}^i$ on V;
- $F^*(\sigma) = \sum_{j=1}^{m} \beta_j \omega^j$ on U,

where α_i and β_j are functions on V and U, respectively, and $\widetilde{\omega}^i, \omega^j$ are the coordinate coframes. Then:

- For $i = 1, \ldots, n$,
$$F^*(\widetilde{\omega}^i) = \sum_{j=1}^{m} \frac{\partial y^i}{\partial x^j} \omega^j;$$
- For $j = 1, \ldots, m$,
$$\beta_j = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^j} \alpha_i.$$

A Special Case

- The formulas
$$F^*(\widetilde{\omega}^i) = \sum_{j=1}^{m} \frac{\partial y^i}{\partial x^j} \omega^j, \quad i = 1, \ldots, n,$$
give the relation of the bases.
- The formulas
$$\beta_j = \sum_{i=1}^{n} \frac{\partial y^i}{\partial x^j} \alpha_i, \quad j = 1, \ldots, m,$$
give the relation of the components.
- Apply this directly to a map of an open subset of $\mathbb{R}^m$ into an open subset of $\mathbb{R}^n$.
- Then we get for $F^*(dy^i)$ the formula
$$F^*(dy^i) = \sum_{j=1}^{m} \frac{\partial y^i}{\partial x^j} dx^j, \quad i = 1, \ldots, n.$$

Remark

- Suppose we apply the above considerations to the diffeomorphism $\varphi : U \to \mathbb{R}^n$ of a coordinate neighborhood U, φ on M.
- Let $V \subseteq \mathbb{R}^n$ denote $\varphi(U)$.
- Let $dx^1, \ldots, dx^n$ be the differentials of the coordinates of $\mathbb{R}^n$.
- That is, $dx^1, \ldots, dx^n$ is the dual basis to $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$.
- By definition, we have $\varphi_*^{-1}(\frac{\partial}{\partial x^i}) = E_i$.
- Hence, $\varphi_*(E_i) = \frac{\partial}{\partial x^i}$, for each i.
- Further, the definition of F_* above gives for $\varphi_*(dx^i)$

$$\langle E_j, \varphi_*(dx^i) \rangle = \langle \varphi_*(E_j), dx^i \rangle = \delta^i_j.$$

- It follows that $\varphi_*(dx^i) = \omega^i$, $i = 1, \ldots, n$, the field of coframes on U dual to the coordinate frames $E_1, \ldots, E_n$.

Notation

- There is a potential source of confusion in notation.
- The coordinates $x^1, \ldots, x^n$ can be considered as functions on U.
- As such, they have differentials dx^i defined by

$$\langle X, dx^i \rangle = Xx^i,$$

  the ith component of X in the coordinate frames.
- In particular, $\langle E_j, dx^i \rangle = E_j x^i = \delta^i_j$.
- So $dx^1, \ldots, dx^n$ are dual to $E_1, \ldots, E_n$.
- Therefore $dx^i = \omega^i$, $i = 1, \ldots, n$.
- Combining this with the formula above gives $dx^i = \varphi^*(dx^i)$.
- This is nonsense, unless we are careful to distinguish x^i as (coordinate) function on $U \subseteq M$, on the left, from x^i as (coordinate) function on $\varphi(U) = V \subseteq \mathbb{R}^n$, on the right.

Example

- We may apply the theorem to obtain examples of covector fields on a submanifold M of a manifold N.
- Let $i : M \to N$ be the inclusion map.
- Suppose σ is a covector field on N.
- Then $i^*(\sigma)$ is a covector field on M called the **restriction** of σ to M.
- It is often denoted σ_M or simply σ.
- Recall that, for each $p \in M$, $T_p(M)$ is identified with a subspace of $T_p(N)$ by the isomorphism i_*.
- So we have for $X_p \in T_p(M)$

$$\sigma_M(X_p) = (i^*\sigma)(X_p) = \sigma(i_*(X_p)) = \sigma(X_p).$$

- The last equality is the identification.

Example (Cont'd)

- As an example, let $M \subseteq \mathbb{R}^n$.
- Let σ be a covector field on $\mathbb{R}^n$, for example take $\sigma = dx^1$.
- Then σ restricts to a covector field σ_M on M.
- Note that in this example dx^1 is never zero as a covector field on $\mathbb{R}^n$.
- But on M it is zero at any point p at which the tangent hyperplane $T_p(M)$ is orthogonal to the x^1-axis.

Subsection 2

Bilinear Forms and The Riemannian Metric

Bilinear Forms

- Let $\boldsymbol{V}$ be a vector space over $\mathbb{R}$.
- A **bilinear form** on $\boldsymbol{V}$ is defined to be a map

$$\Phi : \boldsymbol{V} \times \boldsymbol{V} \to \mathbb{R}$$

  that is linear in each variable separately.
- That is, for $\alpha, \beta \in \mathbb{R}$ and $\boldsymbol{v}, \boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{w}, \boldsymbol{w}_1, \boldsymbol{w}_2 \in \boldsymbol{V}$,

$$\begin{aligned} \Phi(\alpha \boldsymbol{v}_1 + \beta \boldsymbol{v}_2, \boldsymbol{w}) &= \alpha \Phi(\boldsymbol{v}_1, \boldsymbol{w}) + \beta \Phi(\boldsymbol{v}_2, \boldsymbol{w}), \\ \Phi(\boldsymbol{v}, \alpha \boldsymbol{w}_1 + \beta \boldsymbol{w}_2) &= \alpha \Phi(\boldsymbol{v}, \boldsymbol{w}_1) + \beta \Phi(\boldsymbol{v}, \boldsymbol{w}_2). \end{aligned}$$

- A similar definition may be made for a map Φ of a pair of vector spaces $\boldsymbol{V} \times \boldsymbol{W}$ over $\mathbb{R}$.
- Note that the map assigning to each pair $\boldsymbol{v} \in \boldsymbol{V}$, $\sigma \in \boldsymbol{V}^*$ a number $\langle \boldsymbol{v}, \sigma \rangle$, as discussed in the preceding section, is an example.

Bilinear Forms and Matrices

- Bilinear forms on $\boldsymbol{V}$ are completely determined by their n^2 values on a basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ of $\boldsymbol{V}$.
- Suppose $\alpha_{ij} = \Phi(\boldsymbol{e}_i, \boldsymbol{e}_j)$, $1 \leq i, j \leq n$, are given.
- Let $\boldsymbol{v} = \sum \lambda^i \boldsymbol{e}_i$, $\boldsymbol{w} = \sum \mu^j \boldsymbol{e}_j$ be any pair of vectors in $\boldsymbol{V}$.
- Bilinearity requires that

$$\Phi(\boldsymbol{v}, \boldsymbol{w}) = \sum_{i,j=1}^{n} \alpha_{ij} \lambda^i \mu^j.$$

- Conversely, let an $n \times n$ matrix $A = (\alpha_{ij})$ of real numbers be given.
- Then the formula just given determines a bilinear form Φ.
- Thus, there is a one-to-one correspondence between $n \times n$ matrices and bilinear forms on $\boldsymbol{V}$ once a basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is chosen.
- The numbers α_{ij} are called the **components** of Φ **relative to the basis**.

Symmetric and Skew-Symmetric Forms

- A bilinear form, or function, is called **symmetric** if
$$\Phi(\mathbf{v}, \mathbf{w}) = \Phi(\mathbf{w}, \mathbf{v}).$$
- It is called **skew-symmetric** if
$$\Phi(\mathbf{v}, \mathbf{w}) = -\Phi(\mathbf{w}, \mathbf{v}).$$
- It is easily seen that, regardless of the basis chosen, these correspond, respectively, to:
  - Symmetric matrices of components,
  $$A^T = A;$$
  - Skew-symmetric matrices of components,
  $$A^T = -A.$$

Positive Definite Forms and Inner Products

- A symmetric form is called **positive definite** if

$$\Phi(\boldsymbol{v}, \boldsymbol{v}) \geq 0$$

  and equality holds if and only if $\boldsymbol{v} = 0$.
- In this case we often call Φ an **inner product** on $\boldsymbol{V}$.
- A vector space with an inner product is called a **Euclidean vector space**, since Φ allows us to define:
  - The length of a vector,

$$\|\boldsymbol{v}\| = \sqrt{\Phi(\boldsymbol{v}, \boldsymbol{v})}.$$

  - The angle between vectors.

Field of Bilinear Forms

Definition

A **field Φ of C^r-bilinear forms**, $r \geq 0$, **on a manifold** M consists of a function assigning to each point p of M a bilinear form Φ_p on $T_p(M)$. That is, a bilinear mapping

$$\Phi_p : T_p(M) \times T_p(M) \to \mathbb{R},$$

such that for any coordinate neighborhood U, φ the functions

$$\alpha_{ij} = \Phi(E_i, E_j),$$

defined by Φ and the coordinate frames $E_1, \ldots, E_n$ are of class C^r.
Unless otherwise stated, bilinear forms will be C^∞.
To simplify notation we usually write $\Phi(X_p, Y_p)$ for $\Phi_p(X_p, Y_p)$.

Remarks

- The n^2 functions
$$\alpha_{ij} = \Phi(E_i, E_j)$$
on U are called the **components of Φ in the coordinate neighborhood** U, φ.
- Let Φ be a function assigning to each $p \in M$ a bilinear form.
- Then Φ is of class C^r if and only if for every pair of vector fields X, Y on an open set U of M, the function $\Phi(X, Y)$ is C^r on U.
- Φ is $C^\infty(U)$-bilinear as well as $\mathbb{R}$-bilinear.
- That is, for $f \in C^\infty(U)$,
$$\Phi(fX, Y) = f\Phi(X, Y) = \Phi(X, fY).$$

Induced Mappings of Bilinear Forms

- Let $F_* : W \to V$ be a linear map of vector spaces.
- Let Φ be a bilinear form on V.
- Then the formula

$$(F^*\Phi)(\boldsymbol{v}, \boldsymbol{w}) = \Phi(F_*(\boldsymbol{v}), F_*(\boldsymbol{w}))$$

  defines a bilinear form $F^*\Phi$ on W.
- We have the following properties:
  (i) If Φ is symmetric, then $F^*\Phi$ is symmetric.
  If Φ is skew-symmetric, then $F^*\Phi$ is skew-symmetric.
  (ii) If Φ is symmetric, positive definite, and F_* is injective, then $F^*\Phi$ is symmetric, positive definite.
- The latter applies to the identity map i_* of a subspace W into V.
- In this case $i^*\Phi$ is just restriction of Φ to W:

$$(i^*\Phi)(\boldsymbol{v}, \boldsymbol{w}) = \Phi(i_*\boldsymbol{v}, i_*\boldsymbol{w}) = \Phi(\boldsymbol{v}, \boldsymbol{w}).$$

Relation Between Components

- Let $F : M \to N$ be a C^∞ map.
- Suppose that Φ is a field of bilinear forms on N.
- Then, just as in the case of covectors, this defines a field of bilinear forms $F^*\Phi$ on M by the formula for $(F^*\Phi)_p$ at every $p \in M$,

$$(F^*\Phi)(X_p, Y_p) = \Phi(F_*(X_p), F_*(Y_p)).$$

Theorem

Let $F : M \to N$ be a C^∞ map and Φ a bilinear form of class C^r on N. Then $F^*\Phi$ is a C^r-bilinear form on M. Moreover, if Φ is symmetric (skew-symmetric), then $F^*\Phi$ is symmetric (skew-symmetric).

- Suppose U, φ is a coordinate neighborhood of p, V, ψ is a coordinate neighborhood of $F(p)$, such that

$$F(U) \subseteq V.$$

Relation Between Components (Cont'd)

- We may write

$$\beta_{ij}(p) = (F^*\Phi)_p(E_{ip}, E_{jp}) = \Phi(F_*(E_{ip}), F_*(E_{jp})).$$

Applying a previous theorem, we have

$$\beta_{ij}(p) = \sum_{s,t=1}^{n} \frac{\partial y^s}{\partial x^i} \frac{\partial y^t}{\partial x^j} \Phi(\widetilde{E}_{sF(p)}, \widetilde{E}_{tF(p)}).$$

This gives a formula for the matrix of components (β_{ij}) of $F^*\Phi$ at p in terms of the matrix (α_{st}) of Φ at $F(p)$,

$$\beta_{ij} = \sum_{s,t=1}^{n} \frac{\partial y^s}{\partial x^i} \frac{\partial y^t}{\partial x^j} \alpha_{st}(F(p)), \quad 1 \leq i, j \leq m.$$

The functions β_{ij}, thus defined, are of class C^r at least on U. The statements about symmetry and skew-symmetry are obvious consequences of Property (i), mentioned above.

Immersions and Positive Definite Forms

Corollary

If F is an immersion and Φ is a positive definite, symmetric form, then $F^*\Phi$ is a positive definite, symmetric bilinear form.

- We must check that $F^*\Phi$ is positive definite at each $p \in M$.

  Let X_p be any vector tangent to M at p.

  Then
  $$F^*\Phi(X_p, X_p) = \Phi(F_*(X_p), F_*(X_p)) \geq 0.$$

  Moreover, equality holds only if $F_*(X_p) = 0$.

  However, F is an immersion.

  So we have
  $$F_*(X_p) = 0 \quad \text{if and only if} \quad X_p = 0.$$

Riemannian Manifolds

Definition

A manifold M on which there is defined a field of symmetric, positive definite, bilinear forms Φ is called a **Riemannian manifold** and Φ the **Riemannian metric**.

We shall assume always that Φ is of class C^∞.

Example

- The simplest example is $\mathbb{R}^n$ with its natural inner product

$$\Phi_a(X_a, Y_a) = \sum_{i=1}^{n} \alpha^i \beta^i,$$

  where $X = \sum \alpha^i \frac{\partial}{\partial x^i}$ and $Y = \sum \beta^i \frac{\partial}{\partial x^i}$.

- At each point we have

$$\Phi\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij}.$$

- So the matrix of components of Φ, relative to the standard basis, is constant and equals I, the identity matrix.
- It follows that Φ is C^∞.

More Examples

- Any imbedded or immersed sub manifold M of $\mathbb{R}^n$ is endowed with a Riemannian metric from $\mathbb{R}^n$ by virtue of the imbedding (or immersion) $F: M \to \mathbb{R}^n$.
- Thus, for example, a surface M in $\mathbb{R}^3$ has a Riemannian metric.
- The idea of the corollary in this case is very simple.
- Let $i: M \to \mathbb{R}^3$ be the identity.
- Let X_p, Y_p be tangent vectors to M at p.
- Then

$$i^*\Phi(X_p, Y_p) = \Phi(i_*X_p, i_*Y_p) = \Phi(X_p, Y_p).$$

More Examples (Cont'd)

- We got
$$i^*\Phi(X_p, Y_p) = \Phi(X_p, Y_p).$$
- That is, we simply take the value of the form on X_p, Y_p considered as vectors in $\mathbb{R}^3$, using our standard identification of $T_p(M)$ with a subspace of $T_p(\mathbb{R}^3)$.
- In particular S^2, the unit sphere of $\mathbb{R}^3$, has a Riemannian metric induced by the standard inner product in $\mathbb{R}^3$.
- Let X_p, Y_p be tangent to S^2 at p.
- Then $\Phi(X_p, Y_p)$ is just their inner product in $\mathbb{R}^3$.

First Fundamental Form

- Classical differential geometry deals with properties of surfaces in Euclidean space.
- The inner product Φ on the tangent space at each point of the surface, inherited from Euclidean space, is an essential element in the study of the geometry of the surface.
- It is known as the **first fundamental form** of the surface.

Properties of Bilinear Forms: Rank

- We define the **rank** of a form Φ on $\boldsymbol{V}$ to be the codimension of the subspace

$$\boldsymbol{W} = \{\boldsymbol{v} \in \boldsymbol{V} : \Phi(\boldsymbol{v}, \boldsymbol{w}) = 0, \text{ for all } \boldsymbol{w} \in \boldsymbol{V}\}.$$

- That is, $\text{rank}\Phi = \dim \boldsymbol{V} - \dim \boldsymbol{W}$.
- The following facts are often useful:
  - (iii) If Φ is a bilinear form on $\boldsymbol{V}$, then the linear mapping $\varphi : \boldsymbol{V} \to \boldsymbol{V}^*$ defined by $\langle \boldsymbol{w}, \varphi(\boldsymbol{v}) \rangle = \Phi(\boldsymbol{w}, \boldsymbol{v})$ is an isomorphism onto if and only if $\text{rank}\Phi = \dim \boldsymbol{V}$.
  - (iv) Every bilinear form Φ may be written uniquely as the sum of a symmetric and a skew-symmetric bilinear form, namely,

$$\Phi(\boldsymbol{v}, \boldsymbol{w}) = \frac{1}{2}[\Phi(\boldsymbol{v}, \boldsymbol{w}) + \Phi(\boldsymbol{w}, \boldsymbol{v})] + \frac{1}{2}[\Phi(\boldsymbol{v}, \boldsymbol{w}) - \Phi(\boldsymbol{w}, \boldsymbol{v})].$$

  - (v) If a skew-symmetric form Φ has a rank equal to $\dim \boldsymbol{V}$, then $\dim \boldsymbol{V}$ is an even number.

Subsection 3

Riemannian Manifolds as Metric Spaces

Importance of Riemannian Manifolds

- The importance of the Riemannian manifold derives from the fact that it makes the tangent space at each point into a Euclidean space, with inner product defined by $\Phi(X_p, Y_p)$.
- This enables us to define:
  - *Angles* between curves, that is, the angle between their tangent vectors X_p and Y_p at their point of intersection;
  - *Lengths* of curves on M.
- Thus we may study many questions concerning the geometry of these manifolds.
- This forms a large part of the classical differential geometry of surfaces in $\mathbb{R}^3$.

Defining the Length of a Curve

- Let

$$t \to p(t), \quad a \le t \le b,$$

be a curve of class C^1 on a Riemannian manifold M.

- Then its **length** L is defined to be the value of the integral

$$L = \int_a^b \left(\Phi \left(\frac{dp}{dt}, \frac{dp}{dt} \right) \right)^{1/2} dt.$$

- The integrand is a function of t alone.
- So a more precise notation is to denote its value at each t by

$$\Phi_{p(t)} \left(\frac{dp}{dt}, \frac{dp}{dt} \right),$$

where $\frac{dp}{dt} \in T_{p(t)}(M)$ is the tangent vector to the curve at $p(t)$.

- This function is continuous, by the continuity of $\frac{dp}{dt}$ and Φ.

Independence of the Length from Parametrization

- The value of the integral

$$L = \int_a^b \left(\Phi \left(\frac{dp}{dt}, \frac{dp}{dt} \right) \right)^{1/2} dt$$

is independent of the parametrization.

- Consider a new parametrization

$$t = f(s), \quad c \leq s \leq d.$$

- We have given the formula for change of parameter,

$$\frac{dp}{ds} = \frac{dp}{dt} \frac{dt}{ds}.$$

- So we obtain

$$\begin{aligned} \int_c^d (\Phi(\tfrac{dp}{ds}, \tfrac{dp}{ds}))^{1/2} ds &= \int_a^b (\Phi(\tfrac{dp}{dt}, \tfrac{dp}{dt})(\tfrac{dt}{ds})^2)^{1/2} \tfrac{ds}{dt} dt \\ &= \int_a^b (\Phi(\tfrac{dp}{dt}, \tfrac{dp}{dt}))^{1/2} dt. \end{aligned}$$

Parametrization by the Length

- Consider the arc length along the curve from $p(a)$ to $p(t)$, which may be denoted by $s = L(t)$.
- It gives a new parameter by the formula
$$s = L(t) = \int_a^t \left(\Phi\left(\frac{dp}{dt}, \frac{dp}{dt} \right) \right)^{1/2} dt.$$
- This implies
$$\frac{ds}{dt} = \left(\Phi\left(\frac{dp}{dt}, \frac{dp}{dt} \right) \right)^{1/2}.$$
- Equivalently,
$$\left(\frac{ds}{dt} \right)^2 = \Phi\left(\frac{dp}{dt}, \frac{dp}{dt} \right).$$

Parametrization by the Length (Cont'd)

- Let U, φ be a coordinate neighborhood with coordinate frames
$$E_{1p}, \ldots, E_{np}.$$
- Within U, φ, with $\varphi(p) = x = (x^1, \ldots, x^n)$, we have
$$\Phi(E_{ip}, E_{jp}) = g_{ij}(x).$$
- The curve is given by
$$\varphi(p(t)) = (x^1(t), \ldots, x^n(t)).$$
- So $L(t)$ becomes
$$s = L(t) = \int_a^t \left(\sum g_{ij}(x(t)) \frac{dx^i}{dt} \frac{dx^j}{dt} \right)^{1/2} dt.$$
- So, in local coordinates, the Riemannian metric is abbreviated
$$ds^2 = \sum_{i,j=1}^n g_{ij}(x) dx^i dx^j.$$

The Case of $\mathbb{R}^n$

- Consider $\mathbb{R}^n$, with its standard inner product.
- Let
$$p(t) = (x^1(t), \ldots, x^n(t)), \quad a \leq t \leq b,$$
be a curve in $\mathbb{R}^n$.
- Then we have
$$\Phi\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \delta_{ij}.$$
- Moreover,
$$\frac{dp}{dt} = \sum_{i=1}^{n} \dot{x}^i(t) \frac{\partial}{\partial x^i}.$$
- So we have the familiar formula for arc length
$$L = \int_a^b \left(\sum_{i=1}^{n} (\dot{x}^i(t))^2\right)^{1/2} dt.$$

Connected Riemannian Manifolds as Metric Spaces

- Let D^1 be the class of functions that are piecewise C^1.

Theorem

A connected Riemannian manifold is a metric space with the metric

$$d(p,q) = \text{infimum of the lengths of curves of class } D^1 \text{ from } p \text{ to } q.$$

Its metric space topology and manifold topology agree.

- Since M is arcwise connected, $d(p,q)$ is defined.

  By definition $d(p,q)$ is symmetric and nonnegative.

  A curve from p_1 to p_2 and a curve from p_2 to p_3 may be joined to give a curve from p_1 to p_3.

  The length of this curve is the sum of the lengths of the two curves.

  It follows that the triangle inequality is satisfied.

Riemannian Manifolds as Metric Spaces (Cont'd)

- In order to complete the proof we need some inequalities.

  Let p be an arbitrary point of M.

  Let U, φ be a coordinate neighborhood, with $\varphi(p) = (0, \ldots, 0)$.

  Let $a > 0$ be a fixed real number with the property that

$$\varphi(U) \supseteq \overline{B}_a(0),$$

  the closure of the open ball of radius a and center the origin of $\mathbb{R}^n$.

  Let $x^1, \ldots, x^n$ denote the local coordinates.

  Let $g_{ij}(x)$ the components of the metric tensor Φ as functions of these coordinates. These n^2 functions are:
  - C^∞ in their dependence on the coordinates;
  - The coefficients of a positive definite, symmetric matrix for each value of x in $\varphi(U)$.

Riemannian Manifolds as Metric Spaces (Cont'd)

- Consider the compact set defined by

$$\|x\| < r, \quad r \leq a,$$

where $a = (a^1, \ldots, a^n)$ is such that $\sum_{i=1}^n (a^i)^2 = 1$
By the properties of $g_{ij}(x)$, on this compact, the expression

$$\left(\sum_{i,j=1}^n g_{ij}(x) \alpha^i \alpha^j \right)^{1/2}$$

assumes a maximum value M_r and a minimum value $m_r > 0$.
Let m, M denote the min and max corresponding to $r = a$.
Then we have the inequalities

$$0 < m \leq m_r \leq \left(\sum_{i,j=1}^n g_{ij}(x) \alpha^i \alpha^j \right)^{1/2} \leq M_r \leq M.$$

Riemannian Manifolds as Metric Spaces (Cont'd)

- Now let $(\beta^1, \ldots, \beta^n)$ be any n real numbers, such that

$$\left(\sum_{i=1}^{n}(\beta^i)^2\right)^{1/2} = b \neq 0.$$

In the preceding, replace each α^i by $\frac{\beta^i}{b}$.

Then, multiply the inequalities by b.

We get, for every $x \in \overline{B}_r(0)$,

$$0 \leq mb \leq m_r b \leq \left(\sum_{i,j=1}^{n} g_{ij}\beta^i\beta^j\right)^{1/2} \leq M_r b \leq Mb.$$

Intermission: An Assumption Concerning $\mathbb{R}^n$

- Now we shall make the following assumption.
- If x, y are any points of $\mathbb{R}^n$ with its standard Riemannian metric (as defined above), then the infimum of the lengths of all D^1 curves in $\mathbb{R}^n$ from x to y is exactly the length of the line segment $\overline{xy}$.
- In other words, it is $\|y - x\|$ the Euclidean distance from x to y.

Riemannian Manifolds as Metric Spaces (Cont'd)

- Let $p(t)$, $a \leq t \leq b$, be a D^1 curve lying in $\varphi^{-1}(\overline{B}_r(0)) \subseteq U$ which runs from $p = p(a)$ to $q = p(b)$.

  Let its length be

$$L = \int_a^b \left[\sum_{i,j=1}^n g_{ij}(x(t)) \dot{x}^i(t) \dot{x}^j(t) \right]^{1/2} dt.$$

  The last set of inequalities above and the assumption on $\mathbb{R}^n$ imply that, for $p \neq q$,

$$\begin{aligned} 0 < m\|\varphi(q)\| < m_r\|\varphi(q)\| \leq L \\ \leq M_r \int_a^b \left[\sum_{i=1}^n (\dot{x}^i)^2\right]^{1/2} dt \leq M \int_a^b \left[\sum_{i=1}^n (\dot{x}^i)^2\right]^{1/2} dt. \end{aligned}$$

Riemannian Manifolds as Metric Spaces (Cont'd)

- We first use these inequalities to complete the proof that $d(p, q)$ is a metric on M.

  Let q' be any point of M distinct from p.

  Then, for some r, $0 < r \leq a$, q' lies outside of $\varphi^{-1}(B_r(0)) \subseteq U$.

  Consider a curve of class D^1 from $p = p(0)$ to $q' = p(c)$,

$$p(t), \quad 0 \leq t \leq c.$$

  Let L' be the length of $p(t)$, $0 \leq t \leq c$.

  There is a first point $q = p(b)$ on the curve outside $\varphi^{-1}(B_r(0))$.

  That is, such that:
  - $p(t)$ lies inside the neighborhood $\varphi^{-1}(B_r(0))$, for $0 \leq t \leq b$;
  - $q = p(b)$ lies outside $\varphi^{-1}(B_r(0))$.

Riemannian Manifolds as Metric Spaces (Cont'd)

- q is the first point of the curve with $\|\varphi(q)\| = r$.

  Let L denote the length of the curve $p(t)$, $0 \leq t \leq b$.

  Then $L \leq L'$.

  It follows that $L' \geq L \geq mr$.

  But the curve was arbitrarily chosen.

  So we get

  $$d(p,q) \geq mr.$$

  This means that if $q' \neq p$, then $d(p,q') \neq 0$.

  So $d(p,q)$ is a metric as claimed.

Riemannian Manifolds as Metric Spaces (Cont'd)

- We now show the equivalence of:
  - The metric topology on M;
  - The manifold topology on M.

  It is enough to compare the neighborhood systems at an arbitrary point p of M.

  In fact, for the manifold topology, we need only consider the neighborhoods lying inside a single coordinate neighborhood U, φ.

  Thus, we must show that each neighborhood

$$V_r = \varphi^{-1}(B_r(0)) \subseteq U$$

  of the point p contains an ε-ball,

$$S_\varepsilon(P) = \{q \in M : d(p, q) < \varepsilon\}.$$

  of the metric topology, and conversely.

Riemannian Manifolds as Metric Spaces (Cont'd)

- This will follow from the inequalities we have obtained.

  For, given $r \leq a$, choose $\varepsilon > 0$ satisfying $\frac{\varepsilon}{m} < r$.

  Let q be any point of M, such that $d(p,q) < mr$.

  We see that $q \in V_r$, since, otherwise, $d(p,q) \geq mr$ as we have seen.

  But we have chosen $\varepsilon < mr$.

  So we get $S_\varepsilon(p) \subseteq V$.

  Conversely, suppose we consider some metric ball $S_\varepsilon(p)$ about p.

  So $S_\varepsilon(p)$ is a neighborhood of p in the metric topology.

  Choose $r > 0$ so that $r < a$ and $r < \frac{\varepsilon}{M}$.

  Let $q \in V_r = \varphi^{-1}(B_r(0))$.

  Let $(\beta^1, \ldots, \beta^n)$ denote the coordinates of q.

Riemannian Manifolds as Metric Spaces (Cont'd)

- Let $p(t)$, $0 \leq t \leq b$, be the curve from p to q in V_r, defined by the coordinate functions $x^i(t) = \beta^i t$.

  The length L of this curve is given by an integral which yields

$$L = \int_0^1 \left[\sum_{i,j=1}^n g_{ij}(t\beta)\beta^i\beta^j \right]^{1/2} dt \leq M_r \left[\sum_{i=1}^n (\beta^i)^2 \right]^{1/2} \leq Mr < \varepsilon.$$

  Thus $d(p,q) < \varepsilon$ and $q \in S_\varepsilon(p)$.

  It follows that $\varphi^{-1}(B_r(0)) \subseteq S_\varepsilon(p)$.

  That is, each metric neighborhood of p contains a manifold neighborhood of p (lying inside U).

  This completes the proof of the theorem except for the unproved assertion about $\mathbb{R}^n$ (theorem itself in $\mathbb{R}^n$).

Subsection 4

Partitions of Unity

Locally Finite Coverings and Refinements

- A covering $\{A_\alpha\}$ of a manifold M by subsets is said to be **locally finite** if each $p \in M$ has a neighborhood U which intersects only a finite number of sets A_α.
- If $\{A_\alpha\}$ and $\{B_\beta\}$ are coverings of M, then $\{B_\beta\}$ is called a **refinement** of $\{A_\alpha\}$ if each $B_\beta \subseteq A_\alpha$, for some α.
- In these definitions we do not suppose the sets to be open.

Compactness

- Any manifold M is locally compact since it is locally Euclidean.
- It is also σ-**compact**, which means that it is the union of a countable number of compact sets.
- This follows from the local compactness and the existence of a countable basis $P_1, P_2, \ldots$ such that each $\overline{P}_i$ is compact.
- A space with the property that every open covering has a locally finite refinement is called **paracompact**.
- It is a standard result of general topology that a locally compact Hausdorff space with a countable basis is paracompact.

Existence of Countable, Locally Finite Refinements

Lemma

Let $\{A_\alpha\}$ be any covering of a manifold M of dimension n by open sets. Then there exists a countable, locally finite refinement $\{U_i, \varphi_i\}$, consisting of coordinate neighborhoods, with

$$\varphi_i(U_i) = B_3^n(0), \quad i = 1, 2, 3, \ldots,$$

and such that

$$V_i = \varphi^{-1}(B_1^n(0)) \subseteq U_i$$

also cover M.

- We begin with the countable basis of open sets $\{P_i\}$, $\overline{P}_i$ compact. Define a sequence of compact sets $K_1, K_2, \ldots$ as follows.

Countable, Locally Finite Refinements (Cont'd)

- Let $K_1 = \overline{P}_1$.

  Assume that $K_1, \ldots, K_i$ have been defined.

  Let r be the first integer such that

  $$K_i \subseteq \bigcup_{j=1}^{r} P_j.$$

  Define K_{i+1} by

  $$K_{i+1} = \overline{P}_1 \cup \overline{P}_2 \cup \cdots \cup \overline{P}_r = \overline{P_1 \cup \cdots \cup P_r}.$$

  Denote by $\overset{\circ}{K}_{i+1}$ the interior of K_{i+1}.

  It contains K_i.

  For each $i = 1, 2, \ldots$, consider the open set $(\overset{\circ}{K}_{i+2} - K_{i-1}) \cap A_\alpha$.

Countable, Locally Finite Refinements (Cont'd)

- Consider the open set $(\overset{\circ}{K}_{i+2} - K_{i-1}) \cap A_\alpha$.
  Around each p in this set choose a coordinate neighborhood $U_{p,\alpha}, \varphi_{p,\alpha}$ lying inside the set and such that:
  - $\varphi_{p,\alpha}(p) = 0$;
  - $\varphi_{p,\alpha}(U_{p,\alpha}) = B_3^n(0)$.

  Take $V_{p,\alpha} = \varphi_{p,\alpha}^{-1}(B_1^n(0))$.

  Note that these are also interior to $(\overset{\circ}{K}_{i+2} - K_{i-1}) \cap A_\alpha$.

  Moreover allowing p, α to vary, a finite number of the collection of $V_{p,\alpha}$ covers $K_{i+1} - K_i$, a closed compact set.

  Denote these by $V_{i,k}$ with k labeling the sets in this finite collection.

  For each $i = 1, 2, \ldots$, index k takes on just a finite number of values.

  Thus, the collection $V_{i,k}$ is denumerable.

  Renumber these sets as $V_1, V_2, \ldots$.

  Denote by U_1, φ_1, U_2, φ_2, ... the corresponding coordinate neighborhoods containing them.

Countable, Locally Finite Refinements (Cont'd)

- The U_1, φ_1, U_2, φ_2, ... satisfy the requirements of the conclusion.

  For each $p \in M$, there is an index i such that $p \in \overset{\circ}{K}_{i-1}$.

  From the definition of U_j, V_j, it is clear that only a finite number of these neighborhoods meet $\overset{\circ}{K}_{i-1}$.

  Therefore, $\{U_i\}$, and also $\{V_i\}$, are locally finite coverings refining the covering $\{A_\alpha\}$.

  Remark: It is clear that it would be possible to replace the spherical neighborhoods $B_r^n(0)$ by cubical neighborhoods $C_r^n(0)$ in the lemma.
- We shall call the refinement U_i, V_i, φ_i obtained in this lemma a **regular covering by spherical** (or, when appropriate, **cubical**) **coordinate neighborhoods** *subordinate to the open covering* $\{A_\alpha\}$.

Partition of Unity on a Manifold

- Recall that the **support** of a function f on a manifold M is the set

$$\operatorname{supp}(f) = \overline{\{x \in M : f(x) = 0\}}.$$

- That is, the closure of the set on which f vanishes.

Definition

A C^∞ **partition of unity on** M is a collection of C^∞ functions $\{f_\gamma\}$, defined on M, with the following properties:

(1) $f_\gamma \geq 0$ on M;

(2) $\{\operatorname{supp}(f_\gamma)\}$ form a locally finite covering of M;

(3) $\sum_\gamma f_\gamma(x) = 1$, for every $x \in M$.

Partition of Unity on a Manifold (Cont'd)

- Note that, by virtue of Property (2), each point has a neighborhood on which only a finite number of the f_γs are different from zero.
- It follows that the sum in Property (3) is a well-defined C^∞ function on M.
- A partition of unity is said to be **subordinate to an open covering** $\{A_\alpha\}$ of M if, for each γ, there is an A_α, such that

$$\text{supp}(f_\gamma) \subseteq A_\alpha.$$

Regular Coverings and Partitions of Unity

Theorem

Associated to each regular covering $\{U_i, V_i, \varphi_i\}$ of M, there is a partition of unity $\{f_i\}$, such that:

- $f_i > 0$ on $V_i = \varphi_i^{-1}(B_1(0))$;
- $\text{supp} f_i \subseteq \varphi_i^{-1}(\overline{B}_2(0))$.

In particular, every open covering $\{A_\alpha\}$ has a partition of unity which is subordinate to it.

- Exactly as in a previous theorem, we see that there is, for each i, a nonnegative C^∞ function $\widetilde{g}(x)$ on $\mathbb{R}^n$ which is:
  - Identically one on $\overline{B}_1^n(0)$;
  - Zero outside $B_2^n(0)$.

Regular Coverings and Partitions of Unity (Cont'd)

- Consider the function

$$g_i = \begin{cases} \widetilde{g} \circ \varphi_i, & \text{on } U_i, \\ 0, & \text{on } M - U_i. \end{cases}$$

Clearly g_i is C^∞ on M.

It has its support in $\varphi_i^{-1}(\overline{B}_2^n(0))$.

It is $+1$ on $\overline{V}_i$.

Finally, it is never negative.

Consider the functions

$$f_i = \frac{g_i}{\sum_i g_i}, \quad i = 1, 2, \ldots.$$

From the preceding properties and the fact that $\{V_i\}$ is a locally finite covering of M, we can see that the $\{f_i\}$ have the desired properties.

Existence of Riemannian Metrics

Theorem

It is possible to define a C^∞ Riemannian metric on every C^∞ Riemannian manifold.

- Let $\{U_i, V_i, \varphi_i\}$ be a regular covering of M.

  Let f_i be an associated C^∞ partition of unity as defined above.

  By hypothesis, $\varphi_i : U_i \to B_3^n(0)$ is a diffeomorphism.

  Let Ψ denote the usual Euclidean inner product on $\mathbb{R}^n$.

  Then the bilinear form

$$\Phi_i = \varphi_i^* \Psi$$

  defines a Riemannian metric on U_i.

Existence of Riemannian Metrics (Cont'd)

- Taking into account that $f_i > 0$ on V_i, consider

$$f_i \Phi_i.$$

  - It is a Riemannian metric tensor on V_i;
  - It is symmetric on U_i;
  - It is zero outside $\varphi_i^{-1}(\overline{B}_2^n(0))$.

  Hence, it may be extended to a C^∞-symmetric bilinear form on all of M, which:

  - Vanishes outside $\varphi_i^{-1}(\overline{B}_2^n(0))$;
  - Is positive definite at every point of V_i.

  It is easy to check that the sum of symmetric forms is symmetric.

Existence of Riemannian Metrics (Cont'd)

- Therefore $\Phi = \sum f_i \Phi_i$ is symmetric, where Φ is defined by

$$\Phi_p(X_p, Y_p) = \sum_{i=1}^{\infty} f_i(p) \Phi_i(X_p, Y_p), \quad p \in M.$$

We have denoted by $f_i \Phi_i$ its extension to all of M.
Recall that the summation makes sense, since in a neighborhood of each $p \in M$ all but a finite number of terms are zero.
However, Φ is also positive definite.
For every i, $f_i \geq 0$ and each $p \in M$ is contained in at least one V_j.
Then $f_j(p) > 0$.
So, if $0 = \Phi_p(X_p, X_p) = \sum f_i(p) \Phi_i(X_p, X_p)$, then $\Phi_j(X_p, X_p) = 0$.
This means $0 = \varphi_j^* \Psi(X_p, X_p) = \Psi(\varphi_{j*}(X_p), \varphi_{j*}(X_p))$.
However, Ψ is positive definite and φ is a diffeomorphism.
So this implies $X_p = 0$.
Now the proof is complete.

Imbedding a Manifold in a Power of $\mathbb{R}$

Theorem

Any compact C^∞ manifold M admits a C^∞ imbedding as a submanifold of $\mathbb{R}^N$ for sufficiently large N.

- Let $\{U_i, V_i, \varphi_i\}$ be a finite regular covering of M.

  Such a covering exists because of the compactness.

  Recall that we have defined the associated partition of unity $\{f_i\}$ using functions $\{g_i\}$, where $g_i = 1$ on V_i.

  We use here these C^∞ functions $\{g_i\}$ on M rather than the (normalized) $\{f_i\}$.

Imbedding a Manifold (Cont'd)

- Let $\varphi_i : U_i \to B_3^n(0)$ be the coordinate map.

  Consider the mapping

$$\begin{array}{rrcl} g_i\varphi_i : & U_i & \to & B_3^n(0) \\ & p & \mapsto & (g_i(p)x^1(p), \ldots, g_i(p)x^n(p)). \end{array}$$

  It is a C^∞ map on U_i.

  It takes everything outside $\varphi_i^{-1}(B_2^n(0))$ onto the origin.

  It agrees with φ_i on V_i.

  It may be extended to a C^∞ mapping of M into $B_3^n(0)$ by letting it map all of $M - U_i$ onto the origin.

  When we write $g_i\varphi_i$, we will mean this extension.

  On V_i it is a diffeomorphism to $B_1^n(0)$.

  So, on V_i, its Jacobian matrix has rank $n = \dim M$.

Imbedding a Manifold (Cont'd)

- Let $i = 1, \ldots, k$ be the range of indices in our finite regular covering.

  Let $N = (n+1)k$.

  Define

$$F : M \to \mathbb{R}^N \to \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{k} \times \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{k}$$

  by

$$F(p) = (g_1(p)\varphi_1(p); \ldots; g_k(p)\varphi_k(p); g_1(p), \ldots, g_k(p)).$$

  Then F is clearly C^∞ on M.

  Moreover, in any local coordinates on M, the $N \times n$ Jacobian of F breaks up into:

  - k blocks of size $n \times n$;
  - A $k \times n$ matrix.

  So its rank is at most n.

Imbedding a Manifold (Cont'd)

- Now, $p \in M$ implies $p \in V_i$, for some i.

  Further, on V_i, $g_i \equiv 1$.

  So $g_i \varphi_i \equiv \varphi_i$ and the matrix has rank n.

  Thus, $F : M \to \mathbb{R}^N$ is a C^∞ immersion.

  It suffices to show it is one-to-one, since then M is compact and a previous theorem applies.

  Suppose $F(p) = F(q)$.

  Then $g_i(p) = g_i(q)$, $i = 1, \ldots, k$.

  This implies that $g_i(p)\varphi_i(p) = g_i(q)\varphi_i(q)$.

  But $g_i(p) \neq 0$, for some i.

  This means $\varphi_i(p) = \varphi_i(q)$ for that i.

  Since φ_i is one-to-one, we see that $p = q$.

  Thus, F is indeed one-to-one.

Remarks

- It is an obvious disadvantage of this theorem that N may be much larger than we would like it.
- In fact we have no way of giving an effective bound on it from this proof.
- We know, e.g., that it takes at least two coordinate neighborhoods to cover S^2 (using stereographic projections from the north and south poles).
- Hence, $k = 2$ and $n = 2$, which give $N = 6$.
- So we get that S^2 may be imbedded in $\mathbb{R}^6$.
- This is obviously not the best possible!

A "Smoothing" Theorem

Theorem

Let M be a C^∞ manifold.
Let A be a compact subset of M, possibly empty.
Let g be a continuous function on M which is C^∞ on A.
Let ε be a positive continuous function on M.
There exists a C^∞ function h on M, such that:

- $g(p) = h(p)$, for every $p \in A$;
- $|g(p) - h(p)| < \varepsilon(p)$ on all of M.

- In order to prove this we shall need a similar theorem for the case of a closed n-ball in $\mathbb{R}^n$.

Weierstraß Approximation Theorem

Lemma (Weierstraß Approximation Theorem)

Let f be a continuous function on a closed n-ball $\overline{B}^n$ of $\mathbb{R}^n$ and let $\varepsilon > 0$. Then there is a polynomial function p on $\mathbb{R}^n$, such that

$$|f(x) - p(x)| < \varepsilon \quad \text{on } \overline{B}^n.$$

- By hypothesis, g is C^∞ in A.

  By definition of C^∞ function on an arbitrary subset of M, there is a C^∞ extension g^* of $g|_A$ to an open set U which contains A.

  There is no reason to believe that $g(p) = g^*(p)$ on U but not A.

  However, we may replace g by a continuous $\widetilde{g}$ on M, such that:
  (i) $|\widetilde{g}(p) - g(p)| < \frac{1}{2}\varepsilon(p)$;
  (ii) $\widetilde{g} = g$ on A;
  (iii) $\widetilde{g}$ is C^∞ on an open subset W of M which contains A.

Proof of the Theorem

- The procedure is as follows.

  Take any U and g^* as above.

  Use the compactness of A to choose an open set W containing A and such that two further requirements are met:
  - $\overline{W}$ is compact and lies in U;
  - $|g^*(p) - g(p)| < \frac{1}{2}\varepsilon(p)$ on $\overline{W}$.

  Now g^* is C^∞ on U, and, hence, continuous.

  So there is no problem in finding such a set W.

  Using a previous theorem, we define a nonnegative C^∞ function σ which is $+1$ everywhere on $\overline{W}$ and vanishes outside U.

  Finally, we define $\widetilde{g} = \sigma g^* + (1 - \sigma)g$.

  Note that $\widetilde{g}$ satisfies Conditions (i)-(iii).

Proof of the Theorem (Cont'd)

- Choose a regular covering by spherical neighborhoods $\{U_i, V_i, \varphi_i\}$ subordinate to the open covering W, $M - A$ of M.

  Denote by $\{f_i\}$ the corresponding C^∞ partition of unity.

  For every U_i on W, the function $f_i\widetilde{g}$ is:
  - C^∞ on U_i;
  - Vanishes outside $\varphi_i^{-1}(\overline{B}_2^n(0))$.

  Thus, it can be extended to a C^∞ function on M.

  Denote the extended function also by $f_i\widetilde{g}$.

  Then, on M, we have

$$\sum f_i\widetilde{g} \equiv \widetilde{g}.$$

Proof of the Theorem (Cont'd)

- Suppose $U_i \subseteq M - A$.

  Then, on $\overline{B}_2^n(0) \subseteq B_3^n(0) = \varphi_i(U_i)$, we use the Weierstraß Approximation Theorem to obtain a polynomial function p_i, with

  $$|p_i(x) - \widetilde{g} \circ \varphi_i^{-1}(x)| < \frac{1}{2}\varepsilon_i,$$

  where $\varepsilon_i = \inf \varepsilon(p)$ on $\varphi_i^{-1}(\overline{B}_2^n(0))$.

  Each ε_i is defined, since $\overline{B}_2^n(0)$ is compact.

  Let $q_i = p_i \circ \varphi_i$.

  For each i, let $f_i q_i$ be extended to a C^∞ function on all of M, which vanishes outside U_i.

Proof of the Theorem (Cont'd)

- Denote the indices such that U_i is in $M - A$ by i'.
  Denote all other indices by i''.
  Define $h(p)$ by
  $$h(p) = \sum_{i'} f_{i'} q_{i'} + \sum_{i''} f_{i''} \widetilde{g}.$$
  Each point has a neighborhood on which all but a finite number of summands vanish identically.
  So h is well defined and C^∞ on M.
  Suppose $p \in A$.
  We know that:
  - $g = \widetilde{g}$ on A;
  - Each $f_{i'}(p) = 0$ on A;
  - $\sum f_i \equiv 1$ everywhere on M.

  So we obtain
  $$h(p) = \sum_{i''} f_{i''}(p) \widetilde{g}(p) = g(p).$$

Proof of the Theorem (Cont'd)

- On the other hand we have, for $p \notin A$,

$$\begin{array}{rcl} |h(p)-\widetilde{g}(p)| & = & |\sum_{i'} f_{i'}(p)q_{i'}(p)+\sum_{i''} f_i''(p)\widetilde{g}(p) \\ & & -\sum_i f_i(p)\widetilde{g}(p)| \\ & = & |\sum f_{i'}(p)(q_{i'}(p)-\widetilde{g}(p))|. \end{array}$$

Recall that $f_i \geq 0$ for all i.

So, by the preceding, we obtain

$$|h(p)-\widetilde{g}(p)| \leq \sum f_{i'}(p)|q_{i'}(p)-\widetilde{g}(p)| \leq \frac{1}{2}\varepsilon(p)\sum f_{i'}(p).$$

But

$$\sum f_{i'}(p) \leq \sum f_i(p) = 1.$$

We deduce that

$$\begin{array}{rcl} |h(p)-g(p)| & \leq & |h(p)-\widetilde{g}(p)|+|\widetilde{g}(p)-g(p)| \\ & < & \frac{1}{2}\varepsilon(p)+\frac{1}{2}\varepsilon(p)=\varepsilon(p). \end{array}$$

Subsection 5

Tensor Fields

Tensors

Definition

Let $\boldsymbol{V}$ be a vector space over $\mathbb{R}$.
A **tensor** Φ on $\boldsymbol{V}$ is by definition a multilinear map

$$\Phi : \underbrace{\boldsymbol{V} \times \cdots \times \boldsymbol{V}}_{r} \times \underbrace{\boldsymbol{V}^* \times \cdots \times \boldsymbol{V}^*}_{s} \to \mathbb{R},$$

where:

- $\boldsymbol{V}^*$ denotes the dual space to $\boldsymbol{V}$;
- r its **covariant order**;
- s its **contravariant order**.

Tensors (Cont'd)

- By definition, a tensor Φ on $\boldsymbol{V}$ assigns to each r-tuple of elements of $\boldsymbol{V}$ and s-tuple of elements of $\boldsymbol{V}^*$ a real number.
- Moreover, if, for each k, $1 \leq k \leq r+s$, we hold every variable except the kth fixed, then Φ satisfies the linearity condition

$$\begin{aligned}\Phi(\boldsymbol{v}_1, \ldots, \alpha \boldsymbol{v}_k + \alpha' \boldsymbol{v}'_k, \ldots) \\ = \alpha \Phi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k, \ldots) + \alpha' \Phi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}'_k, \ldots),\end{aligned}$$

for all $\alpha, \alpha' \in \mathbb{R}$, and $\boldsymbol{v}_k, \boldsymbol{v}'_k \in \boldsymbol{V}$ (or $\boldsymbol{V}^*$, respectively).

Examples of Tensors

(i) For $r = 1$, $s = 0$, any $\varphi \in \boldsymbol{V}^*$ is a tensor.

(ii) For $r = 2$, $s = 0$, any bilinear form Φ on $\boldsymbol{V}$ is a tensor.

(iii) The natural pairing of $\boldsymbol{V}$ and $\boldsymbol{V}^*$, that is, $(\boldsymbol{v}, \varphi) \to \langle \varphi, \boldsymbol{v} \rangle$ for the case $r = 1$, $s = 1$ is a tensor.

(iv) We have also noted that $\boldsymbol{V}$ and $(\boldsymbol{V}^*)^*$ are naturally isomorphic.

Suppose that they are identified.

Then each $\boldsymbol{v} \in \boldsymbol{V}$ may be considered as a linear map of $\boldsymbol{V}^*$ to $\mathbb{R}$.

So it may be viewed as a tensor with $r = 0$ and $s = 1$.

Vector Space $\mathcal{T}_s^r(\boldsymbol{V})$

- For a fixed (r, s) we let $\mathcal{T}_s^r(\boldsymbol{V})$ be the collection of all tensors on $\boldsymbol{V}$ of covariant order r and contravariant order s.
- We know that as functions from $\boldsymbol{V} \times \cdots \times \boldsymbol{V} \times \boldsymbol{V}^* \times \cdots \times \boldsymbol{V}^*$ to $\mathbb{R}$ they may be added and multiplied by scalars (elements of $\mathbb{R}$).
- Indeed linear combinations of functions from any set to $\mathbb{R}$ are defined and are again functions from that set to $\mathbb{R}$.
- With this addition and scalar multiplication $\mathcal{T}_s^r(\boldsymbol{V})$ is a vector space.
- That is, if $\Phi_1, \Phi_2 \in \mathcal{T}_s^r(\boldsymbol{V})$ and $\alpha_1, \alpha_2 \in \mathbb{R}$, then $\alpha_1\Phi_1 + \alpha_2\Phi_2$, defined by

$$(\alpha_1\Phi_1 + \alpha_2\Phi_2)(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots) = \alpha_1\Phi_1(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots) + \alpha_2\Phi_2(\boldsymbol{v}_1, \boldsymbol{v}_2, \ldots)$$

  is multilinear, and, therefore, is in $\mathcal{T}_s^r(\boldsymbol{V})$.
- Thus $\mathcal{T}_s^r(\boldsymbol{V})$ has a natural vector space structure.

The Vector Space Property

Theorem

With the natural definitions of addition and multiplication by elements of $\mathbb{R}$, the set $\mathcal{T}_s^r(\boldsymbol{V})$ of all tensors of order (r, s) on $\boldsymbol{V}$ forms a vector space of dimension n^{r+s}.

- We consider the case $s = 0$ only, that is, covariant tensors of fixed order r, and we let $\mathcal{T}^r(\boldsymbol{V}) := \mathcal{T}_0^r(\boldsymbol{V})$.

  Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be a basis of $\boldsymbol{V}$.

  Then $\Phi \in \mathcal{T}^r(\boldsymbol{V})$ is completely determined by its n^r values on the basis vectors.

  To see this, suppose

$$\boldsymbol{v}_i = \sum \alpha_i^j \boldsymbol{e}_j, \quad i = 1, \ldots, r.$$

The Vector Space Property (Cont'd)

- By multilinearity, the value of Φ is given by the formula

$$\Phi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r) = \sum_{j_1, \ldots, j_r} \alpha_{i_1}^{j_1} \alpha_{i_2}^{j_2} \cdots \alpha_{i_r}^{j_r} \Phi(\boldsymbol{e}_{j_1}, \ldots, \boldsymbol{e}_{j_r}),$$

  the sum being over all $1 \leq j_1, \ldots, j_r \leq n$.

  The n^r numbers $\{\Phi(\boldsymbol{e}_{j_1}, \ldots, \boldsymbol{e}_{j_r})\}$ are called the **components** of Φ in the basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$.

  We justify the terminology by showing that there is in fact a basis of $\mathcal{T}^r(\boldsymbol{V})$, determined by $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ with respect to which these are components of Φ.

The Vector Space Property (Cont'd)

- Let $\Omega^{j_1 \cdots j_r}$ be that element of $\mathcal{T}^r(\boldsymbol{V})$ whose values on the basis vectors are given by

$$\Omega^{j_1 \cdots j_r}(\boldsymbol{e}_{k_1}, \ldots, \boldsymbol{e}_{k_r}) = \begin{cases} 1, & \text{if } k_i = j_i \text{ for } i = 1, \ldots, r, \\ 0, & \text{if } k_i \neq j_i, \text{ for some } i. \end{cases}$$

  Its values on an arbitrary r-tuple $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r \in \boldsymbol{V}$ is defined by

$$\Omega^{j_1 \cdots j_r}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r) = \alpha_1^{j_1} \alpha_2^{j_2} \cdots \alpha_r^{j_r}.$$

  This definition is linear in the components of each $\boldsymbol{v}_i$.

  Therefore, $\Omega^{j_1 \cdots j_r}$ is indeed a tensor.

The Vector Space Property (Cont'd)

- We show that the n^r tensors so chosen are linearly independent. Suppose

$$\sum_{j_1,\ldots,j_r} \gamma_{j_1\cdots j_r}\Omega^{j_1\cdots j_r} = 0.$$

Then, for any choice of the variables $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$,

$$\sum_{j_1,\ldots,j_r} \gamma_{j_1\cdots j_r}\Omega^{j_1\cdots j_r}(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r) = 0.$$

Now substitute, in turn, each combination $\boldsymbol{e}_{k_1}, \ldots, \boldsymbol{e}_{k_r}$ of basis elements as variables.

By the definition of the $\Omega^{j_1\cdots j_r}$, we see that every coefficient $\gamma_{k_1\cdots k_r} = 0$.

The Vector Space Property (Cont'd)

- Finally, we show that every Φ is a linear combination of these tensors.

  Let
  $$\varphi_{j_1\cdots j_r} = \Phi(\boldsymbol{e}_{j_1}, \ldots, \boldsymbol{e}_{j_r}).$$
  Consider the element
  $$\sum \varphi_{j_1\cdots j_r}\Omega^{j_1\cdots j_r}$$
  of $\mathcal{T}^r(\boldsymbol{V})$.

  Apply again the definition of $\Omega^{j_1\cdots j_r}$.

  We see that this tensor and Φ take the same values on every set of basis elements.

  Hence, they must be equal.
- An easy extension of the argument using both $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ and its dual basis $\omega^1, \ldots, \omega^n$ of $\boldsymbol{V}^*$ gives the general case $\mathcal{T}_s^r(\boldsymbol{V})$.

Covariant Tensor Fields

Definition

A C^∞-**covariant tensor field of order** r on a C^∞ manifold M is a function Φ which:

- Assigns to each $p \in M$ an element Φ_p of $\mathcal{T}^r(T_p(M))$;
- Has the additional property that, given any C^∞-vector fields $X_1, \ldots, X_r$ on an open subset U of M,

$$\Phi(X_1, \ldots, X_r)$$

is a C^∞ function on U.

We denote by $\mathcal{T}^r(M)$ the set of all C^∞-covariant tensor fields of order r on M.

Covariant Tensor Fields (Cont'd)

- A covariant tensor field of order r is not only $\mathbb{R}$-linear but also $C^\infty(M)$-linear in each variable.
- For example, let $f \in C^\infty(M)$.
- Then
$$\Phi(X_1, \ldots, fX_i, \ldots, X_r) = f\Phi(X_1, \ldots, X_i, \ldots, X_r).$$
- This holds at each p by the $\mathbb{R}$-linearity of Φ_p.
- Moreover, the two sides are equal if equality holds for each $p \in M$.
- In the same way, if $f \in C^\infty(U)$, U open in M, the equation holds for Φ_U, the restriction of Φ to U.

The Structure of $\mathcal{T}^r(M)$

- Let U, φ be a coordinate neighborhood.
- Let $E_1, \ldots, E_n$ be the coordinate frames.
- Then $\Phi \in \mathcal{T}^r(M)$ has components

$$\Phi(E_{j_1}, \ldots, E_{j_r}).$$

- These are functions on U whose values at each $p \in U$ are the components of Φ_p relative to the basis of $T_p(M)$ determined by $E_1, \ldots, E_n$.
- By hypothesis, all the components, as functions on the coordinate neighborhoods of some covering of M, are differentiable.
- This implies the differentiability of Φ.
- Linear combinations of covariant tensors of order r (even with C^∞ functions as coefficients) are again covariant tensor fields.
- So $\mathcal{T}^r(M)$ is a vector space over $\mathbb{R}$ [in fact a $C^\infty(M)$ module].

Mappings and Covariant Tensors

- Consider a linear map of vector spaces $F_* : \boldsymbol{V} \to \boldsymbol{W}$.
- It induces a linear map $F^* : \mathcal{T}^r(\boldsymbol{W}) \to \mathcal{T}^r(\boldsymbol{V})$ by the formula

$$F^*\Phi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r) = \Phi(F_*(\boldsymbol{v}_1), \ldots, F_*(\boldsymbol{v}_r)).$$

- Now suppose $F : M \to N$ is a C^∞-map.
- It induces a mapping $F^* : \mathcal{T}^r(N) \to \mathcal{T}^r(M)$, defined, for Φ on N, by

$$F^*\Phi_p(X_{1p}, \ldots, X_{rp}) = \Phi_{F(p)}(F_*(X_{1p}), \ldots, F_*(X_{rp})).$$

- As we have seen, this is a special feature of covariant tensor fields.
- Its analog does not hold for contravariant fields even for $\mathcal{T}_1(M) = \mathfrak{X}(M)$ (vector fields).
- We can show that F^* maps $\mathcal{T}^r(N)$ to $\mathcal{T}^r(M)$ linearly.

Symmetry and Antisymmetry

Definition

Let $\boldsymbol{V}$ be a vector space.
We say $\Phi \in \mathcal{T}^r(\boldsymbol{V})$ is **symmetric** if, for each $1 \leq i, j \leq r$,

$$\Phi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_r) = \Phi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_r).$$

We say Φ is **skew** or **antisymmetric** or **alternating** if, interchanging the ith and jth variables, $1 \leq i, j \leq r$, changes the sign,

$$\Phi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_r) = -\Phi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_j, \ldots, \boldsymbol{v}_i, \ldots, \boldsymbol{v}_r).$$

Alternating covariant tensors are often called **exterior forms**.
A tensor field is **symmetric** (respectively, **alternating**) if it has this property at each point.

Summarizing Theorem

Theorem

Let $F : M \to N$ be a C^∞ map of C^∞ manifolds.
Then each C^∞-covariant tensor field Φ on N determines a C^∞-covariant tensor field $F^*\Phi$ on M by the formula

$$(F^*\Phi)_p(X_{1p}, \ldots, X_{rp}) = \Phi_p(F_*(X_{1p}), \ldots, F_*(X_{rp})).$$

The map $F^* : \mathcal{T}^r(N) \to \mathcal{T}^r(M)$ so defined is linear.
Moreover, it takes symmetric tensors to symmetric tensors and alternating tensors to alternating tensors.

Some Additional Properties

- We may also extend to the case of arbitrary order r:
  - The formula for components of $F^*\Phi$ in terms of those of Φ;
  - The Jacobian of F in local coordinates.
- The same method can also be used to derive formulas for change of components relative to a change of local coordinates.
- These formulas are essentially consequences of the multilinearity at each point of M.

Subspaces of Symmetric and Alternating Tensors

- Let $\Phi_1, \Phi_2 \in \mathcal{T}^r(\boldsymbol{V})$ be symmetric (respectively, alternating) covariant tensors of order r on $\boldsymbol{V}$.
- Then a linear combination

$$\alpha\Phi_1 + \beta\Phi_2, \quad \alpha, \beta \in \mathbb{R},$$

  is also symmetric (respectively, alternating).
- Thus, the symmetric tensors in $\mathcal{T}^r(\boldsymbol{V})$ form a subspace which we denote by $\Sigma^r(\boldsymbol{V})$.
- The alternating tensors (exterior forms) also form a subspace $\bigwedge^r(\boldsymbol{V})$.
- These subspaces have only the 0-tensor in common.

The Signum Homomorphism

- Let σ denote a permutation of $(1, \ldots, r)$, with
$$(1, \ldots, r) \to (\sigma(1), \ldots, \sigma(r)).$$
- We know that any such permutation is a product of transpositions, i.e., permutations interchanging just two elements.
- This representation is not unique.
- But the parity (evenness or oddness) of the number of factors is.
- We let
$$\operatorname{sgn}\sigma = \begin{cases} +1, & \text{if } \sigma \text{ is representable as the product} \\ & \quad \text{of an even number of transpositions,} \\ -1, & \text{otherwise.} \end{cases}$$
- Then, $\sigma \to \operatorname{sgn}\sigma$ is a well-defined map from the group of permutations of r letters $\mathfrak{S}_r$ to the multiplicative group of two elements ± 1.
- It is even a homomorphism, as can be checked from the definition.

Symmetric and Alternating Tensor Fields Revisited

- Now our original definitions may be restated in the following equivalent form.
- $\Phi \in \mathcal{T}^r(\boldsymbol{V})$ is **symmetric** if, for all $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ and permutation σ,
$$\Phi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r) = \Phi(\boldsymbol{v}_{\sigma(1)}, \ldots, \boldsymbol{v}_{\sigma(r)});$$
- Φ is **alternating** if, for all $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ and permutation σ,
$$\Phi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r) = \mathrm{sgn}\sigma \Phi(\boldsymbol{v}_{\sigma(1)}, \ldots, \boldsymbol{v}_{\sigma(r)}).$$

Symmetrization and Antisymmetrization

Definition

We define two linear transformations on the vector space $\mathcal{T}^r(\boldsymbol{V})$:

- The **symmetrizing mapping** $\mathcal{S} : \mathcal{T}^r(\boldsymbol{V}) \to \mathcal{T}^r(\boldsymbol{V})$ by

$$(\mathcal{S}\Phi)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r) = \frac{1}{r!} \sum_{\sigma} \Phi(\boldsymbol{v}_{\sigma(1)}, \ldots, \boldsymbol{v}_{\sigma(r)});$$

- The **alternating mapping** $\mathcal{A} : \mathcal{T}^r(\boldsymbol{V}) \to \mathcal{T}^r(\boldsymbol{V})$ by

$$(\mathcal{A}\Phi)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r) = \frac{1}{r!} \sum_{\sigma} \operatorname{sgn}\sigma \Phi(\boldsymbol{v}_{\sigma(1)}, \ldots, \boldsymbol{v}_{\sigma(r)}).$$

The summation is over all $\sigma \in \mathfrak{S}_r$, the group of all permutations of r letters.

Linearity of $\mathcal{A}$ and $\mathcal{S}$

- It is immediate that these maps are linear transformations on $\mathcal{T}^r(\boldsymbol{V})$.
- First note that $\Phi \to \Phi^\sigma$, defined by

$$\Phi^\sigma(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r) = \Phi(\boldsymbol{v}_{\sigma(1)}, \ldots, \boldsymbol{v}_{\sigma(r)}),$$

  is such a linear transformation;
- Further, any linear combination of linear transformations of a vector space is again a linear transformation.

Properties of $\mathcal{A}$ and $\mathcal{S}$

- We have the following properties of $\mathcal{A}$ and $\mathcal{S}$:
  (i) $\mathcal{A}$ and $\mathcal{S}$ are projections, that is,
  $$\mathcal{A}^2 = \mathcal{A} \quad \text{and} \quad \mathcal{S}^2 = \mathcal{S};$$
  (ii) The following hold:
  $$\mathcal{A}(\mathcal{T}^r(\boldsymbol{V})) = \bigwedge^r(\boldsymbol{V}) \quad \text{and} \quad \mathcal{S}(\mathcal{T}^r(\boldsymbol{V})) = \Sigma^r(\boldsymbol{V});$$
  (iii) Φ is alternating if and only if $\mathcal{A}\Phi = \Phi$;
  Φ is symmetric if and only if $\mathcal{S}\Phi = \Phi$;
  (iv) If $F_* : \boldsymbol{V} \to \boldsymbol{W}$ is a linear map, then both $\mathcal{A}$ and $\mathcal{S}$ commute with $F^* : \mathcal{T}^r(\boldsymbol{W}) \to \mathcal{T}^r(\boldsymbol{V})$.

Proof of the Properties

- We check the properties for $\mathcal{A}$.

  The verification for $\mathcal{S}$ is similar.

  They are also interrelated, so we will not take them in order.

  First note that if Φ is alternating, then the definition implies

  $$\Phi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r) = \text{sgn}\sigma\Phi(\boldsymbol{v}_{\sigma(1)}, \ldots, \boldsymbol{v}_{\sigma(r)}).$$

  There are $r!$ elements of $\mathfrak{S}_r$.

  So, summing both sides over all $\sigma \in \mathfrak{S}_r$, gives

  $$\Phi = \mathcal{A}\Phi.$$

Proof of the Properties (Cont'd)

- On the other hand, suppose we apply a permutation τ to the variables of $\mathcal{A}\Phi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r)$ for an arbitrary $\Phi \in \mathcal{T}^r(\boldsymbol{V})$.

  We obtain

$$\mathcal{A}\Phi(\boldsymbol{v}_{\tau(1)}, \ldots, \boldsymbol{v}_{\tau(r)}) = \frac{1}{r!} \sum_{\sigma} \text{sgn}\sigma \Phi(\boldsymbol{v}_{\sigma\tau(1)}, \ldots, \boldsymbol{v}_{\sigma\tau(r)}).$$

  Now sgn is a homomorphism and $\text{sgn}\tau^2 = 1$.

  So $\text{sgn}\sigma = \text{sgn}\sigma\tau\text{sgn}\tau$.

  From this equation we see that the right side is

$$\frac{1}{r!}\text{sgn}\tau \sum_{\sigma} \text{sgn}\sigma\tau \Phi(\boldsymbol{v}_{\sigma\tau(1)}, \ldots, \boldsymbol{v}_{\sigma\tau(r)}) = \text{sgn}\tau \mathcal{A}\Phi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r).$$

  So $\mathcal{A}\Phi$ is alternating.

  This shows that $\mathcal{A}(\mathcal{T}^r(\boldsymbol{V})) \subseteq \bigwedge^r(\boldsymbol{V})$.

Proof of the Properties (Cont'd)

- Suppose Φ is alternating.

  Then every term in the summation defining $\mathcal{A}\Phi$ is equal.

  So $\mathcal{A}\Phi = \Phi$.

  Thus $\mathcal{A}$ is the identity on $\bigwedge^r(\boldsymbol{V})$ and $\mathcal{A}(\mathcal{T}^r(\boldsymbol{V})) \supseteq \bigwedge^r(\boldsymbol{V})$.

  From these facts Properties (i)-(iii) for $\mathcal{A}$ follow.

  Now consider Property (iv).

  By the definition of F^*, we have

  $$F^*\Phi(\boldsymbol{v}_{\sigma(1)}, \ldots, \boldsymbol{v}_{\sigma(r)}) = \Phi(F_*(\boldsymbol{v}_{\sigma(1)}), \ldots, F_*(\boldsymbol{v}_{\sigma(r)})).$$

  Multiply both sides by $\operatorname{sgn}\sigma$ and sum over all σ.

  Using the linearity of F^*, we get $\mathcal{A}(F^*\Phi)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r)$ on the left and $F^*(\mathcal{A}\Phi)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r)$ on the right.

Extension to Manifolds

- Both of these maps $\mathcal{A}$ and $\mathcal{S}$ can be immediately extended to mappings of tensor fields on manifolds.
- We merely apply them at each point.
- We then verify that both sides of each relation (i)-(iv) give C^∞ functions which agree pointwise on every r-tuple of C^∞-vector fields.
- We summarize (without proof).

Theorem

Let M be a C^∞ manifold. Let $\mathcal{T}^r(M)$ be the space of C^∞-covariant tensor fields of order r over M.
The maps $\mathcal{A}$ and $\mathcal{S}$ are defined on $\mathcal{T}^r(M)$. Moreover, they satisfy Properties (i)-(iv). In the case of Property (iv), $F^* : \mathcal{T}^r(N) \to \mathcal{T}^r(M)$ denotes the linear map induced by a C^∞ mapping $F : M \to N$.

Subsection 6

Multiplication of Tensors

The Setup

- Let $\boldsymbol{V}$ be a vector space and M be a C^∞ manifold.
- We saw that both $\mathcal{T}^r(\boldsymbol{V})$ and $\mathcal{T}^r(M)$ are vector spaces over $\mathbb{R}$.
- In the case of tensor fields, $\mathcal{T}^r(M)$ has also the structure of a $C^\infty(M)$-module.
- We agree, by definition, that

$$\mathcal{T}^0(\boldsymbol{V}) = \mathbb{R} \quad \text{and} \quad \mathcal{T}^0(M) = C^\infty(M).$$

- Recall, next, that our viewpoint is to define tensors as:
  - Functions to $\mathbb{R}$, a field, in the case of $\mathcal{T}^r(\boldsymbol{V})$;
  - Functions to $C^\infty(M)$, an algebra, in the case of $\mathcal{T}^r(M)$.
- In either case it is appropriate to discuss products of such functions.

Multiplication of Tensors on a Vector Space

- Let $\boldsymbol{V}$ be a vector space.
- Let $\varphi \in \mathcal{T}^r(\boldsymbol{V})$, $\psi \in \mathcal{T}^s(\boldsymbol{V})$ be tensors.
- Their product is linear in each of its $r+s$ variables.

Definition

The **product** of φ and ψ, denoted $\varphi \otimes \psi$ is a tensor of order $r+s$ defined by

$$\varphi \otimes \psi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r, \boldsymbol{v}_{r+1}, \ldots, \boldsymbol{v}_{r+s}) = \varphi(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r)\psi(\boldsymbol{v}_{r+1}, \ldots, \boldsymbol{v}_{r+s}).$$

The right-hand side is the product of the values of φ and ψ.
The product defines a mapping

$$\begin{array}{rcl} \mathcal{T}^r(\boldsymbol{V}) \times \mathcal{T}^s(\boldsymbol{V}) & \to & \mathcal{T}^{r+s}(\boldsymbol{V}); \\ (\varphi, \psi) & \to & \varphi \otimes \psi. \end{array}$$

Properties of the Product

Theorem

The mapping $\mathcal{T}^r(\boldsymbol{V}) \times \mathcal{T}^s(\boldsymbol{V}) \to \mathcal{T}^{r+s}(\boldsymbol{V})$ just defined is bilinear and associative. If $\omega^1, \ldots, \omega^n$ is a basis of $\boldsymbol{V}^* = \mathcal{T}^1(\boldsymbol{V})$, then $\{\omega^{i_1} \otimes \cdots \otimes \omega^{i_r}\}$ over all $1 \leq i_1, \ldots, i_r \leq n$ is a basis of $\mathcal{T}^r(\boldsymbol{V})$. Finally, if $F_* : \boldsymbol{W} \to \boldsymbol{V}$ is linear, then $F^*(\varphi \otimes \psi) = (F^*\varphi) \otimes (F^*\psi)$.

- Each statement is proved by straightforward computation.

  For bilinearity, we must show that, if α, β are numbers, $\varphi_1, \varphi_2 \in \mathcal{T}^r(\boldsymbol{V})$ and $\psi \in \mathcal{T}^s(\boldsymbol{V})$, then

$$(\alpha\varphi_1 + \beta\varphi_2) \otimes \psi = \alpha(\varphi_1 \otimes \psi) + \beta(\varphi_2 \otimes \psi).$$

  Similarly for the second variable.

  This is checked by evaluating each side on $r + s$ vectors of $\boldsymbol{V}$.

  In fact basis vectors suffice because of linearity.

Properties of the Product (Cont'd)

- For associativity, we must show

$$(\varphi \otimes \psi) \otimes \theta = \varphi \otimes (\psi \otimes \theta).$$

  The products on both sides being defined in the natural way.

  This is similarly verified.

  This allows us to drop the parentheses.

Properties of the Product (Cont'd)

- Next, we show that $\omega^{i_1} \otimes \cdots \otimes \omega^{i_r}$ form a basis.

  Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be the basis of $\boldsymbol{V}$ dual to $\omega^1, \ldots, \omega^n$.

  Then the tensor $\Omega^{i_1 \cdots i_r}$ previously defined is exactly $\omega^{i_1} \otimes \cdots \otimes \omega^{i_r}$.

  This follows from the two definitions.

  First, we have

$$\Omega^{i_1 \cdots i_r}(\boldsymbol{e}_{j_1}, \ldots, \boldsymbol{e}_{j_r}) = \begin{cases} 0, & \text{if } (i_1, \ldots, i_r) \neq (j_1, \ldots, j_r), \\ 1, & \text{if } (i_1, \ldots, i_r) = (j_1, \ldots, j_r). \end{cases}$$

  Next, we see that

$$\begin{aligned} \omega^{i_1} \otimes \cdots \otimes \omega^{i_r}(\boldsymbol{e}_{j_1}, \ldots, \boldsymbol{e}_{j_r}) &= \omega^{i_1}(\boldsymbol{e}_{j_1}) \omega^{i_2}(\boldsymbol{e}_{j_2}) \cdots \omega^{i_r}(\boldsymbol{e}_{j_r}) \\ &= \delta^{i_1}_{j_1} \delta^{i_2}_{j_2} \cdots \delta^{i_r}_{j_r}. \end{aligned}$$

  So both tensors have the same values on any set of r basis vectors.

  Therefore, they are equal.

Properties of the Product (Cont'd)

- Finally, let $F_* : \boldsymbol{W} \to \boldsymbol{V}$.

  Consider $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{r+s} \in \boldsymbol{W}$.

  Then

$$\begin{aligned}(F^*(\varphi \otimes \psi))(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{r+s}) &= \varphi \otimes \psi(F_*(\boldsymbol{w}_1), \ldots, F_*(\boldsymbol{w}_{r+s})) \\ &= \varphi(F_*(\boldsymbol{w}_1), \ldots, F_*(\boldsymbol{w}_r))\psi(F_*(\boldsymbol{w}_{r+1}), \ldots, F_*(\boldsymbol{w}_{r+s})) \\ &= (F^*\varphi) \otimes (F^*\psi)(\boldsymbol{w}_1, \ldots, \boldsymbol{w}_{r+s}).\end{aligned}$$

  This proves $F^*(\varphi \otimes \psi) = (F^*\varphi) \otimes (F^*\psi)$ and completes the proof.

Reformulation

- Consider the tensor spaces $\mathcal{T}^0(\boldsymbol{V}) = \mathbb{R}, \mathcal{T}^1(\boldsymbol{V}), \cdots, \mathcal{T}^r(\boldsymbol{V}), \ldots$.
- Take the direct sum $\mathcal{T}(\boldsymbol{V})$ over $\mathbb{R}$ of all these tensor spaces,

$$\mathcal{T}(\boldsymbol{V}) = \mathcal{T}^0(\boldsymbol{V}) \oplus \mathcal{T}^1(\boldsymbol{V}) \oplus \cdots \oplus \mathcal{T}^r(\boldsymbol{V}) \oplus \cdots .$$

- We identify each $\mathcal{T}^r(\boldsymbol{V})$ with its (natural) isomorphic image in $\mathcal{T}(\boldsymbol{V})$.
- An element φ of $\mathcal{T}(\boldsymbol{V})$ is said to be of **order** r if it is in $\mathcal{T}^r(\boldsymbol{V})$.
- Every element $\widetilde{\varphi}$ of $\mathcal{T}(\boldsymbol{V})$ is the sum of a finite number of such φ, which we call its **components**.
- Thus $\widetilde{\varphi} \in \mathcal{T}(\boldsymbol{V})$ may be written uniquely

$$\widetilde{\varphi} = \varphi_1^{i_1} + \cdots + \varphi_n^{i_n},$$

where $\varphi^{i_j} \in \mathcal{T}^{i_j}(\boldsymbol{V})$ and $i_1 < i_2 < \cdots < i_r$.

The Tensor Algebra

- If $\widetilde{\varphi}, \widetilde{\psi} \in \mathcal{T}(\boldsymbol{V})$, then they may be added componentwise.
- That is, by adding in $\mathcal{T}^r(\boldsymbol{V})$ any terms in $\mathcal{T}^r(\boldsymbol{V})$.
- They may be multiplied by:
  - Using $\otimes$;
  - Extending it to be distributive on all of $\mathcal{T}(\boldsymbol{V})$.
- This makes $\mathcal{T}(\boldsymbol{V})$ into an associative algebra over $\mathbb{R}$.
- It is called the **tensor algebra**.

Properties of the Tensor Algebra

- The tensor algebra $\mathcal{T}(\boldsymbol{V})$:
  - Contains $\mathbb{R} = \mathcal{T}^0(\boldsymbol{V})$;
  - Has 1 as its unit;
  - Is infinite-dimensional.
- The contents of the preceding theorem (even a little more) immediately yield the following properties:
  - $\mathcal{T}(\boldsymbol{V})$ (direct) is an associative algebra (with unit) over $\mathbb{R} = \mathcal{T}^0(\boldsymbol{V})$.
  - It is generated by $\mathcal{T}^0(\boldsymbol{V})$ and $\mathcal{T}^1(\boldsymbol{V}) = \boldsymbol{V}^*$, the dual space to $\boldsymbol{V}$.
  - Any linear mapping $F_* : \boldsymbol{W} \to \boldsymbol{V}$ of vector spaces induces a homomorphism $F^* : \mathcal{T}(\boldsymbol{V}) \to \mathcal{T}(\boldsymbol{W})$ which is:
    (i) The identity on $\mathbb{R}$;
    (ii) The dual mapping $F^* : \boldsymbol{V}^* \to \boldsymbol{W}^*$ on $\mathcal{T}^1(\boldsymbol{V})$.
  - Properties (i) and (ii) determine F^* uniquely on all of $\mathcal{T}(\boldsymbol{V})$.

Multiplication of Tensor Fields

- We turn to the case of tensor fields on a manifold M.
- Let $\varphi \in \mathcal{T}^r(M)$ and $\psi \in \mathcal{T}^s(M)$.
- Then we may define $\varphi \otimes \psi$ on M by defining it at each point using the definition for tensors on a vector space.
- That is, $(\varphi \otimes \psi)_p$ is defined to be the tensor

$$(\varphi \otimes \psi)_p = \varphi_p \otimes \psi_p$$

  of order $r + s$ on the vector space $T_p(M)$.
- Since this defines a covariant tensor of order $r + s$ on the tangent space at each point of M, it will define a tensor field, if it is C^∞.

Multiplication of Tensor Fields (Cont'd)

- Consider the product $\varphi \otimes \psi$, defined as above.
- According to the definition, in local coordinates the components of $\varphi \otimes \psi$ are the functions of the coordinate frame vectors

$$\varphi \otimes \psi(E_{i_1}, \ldots, E_{i_{r+s}}) = \varphi(E_{i_1}, \ldots, E_{i_r})\psi(E_{i_{r+1}}, \ldots, E_{i_{r+s}})$$

over the coordinate neighborhood.

- The right-hand side is the product of the components in local coordinates of φ and ψ.
- These are two C^∞ functions.
- Thus, the left side is C^∞.
- So $\varphi \otimes \psi$ is indeed a tensor field on M.

Multiplication of Tensors on Manifold

Theorem

The mapping

$$\mathcal{T}^r(M) \times \mathcal{T}^s(M) \to \mathcal{T}^{r+s}(M)$$

just defined is bilinear and associative.

If $\omega^1, \ldots, \omega^n$ is a basis of $\mathcal{T}^1(M)$, then every element of $\mathcal{T}^r(M)$ is a linear combination with C^∞ coefficients of

$$\{\omega^{i_1} \otimes \cdots \otimes \omega^{i_r} : 1 \leq i_1, \ldots, i_r \leq n\}.$$

If $F : N \to M$ is a C^∞ mapping, $\varphi \in \mathcal{T}^r(M)$ and $\psi \in \mathcal{T}^s(M)$, then

$$F^*(\varphi \otimes \psi) = (F^*\varphi) \otimes (F^*\psi),$$

tensor fields on N.

Note on Proof

- Two tensor fields are equal if and only if they are equal at each point.
- So it is only necessary to see that these equations hold at each point.
- This follows at once from the definitions and the preceding theorem.

Tensors in Terms of Local Bases

- In general we do not have a globally defined basis of $\mathcal{T}^1(M)$.
- That is, there may not exist covector fields

$$\omega^1, \ldots, \omega^n,$$

  which are a basis at each point.
- However, we do have a globally defined basis in $\mathbb{R}^n$.
- From this fact, the following corollary is obtained, by applying the theorem to a coordinate neighborhood V, θ of M.
- Let $E_1, \ldots, E_n$ denote the coordinate frames.
- Let $\omega^1, \ldots, \omega^n$ be their duals.
- That is, we have

$$E_i = \theta_*^{-1}\left(\frac{\partial}{\partial x^i}\right) \quad \text{and} \quad \omega^j = \theta^*(dx^j).$$

Tensors in Terms of Local Bases (Cont'd)

Corollary

Each $\varphi \in \mathcal{T}^r(U)$, including the restriction to U of any covariant tensor field on M, has a unique expression of the form

$$\varphi = \sum_{i_1} \cdots \sum_{i_r} a_{i_1 \cdots i_r} \omega^{i_1} \otimes \cdots \otimes \omega^{i_r},$$

where at each point of U,

$$a_{i_1 \cdots i_r} = \varphi(E_{i_1}, \ldots, E_{i_r})$$

are the components of φ in the basis $\{\omega^{i_1} \otimes \cdots \otimes \omega^{i_r}\}$.
Moreover, the $a_{i_1 \cdots i_r}$ are all C^∞ functions on U.

Space of Alternating Tensors

- For each $r > 0$ we have defined the subspace $\bigwedge^r(\boldsymbol{V}) \subseteq \mathcal{T}^r(\boldsymbol{V})$ consisting of alternating covariant tensors of order r.
- It is the image of $\mathcal{T}^r(\boldsymbol{V})$ under the linear mapping $\mathcal{A}$, the alternating mapping.
- We define $\bigwedge^0(\boldsymbol{V})$ to be $\mathbb{R}$, the field.
- Then $\bigwedge^0(\boldsymbol{V}) = \mathcal{T}^0(\boldsymbol{V}) = \mathbb{R}$ and $\bigwedge^1(\boldsymbol{V}) = \mathcal{T}^1(\boldsymbol{V}) = \boldsymbol{V}^*$, but $\bigwedge^r(\boldsymbol{V})$ is properly contained in $\mathcal{T}^r(\boldsymbol{V})$ for $r > 1$.
- We see, therefore, that the direct sum $\bigwedge(\boldsymbol{V})$ of all the spaces $\bigwedge^r(\boldsymbol{V})$ is contained in $\mathcal{T}(\boldsymbol{V})$ as a subspace,

$$\begin{aligned}\bigwedge(\boldsymbol{V}) &= \bigwedge^0(\boldsymbol{V}) \oplus \bigwedge^1(\boldsymbol{V}) \oplus \bigwedge^2(\boldsymbol{V}) \oplus \cdots \\ &\subsetneq \mathcal{T}^0(\boldsymbol{V}) \oplus \mathcal{T}^1(\boldsymbol{V}) \oplus \mathcal{T}^2(\boldsymbol{V}) \oplus \cdots = \mathcal{T}(\boldsymbol{V}).\end{aligned}$$

Space of Alternating Tensors (Cont'd)

- Although $\bigwedge(\boldsymbol{V})$ is a subspace of $\mathcal{T}(\boldsymbol{V})$, it is not a subalgebra.
- Even if $\varphi \in \bigwedge^r(\boldsymbol{V})$ and $\psi \in \bigwedge^s(\boldsymbol{V})$, it may be shown that $\varphi \otimes \psi$ may fail to be an element of $\bigwedge^{r+s}(\boldsymbol{V})$.
- Thus the tensor product of alternating tensors on $\boldsymbol{V}$ is not, in general, an alternating tensor on $\boldsymbol{V}$.
- On the other hand, we know that each tensor determines an alternating tensor, its image under $\mathcal{A}$.

Exterior Multiplication

Definition

The mapping from $\bigwedge^r(\boldsymbol{V}) \times \bigwedge^s(\boldsymbol{V}) \to \bigwedge^{r+s}(\boldsymbol{V})$ defined by

$$(\varphi, \psi) \to \frac{(r+s)!}{r!s!}\mathcal{A}(\varphi \otimes \psi),$$

is called the **exterior product** (or **wedge product**) of φ and ψ and is denoted by $\varphi \wedge \psi$.

Lemma

The exterior product is bilinear and associative.

- Bilinearity is a consequence of the fact that the product is defined by composing the tensor product, a bilinear mapping from $\bigwedge^r(\boldsymbol{V}) \times \bigwedge^s(\boldsymbol{V})$ to $\mathcal{T}^{r+s}(\boldsymbol{V})$, with a linear mapping $\frac{(r+s)!}{r!s!}\mathcal{A}$.

Exterior Multiplication (Cont'd)

- We now show that the product is associative.

  We first prove a property of the alternating mapping $\mathcal{A}$.

  Suppose $\varphi \in \mathcal{T}^r(\boldsymbol{V})$, $\psi \in \mathcal{T}^s(\boldsymbol{V})$ and $\theta \in \mathcal{T}^t(\boldsymbol{V})$.

  Then we show that

$$\mathcal{A}(\varphi \otimes \psi \otimes \theta) = \mathcal{A}(\mathcal{A}(\varphi \otimes \psi) \otimes \theta) = \mathcal{A}(\varphi \otimes \mathcal{A}(\psi \otimes \theta)).$$

  For this purpose let:

  - $\mathfrak{S} = \mathfrak{S}_{r+s+t}$ denote the permutations of $(1, 2, \ldots, r+s+t)$;
  - $\mathfrak{S}'$ denote the subgroup which leaves the last t integers fixed.

  $\mathfrak{S}'$ is isomorphic to the permutation group $\mathfrak{S}_{r+s}$ of $(1, 2, \ldots, r+s)$.

Exterior Multiplication (Cont'd)

- We have

$$\begin{aligned}
&\mathcal{A}(\mathcal{A}(\varphi\otimes\psi)\otimes\theta)(\mathbf{v}_1,\ldots,\ldots,\mathbf{v}_{r+s+t})\\
&=\frac{1}{(r+s+t)!}\sum_{\sigma\in\mathfrak{S}}\mathrm{sgn}\sigma\mathcal{A}(\varphi\otimes\psi)(\mathbf{v}_{\sigma(1)},\ldots,\mathbf{v}_{\sigma(r+s)})\\
&\qquad\qquad\cdot\theta(\mathbf{v}_{\sigma(r+s+1)},\ldots,\mathbf{v}_{\sigma(r+s+t)})\\
&=\frac{1}{(r+s+t)!}\frac{1}{(r+s)!}\sum_{\sigma\in\mathfrak{S}}\sum_{\sigma'\in\mathfrak{S}'}\{\mathrm{sgn}\sigma\sigma'\varphi(\mathbf{v}_{\sigma\sigma'(1)},\ldots,\mathbf{v}_{\sigma\sigma'(r)})\\
&\cdot\psi(\mathbf{v}_{\sigma\sigma'(r+1)},\ldots,\mathbf{v}_{\sigma\sigma'(r+s)})\theta(\mathbf{v}_{\sigma\sigma'(r+s+1)},\ldots,\mathbf{v}_{\sigma\sigma'(r+s+t)})\},
\end{aligned}$$

using the facts that:

  - $\mathrm{sgn}\sigma\mathrm{sgn}\sigma'=\mathrm{sgn}\sigma\sigma'$;
  - σ' is the identity on $r+s+1,\ldots,r+s+t$.

Exterior Multiplication (Cont'd)

- For each σ', as σ runs through $\mathfrak{S}$ and we sum over the outer summation symbol, this expression is equal to

$$\mathcal{A}(\varphi \otimes \psi \otimes \theta)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{r+s+1}).$$

Thus, the expression above reduces to

$$\frac{1}{(r+s)!} \sum_{\sigma' \in \mathfrak{S}'} \mathcal{A}(\varphi \otimes \psi \otimes \theta),$$

evaluated on $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{r+s+t}$.

But there are $(r+s)!$ terms in the summation.

So this gives

$$\mathcal{A}(\varphi \otimes \psi \otimes \theta) = \mathcal{A}(\mathcal{A}(\varphi \otimes \psi) \otimes \theta).$$

The second equality is proved in the same way.

Exterior Multiplication (Cont'd)

- Let φ, ψ, θ be in the subspaces $\bigwedge^r(\boldsymbol{V})$, $\bigwedge^s(\boldsymbol{V})$, $\bigwedge^t(\boldsymbol{V})$, respectively. Then, by definition, we have

$$\varphi \wedge \psi = \frac{(r+s)!}{r!s!}\mathcal{A}(\varphi \otimes \psi)$$

and

$$(\varphi \wedge \psi) \wedge \theta = \frac{(r+s+t)!}{(r+s)!t!}\mathcal{A}((\varphi \wedge \psi) \otimes \theta).$$

A similar expression can be obtained in the other order of associating terms.

From these expressions, we obtain the associativity of the exterior product

$$(\varphi \wedge \psi) \wedge \theta = \varphi \wedge (\psi \wedge \theta).$$

General Associativity

- The following relation allows us to write exterior products without parentheses.

Corollary

Let $\varphi_i \in \bigwedge^{r_i}(\boldsymbol{V})$, $i = 1, \ldots, k$. Then

$$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_k = \frac{(r_1 + r_2 + \cdots + r_k)!}{r_1! r_2! \cdots r_k!} \mathcal{A}(\varphi_1 \otimes \varphi_2 \otimes \cdots \otimes \varphi_k).$$

The Exterior or Grassman Algebra over V

- We define the product
$$\bigwedge(V) \times \bigwedge(V) \to \bigwedge(V)$$
simply by extending the exterior product to be bilinear, so that the distributive law holds.
- Suppose that $\varphi, \psi \in \bigwedge(V)$.
- Then
$$\varphi = \varphi_1 + \cdots + \varphi_k, \quad \varphi_i \in \bigwedge^{r_i}(V),$$
and
$$\psi = \psi_1 + \cdots + \psi_\ell, \quad \psi_i \in \bigwedge^{s_i}(V).$$
- We define
$$\varphi \wedge \psi = \sum_{i=1}^{k} \sum_{j=1}^{\ell} \varphi_i \wedge \psi_j.$$

The Exterior or Grassman Algebra over $\boldsymbol{V}$

Corollary

The set

$$\bigwedge(\boldsymbol{V}) = \bigwedge^0(\boldsymbol{V}) \oplus \bigwedge^1(\boldsymbol{V}) \oplus \bigwedge^2(\boldsymbol{V}) \oplus \cdots,$$

with the exterior product as defined above is an (associative) algebra over $\mathbb{R} = \bigwedge^0(\boldsymbol{V})$.

- The algebra $\bigwedge(\boldsymbol{V})$ is called the **exterior algebra** or **Grassman algebra** over $\boldsymbol{V}$.

Skew Commutativity

Lemma

If $\varphi \in \bigwedge^r(\boldsymbol{V})$ and $\psi \in \bigwedge^s(\boldsymbol{V})$, then

$$\varphi \wedge \psi = (-1)^{rs} \psi \wedge \varphi.$$

- This is equivalent to showing that

$$\mathcal{A}(\varphi \otimes \psi) = (-1)^{rs} \mathcal{A}(\psi \otimes \varphi).$$

To prove this equality we note that

$$\begin{aligned} &\mathcal{A}(\varphi \otimes \psi)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{r+s}) \\ &= \tfrac{1}{(r+s)!} \sum_\sigma \operatorname{sgn}\sigma \varphi(\boldsymbol{v}_{\sigma(1)}, \ldots, \boldsymbol{v}_{\sigma(r)}) \psi(\boldsymbol{v}_{\sigma(r+1)}, \ldots, \boldsymbol{v}_{\sigma(r+s)}) \\ &= \tfrac{1}{(r+s)!} \sum_\sigma \operatorname{sgn}\sigma \psi(\boldsymbol{v}_{\sigma(r+1)}, \ldots, \boldsymbol{v}_{\sigma(r+s)}) \varphi(\boldsymbol{v}_{\sigma(1)}, \ldots, \boldsymbol{v}_{\sigma(r)}). \end{aligned}$$

Skew Commutativity (Cont'd)

- Let τ be the permutation taking $(1, \ldots, s, s+1, \ldots, r+s)$ to $(r+1, \ldots, r+s, 1, \ldots, r)$.

  Then we may write

$$\begin{aligned}
&\mathcal{A}(\varphi \otimes \psi)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{r+s}) \\
&= \tfrac{1}{(r+s)!} \textstyle\sum_{\sigma} \mathrm{sgn}\sigma \mathrm{sgn}\tau \psi(\boldsymbol{v}_{\sigma\tau(1)}, \ldots, \boldsymbol{v}_{\sigma\tau(s)}) \\
&\qquad\qquad \varphi(\boldsymbol{v}_{\sigma\tau(s+1)}, \ldots, \boldsymbol{v}_{\sigma\tau(r+s)}) \\
&= \mathrm{sgn}\tau \mathcal{A}(\psi \otimes \varphi)(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_{r+s}).
\end{aligned}$$

  Now check that $\mathrm{sgn}\tau = (-1)^{rs}$.

  So we get

$$\varphi \wedge \psi = (-1)^{rs} \psi \wedge \varphi.$$

Dimension of $\bigwedge(\boldsymbol{V})$

Theorem

If $r > n = \dim \boldsymbol{V}$, then

$$\bigwedge^r(\boldsymbol{V}) = \{0\}.$$

For $0 \leq r \leq n$,

$$\dim \bigwedge^r(\boldsymbol{V}) = \binom{n}{r}.$$

Let $\omega^1, \ldots, \omega^n$ be a basis of $\bigwedge^1(\boldsymbol{V})$. Then the set

$$\{\omega^{i_1} \wedge \cdots \wedge \omega^{i_r} : 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$$

is a basis of $\bigwedge^r(\boldsymbol{V})$. Finally, we have

$$\dim \bigwedge(\boldsymbol{V}) = 2^n.$$

Dimension of $\bigwedge(\boldsymbol{V})$ (Cont'd)

- Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be any basis of $\boldsymbol{V}$.

  Let φ be an alternating covariant tensor of order $r > \dim \boldsymbol{V}$.

  Then on any set of basis elements

  $$\varphi(\boldsymbol{e}_{i_1}, \ldots, \boldsymbol{e}_{i_r}) = 0.$$

  This is because:
  - Some variable $\boldsymbol{e}_{i_k}$ is repeated;
  - Interchanging two equal variables both changes the sign of φ on the set and leaves it unchanged.

  Now all components of φ are zero.

  So $\varphi = 0$.

  It follows that $\bigwedge^r(\boldsymbol{V}) = \{0\}$.

Dimension of $\bigwedge(\boldsymbol{V})$ (Cont'd)

- Suppose that $0 \leq r \leq n$.

  Let $\omega^1, \ldots, \omega^n$ be the basis of $\boldsymbol{V}^* = \bigwedge^1(\boldsymbol{V})$ dual to $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$.

  $\mathcal{A}$ maps $\mathcal{T}^r(\boldsymbol{V})$ onto $\bigwedge^r(\boldsymbol{V})$.

  So the image of the basis $\{\omega^{i_1} \otimes \cdots \otimes \omega^{i_r}\}$ of $\mathcal{T}^r(\boldsymbol{V})$ spans $\bigwedge^r(\boldsymbol{V})$.

  We have

  $$r!\mathcal{A}(\omega^{i_1} \otimes \cdots \otimes \omega^{i_r}) = \omega^{i_1} \wedge \cdots \wedge \omega^{i_r}.$$

  By the preceding lemma, permuting the order of $i_1, \ldots, i_r$ leaves the right side unchanged, except for a possible change of sign.

  It follows that the set of $\binom{n}{r}$ elements of the form

  $$\omega^{i_1} \wedge \cdots \wedge \omega^{i_r}, \quad 1 \leq i_1 < i_2 < \cdots < i_r \leq n,$$

  span $\bigwedge^r(\boldsymbol{V})$.

Dimension of $\bigwedge(\boldsymbol{V})$ (Cont'd)

- Moreover, these elements are independent.

  Suppose that some linear combination of them is zero, say

$$\sum_{i_1 < \cdots < i_r} \alpha_{i_1 \cdots i_r} \omega^{i_1} \wedge \cdots \wedge \omega^{i_r} = 0.$$

  Then its value on each set of r basis vectors must be zero.

  In particular, given $k_1 < \cdots < k_r$, we have

$$0 = \left(\sum \alpha_{i_1 \cdots i_r} \omega^{i_1} \wedge \cdots \wedge \omega^{i_r} \right) (\boldsymbol{e}_{k_1}, \ldots, \boldsymbol{e}_{k_r}).$$

  This becomes $\alpha_{k_1 \cdots k_r} = 0$ by virtue of the formula of a previous corollary, combined with $\omega^i(\boldsymbol{e}_k) = \delta^i_k$, for $1 \leq i, k \leq n$.

  By suitable choice of $k_1 < \cdots < k_r$, we see that each coefficient must be zero. Therefore the given set of elements of $\bigwedge^r(\boldsymbol{V})$ is linearly independent and a basis.

Dimension of $\bigwedge(\boldsymbol{V})$ (Cont'd)

- To complete the proof we note that

$$\dim \bigwedge(\boldsymbol{V}) = \sum_{r=0}^{n} \dim \bigwedge^{r}(\boldsymbol{V}) = \sum_{r=0}^{n} \binom{n}{r} = 2^n.$$

Theorem

Let $\boldsymbol{V}$ and $\boldsymbol{W}$ be finite-dimensional vector spaces and $F_* : \boldsymbol{W} \to \boldsymbol{V}$ a linear mapping. Then $F^* : \mathcal{T}(\boldsymbol{V}) \to \mathcal{T}(\boldsymbol{W})$ takes $\bigwedge(\boldsymbol{V})$ into $\bigwedge(\boldsymbol{W})$ and is a homomorphism of these (exterior) algebras.

- The theorem is an immediate consequence of:
  - A previous asserted property of F^*;
  - The fact that $\mathcal{A} \circ F^* = F^* \circ \mathcal{A}$;
  - The definition of exterior multiplication.

The Exterior Algebra on Manifolds

- All of these ideas extend to alternating tensor fields on a C^∞ manifold M.

Definition

An alternating covariant tensor field of order r on M will be called an **exterior differential form of degree** r (or sometimes simply r-**form**).

- The set $\bigwedge^r(M)$ of all such forms is a subspace of $\mathcal{T}^r(M)$.
- The following two theorems follow from preceding work.
- We let M, N be manifolds and $F : M \to N$ be a C^∞ mapping.

The Exterior Algebra on Manifolds (Cont'd)

Theorem

Let $\bigwedge(M)$ denote the vector space over $\mathbb{R}$ of all exterior differential forms. Then for $\varphi \in \bigwedge^r(M)$ and $\psi \in \bigwedge^s(M)$ the formula

$$(\varphi \wedge \psi)_p = \varphi_p \wedge \psi_p$$

defines an associative product satisfying

$$\varphi \wedge \psi = (-1)^{rs} \psi \wedge \varphi.$$

With this product, $\bigwedge(M)$ is an algebra over $\mathbb{R}$.

- We shall call $\bigwedge(M)$ the **algebra of differential forms** or **exterior algebra** on M.

The Exterior Algebra on Manifolds (Cont'd)

Theorem (Cont'd)

If $f \in C^\infty(M)$, we also have

$$(f\varphi) \wedge \psi = f(\varphi \wedge \psi) = \varphi \wedge (f\psi).$$

If $\omega^1, \ldots, \omega^n$ is a field of coframes on M (or an open set U of M), then the set

$$\{\omega^{i_1} \wedge \cdots \wedge \omega^{i_r} : 1 \leq i_1 < i_2 < \cdots < i_r \leq n\}$$

is a basis of $\bigwedge^r(M)$ (or $\bigwedge^r(U)$, respectively).

Theorem

If $F : M \to N$ is a C^∞ mapping of manifolds, then $F^* : \bigwedge(N) \to \bigwedge(M)$ is an algebra homomorphism.

Subsection 7

Orientation of Manifolds and the Volume Element

Orientation of Bases of Vector Spaces

- Let $\boldsymbol{V}$ be a vector space.
- Let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$, $\{\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n\}$ be bases of $\boldsymbol{V}$.
- The bases are said to have the **same orientation** if the determinant of the matrix of coefficients expressing one basis in terms of the other is positive,
$$\det(\alpha_i^j) > 0,$$
where
$$\boldsymbol{f}_i = \sum_{j=1}^{n} \alpha_i^j \boldsymbol{e}_j, \quad i = 1, \ldots, n.$$
- It can be checked that:
  - This is an equivalence relation on the set of all bases (or frames) of $\boldsymbol{V}$;
  - There are exactly two equivalence classes.

Oriented Vector Spaces

- Let V be a vector space.
- The equivalence of bases modulo orientation has exactly two equivalence classes.
- A choice of one of these is said to **orient V**.

Definition

An **oriented vector space** is a vector space plus an equivalence class of allowable bases. The selected class consists of all those bases with the same orientation as a chosen one. The bases in this class will be called **oriented** or **positively oriented** bases or frames.

Orientation and Bases of $\bigwedge^n(\boldsymbol{V})$

- Orientation is related to the choice of a basis Ω of $\bigwedge^n(\boldsymbol{V})$.
- Recall that $\dim \bigwedge^n(\boldsymbol{V}) = \binom{n}{n} = 1$.
- So any nonzero element is a basis.

Lemma

Let $\Omega \neq 0$ be an alternating covariant tensor on $\boldsymbol{V}$ of order $n = \dim \boldsymbol{V}$ and let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be a basis of $\boldsymbol{V}$. Then for any set of vectors $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ with $\boldsymbol{v}_i = \sum \gamma_i^j \boldsymbol{e}_j$, we have

$$\Omega(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) = \det(\gamma_j^i)\Omega(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n).$$

- This lemma says that up to a nonvanishing scalar multiple Ω is the determinant of the components of its variables.

Orientation and Bases of $\bigwedge^n(\boldsymbol{V})$ (Cont'd)

- Let $\boldsymbol{V} = \boldsymbol{V}^n$ be the space of n-tuples.

  Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ be the canonical basis.

  The lemma assert that $\Omega(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n)$ is proportional to the determinant whose rows are $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$.
- The proof is a consequence of the definition of determinant.

  Suppose Ω and $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are given.

  Use the linearity and antisymmetry of Ω to write

$$\begin{aligned}
\Omega(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) &= \sum_{j_1,\ldots,j_n} \alpha^{j_1} \cdots \alpha^{j_n} \Omega(\boldsymbol{e}_{j_1}, \ldots, \boldsymbol{e}_{j_n}) \\
&= \sum_{\sigma \in \mathfrak{S}_n} \mathrm{sgn}\sigma \alpha_1^{\sigma(1)} \cdots \alpha_n^{\sigma(n)} \Omega(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) \\
&= \det(\alpha_i^j) \Omega(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n).
\end{aligned}$$

  The last equality is the standard definition of determinant ($\mathfrak{S}_n$ is the symmetric group on n letters).

Using Bases to Determine Orientations

Corollary

A nonvanishing $\Omega \in \bigwedge^n(\boldsymbol{V})$ has the same sign (or opposite sign) on two bases if they have the same (respectively, opposite) orientation.
Thus, choice of an $\Omega \neq 0$ determines an orientation of $\boldsymbol{V}$.
Two such forms Ω_1, Ω_2 determine the same orientation if and only if

$$\Omega_1 = \lambda \Omega_2, \quad \lambda > 0.$$

- From the formula of the lemma we see that Ω has the same sign on equivalent bases and opposite sign on inequivalent bases.

  If $\lambda > 0$, then $\lambda\Omega$ has the same sign on any basis as Ω does.

  The contrary holds if $\lambda < 0$.

Remark

- Suppose $\Omega \neq 0$.
- Then $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$ are linearly independent if and only if

$$\Omega(\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n) \neq 0.$$

- Note, also, that the formula of the lemma can be construed as a formula for change of component of Ω (there is just one component since $\dim \bigwedge^n(\boldsymbol{V}) = 1$), when we change from the basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ of $\boldsymbol{V}$ to the basis $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_n$.

Euclidean Vector Spaces

- Suppose $\boldsymbol{V}$ is a Euclidean vector space.
- So $\boldsymbol{V}$ has a positive definite inner product $\Phi(\boldsymbol{v}, \boldsymbol{w})$.
- Then, in orienting $\boldsymbol{V}$, we may choose an orthonormal basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ to determine the orientation.
- Then, we may choose an n-form Ω whose value on $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ is $+1$.
- Suppose $\boldsymbol{f}_i = \sum \alpha_i^j \boldsymbol{e}_j$ is another orthonormal basis.
- Then
$$\Omega(\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n) = \det(\alpha_i^j)\Omega(\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) = \pm 1,$$
depending on whether $\boldsymbol{f}_1, \ldots, \boldsymbol{f}_n$ is similarly or oppositely oriented.
- Thus, the value of Ω on any orthonormal basis is ± 1.
- Ω is uniquely determined up to its sign by this property.
- In this case, Ω may be given a geometric meaning when $n = 2$ or 3.
- $\Omega(\boldsymbol{v}_1, \boldsymbol{v}_2)$ or $\Omega(\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$ is the area or volume, respectively, of the parallelogram or parallelepiped of which the given vectors are the sides from the origin.

Orientable Manifolds

- To extend the concept of orientation to a manifold M we must try to orient each of the tangent spaces $T_p(M)$ in such a way that orientation of nearby tangent spaces agree.

Definition

We shall say that M is **orientable** if it is possible to define a C^∞ n-form Ω on M which is not zero at any point. In this case, M is said to be **oriented** by the choice of Ω.

- By the preceding corollary, any such Ω orients each tangent space.
- Of course any form $\Omega' = \lambda\Omega$, where $\lambda > 0$ is a C^∞ function, would give M the same orientation.

Natural Orientation

- $\mathbb{R}^n$, with the form
$$\widetilde{\Omega} = dx^1 \wedge \cdots \wedge dx^n,$$
is an example.
- This is known as the **natural orientation** of $\mathbb{R}^n$.
- It corresponds to the orientation of the frames
$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}.$$
- If $U \subseteq \mathbb{R}^n$ is an open set, it is oriented by
$$\widetilde{\Omega}_U = \widetilde{\Omega}|_U.$$

Orientation-Preserving Diffeomorphisms

- We say that a diffeomorphism $F : U \to V \subseteq \mathbb{R}^n$ is **orientation preserving** if
$$F^*\widetilde{\Omega}_V = \lambda\widetilde{\Omega}_U,$$
where $\lambda > 0$ a C^∞ function on U.
- More generally a diffeomorphism $F : M_1 \to M_2$ of manifolds oriented by Ω_1, Ω_2, respectively, is **orientation-preserving** if
$$F^*\Omega_2 = \lambda\Omega_1,$$
where $\lambda > 0$ is a C^∞ function on M.

Alternative Definition of Orientability

- A second, perhaps more natural definition of orientability can be given as follows.
- M is **orientable** if it can be covered with *coherently oriented* coordinate neighborhoods

$$\{U_\alpha, \varphi_\alpha\}.$$

- These are neighborhoods such that, if $U_\alpha \cap U_\beta \neq \emptyset$, then $\varphi_\alpha \circ \varphi_\beta^{-1}$ is orientation-preserving.
- We will now see that this second definition is equivalent to the one given previously.

Equivalence of the Definitions

Theorem

A manifold M is orientable if and only if it has a covering $\{U_\alpha, \varphi_\alpha\}$ of coherently oriented coordinate neighborhoods.

- First suppose that M is orientable.

  Let Ω be a nowhere vanishing n-form, determining the orientation.

  Choose any covering $\{U_\alpha, \varphi_\alpha\}$ by coordinate neighborhoods.

  Let $x_\alpha^1, \ldots, x_\alpha^n$ be local coordinates, such that for Ω, restricted to U_α, we have the expression in local coordinates

$$\varphi_\alpha^{-1*}\Omega_{U_\alpha}\lambda_\alpha(x)dx_\alpha^1 \wedge \cdots \wedge dx_\alpha^n, \text{ with } \lambda_\alpha > 0.$$

Equivalence of the Definitions (Cont'd)

- Replacing coordinates $(x^1, \ldots, x^n)$ by $(-x^1, \ldots, x^n)$, that is, changing the sign of one coordinate, changes the sign of λ.

  So we may easily choose coordinates so that the scalar function λ_α, component of Ω, is positive on U_α.

  An easy computation, using a previous lemma and remark, shows that if $U_\alpha \cap U_\beta \neq \emptyset$, then on this set the formula for change of component is

$$\lambda_\alpha \det\left(\frac{\partial x_\alpha^i}{\partial x_\beta^j}\right) = \lambda_\beta.$$

  Since $\lambda_\alpha > 0$ and $\lambda_\beta > 0$, the determinant of the Jacobian is positive.

  So the chosen coordinate neighborhoods are coherently oriented.

Equivalence of the Definitions (Converse)

- Now suppose that M has a covering by coherently oriented coordinate neighborhoods $\{U_\alpha, \varphi_\alpha\}$.

  We use a subordinate partition of unity $\{f_i\}$ to construct an n-form Ω on M which does not vanish at any point.

  For each $i = 1, 2, \ldots$ we choose a coordinate neighborhood $U_{\alpha_i}, \varphi_{\alpha_i}$ of the covering, such that $U_{\alpha_i} \supseteq \text{supp} f_i$. These neighborhoods, which we relabel U_i, φ_i, cover M.

  If $U_i \cap U_j \neq \emptyset$, then, by assumption, the determinant of the Jacobian matrix of $\varphi_i \circ \varphi_j^{-1}$ is positive on $U_i \cap U_j$.

Equivalence of the Definitions (Converse Cont'd)

- Define $\Omega \in \bigwedge^n(M)$ by

$$\Omega = \sum_i f_i \varphi_i^*(dx_i^1 \wedge \cdots \wedge dx_i^n),$$

where each summand is extended to all of M by defining it to be zero outside the closed set suppf_i.

Let $p \in M$ be arbitrary.

We show that $\Omega_p \neq 0$.

Recall that $\{\text{supp} f_i\}$ is locally finite.

So we may choose a coordinate neighborhood V, ψ of p which:
  - Is coherently oriented to the U_i, φ_i;
  - Intersects only a finite number of the sets suppf_i, say for $i = i_1, \ldots, i_k$.

Equivalence of the Definitions (Converse Cont'd)

- Let $y^1, \ldots, y^n$ be the local coordinates in V.

  Use the same formula as above on each summand to change components,

$$\begin{aligned} \Omega_p &= \textstyle\sum_{j=1}^{k} f_{ij}(p) \varphi_{i_j}^*(dx_{i_j}^1 \wedge \cdots \wedge d_{i_j}^n) \\ &= \textstyle\sum f_{i_j}(p) \det \left(\dfrac{\partial x_{i_j}^k}{\partial y^\ell} \right)_{\psi(p)} \psi^*(dy^1 \wedge \cdots \wedge dy^n). \end{aligned}$$

  Now each $f_{i_j} \geq 0$ on M.

  Moreover, at least one of them is positive at p.

  Finally, the Jacobian determinants are all positive.

  This implies $\Omega_p \neq 0$ and, since p was arbitrary, Ω is never zero on M.

The Case of Riemannian Manifolds

- A Riemannian manifold has the special property that the tangent space $T_p(M)$ at every point p has an inner product.
- We apply our remarks about n-forms on a Euclidean vector space of dimension n.

Theorem

Let M be an orientable Riemannian manifold with Riemannian metric Φ. Corresponding to an orientation of M, there is a uniquely determined n-form Ω which:

- Gives the orientation;
- Has the value $+1$ on every oriented orthonormal frame.

The Case of Riemannian Manifolds (Cont'd)

- It is clear from our earlier discussion that at each point $p \in M$, Ω_p is determined uniquely by the requirement that, on any oriented orthonormal basis $F_{1p}, \ldots, F_{np}$ of $T_p(M)$, we have

$$\Omega_p(F_{1p}, \ldots, F_{np}) = +1.$$

  Let U, φ be any coordinate neighborhood.

  Let $E_1, \ldots, E_n$ be be coordinate frames.

  The functions

$$g_{ij}(P) = \Phi_p(E_{ip}, E_{jp}), \quad p \in U,$$

  define the components of Φ relative to these local coordinates.

  They are C^∞, by definition.

  We derive an expression for the component $\Omega(E_1, \ldots, E_n)$ on U in terms of the matrix (g_{ij}).

  From this, it will be apparent that Ω is a C^∞ n-form.

The Case of Riemannian Manifolds (Cont'd)

- Choose at $p \in U$ any oriented, orthonormal basis $F_{1p}, \ldots, F_{np}$.

  Let the $n \times n$ matrix (α_i^k) denote the components of $E_{1p}, \ldots, E_{np}$ with respect to this basis,

  $$E_{ip} = \sum_{k=1}^{n} \alpha_i^k F_{kp}, \quad i = 1, \ldots, n.$$

  Now we have

  $$\Phi(F_{kp}, F_{ip}) = \delta_{ki}.$$

  Hence, we obtain, for $1 \leq i, j \leq n$,

  $$g_{ij}(P) = \Phi_p(E_{ip}, E_{jp}) = \left(\sum_k \alpha_i^k F_{kp}, \sum_\ell \alpha_j^\ell F_{\ell p}\right) = \sum_{k=1}^{n} \alpha_i^k \alpha_j^k.$$

The Case of Riemannian Manifolds (Cont'd)

- The equation $g_{ij}(p) = \sum_{k=1}^{n} \alpha_i^k \alpha_j^k$, $1 \leq i, j \leq n$, may be written as a matrix equation:
$$(g_{ij}(p)) = A^T A,$$
the product of the transpose of $A = (\alpha_i^k)$ with A itself.
On the other hand:
    - $\Omega_p(E_{1p}, \ldots, E_{np}) = \det(\alpha_i^k)\Omega_p(F_{1p}, \ldots, F_{np})$, by a previous lemma;
    - $\Omega_p(F_{1p}, \ldots, F_{np}) = +1$, by our definitions.

  Since $\det(A^T A) = (\det A)^2 = \det(g_{ij})$, this gives for the component of Ω in local coordinates
$$\Omega_p(E_{1p}, \ldots, E_{np}) = (\det(g_{ij}(p)))^{1/2}.$$
So the component is the square root of a positive C^∞ function of $p \in U$. So it is itself a C^∞ function on the local coordinate neighborhood U.
Since U, φ is arbitrary, Ω is a C^∞ n-form on M.

Volume Element

- This form Ω is called the (natural) **volume element** of the oriented Riemannian manifold.
- We have just seen that in local coordinates we have the following expression for Ω:

$$\varphi^{-1*}\Omega = \sqrt{g}dx^1 \wedge \cdots \wedge dx^n,$$

  where $g(x) = \det(g_{ij}(x))$ (we use the same notation for g_{ij} as functions on U and on $\varphi(U)$).
- When $M = \mathbb{R}^n$, with the usual coordinates and metric, this becomes

$$\Omega = dx^1 \wedge \cdots \wedge dx^n.$$

- In this case, as seen, the value of Ω_p on a set of vectors is the volume of the parallelepiped whose edges from p are these vectors.

Volume Element (Cont'd)

- In particular, on the unit cube with vertex at p and sides
$$\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n},$$
Ω has the value $+1$.
- The existence of the form Ω on a Riemannian manifold will enable us to define the volume of suitable subsets of the manifold.
- Moreover, we will be able to extend to these manifolds the volume integrals defined in $\mathbb{R}^n$ in integral calculus.

Subsection 8

Exterior Differentiation

Local Representations of k-Forms

- Let U be an open subset of a manifold M.
- We shall denote by θ_U the restriction of an exterior form on M to U.
- Of course $\theta_U = i^*\theta$, $i : U \to M$ being the inclusion map.
- Let U, φ be a coordinate neighborhood, with $x^1, \ldots, x^n$ as coordinate functions on U, i.e.,

$$\varphi(q) = (x^1(q), \ldots, x^n(q)).$$

- Then the differentials of these functions $dx^1, \ldots, dx^n$:
  - Are linearly independent elements of $\bigwedge^1(U)$;
  - Constitute a C^∞ field of coframes on U.
- It follows that they, with 1, generate $\bigwedge(U)$ over $C^\infty(U)$.
- Equivalently, $C^\infty(U) = \bigwedge^0(U)$ and $\bigwedge^1(U)$ generate the algebra $\bigwedge(U)$ over $\mathbb{R}$.

Local Representations of k-Forms (Cont'd)

- Thus, locally every k-form θ on M has a unique representation on U

$$\theta_U = \sum_{i_1 < \cdots < i_k} a_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}, \quad a_{i_1 \cdots i_k} \in C^\infty(U),$$

  the sum over all sets of indices such that $1 \leq i_1 < i_2 < \cdots < i_k \leq n$.
- Define $b_{i_1 \cdots i_k}$ for all values of the indices so as:
  - To change sign whenever two indices are permuted;
  - To equal $a_{i_1 \cdots i_k}$, if $i_1 < \cdots < i_k$.
- The we get the representation

$$\theta_U = \sum \frac{1}{k!} b_{i_1 \cdots i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

  the summation being over all values of the indices.
- The use of $dx^1, \ldots, dx^n$, rather than $\omega^1, \ldots, \omega^n$, is to emphasize that the dx^i are differentials of functions on $U \subseteq M$.

Operator d_M

Theorem

Let M be any C^∞ manifold. Let $\bigwedge(M)$ be the algebra of exterior differential forms on M. Then there exists a unique $\mathbb{R}$-linear map

$$d_M : \bigwedge(M) \to \bigwedge(M),$$

such that:

(1) If $f \in \bigwedge^0(M) = C^\infty(M)$, then $d_M f = df$, the differential of f;

(2) For $\theta \in \bigwedge^r(M)$, $\sigma \in \bigwedge^s(M)$,

$$d_M(\theta \wedge \sigma) = d_M\theta \wedge \sigma + (-1)^r \theta \wedge d_M\sigma;$$

(3) $d_M^2 = 0$.

- We give the proof in a series of steps.

Operator d_M (Step (A))

(A) Suppose that d_M exists.
Let $g, f^1, \ldots, f^r \in C^\infty(M)$.
Properties (1)-(3) imply that, for $\theta = g\ df^1 \wedge \cdots \wedge df^r$, we must have

$$d_M\theta = dg \wedge df^1 \wedge \cdots \wedge df^r.$$

Now suppose that M is covered by a single coordinate neighborhood U, φ with coordinate functions $x^1, \ldots, x^n$.
The above remark and linearity imply that d_M must be given by

$$d_M \left(\sum a_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r} \right) = \sum da_{i_1 \cdots i_r} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r},$$

where

$$da_{i_1 \cdots i_r} = \sum_{j=1}^{n} \frac{\partial a_{i_1 \cdots i_r}}{\partial x^j} dx^j$$

and the summation is over $1 \leq i_1 < i_2 < \cdots < i_r \leq n$.
Therefore, if defined at all, d_M is unique in this case.

Operator d_M (Step (A) Cont'd)

- Conversely, suppose d_M is defined by this sum.
  Then it is linear and trivially satisfies Properties (1) and (3).
  To check Property (2) it is enough to consider forms
  $$\theta = adx^{i_1} \wedge \cdots \wedge dx^{i_r} \quad \text{and} \quad \sigma = bdx^{j_1} \wedge \cdots \wedge dx^{j_s}.$$
  The general statement is then a consequence of linearity.
  $$\begin{aligned} & d_M[(adx^{i_1} \wedge \cdots \wedge dx^{i_r}) \wedge (bdx^{j_1} \wedge \cdots \wedge dx^{j_s})] \\ & = d_M(ab)(dx^{i_1} \wedge \cdots \wedge dx^{i_r}) \wedge (dx^{j_1} \wedge \cdots \wedge dx^{j_s}) \\ & = [(d_M a)b + a(d_M b)] \wedge (dx^{i_1} \wedge \cdots \wedge dx^{i_s}) \wedge (dx^{j_1} \wedge \cdots \wedge dx^{j_s}) \\ & = (d_M a \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}) \wedge (bdx^{j_1} \wedge \cdots \wedge dx^{j_s}) \\ & \qquad + (-1)^r (adx^{i_1} \wedge \cdots \wedge dx^{i_r}) \wedge (db \wedge dx^{j_1} \wedge \cdots \wedge dx^{j_s}). \end{aligned}$$
  The $(-1)^r$ is due to the fact that
  $$db \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r} = (-1)^r dx^{i_1} \wedge \cdots \wedge dx^{i_r} \wedge db.$$

Operator d_M (Step (B))

(B) Suppose $d_M : \bigwedge(M) \to \bigwedge(M)$, with Properties (1)-(3) is defined.

Let $U \subseteq M$ be a coordinate neighborhood on M.

Suppose its coordinate functions are $x^1, \ldots, x^n$.

According to Step (A),

$$d_U : \bigwedge(U) \to \bigwedge(U)$$

is uniquely defined.

We will show that, for any $\theta \in \bigwedge(M)$, the restriction of $d_M\theta$ to U is equal to d_U applied to θ restricted to U,

$$(d_M\theta)_U = d_U\theta_U.$$

Operator d_M (Step (B) Cont'd)

- We may suppose that $\theta \in \bigwedge^r(M)$ and that

$$\theta_U = \sum a_{i_1 \cdots i_r} dx^{i_1} \wedge \cdots \wedge dx^{i_r}, \quad a_{i_1 \cdots i_r} \in C^\infty(U).$$

Suppose p is an arbitrary point of U.

Apply a previous corollary to an open set W, $p \in W$ and $\overline{W} \in U$.

We find a neighborhood V of p, with $V \subseteq W$, and C^∞ functions $y^1, \ldots, y^n$ and $b_{i_1 \cdots i_r}$ on M, which:
  - Vanish outside W;
  - Are identical to $x^1, \ldots, x^n$, respectively, on V.

Define $\sigma \in \bigwedge^r(M)$ by

$$\sigma = \sum b_{i_1 \cdots i_r} dy^{i_1} \wedge \cdots \wedge dy^{i_r}.$$

Then σ is an r-form on M which:
  - Vanishes outside W;
  - Is identical to θ on V.

Operator d_M (Step (B) Cont'd)

- Now let g be a C^∞ function on M which:
  - Has the value $+1$ at p;
  - Is zero outside V.

  The r-form $g(\theta - \sigma)$ vanishes everywhere on M as does $dg \wedge (\theta - \sigma)$.

  Therefore, using (A),

$$gd_M\theta = gd_M\sigma = g \sum da_{i_1 \cdots i_r} \wedge dy^{i_1} \wedge \cdots \wedge dy^{i_r}.$$

  On V we have

$$\sum da_{i_1 \cdots i_r} \wedge dy^{i_1} \wedge \cdots \wedge dy^{i_r} = \sum da_{i_1 \cdots a_r} \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_r}.$$

  So at the point p, where $g(p) = 1$, $d_M\theta = d_U\theta_U$.

  Since p is arbitrary, this holds throughout U.

Operator d_M (Step (C))

(C) Suppose $d_M : \bigwedge(M) \to \bigwedge(M)$ satisfying Properties (1)-(3) exists.

We show that it is unique.

Let $\{U_\alpha, \varphi_\alpha\}$ be a covering of M by coordinate neighborhoods.

By Step (A), each d_{U_α} exists.

By Step (B), for any $\theta \in \bigwedge(M)$, we have, for any U_α,

$$(d_M\theta)_{U_\alpha} = d_{U_\alpha}\theta_{U_\alpha}.$$

Every $p \in M$ lies in a neighborhood U_α.

So this would determine d_M completely.

On the other hand, we may use this formula to define d_M.

To do so we must verify that, if $p \in U_\alpha \cap U_\beta$, then $d_M\theta$ is uniquely determined at p.

Operator d_M (Step (C) Cont'd)

- Let $U = U_\alpha \cap U_\beta$.
- We apply Steps (A) and (B) to U, an open subset and coordinate neighborhood with coordinate map φ_β cut down to U.

  We obtain

$$(d_{U_\alpha}\theta_{U_\alpha})_U = d_U\theta_U = (d_{U_\beta}\theta_{U_\beta})_U.$$

  Therefore, $(d_M\theta)_{U_\alpha}$ is determined on every U_α in such a manner that $(d_M\theta)_{U_\alpha} = (d_M\theta)_{U_\beta}$ on points common to U_α and U_β.

  This determines d_M.

  Properties (1)-(3) hold on each U_α.

  Moreover, the other operations of exterior algebra commute with restriction.

  That is, $(\theta \wedge \sigma)_U = \theta_U \wedge \sigma_U$, and so on.

  So d_M has the required properties as an operator on $\bigwedge(M)$.

Notation

- Since d_M is uniquely defined for every C^∞ manifold M, we can drop the subscript M and use d to denote all of these operators.
- We know from the above proof that d commutes with restriction of differential forms to coordinate neighborhoods.
- We investigate how it behaves relative to a C^∞ mapping $F : M \to N$.
- Any such mapping, as we know, induces a homomorphism

$$F^* : \bigwedge(N) \to \bigwedge(M).$$

- The following theorem gives the relation between F^* and d.

Mappings and Differential Operators

Theorem

F^* and d commute, that is, $F^* \circ d = d \circ F^*$.

- We know that:
  - Both F^* and d are $\mathbb{R}$-linear;
  - The equality $F^*(d\varphi) = d(F^*\varphi)$ holds on M, if it holds locally.

  By the facts concerning d, determined above, it suffices to establish the theorem for pairs V, ψ, U, θ of coordinate neighborhoods on M, N, respectively, such that $F(V) \subseteq U$.

  Let $m = \dim M$ and $n = \dim N$ and $x^1, \ldots, x^m$ and $y^1, \ldots, y^n$ be the coordinate functions on V, U, respectively.

  Let $y^j = y^j(x^1, \ldots, x^m)$, $j = 1, \ldots, n$, give F in local coordinates.

  Then it is enough to establish $F^* \circ d = d \circ F^*$ on forms of type

$$\varphi = a(x) dx^{i_1} \wedge \cdots \wedge dx^{i_k},$$

  since any other forms are sums of such forms.

Mappings and Differential Operators (Cont'd)

- We proceed by induction on the degree of the forms.

  Consider a forms $a(x)$ of degree zero, i.e., a C^∞ function.

  For $X_p \in T_p(M)$, we have

$$\begin{array}{rcl} F^*(da)(X_p) & = & da(F_* X_p) \\ & = & (F_* X_p)a \\ & = & X_p(a \circ F) \\ & = & X_p(F^* a) \\ & = & d(F^* a)(X_p). \end{array}$$

  Therefore, $F^*(da) = d(F^* a)$.

Mappings and Differential Operators (Cont'd)

- Suppose the theorem to be true for all forms of degree less than k.

  Let φ be a k-form of the type above.

  Let $\varphi_1 = adx^{i_1}$ and $\varphi_2 = dx^{i_2} \wedge \cdots \wedge dx^{i_k}$.

  So $\varphi = \varphi_1 \wedge \varphi_2$, with both φ_1 and φ_2 of degree less than k.

  Moreover, since $d^2 = 0$, we have $d\varphi_2 = 0$.

  Thus,

$$\begin{array}{rcl} d(F^*(\varphi_1 \wedge \varphi_2)) & = & d[(F^*\varphi_1) \wedge (F^*\varphi_2)] \\ & = & (dF^*\varphi_1) \wedge (F^*\varphi_2) - (F^*\varphi_1) \wedge (dF^*\varphi_2) \\ & = & F^*(d\varphi_1) \wedge F^*\varphi_2 \\ & = & F^*(d\varphi_1 \wedge \varphi_2) \\ & = & F^*d(\varphi_1 \wedge \varphi_2). \end{array}$$

Defining a Subspace

- On a vector space $\boldsymbol{V}$ of dimension n, a k-dimensional subspace $\boldsymbol{D}$ may be determined in either of two equivalent ways:
  (i) By giving a basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k$ of $\boldsymbol{D}$;
  (ii) By giving $n - k$ linearly independent elements of $\boldsymbol{V}^*$, say $\varphi^{k+1}, \ldots, \varphi^n$ which are zero on $\boldsymbol{D}$.
- In fact we may extend $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_k$ to a basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ of $\boldsymbol{V}$ so that $\varphi^{k+1}, \ldots, \varphi^n$ is part of a dual basis $\varphi^1, \ldots, \varphi^n$ of $\boldsymbol{V}^*$.

An Auxiliary Lemma

Lemma

Let $\omega \in \bigwedge^1(M)$ and $X, Y \in \mathfrak{X}(M)$. Then we have

$$d\omega(X, Y) = X\omega(Y) - Y\omega(X) - \omega([X, Y]).$$

- It is enough to prove that it is true locally, say in a coordinate neighborhood of each point.
  In any such neighborhood with coordinates $x^1, \ldots, x^n$,
  $$\omega = \sum_{i=1}^{n} a_i dx^i.$$
  The equation of the lemma holds for all ω if it holds for every ω of the form fdg, where f, g are C^∞ functions on the neighborhood.
  Suppose, then, that $\omega = fdg$.
  Let X, Y be C^∞-vector fields.

An Auxiliary Lemma

- We evaluate both sides of the equation of the lemma separately.
  We get

$$\begin{aligned} d\omega(X, Y) &= df \wedge dg(X, Y) \\ &= df(X)dg(Y) - dg(X)df(Y) \\ &= (Xf)(Yg) - (Xg)(Yf); \end{aligned}$$

- Moreover,

$$\begin{aligned} &X\omega(Y) - Y\omega(X) - \omega([X, Y]) \\ &\qquad = X(fdg(Y)) - Y(fdg(X)) - fdg([X, Y]) \\ &\qquad = X(f(Yg)) - Y(f(Xg)) - f(XYg - YXg) \\ &\qquad = (Xf)(Yg) - (Xg)(Yf) \end{aligned}$$

  after cancelation.

  This proves the lemma.

Involutiveness of a Distribution

Theorem

Let Δ be a C^∞ distribution of dimension k on M, $\dim M = n$.
Then Δ is involutive if and only if, in a neighborhood V of each $p \in M$, there exist $n-k$ linearly independent one-forms $\varphi^{k+1}, \varphi^{k+2}, \ldots, \varphi^n$ which vanish on Δ and satisfy the condition

$$d\varphi^r = \sum_{\ell=k+1}^{n} \theta_\ell^r \wedge \varphi^\ell, \quad r = k+1, \ldots, n,$$

for suitable 1-forms θ_ℓ^r.

- This may be considered a sort of dual statement to our earlier condition on Δ in terms of the existence of a local basis $X_1, \ldots, X_k$ at each point.

Involutiveness of a Distribution (Cont'd)

- Suppose a distribution Δ is given.

  Consider an arbitrary point.

  Let V be a neighborhood.

  In V, a local basis $X_1, \ldots, X_k$ of Δ can be completed to a field of frames

  $$X_1, \ldots, X_k, \ldots, X_n.$$

  Let

  $$\varphi^1, \ldots, \varphi^k, \varphi^{k+1}, \ldots, \varphi^n$$

  be the uniquely determined dual field of coframes.

  Then $\varphi^{k+1}, \ldots, \varphi^n$ vanish on $X_1, \ldots, X_k$ and hence on Δ.

Involutiveness of a Distribution (Cont'd)

- Now consider the expressions

$$[X_i, X_j] = \sum_{i=1}^{n} c_{ij}^{\ell} X_{\ell},$$

giving $[X_i, X_j]$ as linear combinations of the basis.
The distribution Δ is involutive if and only if, in the preceding expressions, we have

$$c_{ij}^{\ell} = 0, \quad 1 \le i, j \le k, \quad k+1 \le \ell \le n.$$

Using the preceding lemma and recalling that $\varphi^i(X_j)$ is constant for $1 \le i, j \le n$, we compute $d\varphi^r$,

$$\begin{array}{rcl} d\varphi^r(X_i, X_j) & = & -\varphi^r([X_i, X_j]) \\ & = & -\sum_{\ell=1}^{n} c_{ij}^{\ell} \varphi^r(X_{\ell}) \\ & = & -c_{ij}^{r}, \quad 1 \le i, j, r \le n. \end{array}$$

Involutiveness of a Distribution (Cont'd)

- On the other hand

$$d\varphi^r = \frac{1}{2}\sum_{s,t}^{n} b^r_{st}\varphi^s \wedge \varphi^t, \quad 1 \leq r \leq n,$$

where b^r_{st} are uniquely determined if we assume $b^r_{st} = -b^r_{ts}$.
Hence,

$$\begin{aligned} d\varphi^r(X_i, X_j) &= \tfrac{1}{2}\textstyle\sum_{s,t} b^r_{st}[\varphi^s(X_i)\varphi^t(X_j) - \varphi^t(X_i)\varphi^s(X_j)] \\ &= \tfrac{1}{2}(b^r_{ij} - b^r_{ji}) \\ &= b^r_{ij}. \end{aligned}$$

From this we have $b^r_{ij} = -c^r_{ij}$.

Involutiveness of a Distribution (Cont'd)

- So the system is involutive if and only if, for each $r > k$,

$$d\varphi^r = \sum_{i=k+1}^{n} \left\{ \sum_{i=1}^{k} b_{i\ell}^r \varphi^i + \sum_{j=k+1}^{n} \frac{1}{2} b_{j\ell}^r \varphi^j \right\} \wedge \varphi^\ell.$$

That is, the terms involving b_{ij}^r, with $1 \leq i, j \leq k$ and $r > k$, vanish. Taking the terms in $\{\}$ as θ_ℓ^r, we have completed the proof.

Ideals

- We can state the preceding theorem in a more elegant way if we introduce the concept of an ideal of $\bigwedge(M)$.

Definition

An **ideal** of $\bigwedge(M)$ is a subspace $\mathcal{I}$ which has the property that whenever $\varphi \in \mathcal{I}$ and $\theta \in \bigwedge(M)$, then

$$\varphi \wedge \theta \in \mathcal{I}.$$

Example: Let $\mathcal{I}$ be a subspace of $\bigwedge^1(M)$, that is, a collection of one-forms closed under addition and multiplication by real numbers. Then the set

$$\bigwedge(M) \wedge \mathcal{I} = \{\theta \wedge \varphi : \varphi \in \mathcal{I}\}$$

is an ideal, the ideal generated by $\mathcal{I}$.

Rephrasing the Theorem in Terms of Ideals

- Now suppose Δ is a distribution on M.
- Suppose, also, that $\mathcal{I}$ is the collection of 1-forms φ on M which vanish on Δ, that is, for each $p \in M$,

$$\varphi_p(X_p) = 0, \quad \text{for all } X_p \in \Delta_p.$$

- $\mathcal{I}$ is a subspace.
- In fact, if $f \in C^\infty(M)$ and $\varphi \in \mathcal{I}$, then $f\varphi \in \mathcal{I}$.
- The we have the following characterization.
- Δ is in involution if and only if

$$d\mathcal{I} = \{d\varphi : \varphi \in \mathcal{I}\}$$

  is in the ideal generated by f.