Introduction to Differential Geometry

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1 Integration on Manifolds

Subsection 1

Integration on $\mathbb{R}^n$ and Domains

Sets of Content Zero and of Measure Zero

- Let A be a subset of $\mathbb{R}^n$.
- We say that A has (n-**dimensional**) **Jordan content zero**, $c(A) = 0$, if for any $\varepsilon > 0$, there exists a finite collection of cubes $C_1, \ldots, C_s$ which cover A and the sum of whose volumes is less than ε,

$$\sum_{i=1}^{s} \operatorname{vol} C_i < \varepsilon.$$

- We say that A has **Lebesgue measure zero**, $m(A) = 0$, if, for $\varepsilon > 0$, there exists a countable set of cubes covering A, with

$$\sum_{i=1}^{\infty} \operatorname{vol} C_i < \varepsilon.$$

Content Zero versus Measure Zero

- These are not equivalent concepts.
- It is easy to see that the subset of rational numbers in $\mathbb{R}$ has measure zero but not content zero.
- We have $c(A) = 0$ implies $m(A) = 0$.
- Moreover, if A is compact, the converse also holds.
- More generally, $m(A) = 0$ if and only if A is a countable union of sets of content zero.

Domains of Integration in $\mathbb{R}^n$

Definition

A bounded subset D of $\mathbb{R}^n$ is said to be a **domain of integration** if its boundary BdD has content zero.
A function f on $\mathbb{R}^n$ is said to be **almost continuous** if the set of points at which it fails to be continuous has content zero.

- The most obvious example of a domain of integration is a cube, or an n-ball.
- The usual domains of integration in $\mathbb{R}^2$ or $\mathbb{R}^3$, bounded by piecewise differentiable curves or surfaces, are also examples.

Integrability of Bounded and Almost Continuous Functions

Theorem

Let D be a domain of integration in $\mathbb{R}^n$ and let f be a real-valued function on D. Suppose that f is bounded and almost continuous on D. Then the Riemann integral

$$\int_D f dv$$

exists.

- We shall refer to a function with these properties as **integrable on** D.
- To say that the integral exists means, of course, that it is a limit of approximating sums in the usual sense.
- The proof is essentially the same as that which is at least outlined in every calculus book.

Basic Properties of Domains

- Let D, D_1 and D_2 denote domains of integration in $\mathbb{R}^n$.
- Let f, g be bounded almost continuous functions on $\mathbb{R}^n$.
- It is not too difficult to show that the following sets are also domains of integration:
  - $\overline{D}$, the closure of D;
  - $\overset{\circ}{D}$, the interior of D;
  - $D_1 \cup D_2$;
  - $D_1 \cap D_2$;
  - $D_1 - D_2$.

Basic Properties of the Riemann Integral

- We further have the following standard properties.
- If $c(D) = 0$, then
$$\int_D f dv = 0.$$
- The following equations holds
$$\int_{D_1 \cup D_2} f dv = \int_{D_1} f dv + \int_{D_2} f dv - \int_{D_1 \cap D_2} f dv.$$
- For all $a, b \in \mathbb{R}$,
$$\int_D (af + bg) dv = a \int_D f dv + b \int_D g dv.$$
- If $f \geq 0$ on D and $c(D) \neq 0$, then
$$\int_D f dv \geq 0.$$
Equality holds iff $f = 0$ at every point at which it is continuous.

Characteristic Functions and Integration

- Recall that the **characteristic function** k_A of a subset A of a space X is defined to be

$$k_A(x) = \begin{cases} 1, & \text{if } x \in A, \\ 0, & \text{if } x \notin A. \end{cases}$$

- Therefore k_A is bounded and its discontinuities are exactly the set $\text{Bd}A$ of boundary points of A.
- In particular, if D is a domain of integration, we have $c(\text{Bd}D) = 0$ so that k_D is integrable.
- If D' is a domain of integration, $D' \supseteq D$, then

$$\int_{D'} k_D f dv = \int_D f dv.$$

Volumes

- Thus, if f on $\mathbb{R}^n$ is bounded, has compact support, and is almost continuous, then we define $\int_{\mathbb{R}^n} fdv$ unambiguously by

$$\int_{\mathbb{R}^n} fdv = \int_D fdv,$$

using any domain of integration D such that $D \supseteq \text{supp} f$.

Definition

Let D be any domain of integration. Then we define the **volume** of D, $\text{vol} D$, by

$$\text{vol} D = \int_{\mathbb{R}^n} k_D dv = \int_D k_D dv.$$

The Mean Value Property

- The following property is an easy consequence of the definitions:

$$(\inf_D f)\mathrm{vol}D \leq \int_D f dv \leq (\sup_D f)\mathrm{vol}D.$$

- When D is connected and f is continuous, we obtain the **mean value property**

$$\int_D f dv = f(a)\mathrm{vol}D,$$

for some point $a \in D$.

A Version of Fubini's Theorem

- The following theorem, a special case of Fubini's theorem, justifies the usual evaluation of multiple integrals by repeated single integrations of functions of one variable (iterated integrals).

Theorem

Suppose f is a continuous function on the domain of integration

$$D = \{x \in \mathbb{R}^n : a^i \leq x^i \leq b^i,\ i = 1, \ldots, n\}.$$

Then

$$\int_D f dv = \int_{a^n}^{b^n} \cdots \int_{a^1}^{b^1} f(x^1, \ldots, x^n) dx^1 \cdots dx^n,$$

the expression on the right denoting repeated single integrations.

Change of Variables

- Let $G : U \to U'$ be a diffeomorphism of $U \subseteq \mathbb{R}^n$ onto $U' \subseteq \mathbb{R}^n$.
- Let ΔG be the determinant of its Jacobian.
- Let G be given by coordinate functions

$$y^i = y^i(x), \quad i = 1, \ldots, n.$$

- Then

$$\Delta G = \det\left(\frac{\partial y^i}{\partial x^j}\right).$$

Change of Variables (Cont'd)

- A function f' on U' determines a function on U,

$$f = f' \circ G.$$

- We have the following relation between their integrals.

Theorem (Change of Variables)

Suppose $D \subseteq U$ and $D' = G(D) \subseteq U'$ are domains of integration.
Suppose, also, that f' is integrable on D'.
Let $f = f' \circ G$, that is,

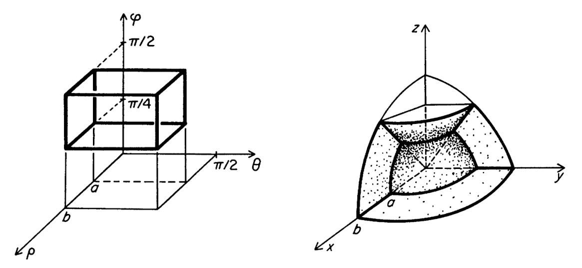
$$f(x^1, \ldots, x^n) = f'(g^1(x), \ldots, g^n(x)).$$

Then f is integrable on D and

$$\int_{D'} f'(y) dv' = \int_D f'(G(x)) |\Delta G| dv = \int_D f(x) |\Delta G| dv.$$

Example

- Let
$$D = \left\{(\rho, \theta, \varphi) : 0 < a \leq \rho \leq b, 0 \leq \theta \leq \frac{\pi}{2}, \frac{\pi}{4} \leq \varphi \leq \frac{\pi}{2}\right\}.$$



- Let D' be the first quadrant region of xyz-space:
  - Between the spheres with center at the origin and radii a and b;
  - Outside the inverted cone $z^2 = x^2 + y^2$.

Example (Cont'd)

- Let G be given by the coordinate functions

$$x = \rho \sin\varphi \cos\theta, \quad y = \rho \sin\varphi \sin\theta, \quad z = \rho \cos\varphi.$$

- Given $f'(x, y, z) = x^2 + y^2 + z^2$, then $f = f' \circ G$ is

$$f(\rho, \theta, \varphi) = f'(\rho \sin\varphi \cos\theta, \rho \sin\varphi \sin\theta, \rho \cos\varphi) = \rho^2.$$

- Also

$$\Delta = \left| \begin{matrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \varphi} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \varphi} \end{matrix} \right| = |\rho^2 \sin\varphi|.$$

- So

$$\int_{D'} (x^2 + y^2 + z^2) dx dy dz = \int_D \rho^2 |\rho^2 \sin\varphi| d\rho d\varphi d\theta.$$

Images of Sets of Content Zero

- Recall that a set is **relatively compact** if its closure is compact.

Lemma

Let A be a relatively compact subset of $\mathbb{R}^n$ of content zero. Let

$$F : A \to \mathbb{R}^m, \quad n \leq m,$$

be a C^1 mapping. Then $F(A)$ has content zero.

- By definition F is C^1 on an open set $U \supseteq A$.

  Choose an open set $V \supseteq A$, such that $\overline{V}$ is a compact subset of U.

  Let

$$K = \sup_{x \in \overline{V}} \left| \frac{\partial f^i}{\partial x^j} \right|$$

  be a bound of the derivatives on $\overline{V}$ of the coordinate functions of F.

Images of Sets of Content Zero (Cont'd)

- Choose δ_1, $0 < \delta_1 \leq 1$, so that every cube of side δ_1 whose center is in A lies inside V.

  By the Mean Value Theorem, for any x in a cube of side δ_1 and center $a \in A$,
  $$\|F(x) - F(a)\| < \sqrt{nm} K \|x - a\|.$$
  Take $0 < \delta < \delta_1$.

  Consider a cube C of side δ and center $a \in A$.

  C must map into a cube C' of center $F(a)$ and side length less than or equal to $\sqrt{nm} K \delta$.

Images of Sets of Content Zero (Cont'd)

- Thus, $F(C)$ lies in a cube C' whose volume satisfies

$$\begin{array}{rcl} \text{vol}C' & \leq & (\sqrt{nm}K\delta)^m \quad (\delta < \delta_1 \leq 1) \\ & = & (nm)^{m/2}K^m\delta^{m-n}\delta^n \\ & \leq & k\text{vol}C. \quad (\text{vol}C = \delta^n) \end{array}$$

where $k = K^m(nm)^{m/2}$ is independent of $a \in A$.

From this, it follows at once that, given any $\varepsilon > 0$, we may cover $F(A)$ with a finite number of cubes $C_1', \ldots, C_s'$ whose total volume is less than ε.

We need only cover A with cubes $C_1, \ldots, C_s$ whose:
  - Volume is less than $\frac{\varepsilon}{k}$;
  - Side is less than δ_1.

This shows that the content of $F(A)$ is zero.

Sets of Zero Content and Zero Measure in Manifolds

Definition

A relatively compact subset $A \subseteq M$ is said to have **content zero**, written $c(A) = 0$, if it is the union of a finite number of subsets

$$A = A_1 \cup \cdots \cup A_s,$$

each of which lies in a coordinate neighborhood U_i, φ_i, such that, in $\mathbb{R}^n$,

$$c(\varphi_i(A_i)) = 0, \quad i = 1, \ldots, s.$$

An arbitrary subset $B \subseteq M$ is said to have **measure zero**, written $m(B) = 0$, if B is the union of a countable collection of subsets $B = \bigcup_{i=1}^{\infty} B_i$, such that each B_i has content zero.

Properties of Sets of Zero Content or Zero Measure

Corollary

Suppose $A \subseteq M$ has content (respectively, measure) zero.
Let

$$F : M \to N$$

be a C^1 map with $\dim M \leq \dim N$.
Then $F(A)$ has content (respectively, measure) zero.
In particular, this holds if F is a diffeomorphism.

- This is an obvious application of the preceding lemma to the definition.

Domains of Integration in Manifolds

- If M is a manifold, $D \subseteq M$ is a **domain of integration** if D is relatively compact and the boundary of D has content zero, $c(\text{Bd}D) = 0$.

Theorem

If D is a domain of integration in M, so are its closure and its interior. Finite unions and intersections of domains of integration are domains of integration. Finally, the image of a domain of integration under a dfffeomorphism is a domain of integration.

- These are all immediate consequences of the definition and of the corresponding statements for:
  - Subsets of content zero;
  - Domains of integration in $\mathbb{R}^n$.

  For the last statement we must note that a diffeomorphism takes boundary points to boundary points.

Subsection 2

A Generalization to Manifolds

Oriented Manifolds Revisited

- Suppose that M is an oriented manifold and $\dim M = n$.
- By definition, this means that there is a C^∞ n-form

$$\Omega$$

on M which is not zero at any point of M.
- $\{\Omega\}$ is a basis of $\bigwedge^n(M)$.
- That is, any other n-form ω is given by

$$\omega = f\Omega,$$

where f is a function on M.
- Since Ω is C^∞, ω will have the differentiability class of f.

Integrable Functions on a Manifold

Definition

A function f on M is **integrable** if:

- It is bounded;
- Has compact support (vanishes outside a compact set);
- Is almost continuous (that is, continuous except possibly on a set of content zero).

An n-form ω on M, in the very general sense of a function assigning to each $p \in M$ an element ω_p of $\bigwedge^n(T_p(M))$, is said to be **integrable** if

$$\omega = f\Omega,$$

where f is an integrable function (we are not requiring ω to be C^∞ or even C^1).

Integrable Functions on a Manifold (Remark)

- The definition of integrable n-form does not depend on the particular Ω we use.
- Any other $\widetilde{\Omega}$ giving the orientation is of the form $\widetilde{\Omega} = g\Omega$, where g is a positive C^∞ function on M.
- Thus,

$$f\Omega = \frac{f}{g}\widetilde{\Omega}.$$

- If f has compact support, is bounded, and is almost continuous, then the same will be true of $\frac{f}{g}$.
- We denote by $\bigwedge_0^n(M)$ the set of integrable n-forms.
- Like $\bigwedge^n(M)$, it is a vector space over $\mathbb{R}$.
- Moreover, it is closed under multiplication by continuous or integrable functions on M.

Definition of Integral of $\omega \in \bigwedge_0^n(M)$

- A subset $Q \subseteq M$ is called a **cube** of M if it lies in the domain of an associated, oriented, coordinate neighborhood U, φ and

$$\varphi(Q) = C = \{x \in \mathbb{R}^n : 0 \leq x^i \leq 1, i = 1, \ldots, n\},$$

the unit cube of $\mathbb{R}^n$.

- Thus a cube is a compact set and is coordinatized in a definite way.
- We first define the integral over M of any $\omega \in \bigwedge_0^n(M)$ whose support lies interior to some cube Q.

Definition of Integral of $\omega \in \bigwedge_0^n(M)$ (Cont'd)

- Let U, φ be the coordinate neighborhood associated with Q.
- Suppose
$$\varphi^{-1*}(\omega) = f(x) dx^1 \wedge \cdots \wedge dx^n$$
represents ω in the local coordinates.
- Then f is bounded and almost continuous on C.
- So $\int_C f dv$ is defined.
- We define
$$\int_M \omega = \int_C f dv.$$

Independence of Choice of Cube

- We must show that the value of this integral is independent of the particular cube we have used.
- Suppose Q' is another cube containing suppω.
- Let U', φ' be the associated coordinate neighborhood.
- We denote the local coordinates for this neighborhood by

$$y^1, \ldots, y^n.$$

- Suppose that

$$\varphi'^{-1*}(\omega) = f'(y) dy^1 \wedge \cdots \wedge dy^n$$

represents ω on $\varphi'(U')$.

Independence of Choice of Cube (Cont'd)

- Consider the diffeomorphism

$$G = \varphi' \circ \varphi^{-1} : \varphi(U \cap U') \to \varphi'(U \cap U').$$

- Let ΔG be the determinant of its Jacobian matrix.
- ΔG is positive, since the neighborhoods are oriented.
- By the rules for change of components of an n-form, we have

$$f(x) = f'(G(x))\Delta G.$$

- On the other hand, since Q, Q' are domains of integration.
- Therefore, so are $Q \cap Q'$ and its images $D = \varphi(Q \cap Q')$ and $D' = \varphi'(Q \cap Q')$, which lie in the unit cube of the x-coordinate space and the y-coordinate space, respectively.

Independence of Choice of Cube (Cont'd)

- Now supp$\omega \subseteq Q \cap Q'$.
- So supp$f \subseteq D$ and supp$f' \subseteq D'$.
- Therefore
$$\int_C f(x)dv = \int_D f(x)dv \quad \text{and} \quad \int_{C'} f'(y)dv' = \int_{D'} f'(y)dv'.$$
- According to the Change of Variable Theorem, since $D' = G(C)$,
$$\int_{D'} f'(y)dv' = \int_D f'(G(x))|\Delta G|dv.$$
- However, $\Delta G > 0$ so that $|\Delta G| = \Delta G$.
- So, by the formula for change of components, the integral on the right must equal
$$\int_D f(x)dv.$$
- This shows that $\int_M \omega$ is uniquely determined for every integrable ω which vanishes outside of some cube.

Linearity Property

- We note, in particular, the following linearity property.
- Suppose ω_1, ω_2 vanish outside a cube Q.
- Then, for all real numbers a_1, a_2,

$$\int_M a_1\omega_1 + a_2\omega_2 = a_1 \int_M \omega_1 + a_2 \int_M \omega_2.$$

Integral of Integrable n-Forms

- Suppose that ω is an arbitrary integrable n-form.
- Let $K = \text{supp}\omega$.
- Choose a finite covering of K by the interiors $\overset{\circ}{Q}_1, \ldots, \overset{\circ}{Q}_s$ of cubes $Q_1, \ldots, Q_s$ associated with coordinate neighborhoods U_1, φ_1, ..., U_s, φ_s, respectively.
- The open sets $M - K$, $\overset{\circ}{Q}_1, \ldots, \overset{\circ}{Q}_s$ cover M.
- Take a suitable partition of unity $\{f_i\}$ subordinate to this covering.
- We may assume that:
  - For $j > s$, $f_j = 0$ on K;
  - For $j = 1, \ldots, s$, $\text{supp} f_j \subseteq \overset{\circ}{Q}_j$, the interior of the cube Q_j.

Integral of Integrable n-Forms (Cont'd)

- Since $\sum f_j \equiv 1$, we have
$$\omega = f_1\omega + \cdots + f_s\omega.$$
- Each f_j has its support on the interior $\mathring{Q}_j$ of the cube Q.
- So each of the integrals
$$\int_M f_j\omega$$
is defined.
- We define
$$\int_M \omega = \int_M f_1\omega + \cdots + \int_M f_s\omega.$$

Independence of Covering and Functions

- Let $Q'_1, \ldots, Q'_r$ be another set of cubes whose interiors cover K.
- Choose again a partition of unity $\{g_k\}$ such that:
  - $\text{supp} g_k \subseteq \overset{\circ}{Q'}_k$, $k = 1, \ldots, r$;
  - $g_k = 0$ on K for $k > r$.
- Then
$$\sum_{i,k} f_i g_k \equiv \sum_i f_i \sum_k g_k \equiv 1.$$
- Moreover, for fixed k, $1 \leq k \leq r$, we have
$$\text{supp} f_i g_k \subseteq Q'_k.$$

Independence of Covering and Functions (Cont'd)

- By the linearity of the integral with respect to forms with support in the same cube,

$$\int_M g_k\omega = \int_M f_1 g_k\omega + \cdots + \int_M f_s g_k\omega.$$

- We compute $\int_M \omega$ using this second covering by cubes.
- We have

$$\int_M \omega = \sum_{k=1}^{r} \int_M g_k\omega = \sum_{k=1}^{r}\sum_{i=1}^{s} \int_M f_i g_k\omega.$$

- By a symmetric argument, the sum on the right is also equal to

$$\sum_{i=1}^{s} \int_M f_i\omega.$$

- Hence, both choices assign the same value to $\int_M \omega$.

Properties of Integrals

Theorem

The process just defined assigns to each integrable n-form ω on an oriented manifold M a real number $\int_M \omega$.
We have the following properties:

(i) If $-M$ denotes the same underlying manifold, with opposite orientation, then

$$\int_{-M} \omega = -\int_M \omega.$$

(ii) The mapping $\omega \to \int_M \omega$ is an $\mathbb{R}$-linear mapping on $\bigwedge_0^n(M)$, that is, for all $a_1, a_2 \in \mathbb{R}$ and all $\omega_1, \omega_2 \in \bigwedge_0^n(M)$,

$$\int_M a_1\omega_1 + a_2\omega_2 = a_1 \int_M \omega_1 + a_2 \int_M \omega_2.$$

Properties of Integrals (Cont'd)

Theorem (Cont'd)

(iii) Let Ω be a nowhere vanishing n-form giving the orientation of M. If $\omega = g\Omega$, with $g \geq 0$, then

$$\int_M g\Omega \geq 0,$$

and equality holds if and only if $g = 0$, wherever it is continuous.

(iv) Let $F : M_1 \to M_2$ be a diffeomorphism and $\omega \in \bigwedge_0^n(M_2)$. Then

$$\int_{M_1} F^*\omega = \pm \int_{M_2} \omega,$$

with sign depending on whether F preserves or reverses orientation.

Proof of the Theorem

- Because of the definition, we need to verify these properties only for forms ω whose support lies in a cube Q associated with the oriented coordinate neighborhood U, φ and coordinates $x^1, \ldots, x^n$.
  Suppose
  $$\varphi^{-1*}(\omega) = f(x) dx^1 \wedge \cdots \wedge dx^n.$$
  Then, by definition,
  $$\int_M \omega = \int_C f(x) dv.$$
  Suppose that the orientation of M is reversed.
  Then the map φ assigning coordinates in U must be replaced by a map φ', such that the Jacobian of $\varphi' \circ \varphi^{-1}$ has negative determinant.
  For example, by interchanging the first and second variables.
  f is the component of ω in the local coordinates.
  So the interchange changes the sign of f.
  Hence, it changes the sign of the integral.

Proof of the Theorem (Cont'd)

- Property (ii) was previously noted.

  It is a consequence of the corresponding property for the Riemann integral on $\mathbb{R}^n$.

  Next, note that in (oriented) local coordinates

  $$\varphi^{-1*}\Omega = p(x)dx^1 \wedge \cdots \wedge dx^n, \quad p(x) > 0.$$

  So

  $$\int_M g\Omega = \int_C g(x)p(x)dv.$$

  Now $g(x)p(x) \geq 0$, and vanishes exactly where $g(x)$ vanishes.

  The assertion now follows from the corresponding property in $\mathbb{R}^n$.

  This proves Property (iii).

Proof of the Theorem (Cont'd)

- Suppose $F : M_1 \to M_2$ is a diffeomorphism preserving orientation.

  Let ω on M_2 have support in a cube Q associated with the coordinate neighborhood U, φ.

  Then $Q' = F^{-1}(Q)$ is a cube on M_1 associated with

  $$U' = F^{-1}(U) \quad \text{and} \quad \varphi' = \varphi \circ F^{-1}.$$

  This cube contains the support of $F^*\omega$.

  With respect to it, we have precisely the same expression

  $$f(x)dx^1 \wedge \cdots \wedge dx^n$$

  for both ω and $F^*\omega$ in local coordinates.

Proof of the Theorem (Cont'd)

- Hence,

$$\int_{M_2} \omega = \int_{M_1} F^* \omega = \int_C f dv.$$

Assume, on the other hand, that F does not preserve orientation.

Then the equation

$$\int_{M_1} F^* \omega = -\int_{M_2} \omega$$

follows from the orientation-preserving case and Property (i).

Remark

- Note that a special case of the definition above, namely $M = \mathbb{R}^n$, defines
$$\int_{\mathbb{R}^n} f(x^1, \ldots, x^n) dx^1 \wedge \cdots \wedge dx^n$$
for any bounded function f on $\mathbb{R}^n$ which has compact support and is almost continuous.
- We can also show that, if $\text{supp} f \subseteq D$, a domain of integration, then
$$\int_{\mathbb{R}} f(x) dx^1 \wedge \cdots \wedge dx^n = \int_D f(x) dv,$$
the usual Riemann integral.

Volume Elements in Riemannian Manifolds

- A **volume element** is, by definition, a nowhere vanishing n-form Ω on M which is in that class which determines the orientation.
- On an arbitrary oriented manifold there is such a form Ω.
- It is determined only to within a multiple by a positive C^∞ function.
- This is not enough to define volumes.
- We must have a unique Ω given, say, by the structure of M.
- One case in which this occurs, according to a previous theorem, is on an oriented Riemannian manifold M.
- In this case there is a unique Ω whose value on any orthonormal frame is $+1$.
- We shall always use this Ω on the Riemannian manifold.
- In this section, we shall discuss only the Riemannian case.
- Then, using Ω and the characteristic function k_D of a domain of integration D we are able to parallel the theory for $\mathbb{R}^n$.

Integration in Riemannian Manifolds

Definition

Let D be a domain of integration on an oriented Riemannian manifold M.
Let k_D be the characteristic function of D.
We define the **volume** of D, denoted by vol$D$, by

$$\text{vol}D = \int_M k_D \Omega.$$

If f is any integrable function on M, we define the integral of f over D, denoted $\int_D f$, by

$$\int_D f = \int_M f k_D \Omega.$$

Integration in Riemannian Manifolds (Cont'd)

Definition (Cont'd)

When M is compact, we may take $D = M$ and obtain

$$\text{vol} M = \int_M \Omega$$

and

$$\int_M f = \int_M f\Omega.$$

- These integrals are defined, since k_D is continuous except on $\text{Bd} D$ which has content zero.

Properties of the Integral

Lemma

With the preceding definitions the integral of f on a domain of integration on M satisfies the following properties of the Riemann integral on $\mathbb{R}^n$.

- If $c(D) = 0$, then $\int_D f dv = 0$;
- $\int_{D_1 \cup D_2} f dv = \int_{D_1} f dv + \int_{D_2} f dv - \int_{D_1 \cap D_2} f dv$;
- $\int_D (af + bg) dv = a \int_D f dv + b \int_D g dv$, for all $a, b \in \mathbb{R}$;
- If $f \geq 0$ on D and $c(D) \neq 0$, then
$$\int_D f dv \geq 0,$$
with equality iff $f = 0$ at every point at which it is continuous.

It is equal to the Riemann integral when $M = \mathbb{R}^n$ (with its standard metric).

Outline of the Steps

- The lemma is a consequence of the definitions and of the corresponding properties of the Riemann integral.
- We choose a covering of D by the interiors of cubes.
- We take a corresponding partition of unity as in the definition of $\int_M \omega$.
- We then show that it is possible to reduce the proof to verifying each property for the special case in which $\omega = f\Omega$ has its support in a single cube.
- In this case, the properties coincide with the properties of the integral on $\mathbb{R}^n$.
- For the last statement we use a previous remark.

Components of Riemannian Metric Tensor

- Let U, φ be local coordinates.
- Let $E_1, \ldots, E_n$ be coordinate frames.
- Let $\Phi(X, Y)$ be a Riemannian metric tensor.
- The matrix components $\Phi(E_i, E_j)$ on U are customarily denoted by

$$g_{ij}, \quad i, j = 1, \ldots, n.$$

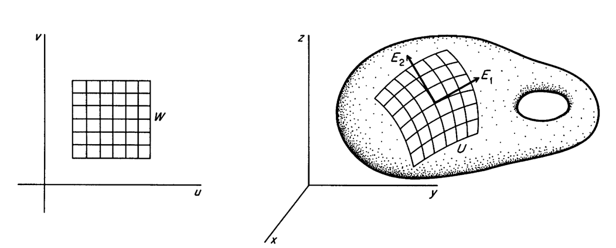
- The same symbols g_{ij} are frequently used to denote:
  - $g_{ij}(p) = \Phi_p(E_{ip}, E_{jp})$, the components considered as functions on $U \subseteq M$;
  - $\widehat{g}_{ij}(x^1, \ldots, x^n) = g_{ij}(\varphi(p))$, the components considered as the corresponding functions on $\varphi(U) \subseteq \mathbb{R}^n$.
- In a previous section we found that the local expression for Ω on an oriented neighborhood was

$$\varphi^{-1*}\Omega = \sqrt{g}\, dx^1 \wedge \cdots \wedge dx^n, \quad g = \det(g_{ij}).$$

Example

- Let M be a surface in $\mathbb{R}^3$ with the Riemannian metric induced by the standard metric of $\mathbb{R}^3$.
- Let U, φ be a coordinate neighborhood with coordinates (u, v).
- Suppose $\varphi(U) = W$, an open subset of the uv-plane.
- Let $F = \varphi^{-1}$ so that $F : W \to M$ has image U.
- Let the C^∞-coordinate functions for the mapping be

$$F(u, v) = (f(u, v), g(u, v), h(u, v)).$$



Example (Cont'd)

- As in a previous example the coordinate frames E_1, E_2 on U are

$$\begin{aligned} E_1 &= F_*(\frac{\partial}{\partial u}) = \frac{\partial f}{\partial u}\frac{\partial}{\partial x} + \frac{\partial g}{\partial u}\frac{\partial}{\partial y} + \frac{\partial h}{\partial u}\frac{\partial}{\partial z}, \\ E_2 &= F_*(\frac{\partial}{\partial v}) = \frac{\partial f}{\partial v}\frac{\partial}{\partial x} + \frac{\partial g}{\partial v}\frac{\partial}{\partial y} + \frac{\partial h}{\partial v}\frac{\partial}{\partial z}. \end{aligned}$$

- Hence we have

$$\begin{aligned} g_{11}(u,v) &= (\frac{\partial f}{\partial u})^2 + (\frac{\partial g}{\partial u})^2 + (\frac{\partial h}{\partial u})^2 = (E_1, E_1), \\ g_{12}(u,v) &= \frac{\partial f}{\partial u}\frac{\partial f}{\partial v} + \frac{\partial g}{\partial u}\frac{\partial g}{\partial v} + \frac{\partial h}{\partial u}\frac{\partial h}{\partial v} \\ &= (E_1, E_2) = (E_2, E_1) = g_{21}(u,v), \\ g_{22}(u,v) &= (\frac{\partial f}{\partial v})^2 + (\frac{\partial g}{\partial v})^2 + (\frac{\partial h}{\partial v})^2 = (E_2, E_2). \end{aligned}$$

Example (Cont'd)

- These are denoted E, F, G, respectively, and we have then

$$\begin{array}{rcl} \varphi^{-1*}\Omega & = & F^*\Omega \\ & = & (g_{11}g_{22} - g_{12}^2)^{1/2} du \wedge dv \\ & = & (EG - F^2)^{1/2} du \wedge dv. \end{array}$$

- Let D be a domain of integration on M such that $D \subseteq U$.
- Let h be an integrable function on D.
- Then

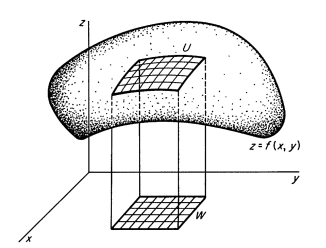
$$\begin{array}{rcl} \int_D h & = & \int_D h\Omega \\ & = & \int_{\varphi(D)} h(u,v)(EG - F^2)^{1/2} du \wedge dv \\ & = & \int_{\varphi(D)} h(u,v)(EG - F^2)^{1/2} dudv. \end{array}$$

Example (Cont'd)

- Suppose that φ is the (diffeomorphic) projection of an open set U of M onto an open set W of the xy-plane, which we identify with the parameter plane.
- In this case $F : W \to U$ is given by

$$F(x, y) = (x, y, f(x, y)).$$

The graph of $z = f(x, y)$ lying over W is the subset U of M.



- The coordinate frames are

$$E_1 = \frac{\partial}{\partial x} + f_x \frac{\partial}{\partial z} \quad \text{and} \quad E_2 = \frac{\partial}{\partial y} + f_y \frac{\partial}{\partial z}.$$

Example (Cont'd)

- So
$$E = 1 + f_x^2, \quad F = f_x f_y, \quad G = 1 + f_y^2.$$
- Hence,
$$F^{\Omega} = (EG - F^2)^{1/2} dx \wedge dy = (1 + f_x^2 + f_y^2)^{1/2} dx \wedge dy.$$
- Let $D \subseteq U$ be a domain of integration.
- Let $A \subseteq W$ be its projection to the xy-plane.
- Then for any integrable function h on M we have
$$\int_D h = \int_A h(x, y, z)(1 + f_x^2 + f_y^2)^{1/2} dxdy.$$
- When $h = 1$, the value of this integral is the area of D ($= \text{vol}D$).

Example (Cont'd)

- Suppose, for example, $M = S^2$, the unit sphere.
- Let U be the upper hemisphere and $D = U$.
- Then
$$A = W = \{(x, y) : x^2 + y^2 < 1\}.$$
- Moreover,
$$F(x, y) = (x, y, (1 - x^2 - y^2)^{1/2}).$$
- The area of U is
$$\begin{aligned} \int_U \Omega &= \int_A (1 - x^2 - y^2)^{-1/2} dx \wedge dy \\ &= \int_{-1}^{+1} \int_{-(1-y^2)^{1/2}}^{(1-y^2)^{1/2}} (1 - x^2 - y^2)^{-1/2} dx dy \\ &= 2\pi. \end{aligned}$$

Remark

- Let M be a compact manifold.
- In practice (or for theoretical purposes) one might hope that M could be covered by a finite number of domains of integration $D_1, \ldots, D_s$, such that:
  (i) $c(D_i \cap D_j) = 0$, $i \neq j$, $i, j = 1, \ldots, s$;
  (ii) Each D_i lies in a coordinate neighborhood U_i, φ_i.
- We use the fact that

$$\int_M f = \int_{D_1} f + \cdots + \int_{D_s} f.$$

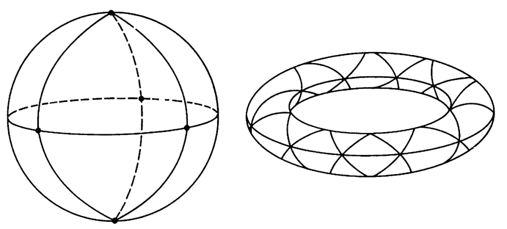
Remark (Cont'd)

- It is then possible to evaluate each integral on the right separately as an integral on $\varphi_i(D_i) \subseteq \mathbb{R}^n$.
- Let $f(x)$ denote the expression for f in local coordinates.
- Let $g = \det(g_{ij})$.
- Then we have

$$\begin{aligned} \int_{D_i} f &= \int_{\varphi_i(D_i)} f(x)\sqrt{g}\, dx^1 \wedge \cdots \wedge dx^n \\ &= \int_{\varphi_i(D_i)} f(x)\sqrt{g}\, dv. \end{aligned}$$

Remark (Cont'd)

- It can be shown that any differentiable manifold M (compact or not) can be covered with a collection of domains of integration $D_1, D_2, \ldots$, each the diffeomorphic image of a simplex (for $n = 2$ a triangle, for $n = 3$ a tetrahedron, and so on).
- Moreover these domains intersect in sets of content zero. [This is part of a theorem which asserts that any C^∞ manifold is triangulable.]



- When M is compact the number of D_i is finite.
- This is not a complete description of a triangulation, but it shows that for both practical and theoretical purposes a technique of evaluation of $\int_M f$ or $\int_M \Omega$ is available.

Subsection 3

Integration on Lie Groups

Translations and Inner Automorphisms of Lie Groups

- Let G be an arbitrary Lie group of dimension n.
- Given $a, b \in G$, we denote by:
  - L_a left translation by a;
  - R_b right translation by b;
  - $I_a = L_a \circ R_{a^{-1}}$ the inner automorphism, $I_a(x) = axa^{-1}$, of G.
- These are C^∞ mappings, with inverses

$$L_a^{-1} = L_{a^{-1}}, \quad R_a^{-1} = R_{a^{-1}}, \quad I_a^{-1} = I_{a^{-1}}.$$

- Hence, they are diffeomorphisms.
- So they induce $\mathbb{R}$-linear mappings of $\mathfrak{X}(G)$ - the C^∞-vector fields on G - onto itself, which preserve the bracket operation.
- However, on G our main interest is in the subspace $\mathfrak{g}$ of $\mathfrak{X}(G)$ consisting of all left-invariant vector fields on G.
- We have seen $\mathfrak{g}$ is a Lie algebra, the Lie algebra of G, with respect to the product $[X, Y]$.

Translations and Inner Automorphisms (Cont'd)

- Given $a, b \in G$, we have, by associativity, $a(xb) = (ax)b$.
- Thus, the left and right translations L_a and R_b commute.
- From this we deduce that if $X \in \mathfrak{g}$, then $R_{b*}X \in \mathfrak{g}$.
- Moreover,
$$L_{g*}(R_{b*}X) = R_{b*}(L_{g*}X) = R_{b*}X.$$
- Similarly,
$$I_{a*}X = L_{a*}R_{a^{-1}*}X = R_{a^{-1}*}X \in \mathfrak{g}.$$
- Thus $I_{a*} : \mathfrak{g} \to \mathfrak{g}$.

Translations and Inner Automorphisms (Cont'd)

- Now I_{a*} is both a linear mapping and preserves the product,

$$I_{a*}[X, Y] = [I_{a*}X, I_{a*}Y].$$

- So I_{a*} is an automorphism of the Lie algebra $\mathfrak{g}$.
- Finally, note that $I_{ab} = I_a \circ I_b$.
- So, by the chain rule,

$$I_{ab*} = I_{a*} \circ I_{b*}.$$

- Denote I_{g*} by $\mathrm{Ad}g$, for g any element of G.
- Putting the preceding facts together, we have proved most of the following:
  - The mapping of G into the group of all automorphisms of $\mathfrak{g}$ defined by $g \to \mathrm{Ad}g$ is a homomorphism.
  - Let $Gl(\mathfrak{g})$ denote the group of all nonsingular linear transformations of $\mathfrak{g}$ as a vector space. Then $\mathrm{Ad} : G \to Gl(\mathfrak{g})$ is C^∞.

Translations and Inner Automorphisms (Cont'd)

- We prove and interpret the last statement.
- In general, if V is a finite-dimensional vector space over $\mathbb{R}$, then the group $Gl(\boldsymbol{V})$ of all nonsingular linear transformations of $\boldsymbol{V}$ onto $\boldsymbol{V}$ is isomorphic to $Gl(n, \mathbb{R})$, $n = \dim \boldsymbol{V}$.
- The isomorphism depends on the choice of a basis $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n$ of $\boldsymbol{V}$.
- It is given by letting $A \in Gl(\boldsymbol{V})$ correspond to the matrix (α_{ij}) defined by

$$A(\boldsymbol{e}_j) = \sum_{i=1}^{n} \alpha_{ij} \boldsymbol{e}_i, \quad j = 1, \ldots, n.$$

Translations and Inner Automorphisms (Cont'd)

- We take the topology and C^∞ structure on $Gl(\boldsymbol{V})$ obtained by identifying it with the Lie group $Gl(n, \mathbb{R})$.
- It may be shown that this C^∞ structure is independent of the choice of basis.
- Suppose we choose a basis of $\mathfrak{g}$,
$$X_1, \ldots, X_n.$$
- Let the matrix corresponding in this way to Adg be
$$(\alpha_{ij}(g)).$$
- The last statement asserts that
$$g \mapsto (\alpha_{ij}(g))$$
is a C^∞ mapping.

Translations and Inner Automorphisms (Cont'd)

- Note that $I_g(e) = e$.
- Hence $I_{g*} : T_e(G) \to T_e(G)$.
- Now $\mathfrak{g}$ may be naturally identified with $T_e(G)$ by identifying each $X \in \mathfrak{g}$ with its value X_e at e.
- So we may think of Adg as a linear transformation on $\mathfrak{g}$ - the left-invariant vector fields - or on $T_e(G)$.
- On $T_e(G)$, Adg coincides with the transformation induced by I_g according to the definition.
- Taking this point of view, the matrix $(\alpha_{ij}(g))$ is a submatrix of the Jacobian matrix, evaluated at (g, e), of the C^∞ mapping of $G \times G \to G$ defined by

$$(g, x) \mapsto gxg^{-1} = I_g(x).$$

- Hence $g \mapsto (\alpha_{ij}(g))$ is C^∞.

Representations of Lie Groups

Definition

A **representation** of a Lie group G on a vector space $\boldsymbol{V}$ is a Lie group homomorphism of G into the group $Gl(\boldsymbol{V})$ of nonsingular linear transformations of $\boldsymbol{V}$ onto $\boldsymbol{V}$.

The **degree** (**dimension**) of the representation is the dimension of $\boldsymbol{V}$.

A **matrix representation** of G of degree n is a Lie group homomorphism of G into $Gl(n, \mathbb{R})$.

The representation $g \mapsto \text{Ad}g$ is called the **adjoint representation** of G.

- We remark again that we interpret $\text{Ad}g$ both as a linear mapping on $\mathfrak{g}$, the space of invariant vector fields, and on $T_e(G)$, the tangent space at the identity.
- This is by virtue of the identification of $\mathfrak{g}$ with $T_e(G)$.
- $\text{Ad}g$ is induced by the diffeomorphism $I_g(x) = gxg^{-1}$ of G onto G.

Invariant Tensor Fields

Definition

A covariant tensor field Φ of order r on G is:

- **Left-invariant** if $L_a^*\Phi_{ag} = \Phi_g$;
- **Right-invariant** if $R_a^*\Phi_{ga} = \Phi_g$.

It is **bi-invariant** if it is both left- and right-invariant.

- We remark that any left- (or right-) invariant covariant tensor field $\Phi \in \mathscr{T}^r(G)$ is necessarily C^∞.
- Let $X_1, \ldots, X_n$ be a basis of C^∞ left- (right-) invariant vector fields.
- Then $\Phi(X_{i_1}, \ldots, X_{i_r})$ is constant - hence C^∞ - on G for any $1 \leq i_1, \ldots, i_r \leq n$.
- Therefore, the components of Φ with respect to a C^∞-frame field are C^∞, and Φ is thus C^∞.

Existence of Invariant Tensor Fields

Lemma

Let Φ_e be a covariant tensor of order r on the tangent space $T_e(G)$ at the identity. Then there is a unique left-invariant tensor field and a unique right-invariant tensor field coinciding at e with Φ_e. These two agree everywhere on G. That is, Φ_e determines a bi-invariant tensor field if and only if

$$(\mathrm{Ad}g)^* \Phi_e = \Phi_e, \quad \text{for all } g \in G.$$

- Let Φ_e be a covariant tensor on $T_e(G)$.

  For each $g \in G$, there exists a unique left translation $L_g : G \to G$ which takes e to g.

  Define $\Phi \in \mathscr{T}^r(G)$ by

$$\Phi_g = L^*_{g^{-1}} \Phi_e.$$

Existence of Invariant Tensor Fields (Cont'd)

- We have

$$L_a^*\Phi_{ag} = L_a^*(L^*_{g^{-1}a^{-1}}\Phi_e) = L_a^* \circ L^*_{a^{-1}} \circ L^*_{g^{-1}}\Phi_e = L^*_{g^{-1}}\Phi_e.$$

Since this is just Φ_g, we see that Φ is left-invariant.
Similarly, $R^*_{g^{-1}}\Phi_e$ is a right-invariant tensor field.
If Φ is bi-invariant, then

$$(\mathrm{Ad}g)^*\Phi_e = (L_g \circ R_{g^{-1}})^*\Phi_e = L_g^* \circ R^*_{g^{-1}}\Phi_e = \Phi_e.$$

Conversely, if this relation holds, then

$$L^*_{g^{-1}}\Phi_e = L^*_{g^{-1}} \circ L_g^* \circ R^*_{g^{-1}}\Phi_e = R^*_{g^{-1}}\Phi_e.$$

So the left- and right-invariant tensor fields determined by Φ_e agree at every $g \in G$.

Existence of Invariant Tensor Fields (Cont'd)

- It is immediate that an invariant field must be determined by its value at anyone element, say e, of G.

Corollary

Every Lie group has a left-invariant Riemannian metric and a left-invariant volume element. In particular every Lie group is orientable.

- Take any inner product Φ_e on $T_e(G)$.

  Apply the lemma to:
  - Φ_e;
  - The volume element Ω_e determined by Φ_e, with a choice of orientation of $T_e(G)$.

  We get a left-invariant Riemannian metric Φ and volume element Ω.

The Case of Compact Connected Lie Groups

Theorem

An oriented, compact, connected Lie group G has a unique bi-invariant volume element Ω, such that $\mathrm{vol} G = 1$.

- Let Ω be a left-invariant volume element on G.
  We claim that Ω is necessarily right-invariant also.
  In order to prove this, it is enough to show that
  $$(\mathrm{Ad} g)^* \Omega_e = \Omega_e, \quad \text{for all } g \in G.$$
  Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}$.
  Let X_{ie}, $i = 1, \ldots, n$, be the corresponding basis of $T_e(G)$.
  We have seen that
  $$(\mathrm{Ad} g) X_j = \sum_{i=1}^{n} \alpha_{ij}(g) X_i.$$
  Also, $g \mapsto (\alpha_{ij}(g))$ defines a C^∞ homomorphism of $G \to Gl(n, \mathbb{R})$.

The Case of Compact Connected Lie Groups (Cont'd)

- The linear transformation $(\mathrm{Ad}g)^*$ on $\bigwedge^n(T_e(G))$, determined by $\mathrm{Ad}g$, acts on Ω_e by
$$(\mathrm{Ad}g)^*\Omega_e = \det(\alpha_{ij}(g))\Omega_e.$$
By hypothesis, G is compact and connected.
The same applies to its image under the C^∞-homomorphism
$$g \to \det(\alpha_{ij}(g))$$
of G to R^*, the multiplicative group of nonzero real numbers.
However, the only compact connected subgroup of R^* is $\{+1\}$, the trivial group consisting of the identity.
Hence
$$\det(\alpha_{ij}(g)) = 1.$$
This shows that $(\mathrm{Ad}g)^*\Omega_e = \Omega_e$, for all $g \in G$.

The Case of Compact Connected Lie Groups (Cont'd)

- By the preceding lemma, this proves that Ω is bi-invariant.

  Any other bi-invariant Ω must be of the form
  $$\lambda\Omega, \quad \lambda \text{ a positive constant.}$$
  But then
  $$\text{vol}\, G = \int_G \lambda\Omega = \lambda \int_G \Omega.$$
  Hence, it is possible to choose just one $\lambda \neq 0$, such that
  $$\text{vol}\, G = +1.$$
  For the opposite orientation on G, we would have $-\Omega$ as the corresponding unique bi-invariant volume element.

Bi-Invariant Riemannian Metric

Corollary

On a compact connected Lie group G it is possible to define a bi-invariant Riemannian metric $\widetilde{\Phi}$.

- Let Φ_e be a symmetric, positive definite, bilinear form on $T_e(G)$.
  Let Ω be the bi-invariant volume element.
  Given $X_e, Y_e \in T_e(G)$, we define a function on G by
  $$f(g) = ((\mathrm{Ad}g)^*\Phi_e)(X_e, Y_e) = \Phi_e((\mathrm{Ad}g)X_e, (\mathrm{Ad}g)Y_e).$$
  The last equality is just the usual definition of $(\mathrm{Ad}g)^*$.
  Then define the bilinear form $\widetilde{\Phi}_e$ on $T_e(G)$ by
  $$\widetilde{\Phi}_e(X_e, Y_e) = \int_G f(g)\Omega.$$

Bi-Invariant Riemannian Metric (Cont'd)

- According to a previous lemma, $\widetilde{\Phi}_e$ determines a bi-invariant form if, for every $a \in G$,

$$(\mathrm{Ad}a)^*\widetilde{\Phi}_e(X_e, Y_e) = \widetilde{\Phi}_e(X_e, Y_e).$$

The left-hand term may be written $\widetilde{\Phi}_e((\mathrm{Ad}a)X_e, (\mathrm{Ad}a)Y_e)$. Applying the definition of $\widetilde{\Phi}_e$ to this expression, we find that

$$\begin{array}{rcl} (\mathrm{Ad}a)^*\widetilde{\Phi}_e(X_e, Y_e) & = & \int_G (\mathrm{Ad}g)^*\Phi_e((\mathrm{Ad}a)X_e, (\mathrm{Ad}a)Y_e)\Omega \\ & = & \int_G (\mathrm{Ad}g)^*(\mathrm{Ad}a)^*\Phi_e(X_e, Y_e)\Omega \\ & = & \int_G (\mathrm{Ad}(ag))^*\Phi_e(X_e, Y_e)\Omega. \end{array}$$

This shows that

$$(\mathrm{Ad}a)^*\widetilde{\Phi}(X_e, Y_e) = \int_G f(R_a(g))\Omega.$$

Bi-Invariant Riemannian Metric (Cont'd)

- On the other hand, $I_a : G \to G$ is a diffeomorphism.
  Moreover, a previous theorem asserts that
  $$\int_{I_a(G)} f(g)\Omega = \int_G f(R_a(g))R_a^*\Omega.$$
  Since $I_a(G) = G$ and $R_a^*\Omega = \Omega$, we see that
  $$(\mathrm{Ad}a)^*\widetilde{\Phi}(X_e, Y_e) = \int_G f(g)\Omega = \widetilde{\Phi}(X_e, Y_e).$$
  It follows that $\widetilde{\Phi}$ is a bi-invariant bilinear form on G.
  It is symmetric and we can check that it is positive definite.
  Since we do so in a more general case below, we will omit this verification here.
  Remark: When we use this Riemannian metric on G, we see that both right and left translations are isometries, that is, they preserve the Riemannian metric (and also its associated distance function).

Representations and Invariant Inner Products

- Let $(\rho, \boldsymbol{V})$ be a representation of G on a finite-dimensional real vector space $\boldsymbol{V}$, with
$$\rho : G \to Gl(\boldsymbol{V}).$$
- Suppose a basis is chosen in $\boldsymbol{V}$.
- This determines a C^∞ homomorphism of G into $Gl(n, \mathbb{R})$, $n = \dim \boldsymbol{V}$.
- A special case is $\rho = \mathrm{Ad}$ with $\boldsymbol{V} = \mathfrak{g}$.

Theorem

Let G be compact and connected and ρ a representation of G on $\boldsymbol{V}$. Then there is an inner product $(\boldsymbol{u}, \boldsymbol{v})$ on $\boldsymbol{V}$, such that every $\rho(g)$ leaves the inner product invariant,
$$(\rho(g)\boldsymbol{u}, \rho(g)\boldsymbol{v}) = (\boldsymbol{u}, \boldsymbol{v}).$$

Representations and Invariant Inner Products (Cont'd)

- Let $\Phi(\boldsymbol{u}, \boldsymbol{v})$ be an arbitrary inner product on $\boldsymbol{V}$.

  Given a fixed $\boldsymbol{u}, \boldsymbol{v} \in \boldsymbol{V}$, let

  $$f(g) = \Phi(\rho(g)\boldsymbol{u}, \rho(g)\boldsymbol{v}).$$

  This defines a C^∞ function on G.

  Then we define

  $$(\boldsymbol{u}, \boldsymbol{v}) = \int_G f(g)\Omega$$

  with Ω denoting the bi-invariant volume element.

  The linearity of the integral implies at once that $(\boldsymbol{u}, \boldsymbol{v})$ is bilinear.

  It is clearly symmetric in $\boldsymbol{u}, \boldsymbol{v}$ since the integrand is.

  Moreover, $(\boldsymbol{u}, \boldsymbol{v}) \geq 0$, and equality implies $\boldsymbol{u} = \boldsymbol{0}$, since $f(g) \geq 0$ on G, with equality holding if and only if the integral vanishes.

Representations and Invariant Inner Products (Cont'd)

- Finally, for $a \in G$ we have

$$\begin{aligned}(\rho(a)\boldsymbol{u}, \rho(a)\boldsymbol{v}) &= \int_G \Phi(\rho(g)\rho(a)\boldsymbol{u}, \rho(g)\rho(a)\boldsymbol{v})\Omega \\ &= \int_G \Phi(\rho(ga)\boldsymbol{u}, \rho(ga)\boldsymbol{v})\Omega \\ &= \int_G f(ga)\Omega.\end{aligned}$$

  But by the same argument as in the previous proof, this is equal to

$$\int_G f(g)\Omega = (\boldsymbol{u}, \boldsymbol{v}).$$

- Note that, if we let $\rho = \mathrm{Ad}$ and $\boldsymbol{V} = \mathfrak{g}$, we obtain the preceding corollary as a special case.

Matrix Representation

- The preceding result could be stated by saying that each $\rho(g)$ is an isometry of the vector space $\boldsymbol{V}$ with the inner product $(\boldsymbol{u}, \boldsymbol{v})$.
- Since the matrix of an isometry of $\boldsymbol{V}$ relative to an orthonormal basis is an orthogonal matrix, we have the following corollary concerning the representations of a compact group.

Corollary

Relative to a suitable basis of $\boldsymbol{V}$, the matrices representing every $\rho(g)$ are orthogonal.

Invariance, Irreducibility and Semisimplicity

- We shall say that $\boldsymbol{W} \subseteq \boldsymbol{V}$ is **invariant** if it is invariant for every linear transformation $\rho(g)$.
- The representation is **irreducible** if $\boldsymbol{V}$ contains no nontrivial invariant subspaces.
- If each invariant subspace $\boldsymbol{W}$ has a complementary invariant subspace $\boldsymbol{W}'$, such that

$$\boldsymbol{V} = \boldsymbol{W} \oplus \boldsymbol{W}',$$

  then the representation is said to be **semisimple**.
- In the case of a semisimple representation, it is easily verified that

$$\boldsymbol{V} = \boldsymbol{W}_1 \oplus \cdots \oplus \boldsymbol{W}_r,$$

  where the $\boldsymbol{W}_i$ are invariant irreducible subspaces.

Semisimplicity and Decomposition

Corollary

If ρ is a representation of a compact connected Lie group G on a finite-dimensional vector space $\boldsymbol{V}$, then it is semisimple. Moreover

$$\boldsymbol{V} = \boldsymbol{W}_1 \oplus \cdots \oplus \boldsymbol{W}_r,$$

where:

- For $i \neq j$, the subspaces are mutually orthogonal;
- Each is a nontrivial irreducible subspace.

- If $\boldsymbol{V}$ is irreducible, there is nothing to prove.

  Suppose $\boldsymbol{V}$ contains a nontrivial invariant subspace $\boldsymbol{W}$.

  We show its orthogonal complement $\boldsymbol{W}^\perp$ is also invariant.

Semisimplicity and Decomposition (Cont'd)

- Let $\boldsymbol{V}$ be a nontrivial invariant subspace $\boldsymbol{W}$.

  Consider its orthogonal complement $\boldsymbol{W}^{\perp}$.

  Let $\boldsymbol{w} \in \boldsymbol{W}^{\perp}$ and let $\boldsymbol{v} \in \boldsymbol{W}$.

  Then
  $$(\rho(g)\boldsymbol{v}, \rho(g)\boldsymbol{w}) = (\boldsymbol{v}, \boldsymbol{w}) = 0.$$

  Thus, $\rho(g)\boldsymbol{w}$ is orthogonal to $\rho(g)\boldsymbol{v}$, for every $\boldsymbol{v} \in \boldsymbol{W}$.

  Since $\rho(g)$ is nonsingular, this means that $\rho(g)\boldsymbol{w}$ is orthogonal to every element of $\boldsymbol{W}$.

  So it must be in $\boldsymbol{W}^{\perp}$.

  Hence $\boldsymbol{V} = \boldsymbol{W} \oplus \boldsymbol{W}^{\perp}$, a direct sum of complementary invariant subspaces.

  Repeated application of this argument gives the final statement.

Example

- There exist representations of noncompact connected groups which do not have the property of complete reducibility.
- As a result, they cannot leave an inner product invariant.
- Consider, e.g., $\rho : \mathbb{R} \to Gl(2, \mathbb{R})$ acting on $\boldsymbol{V}^2$ defined by

$$\rho(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}.$$

- $\rho(t)$ acts on $\boldsymbol{V}^2$, the space of all $\begin{pmatrix} x \\ y \end{pmatrix}$, $x, y \in \mathbb{R}$,

$$\rho(t) \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} x + ty \\ y \end{pmatrix}.$$

- The subspace $\begin{pmatrix} x \\ 0 \end{pmatrix}$ is invariant.
- But it has no complementary invariant subspace.

Subsection 4

Manifolds With Boundary

Half-Spaces in $\mathbb{R}^n$

- Consider the closed half-space

$$H^n = \{x = (x^1, \ldots, x^n) \in \mathbb{R}^n : x^n \geq 0\},$$

with the relative topology of $\mathbb{R}^n$.
- Denote by ∂H^n the subspace defined by

$$\partial H^n = \{x \in H^n : x^n = 0\}.$$

- Then ∂H^n is the same space whether considered as a subspace of $\mathbb{R}^n$ or H^n.
- It is called the **boundary** of H^n.
- All of these spaces carry the metric topology derived from the metric of $\mathbb{R}^n$.
- ∂H^n is obviously homeomorphic to $\mathbb{R}^{n-1}$ by the map

$$(x^1, \ldots, x^{n-1}) \to (x^1, \ldots, x^{n-1}, 0).$$

Diffeomorphisms Generalized

- Recall that differentiability has been defined for functions and mappings to $\mathbb{R}^m$ of arbitrary subsets of $\mathbb{R}^n$.
- We see that the notion of diffeomorphism applies at once to (relatively) open subsets U, V of H^n.
- U, V are **diffeomorphic** if there exists a one-to-one map $F : U \to V$ (onto) such that F and F^{-1} are both C^∞ maps.
- This is broader than the earlier definition.
- Here, U, V are not necessarily open subsets of $\mathbb{R}^n$, but are in fact the intersections of such sets with H^n.
- If $U, V \subseteq \mathbb{R}^n - \partial H^n$, then U and V are actually open in $\mathbb{R}^n$.
- In this case, this definition of diffeomorphism coincides with the previous one.

Diffeomorphisms and Boundaries

- We show that if $U \cap \partial H^n \neq \emptyset$, then:
  - $V \cap \partial H^n \neq \emptyset$;
  - $F(U \cap \partial H^n) \subseteq V \cap \partial H^n$.
- Similarly, $F^{-1}(V \cap \partial H^n) \subseteq U \cap \partial H^n$.
- In other words, diffeomorphisms on open sets of H^n take boundary points to boundary points and interior points to interior points.
- This follows at once from the Inverse Function Theorem, which asserts that $U - \partial H^n$ is open in $\mathbb{R}^n$.
- Hence, F must map it diffeomorphically onto an open subset of $\mathbb{R}^n$.
- But no open subset of H^n which contains a boundary point, that is, a point of ∂H^n, can be open in $\mathbb{R}^n$.
- Thus,

$$F(U - \partial H^n) \subseteq V - \partial H^n \quad \text{and} \quad F^{-1}(V - \partial H^n) \subseteq U - \partial H^n.$$

- Since F and F^{-1} are one-to-one on U and V, the result follows.

Additional Properties of Diffeomorphisms

- The sets $U \cap \partial H^n$ and $V \cap \partial H^n$ are open subsets of ∂H^n, a submanifold of $\mathbb{R}^n$ diffeomorphic to $\mathbb{R}^{n-1}$.
- F, F^{-1} restricted to these open sets in ∂H^n are diffeomorphisms.
- Both F and F^{-1} can be extended to open sets U', V' of $\mathbb{R}^n$ having the property that $U = U' \cap H^n$ and $V = V' \cap H^n$.
- These extensions will not be unique nor are the extensions in general inverses throughout these larger domains.
- However, the derivatives of F and F^{-1} on U and V are independent of the extensions chosen and we may suppose that even on the extended domains the Jacobians are of rank n.
- These statements are immediate consequences of:
    - The definition of differentiability for arbitrary subsets of $\mathbb{R}^n$;
    - The fact that the Jacobian of a C^∞ mapping has its maximum rank on an open subset of its domain.

Manifolds With Boundary

Definition

A C^∞ **manifold with boundary** is a Hausdorff space M with a countable basis of open sets and a differentiable structure $\mathscr{U}$ in the following (generalized) sense.

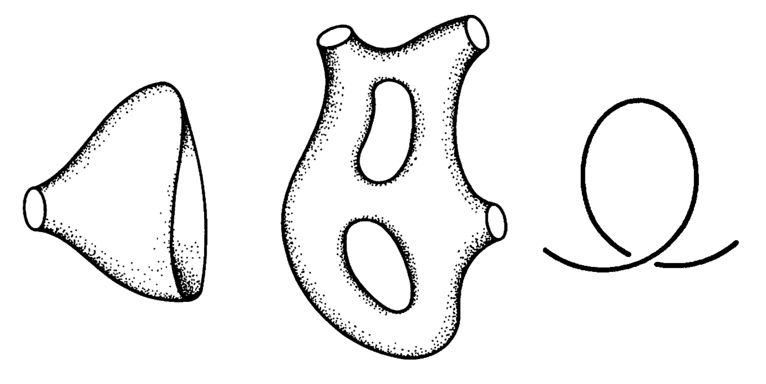
$\mathscr{U} = \{U_\alpha, \varphi_\alpha\}$ consists of a family of open subsets U_α of M each with a homeomorphism φ_α onto an open subset of H^n (topologized as a subspace of $\mathbb{R}^n$) such that:

(1) The U_α cover M;

(2) If U_α, φ_α and U_β, φ_β are elements of $\mathscr{U}$, then $\varphi_\beta \circ \varphi_\alpha^{-1}$ and $\varphi_\alpha \circ \varphi_\beta^{-1}$ are diffeomorphisms of $\varphi_\alpha(U \cap V)$, $\varphi_\beta(U \cap V)$, open subsets of H^n;

(3) $\mathscr{U}$ is maximal with respect to Properties (1) and (2).

Examples of Manifolds With Boundary



Boundary of M

- The U, φ are coordinate neighborhoods on M.
- If $\varphi(p) \in \partial H^n$ in one coordinate system, then this holds for all coordinate systems.
- The collection of such points is called the **boundary** of M, denoted

$$\partial M.$$

- $M - \partial M$ is a manifold (in the ordinary sense).
- It is denoted by $\text{Int} M$.
- If $\partial M = \emptyset$, then M is a manifold of the familiar type.
- We call it a **manifold without boundary** when it is necessary to make the distinction.

Differentiable Structure on Boundary

Theorem

Let M be a C^∞ manifold (of dimension n) with boundary.
Then the differentiable structure of M determines a C^∞-differentiable structure of dimension $n-1$ on the subspace ∂M of M.
The inclusion $i : \partial M \to M$ is an imbedding.

- For a coordinate neighborhood U, φ of M which contains points of ∂M, consider the coordinate neighborhood $\widetilde{U}, \widetilde{\varphi}$ of ∂M, given by
  - $\widetilde{U} = U \cap \partial M$;
  - $\widetilde{\varphi} = \varphi|_{U \cap \partial M}$.
- The differentiable structure $\widetilde{\mathscr{U}}$ on ∂M is determined by the coordinate neighborhoods $\widetilde{U}, \widetilde{\varphi}$, where U, φ ranges over coordinate neighborhoods of M containing points of ∂M.

Differentiability

- Differentiable functions, differentiable mappings, rank, and so on, may now be defined on M exactly as before by using local coordinates.
- By virtue of the C^∞ compatibility of such coordinate systems these concepts are independent of the choice of coordinates.
- We also define $T_p(M)$ at boundary points of M.
- This could be done using derivations on $C^\infty(p)$ as before, but to avoid some slight complications we use an alternative definition.
- First note that in the case of $H^n \subseteq \mathbb{R}^n$, upon which manifolds with boundary are modeled, we identify $T_x(H^n)$ with $T_x(\mathbb{R}^n)$.
- We may think of this identification as being given by the inclusion mapping.
- For $x \in \partial H^n$, this defines what we mean by $T_x(H^n)$.

Vectors and Components

- Consider a general manifold M.
- For $p \in \partial M$, we define a **vector** $X_p \in T_p(M)$ to be an assignment, to each coordinate neighborhood U, φ, of an n-tuple of numbers $(\alpha^1, \ldots, \alpha^n)$, the U, φ **components of** X_p, satisfying the following condition:

  If $(x^1, \ldots, x^n)$ and $(y^1, \ldots, y^n)$ are coordinates around p in neighborhoods U, φ and V, ψ, then the components $(\alpha^1, \ldots, \alpha^n)$ and $(\beta^1, \ldots, \beta^n)$ relative to U and V are related by

$$\beta^i = \sum_{j=1}^{n} \left(\frac{\partial y^i}{\partial x^j} \right)_{\varphi(p)} \alpha^j, \quad i = 1, \ldots, n.$$

- What this does is attach, to each $p \in M$, a $T_p(M)$ such that each coordinate system U, φ determines an isomorphism φ_* taking X_p with components $(\alpha^1, \ldots, \alpha^n)$ to the vector $\sum \alpha^i (\frac{\partial}{\partial x^i}) \in T_{\varphi(p)}(H^n)$.

Bases of Tangent Spaces

- As before, let $E_1, \ldots, E_n$ will denote the basis determined by
$$\varphi_*(E_i) = \frac{\partial}{\partial x^i}, \quad i = 1, \ldots, n.$$
- Having defined $T_p(M)$ on ∂M [it is already known on $\mathrm{Int} M$, which is an ordinary manifold], we may extend all of our definitions and theorems to manifolds with boundary.
- In particular, exterior differential forms and the exterior calculus is still valid on manifolds with boundary, without any essential change in the definitions or proofs.

Regular Domains

Definition

A **regular domain** D on a manifold M is a closed subset of M, with nonempty interior $\overset{\circ}{D}$, such that if

$$p \in \partial D = D - \overset{\circ}{D},$$

then p has a cubical coordinate neighborhood U, φ, such that:

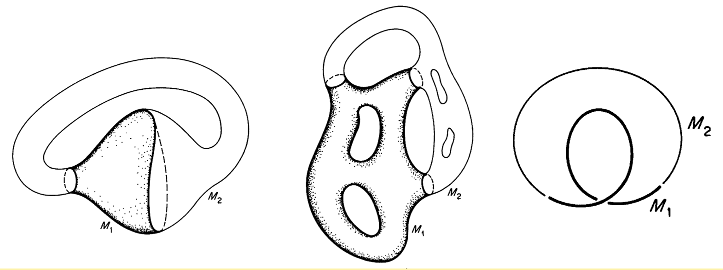
- $\varphi(p) = (0, \ldots, 0)$;
- $\varphi(U) = C^n_\varepsilon(0)$;
- $\varphi(U \cap D) = \{x \in C^n_\varepsilon(0) : x^n \geq 0\}$ on ∂D.

Prroperties of Regular Domains

- Let M be a manifold and D a regular domain on M.
- If D is compact, then it is a domain of integration on M.
- We may check that D, with the topology and differentiable structure induced by M, is a manifold with boundary.
- All preceding examples can be seen to be of this type.
  - H^n and the closed unit ball $\overline{B}^n$ are regular domains of $M = \mathbb{R}^n$;
  - $N \times I$ is a regular domain of $N \times \mathbb{R}$;
  - The set D obtained by removing from a manifold M a diffeomorphic image of an open ball is a regular domain.

Manifolds with Boundaries and Regular Domains

- It is a fact, somewhat difficult to prove, that any manifold M with boundary can be realized as a regular domain of a larger manifold M'.
- The basic idea is simple:
  - Take two copies of M, say M_1 and M_2;
  - "Glue" them together along their boundaries, while identifying corresponding boundary points.
- The resulting manifold is called the **double** of M.
- It contains M as a regular domain.



Orientability

- Let M be a manifold with non-empty boundary.
- M is **orientable** provided that it has a covering of coordinate neighborhoods $\{U_\alpha, \varphi_\alpha\}$ which are *coherently oriented*.
- That is, if $U_\alpha \cap U_\beta \neq \emptyset$, then $\varphi_\beta \circ \varphi_\alpha^{-1}$ has positive Jacobian determinant (or equivalently, preserves the natural orientation of H^n).
- This is equivalent to the existence of a nowhere vanishing n-form Ω on M.

Orientability (Cont'd)

- The proof of this equivalence is the same except that, when we speak of a partition of unity on M associated to a regular covering

$$\{U_i, V_i, \varphi_i\},$$

we limit ourselves to a regular covering by cubical coordinate neighborhoods, concerning which we impose the following slight restriction:

If $U_i \cap \partial M \neq \emptyset$, then

$$\varphi_i(U_i) = C_3^n(0) \cap H^n \quad \text{and} \quad \varphi_i(V_i) = C_1^n(0) \cap H^n.$$

- With this modified definition of regular covering we still have:
  - A regular covering (by definition locally finite) refining any open covering $\{A_\alpha\}$ of M;
  - An associated C^∞ partition of unity $\{f_i\}$ on M.

Induced Regular Covering of the Boundary

- Consider those
$$U_i, V_i, \varphi_i$$
of the regular covering that intersect ∂M.
- They determine a regular covering
$$\widetilde{U}_i = U_i \cap \partial M, \quad \widetilde{V}_i = V_i \cap \partial M, \quad \widetilde{\varphi}_i = \varphi|_{\widetilde{U}_i}$$
of ∂M.
- Moreover, the associated partition of unity restricts to an associated partition of unity on ∂M,
$$\{\widetilde{f}_i = f_i|_{\partial M}\}.$$

Induced Orientation of the Boundary

Theorem

Let M be an oriented manifold and suppose ∂M is not empty. Then ∂M is orientable and the orientation of M determines an orientation of ∂M.

- ∂M is an $(n-1)$-dimensional submanifold of M.

  So its tangent space at each point may be identified with an $(n-1)$-dimensional subspace of $T_p(M)$.

  We denote this subspace by $T_p(\partial M)$.

  We show that there is a distinction between the two half-spaces into which $T_p(\partial M)$ divides $T_p(M)$ which is independent of coordinates.

  Suppose that U, φ and V, ψ are coordinate neighborhoods of $p \in \partial M$ with respective local coordinates

$$(x^1, \ldots, x^n) \quad \text{and} \quad (y^1, \ldots, y^n).$$

Induced Orientation of the Boundary (Cont'd)

- By our definitions of coordinates of boundary points, the last coordinate x^n or y^n is equal to zero if the point in U or V, respectively, is on ∂M, and positive otherwise.

  Let the change of coordinate functions be

  $$y^i = y^i(x^1, \ldots, x^n), \quad i = 1, \ldots, n.$$

  Then we have

  $$0 = y^n(x^1, \ldots, x^{n-1}, 0).$$

  So, for every $q \in U \cap \partial M$,

  $$\left(\frac{\partial y^n}{\partial x^1}\right)_{\varphi(q)} = \cdots = \left(\frac{\partial y^n}{\partial x^{n-1}}\right)_{\varphi(q)} = 0.$$

Induced Orientation of the Boundary (Cont'd)

- It follows that the Jacobian matrix then has the form

$$D(\psi \circ \varphi^{-1}) = \begin{pmatrix} \frac{\partial y^1}{\partial x^1} & \cdots & \frac{\partial y^{n-1}}{\partial x^1} & 0 \\ \vdots & & \vdots & \vdots \\ \frac{\partial y^1}{\partial x^{n-1}} & \cdots & \frac{\partial y^{n-1}}{\partial x^{n-1}} & 0 \\ \frac{\partial y^1}{\partial x^n} & \cdots & \frac{\partial y^{n-1}}{\partial x^n} & \frac{\partial y^n}{\partial x^n} \end{pmatrix}_{\varphi(q)}.$$

  Since the Jacobian is nonsingular, $\frac{\partial y^n}{\partial x^n} \neq 0$ at $\varphi(q)$.

  In fact, it must be positive.

  Let $\varphi(q) = (a^1, a^2, \ldots, a^{n-1}, 0)$.

  Consider $f(t)$, defined by $f(t) = y^n(a^1, \ldots, a^{n-1}, t)$.

  We have $f(0) = 0$ and $f(t) > 0$ in some interval $0 < t < \delta$.

  Therefore, $f'(0) = (\frac{\partial y^n}{\partial x^n})_{\varphi(q)}$ can certainly not be negative.

  Therefore $\frac{\partial y^n}{\partial x^n} > 0$ at $\varphi(q)$ as claimed.

Induced Orientation of the Boundary (Cont'd)

- If U, φ and V, ψ are oriented neighborhoods of M, then the preceding matrix has positive determinant.

  So $\frac{\partial y^n}{\partial x^n}$ and the $(n-1) \times (n-1)$ minor determinant obtained by striking out the last row and column has the same sign.

  This minor is exactly the determinant of $D(\widetilde{\psi} \circ \widetilde{\varphi}^{-1})$, the change of coordinates from $\widetilde{U} = U \cap \partial M$, $\widetilde{\varphi} = \varphi|_{\widetilde{U}}$ to $\widetilde{V} = V \cap \partial M, \widetilde{\psi} = \psi|_{\widetilde{V}}$ on the submanifold ∂M.

  Thus the neighborhoods on ∂M determined by oriented neighborhoods on M are coherent.

  It follows that they determine an orientation on ∂M.

Remark I

- Let $q \in U \cap V$ be a boundary point of M.
- Let $X_q \in T_q(M)$.
- Suppose we express X_q in the coordinate frames of either U, φ or V, ψ,

$$\begin{aligned} X_q &= \alpha^1 E_1 + \cdots + \alpha^{n-1} E_{n-1} + \alpha^n E_n \\ &= \beta^1 F_1 + \cdots + \beta^{n-1} F_{n-1} + \beta^n F_n. \end{aligned}$$

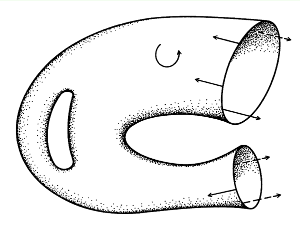
- We saw that

$$\left(\frac{\partial y^n}{\partial x^m} \right)_{\varphi(q)} > 0.$$

- It follows that α^n and β^n have the same sign.
- This fact does not depend on the coordinates being oriented.

Remark I (Cont'd)

- It follows that the vectors of $T_p(M) - T_p(\partial M)$ fall into two classes.



  - Those whose last component is positive, which we call **inward pointing vectors** at $p \in \partial M$;
  - Those for which the last component is negative, which we call **outward pointing vectors**.
- Those for which the last component vanishes are tangent to ∂M.
- Moreover, this classification is independent of the orientation of M.

Remark II

- We describe a special case of gluing two manifolds with identical boundaries together along their boundaries.
- Let M_1, M_2 be two manifolds (without boundary) of dimension n.
- Let U_i, φ_i be coordinate neighborhoods of points $p_i \in M_i$, $i = 1, 2$.
- We suppose that in each case we have

$$\varphi_i(p_i) = (0, \ldots, 0) \quad \text{and} \quad \varphi_i(U_i) = B_2^n(0).$$

- We set

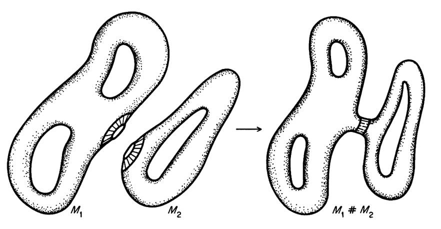
$$V_i = \varphi_i^{-1}(B_1(0)).$$

- Then $M_i' = M_i - V_i$, $i = 1, 2$, is a manifold with boundary.
- Indeed, one has

$$\varphi_i(\partial M_i') = S^{n-1}.$$

Remark II (Cont'd)

- The manifold obtained by gluing M_1' to M_2' along the boundaries is called the **connected sum** of M_1 and M_2, denoted $M_1 \# M_2$.



- We would like to define $M_1 \# M_2$ without loss of differentiability.
- So we actually remove only $\varphi^{-1}(\overline{B}_{1/2}(0))$ from each M_i to get M_i''.
- Then we identify points $q_i \in U_i - \varphi_i^{-1}(\overline{B}_{1/2}(0))$, $i = 1, 2$, whenever

$$\varphi_1(q_1) = \frac{\varphi_2(q_2)}{\|\varphi_2(p_2)\|^2}.$$

Remark II (Cont'd)

- So $q_1 \in M_1''$ and $q_2 \in M_2''$ are identified if their images $\varphi_1(p_1)$ and $\varphi_2(p_2)$ in $\mathbb{R}^n$ are "reflections" of one another in the unit sphere (lie on the same ray and have reciprocal distance from the origin).
- It turns out that any closed surface (compact 2-manifold) can be obtained as:
  - The connected sum of copies of S^2 and T^2 if it is orientable;
  - The connected sum of copies of P^2 and T^2 if it is nonorientable.

Subsection 5

Stokes's Theorem for Manifolds With Boundary

Setup

- We consider an oriented manifold M with possibly nonempty boundary ∂M, oriented by the orientation of M.
- We consider only oriented coordinate neighborhoods U, φ.
- If $U \cap \partial M \neq 0$, then we denote by $\widetilde{U}, \widetilde{\varphi}$ the corresponding neighborhood on ∂M,

$$\widetilde{U} = U \cap \partial M, \widetilde{\varphi} = \varphi|_{\widetilde{U}}.$$

- All of the concepts used in defining the integral extend to M, e.g., the definitions of content zero, domain of integration, and so on.
- In particular $\partial \widetilde{M}$ has measure zero and, if compact, has content zero.
- This follows from corresponding properties of ∂H^n.

Cubes

- A cube Q associated with U, φ is as before, unless $U \cap \partial M \neq \emptyset$.
- If $U \cap \partial M \neq \emptyset$, then we assume that Q has a "face" on ∂M.
- That is, we assume

$$\varphi(Q \cap \partial M) = \{x \in \mathbb{R}^n : 0 \leq x^i \leq 1 \text{ and } x^n \equiv 0\}.$$

- In this case we note two facts:
  - (a) $\widetilde{Q} = Q \cap \partial M$ is a cube of ∂M associated with $\widetilde{U}, \widetilde{\varphi}$;
  - (b) The interior of Q has a different image in $\mathbb{R}^n$ than it has when $U \subseteq \text{Int} M$, namely,

$$\overset{\circ}{Q} = \varphi^{-1}(\{x \in \mathbb{R}^n : 0 < x^i < 1,\ 1 \leq i \leq n-1;\ 0 \leq x < 1\}).$$

Integrals

- Taking these modifications into account, the definition of

$$\int_M \Omega$$

is exactly as before.

- The integral of an integrable n-form has the same properties as before.
- Indeed, if M is a compact regular domain in a manifold N, then it is necessarily a domain of integration in N and

$$\int_M \Omega = \int_N k_M \Omega.$$

- So there is nothing new to define in this case!
- The same comments apply to the integral over a Riemannian manifold with boundary and to the definition of vol$M$ when M is compact.

Notation for Stokes' Theorem

- Now suppose M is both oriented and compact.
- Let ω be an $(n-1)$ form of class C^1 at least on M.
- We have an important relation between:
  - The integral of $d\omega$ over M;
  - $i^*\omega$, the restriction of ω to ∂M ($i : \partial M \to M$ the inclusion mapping).
- To simplify the statement of the theorem we let $\partial\widetilde{M}$ denote:
  - ∂M, the boundary with the orientation induced by M, when n is even;
  - $-\partial M$, the boundary with the opposite orientation, when n is odd.
- Thus

$$\partial\widetilde{M} = (-1)^n \partial M.$$

Stokes's Theorem

Theorem (Stokes's Theorem)

Let M be an oriented compact manifold of dimension n and let ∂M have the induced orientation. Then we have

$$\int_M d\omega = \int_{\partial \widetilde{M}} i^* \omega.$$

When $\partial M = \emptyset$, the integral over M vanishes.

- According to our definitions, it is enough to establish the theorem for an ω whose support is contained in the interior $\mathring{Q}$ of a cube Q associated to a coordinate neighborhood U, φ.

  Suppose ω has its support in Q.

  Let $x^1, \ldots, x^n$ be the local coordinates.

Stokes's Theorem (Cont'd)

- We may suppose that, in these coordinates, ω is expressed as

$$\varphi^{-1*}(\omega) = \sum_{j=1}^{n} (-1)^{j-1} \lambda^j dx^1 \wedge \cdots \wedge dx^{j-1} \wedge dx^{j+1} \wedge \cdots \wedge dx^n.$$

Then we have

$$\varphi^{-1*}(d\omega) = d\varphi^{-1*}(\omega) = \left(\sum_{j=1}^{n} \frac{\partial \lambda^j}{\partial x^j} \right) dx^1 \wedge \cdots \wedge dx^n.$$

So

$$\int_M d\omega = \int_Q \left(\sum_{j=1}^{n} \frac{\partial \lambda^j}{\partial x^j} \right) dv = \sum_j \int_0^1 \cdots \int_0^1 \frac{\partial \lambda^j}{\partial x^j} dx^1 \cdots dx^n.$$

This follows from the definition of integration on M and the Iterated Integral Theorem.

Stokes's Theorem (Cont'd)

- We obtained

$$\int_M d\omega = \sum_j \int_0^1 \cdots \int_0^1 \frac{\partial \lambda^j}{\partial x^j} dx^1 \cdots dx^n.$$

On the right consider the jth summand only.

Integrate first with respect to the variable x^j.

This gives an $(n-1)$-fold iterated integral

$$\int_0^1 \cdots \int_0^1 [\lambda^j(x^1, \ldots, x^{j-1}, 1, x^{j+1}, \ldots, x^n) \\ -\lambda^j(x^1, \ldots, x^{j-1}, 0, x^{j+1}, \ldots, x^n)] dx^1 \cdots \widehat{dx^j} \cdots dx^n,$$

where $\widehat{dx^j}$ indicates that this differential is to be omitted.

Sum these $(n-1)$-fold iterated integrals, for $j = 1, \ldots, n$.

Stokes's Theorem (Cont'd)

- The sum shows that, if $\text{supp}(\omega) \subseteq \overset{\circ}{Q}$, two cases can occur regarding $\int_M d\omega$.
  - $Q \cap \partial M = \emptyset$. In this case
  $$\varphi(\overset{\circ}{Q}) = \{x : 0 < x^i < 1, i = 1, \ldots, n\};$$
  - $Q \cap \partial M \neq \emptyset$. In this case,
  $$\varphi(\overset{\circ}{Q}) = \{x : 0 < x^i < 1, i = 1, \ldots, n-1; 0 \leq x^n < 1\}.$$

  Consider the first case.

  Using $\text{supp}\omega \subseteq \overset{\circ}{Q}$, we see that $\lambda^j = 0$, if any $x^j = 0, 1$.

  Hence, each of the integrands above vanish and $\int_M d\omega = 0$.

  On the other hand, $\text{supp}\omega \subseteq \overset{\circ}{Q}$ which has no points on ∂M.

  So ω restricted to ∂M is the zero $(n-1)$-form.

  Thus, $\int_M d\omega = 0 = \int_{\partial M} i^*\omega$ and Stokes's Theorem holds.

Stokes's Theorem (Cont'd)

- In the second case we again have all of the integrands equal to zero except the one corresponding to $j = n$. Therefore

$$\int_M d\omega = -\int_0^1 \cdots \int_0^1 \lambda^n(x^1, \ldots, x^{n-1}, 0) dx^1 \cdots dx^{n-1}.$$

On the other hand, we may evaluate $\int_{\partial M} i^* \omega$ using the fact that $i^*\omega$ has its support in $\widetilde{Q} = Q \cap \partial M$.

To obtain an expression of $i^*\omega$ in local coordinates, we apply the corresponding inclusion

$$i : (x^1, \ldots, x^{n-1}) \to (x^1, \ldots, x^{n-1}, 0).$$

We note that $i^* dx^n = 0$.

Stokes's Theorem (Cont'd)

- So, in the local coordinates $\widetilde{U}, \widetilde{\varphi}$, $i^*\omega$ collapses to
$$\widetilde{\varphi}^{-1*}(i^*\omega) = (-1)^{n-1}\lambda^n(x^1, \ldots, x^{n-1}, 0)dx^1 \wedge \cdots \wedge dx^{n-1}.$$
This gives
$$\int_{\partial M} i^*\omega = (-1)^{n-1} \int_0^1 \cdots \int_0^1 \lambda^n(x^1, \ldots, x^{n-1}, 0)dx^1 \cdots dx^{n-1}.$$
We are considering the case where $\text{supp}\omega \subseteq \mathring{Q}$ and $Q \cap \partial M \neq \emptyset$. We find that
$$\int_M d\omega = (-1)^n \int_{\partial M} i^*\omega = \int_{\pm\partial M} i^*\omega,$$
with:
  - The right-hand integral over ∂M, when n is even;
  - The right-hand integral over $-\partial M$, when n is odd.

That is, the right-hand integral is over $\partial\widetilde{M}$.

Example: Green's Theorem

- Let M be a bounded regular domain of $\mathbb{R}^2$.
- That is, M is the closure of a bounded open subset of the plane, bounded by simple closed curves of class C^∞.
- For example, let M be a circular disk or annulus.
- Then ∂M is the union of these curves.
- In the example, a circle or a pair of concentric circles.
- Let ω be a one-form of class C^1 on M.
- Using the natural Cartesian coordinates, we have

$$\omega = adx + bdy.$$

- By definition of differentiability on arbitrary sets, a, b can be taken as restrictions of C^1 functions on some open set containing M.
- We have

$$d\omega = \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy.$$

Example: Green's Theorem (Cont'd)

- By Stokes's Theorem

$$\int_M \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy = \int_{\partial M} adx + bdy.$$

- According to a previous remark, the left-hand side is the ordinary Riemann integral over the domain of integration $M \subseteq \mathbb{R}^2$.
- On the other hand, if we think of ∂M as a one-dimensional manifold and cover it with (oriented) neighborhoods, it is clear that its value is that of the usual line integral along a curve C (or curves C_i) oriented so that as we traverse the curve the region is on the left.
- Thus the equality above may be written

$$\iint_M \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dxdy = \sum_i \int_{C_i} adx + bdy,$$

which is the usual statement of Green's Theorem.

Example: Divergence Theorem

- Let M be a regular domain of $\mathbb{R}^3$.
- That is, M is the closure of a bounded open set, bounded by closed C^∞ surfaces.
- Examples are:
  - The ball of radius 1, which is bounded by the sphere S^2;
  - The region interior to a torus T^2, obtained by rotating a circle around a line exterior to it.
- Consider the two-form

$$\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy,$$

where P, Q, R are C^1 functions on some open set of $\mathbb{R}^3$ containing M.
- We have

$$d\omega = \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}\right) dx \wedge dy \wedge dz.$$

Example: Divergence Theorem (Cont'd)

- Stokes's Theorem asserts that

$$\int_M \left(\frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} \right) dx \wedge dy \wedge dz = \int_{-\partial M} P dy \wedge dz + Q dz \wedge dx + R dx \wedge dy.$$

- Translate these, respectively, into:
  - A Riemann integral over a domain;
  - A surface integral over the boundary.
- Then we obtain the Divergence Theorem of Advanced Calculus.

Example: Stokes' Theorem of Advanced Calculus

- Let M be a piece of surface imbedded in $\mathbb{R}^3$ and bounded by smooth simple closed curves.
- For example, a sphere with one or more open circular disks removed.
- Thus, ∂M consists of boundary circles.
- Now dx, dy and dz may be considered, by restriction, as one-forms on M or on ∂M.
- So any one-form ω on M may be written

$$\omega = Adx + Bdy + Cdz,$$

where A, B and C are C^1 functions on M.
- Then

$$\begin{aligned} d\omega &= \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz \wedge dx \\ &\quad + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy. \end{aligned}$$

Example: Stokes' Theorem of Advanced Calculus (Cont'd)

- In this case Stokes's Theorem asserts that

$$\int_M \left(\frac{\partial C}{\partial y} - \frac{\partial B}{\partial z}\right) dy \wedge dz + \left(\frac{\partial A}{\partial z} - \frac{\partial C}{\partial x}\right) dz \wedge dx + \left(\frac{\partial B}{\partial x} - \frac{\partial A}{\partial y}\right) dx \wedge dy = \int_{\partial M} Adx + Bdy + Cdz.$$

- The left integral can be converted to an ordinary surface integral over the surface M in $\mathbb{R}^3$.
- The right integral can be converted to a line integral.
- In this way, one obtains Stokes Theorem of Advanced Calculus.

Deficiencies of the Version of Stokes' Theorem

- The version of Stokes's Theorem proved above holds only for smooth manifolds with smooth boundary.
- Thus, for example, our proof does not even include the case of a square in $\mathbb{R}^2$ or an open set of $\mathbb{R}^3$ bounded by a polyhedron.
- The difficulty in these cases is not so much with the analysis and integration theory, as with:
  - Describing the regions of integration to be admitted;
  - Giving precise definitions of orientability and induced orientation of the boundary.

Generalizing Stokes' Theorem

- The search for reasonable domains of integration to validate Stokes's theorem usually leads to the concept of a simplicial or polyhedral complex.
- This is a space made up by fastening together along their faces a number of simplices (line segments, triangles, tetrahedra, and their generalizations) or more general polyhedra (cubes, for example).
- It can be shown that any C^∞ manifold M may be "triangulated", i.e., it is homeomorphic (even with considerable smoothness) to such a complex.
- One infers that the integral over M becomes the sum of the integrals over the pieces, which are images of simplices, cubes, or other polyhedra as the case may be.
- The strategy is then to reduce the theory (including Stokes's Theorem) to the case of polyhedral domains of $\mathbb{R}^n$.

Example: Line Integrals in a Manifold

- Let $[a,b] = \{t \in \mathbb{R} : a \leq t \leq b\}$.
- Consider a C^1 mapping

$$F : [a,b] \to M.$$

- Its image is a C^1 curve S on M.
- Let ω be a one-form on M.
- We define $\int_S \omega$ by

$$\int_S \omega = \int_{[a,b]} F^* \omega.$$

- This is called the **line integral of ω along S**.
- In general, S is not a submanifold of M.

Example: Line Integrals in a Manifold (Cont'd)

- The right-hand side $\int_{[a,b]} F^*\omega$ is the integral of a one-form, $F^*\omega = f(t)dt$, on a one-dimensional manifold with boundary.
- Thus
$$\int_S \omega = \int_a^b f(t)dt.$$
- Exactly as for line integrals in $\mathbb{R}^n$, we may prove that the value of the integral does not depend on the parameter as long as the orientation of S is preserved.
- Thus the integral of ω over an oriented C^1 curve S of M is defined.
- A reverse orientation, i.e., traversing S in the opposite sense, changes the sign of the integral,
$$\int_{-S} \omega = -\int_S \omega.$$

Line Integrals: A Generalization

- Let $\widetilde{S}$ be an oriented continuous and piecewise differentiable curve.
- That is, $\widetilde{S}$ is a union of curves $S_1, S_2, \ldots, S_r$ such that each S_i is C^1 and the terminal point of S_i is the initial point of S_{i+1}.
- Then we define the integral over $\widetilde{S}$ by

$$\int_{\widetilde{S}} \omega = \sum_{i=1}^{r} \int_{S_i} \omega.$$

- This extends the definition of line integral on a manifold.
- The definition reduces to the usual one when $M = \mathbb{R}^n$.
- In fact we could have used that as a starting point by:
  - Subdividing the curve $\widetilde{S}$ on an arbitrary manifold into a finite union of C^1 curves S_i, each in a single coordinate neighborhood;
  - Evaluating the integral over each S_i in local coordinates, i.e., in $\mathbb{R}^n$.

Example

- Consider the special case $\omega = df$, where f is a C^∞ function on M (this implies that $d\omega = 0$).
- In this case the value of the line integral along the piecewise differentiable curve S from p to q is given by

$$\int_S df = f(q) - f(p).$$

- In particular, it is independent of the path chosen.
- Suppose p is held fixed.
- Then $f(q)$ is given, at each q, by adding $f(p)$ to the value of the line integral along any piecewise C^1 curve from p to q.
- Thus, f is determined to within an additive constant by the line integral (assuming M connected).

Application on the Unit Square

- We have a (line) integral of a one-form ω over an oriented piecewise differentiable curve $\widetilde{S}$.
- We can now state Stokes's Theorem for a polygonal region Q of $\mathbb{R}^2$.
- Such a region is bounded by an oriented piecewise linear (simple closed) curve $\widetilde{S} = \partial Q$.
- We carry this out for the unit square.

Theorem

Let ω be a C^1 one-form defined on

$$Q = \{(x, y) : 0 \leq x \leq 1, 0 \leq y \leq 1\}.$$

Let $\widetilde{S}$ be the boundary of Q traversed in the counterclockwise sense. Then

$$\int_Q d\omega = \int_{\widetilde{S}} \omega.$$

Application on the Unit Square (Cont'd)

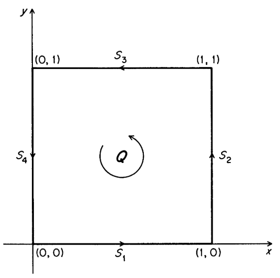
- Let
$$\omega = adx + bdy,$$
where a, b vanish outside Q and are C^1 functions on Q.
Then, on Q,
$$d\omega = \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dx \wedge dy.$$
By a previous remark,
$$\begin{aligned} \int_Q d\omega &= \int_0^1 \int_0^1 \left(\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}\right) dxdy \\ &= \int_0^1 [b(1,y) - b(0,y)]\, dy - \int_0^1 [a(x,1) - a(x,0)]\, dx. \end{aligned}$$
The orientation is that given by the standard coordinate system in $\mathbb{R}^2$.

Application on the Unit Square (Cont'd)

- On the other hand, consider the integral over the boundary.
  Note that:
  - $dy = 0$ on the horizontal sides;
  - $dx = 0$ on the vertical sides.

  So we have

$$\int_{\widetilde{S}} \omega = \sum_{i=1}^{4} \int_{S_i} a dx + b dy$$
$$= \int_0^1 a(x,0) dx + \int_0^1 b(1,y) dy + \int_1^0 a(x,1) dx + \int_1^0 b(0,y) dy.$$



Comparing the values of the integrals, shows that the theorem is true.

Subsection 6

Homotopy of Mappings and The Fundamental Group

Homotopy

Definition

Let F, G be continuous mappings from a topological space X to a topological space Y and let $I = [0, 1]$, the unit interval.
Then F is **homotopic** to G if there is a continuous mapping (the **homotopy**)

$$H : X \times I \to Y$$

which satisfies the conditions:

$$F(x) = H(x, 0) \quad \text{and} \quad G(x) = H(x, 1), \quad \text{for all } x \in X.$$

If X and Y are manifolds and F, G are C^∞, we define a C^∞ or **smooth homotopy** by requiring that H be C^∞ in addition to the conditions above.

Remarks

- We remark that
$$H_t(x) = H(x,t)$$
defines a one-parameter family of mappings
$$H_t : X \to Y, \quad 0 \le t \le 1,$$
such that
$$F = H_0 \quad \text{and} \quad G = H_1.$$
- The formulation of the definition emphasizes the simultaneous continuity in both variables t and x.

The C^∞ Case and Boundaries

- If $\partial X = \emptyset$, then $X \times I$ is a regular domain of $X \times \mathbb{R}$ and is a manifold with boundary.

  Indeed, $\partial(X \times I) = X \times \{0\} \cup X \times \{1\}$.

  So C^∞ is perfectly well defined.
- If $\partial X \neq \emptyset$, then $X \times I$ is not a manifold with boundary [consider, e.g., $X = \overline{B}_1^2(0)$, the closed unit disk].

  However, it is a reasonably nice domain of $X \times \mathbb{R}$ which is a manifold (with nonempty boundary).

  So only minor technical problems arise.
- We remark however, that when both X and Y have nonempty boundaries, there are cases in which it is natural to require that

$$H_t(\partial X) \subseteq \partial Y, \quad \text{for } 0 \leq t \leq 1.$$

Relative Homotopy

- Suppose (X, A) and (Y, B) are pairs consisting of:
  - Spaces X and Y;
  - Closed subspaces $A \subseteq X$ and $B \subseteq Y$.
- Consider continuous maps

$$F, G : X \to Y,$$

  such that:
  - $F(A) \subseteq B$;
  - $G(A) \subseteq B$.
- F and G map the pair (X, A) into the pair (Y, B) continuously.

Relative Homotopy (Cont'd)

- We say that F and G are **relatively homotopic** if there exists a continuous map
$$H : X \times I \to Y,$$
such that:
  - $H(A \times I) \subseteq B$;
  - $H(x, 0) \equiv F(x)$;
  - $H(x, 1) \equiv G(x)$.
- We have added to the original definition the requirement that
$$H_t(A) \subseteq B, \quad \text{for } 0 \leq t \leq 1.$$
- When $A = \emptyset = B$, the definition reduces to the original one.
- We will write $F \sim G$ to indicate that F and G are (relatively) homotopic.

The Equivalence Property

Theorem

Relative homotopy is an equivalence relation on the continuous maps of (X, A) into (Y, B), for any topological spaces X and Y and closed subspaces A and B, respectively.

- Note that $H(x, t) \equiv F(x)$ is a homotopy of $F(x)$ with $F(x)$.

  So the relation $\sim$ is reflexive.

  Let $H(x, t)$ be a homotopy of F to G.

  Then

  $$\widetilde{H}(x, t) = H(x, 1 - t)$$

  is a homotopy of G to F.

  So $\sim$ is symmetric as well.

The Equivalence Property (Cont'd)

- Finally, suppose that:
  - $F_1 \sim F_2$ via a homotopy H_1;
  - $F_2 \sim F_3$ via a homotopy H_2.

  Then we define $H(x,t)$, a homotopy of F_1 and F_3, by

$$H(x,t) = \begin{cases} H_1(x,2t), & \text{if } 0 \leq t \leq \frac{1}{2}, \\ H_2(x,2t-1), & \text{if } \frac{1}{2} \leq t \leq 1. \end{cases}$$

  It is easily verified that $H(x,t)$ is continuous.

  Moreover, all these maps take A into B, for every t between 0 and 1 inclusive.

  Finally, it can be shown that the constructed homotopies are C^∞, provided the given ones are C^∞.

Paths

- A continuous map

$$f : I \to M$$

  of the interval $I = [0, 1]$ into a manifold M is called a **path**, with:
  - $f(0)$ its **initial point**;
  - $f(1)$ its **terminal point**.
- We shall consider homotopy classes of paths under the additional restriction that the homotopy keep initial and terminal points fixed.
- Formally, we require that $H(t, 0)$ and $H(t, 1)$ are constant functions.
- This is exactly relative homotopy for $(I, \{0, 1\})$ and $(X, \{b, d\})$, with

$$b = f(0) \quad \text{and} \quad d = f(1).$$

Loops

- Given a manifold M, fix a **basepoint** b on M.
- Consider the paths with b as initial point.
- If b is also the terminal point, then the path is called a **loop**.
- Thus a **loop** is a continuous map
$$f : I \to M$$
such that $f(0) = b = f(1)$.
- We denote its homotopy class by
$$[f],$$
meaning always relative homotopy.

Simple Connectedness

- Among the homotopy classes of loops is that of the **constant loop**

$$e_b(s) = b, \quad 0 \leq s \leq 1.$$

- If this is the only homotopy class and M is connected, then we say M is **simply connected**.
- This means that every loop at b can be deformed over M to the constant loop.
- This property does not depend on the choice of b.
- Moreover, it is equivalent to the statement that any closed curve (continuous image of S^1) may be continuously deformed to a point on M.

Product of Paths

- Let M be a connected manifold.
- Let f, g be paths on M with the terminal point $f(1)$ coinciding with the initial point $g(0)$.
- We may combine these to a single path h after readjusting the parametrization.
- In fact, consider the continuous map

$$h : I \to M,$$

defined by

$$h(s) = \begin{cases} f(2s), & \text{if } 0 \leq s \leq \frac{1}{2}, \\ g(2s-1), & \text{if } \frac{1}{2} \leq s \leq 1. \end{cases}$$

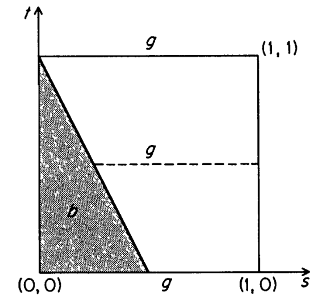
- It traverses the image of f followed by that of g.
- We shall call this the **product** of f and g, denoted $f * g$.

Properties of the Product of Paths

- The product of paths has the following properties with respect to (relative) homotopy:
  (i) $f * (g * h) \sim (f * g) * h$;
  (ii) Let $f(1) = b = g(0)$ and suppose $f = e_b$. Then
  $$e_b * g \sim g.$$
  Similarly, if $g = e_b$, then
  $$f * e_b \sim f;$$
  (iii) If $f_1 \sim f_2$ and $g_1 \sim g_2$, then
  $$f_1 * g_1 \sim f_2 * g_2;$$
  (iv) If $g(s) = f(1 - s)$ and $a = f(0)$, $b = f(1)$, then
  $$f * g \sim e_b \quad \text{and} \quad g * f \sim e_a;$$
  (v) If $F : M \to N$ is continuous and $f' = F \circ f$, $g' = F \circ g$, then
  $$(f * g)' = f' * g'.$$

Verification of Property (ii)

- By definition, we have:
  - $e_b * g(s) = b$, for $s \in [0, \frac{1}{2}]$;
  - $e_b * g(s) = g(2s - 1)$, for $s \in [\frac{1}{2}, 1]$.
- We wish to construct a homotopy H, showing that $e_b * g \sim g$.
- We use the idea captured in the figure.

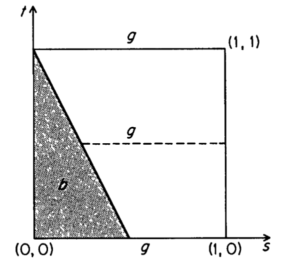


Verification of Property (ii) (Cont'd)

- We use the idea captured in the figure.
- Define $H(s,t)$ in the following way:

$$H(s,t)=\begin{cases} b, & \text{if } 0\le s\le \frac{1}{2}(1-t) \text{ and } 0\le t\le 1,\\ g(\frac{2s-1+t}{1+t}), & \text{if } \frac{1}{2}(1-t)\le s\le 1.\end{cases}$$

- The diagram shows how $H : I\times I\to M$ maps various portions of the unit square.
  - The shaded portion is mapped onto $b=g(0)$;
  - Each horizontal segment in the unshaded part is mapped onto the image of g with the parametrization modified proportionately.



The Fundamental Group of a Manifold

Theorem

Let $\pi_1(M, b)$ denote the homotopy classes of all loops at $b \in M$.
Then $\pi_1(M, b)$ is a group with product

$$[f][g] = [f * g].$$

If $F : M \to N$ is continuous, then F determines a homomorphism

$$F_* : \pi_1(M, b) \to \pi_1(N, F(b))$$

by

$$F_*[f] = [F \circ f].$$

The Fundamental Group of a Manifold (Cont'd)

Theorem (Cont'd)

If G is homotopic to F relative to the pairs (M, b) and $(N, F(b))$, then

$$F_* = G_*.$$

When F is the identity mapping on M, F_* is the identity isomorphism. Finally, for compositions of continuous mappings,

$$(F \circ G)_* = F_* \circ G_*.$$

The Fundamental Group of a Manifold (Cont'd)

- Property (iii) assures us that $[f * g]$ is independent of the representatives f and g chosen from $[f]$ and $[g]$.

  So the product is well defined.

  By Property (i), the product is associative.

  Property (ii) gives the existence of an identity $[e_b]$.

  Property (iv) gives the existence of inverses.

  Thus $\pi_1(M, b)$ is a group.

  Property (v) shows that $F : M \to N$ induces a homomorphism F_*.

  The last statement of the theorem is immediate from the definitions.

  Finally, suppose $H : M \times I \to N$ is a homotopy of F and G.

  Then $H(f(x), t)$ is a homotopy of $F_* f = F \circ f$ and $G_* f = F \circ g$.

Topological Invariance of Fundamental Group

Corollary

Suppose M_1 and M_2 are homeomorphic and b_1, b_2 correspond under the homeomorphism. Then the mapping F_* is an isomorphism of the corresponding fundamental groups

$$\pi_1(M_1, b_1) \cong \pi_1(M_2, b_2).$$

- Let $F : M_1 \to M_2$ be the homeomorphism.

  Let $G : M_2 \to M_1$ be its inverse.

  By the last statement of the theorem,

  $$F_* \circ G_* \quad \text{and} \quad G_* \circ F_*$$

  are the identity isomorphisms.

  So F_* and G_* are isomorphisms.

Contractibility

- If the identity map of M to M is homotopic to the constant map of M onto one of its points b, then M is said to be **contractible** (**to** b).

  Example: Consider any open subset of $\mathbb{R}^n$ which is star-shaped with respect to a point b.

  Then
  $$H(x,t) = (1-t)x + tb$$
  is such a homotopy.

  It follows that such a subset is contractible.

Contractibility and Simple Connectedness

Corollary

If M is contractible to b, then $\pi_1(M, b) = \{e\}$, the identity element alone. It follows that M is simply connected.

- Let f be a loop at b.

  It is homotopic to the constant loop e_b by

  $$H(f(s), t), \quad 0 \leq s, t \leq 1.$$

  This shows that $\pi_1(M, b) = \{1\}$.

  From this, we can deduce simple connectedness.

  We may also prove it directly from the definition using again the mapping H.
- There are simply connected spaces which are not contractible.
- The sphere S^n, $n > 1$, is the simplest example.

Integrals Along Differentiable Paths

- Let M be a manifold and ω be a one-form on M.
- Suppose $p, q \in M$.
- Let S_1, S_2 be two piecewise differentiable paths of M from p to q.
- It is natural to ask whether or not

$$\int_{S_1} \omega = \int_{S_2} \omega.$$

- In general they are not equal, even in very simple cases.
- But the standard theorems of Advanced Calculus on independence of path may be generalized to manifolds with essentially the same proofs.

A Special Case

Theorem

Let ω be a one-form on a manifold M, such that $d\omega = 0$ everywhere. Let S_1, S_2 be homotopic piecewise differentiable paths from $p \in M$ to $q \in M$. Then

$$\int_{S_1} \omega = \int_{S_2} \omega.$$

- Let S_1 and S_2 be C^1 curves homotopic by a differentiable mapping

$$H : I \times I \to M.$$

Then the result is a straightforward application of Stokes's Theorem for the unit square.

A Special Case (Cont'd)

- In the general case the (continuous) homotopy H of the piecewise differentiable curves must be altered as follows.

  First $I \times I$ is subdivided by vertical and horizontal lines so that:
  - It is differentiable on each boundary segment;
  - H carries each subrectangle Q_{ij} into a single coordinate neighborhood U.

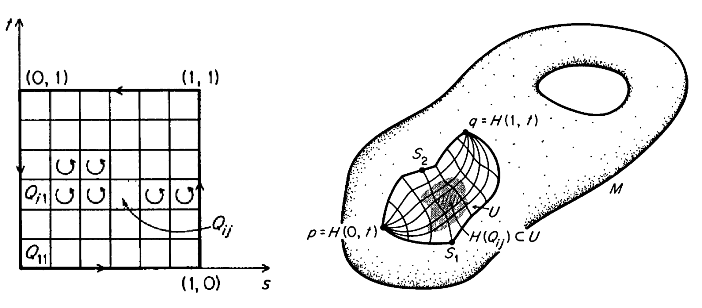
  Then the techniques of a previous section are used to alter H successively to a homotopy $\widetilde{H}$ which is differentiable on each Q_{ij}.

  From this point the proof follows the usual one of Advanced Calculus.

  The new homotopy $\widetilde{H}$ maps the edges of the square $Q = I \times I$ into the paths $S_1, q, -S_2, p$, respectively, as we go around ∂Q counterclockwise.

A Special Case (Cont'd)

- The images of the left and right vertical edges are the constant paths p and q.



Since the line integral of ω over a constant path is zero, we have

$$\int_{\partial Q} \widetilde{H}^* \omega = \int_{S_1} \omega + \int_{-S_2} \omega = \int_{S_1} \omega - \int_{S_2} \omega.$$

A Special Case (Cont'd)

- On the other hand, we can check that, if we denote the oriented squares of the subdivision by Q_{ij}, then line integrals over the same path in opposite directions cancel out,

$$\int_{\partial Q} \widetilde{H}^* \omega = \sum_{i,j} \int_{\partial Q_{ij}} \widetilde{H}^* \omega.$$

By a previous theorem and remarks,

$$\int_{\partial Q_{ij}} \widetilde{H}^* \omega = \int_{Q_{ij}} d\widetilde{H}^* \omega.$$

Since $d\widetilde{H}^* \omega = \widetilde{H}^* d\omega = 0$, we see that

$$\int_{S_1} \omega - \int_{S_2} \omega = 0.$$

Consequence

Corollary

Let ω be a C^∞ one-form on a simply connected manifold M. Suppose that $d\omega = 0$ everywhere. Then there is a C^∞ function f on M, such that

$$\omega = df.$$

If f and g are two such functions, then $f - g$ is constant.

- We choose a fixed basepoint $b \in M$.

  Define f at any $p \in M$ by choosing a piecewise differentiable curve S from b to p and setting

  $$f(p) = \int_S \omega.$$

  The theorem assures us that this defines a function on M.

  The remainder of the proof deals with purely local properties.

Consequence (Cont'd)

- We show that f is a C^∞ function with the property that $df = \omega$.

  If we show the latter fact, it will follow that f is C^∞, because we have assumed ω to be C^∞.

  Changing the basepoint changes f by an additive constant, the value of the integral of ω along the path between the old and new basepoints.

  Hence, it does not change df at all.

  Therefore it is enough to show that $df = \omega$ at the basepoint.

  Let U, φ be a coordinate neighborhood of the basepoint b.

  We suppose that $x^1, \ldots, x^n$ are the local coordinates, such that:
  - $\varphi(b) = (0, \ldots, 0)$;
  - $\varphi(U) = B_1^n(0)$.

Consequence (Cont'd)

- Let $f(x^1, \ldots, x^n)$ denote the expression for f in local coordinates. Denote ω in local coordinates by

$$\omega = \alpha_1(x)dx^1 + \cdots + \alpha_n(x)dx^n.$$

We have, by definition,

$$f(x) = \int_C \alpha_1(x)dx^1 + \cdots + \alpha_n(x)dx^n,$$

the line integral along any path C from $(0, \ldots, 0)$ to $(x^1, \ldots, x^n)$. We must show that, at $x = (0, \ldots, 0)$,

$$\frac{\partial f}{\partial x^j} = \alpha_j, \quad j = 1, \ldots, n.$$

Consequence (Cont'd)

- We must show that, at $x = (0, \ldots, 0)$,

$$\frac{\partial f}{\partial x^j} = \alpha_j, \quad j = 1, \ldots, n.$$

However, this is immediate from the definitions,

$$\begin{aligned} \left(\frac{\partial f}{\partial x^j}\right)_0 &= \lim_{h \to 0} \frac{1}{h}(f(0, \ldots, h, \ldots, 0) - f(0, \ldots, 0)) \\ &= \lim_{h \to 0} \frac{1}{h} \int_0^h \alpha_j(0, \ldots, x^j, \ldots, 0) dx^j \\ &= \alpha_j(0, \ldots, 0). \end{aligned}$$

For the last statement, note that $d(f - g) = \omega - \omega = 0$ so that $f - g =$ constant on the (connected) manifold M.

Subsection 7

Applications of Differential Forms and de Rham Groups

Closed and Exact k-Forms

Definition

A k-form ω on a manifold M (with possibly nonempty boundary) is said to be **closed** if

$$d\omega = 0$$

everywhere.
It is said to be **exact** if there is a $(k-1)$-form η, such that

$$d\eta = \omega.$$

- We recall some facts about the operator d and apply them here.
- We denote by $Z^k(M)$ the set of closed k-forms on M.
- We denote by $B^k(M)$ the set of exact k-forms on M.

Properties of Closed and Exact Forms

- $Z^k(M)$ is the kernel of the homomorphism

$$d : \bigwedge^{k}(M) \to \bigwedge^{k+1}(M).$$

- So it is a linear subspace of $\bigwedge^k(M)$.
- $B^k(M)$ is the image of

$$d : \bigwedge^{k-1}(M) \to \bigwedge^{k}(M).$$

- So it is also a linear subspace.
- We know that $d^2 = 0$.
- Therefore,

$$B^k(M) \subseteq Z^k(M).$$

- This allows us to form the quotient

$$H^k(M) := Z^k(N)/B^k(M).$$

The de Rham Groups

Definition

The **de Rham group of dimension** k of M is the quotient space

$$H^k(M) = Z^k(M)/B^k(M).$$

If $n = \dim M$, we denote by $H^*(M)$ the direct sum

$$H^*(M) = H^0(M) \oplus \cdots \oplus H^n(M).$$

- Note that
$$H^*(M) = Z(M)/B(M),$$
where:
  - $Z(M)$ is the kernel of $d : \bigwedge(M) \to \bigwedge(M)$ and the direct sum of the $Z^k(M)$, $k = 0, \ldots, n$;
  - $B(M)$ is the image of $d : \bigwedge(M) \to \bigwedge(M)$ and the direct sum of the $B^k(M)$, $k = 0, 1, \ldots, n$.

Properties of the de Rham Groups

- Although called de Rham groups,

$$H^k(M), \quad k = 0, \ldots, n = \dim M,$$

  are actually vector spaces over $\mathbb{R}$.
- In fact, $H^*(M)$ is an algebra, with the multiplication being that naturally induced by the exterior product of differential forms.
- This follows directly from the property of d asserting that when $\varphi \in \bigwedge^r(M)$, $\psi \in \bigwedge^s(M)$, then

$$d(\varphi \wedge \psi) = d\varphi \wedge \psi + (-1)^r \varphi \wedge d\psi.$$

- From this, it follows that $Z(M)$ is an algebra containing $B(M)$ as an ideal.

de Rham's Theorem and Remarks

Theorem (de Rham's Theorem)

There is a natural isomorphism of $H^*(M)$ and the cohomology ring of M, under which $H^k(M)$ corresponds to the kth cohomology group.

- This requires knowledge of Algebraic topology and cohomology groups.
- Among the consequences, we get:
  - Whenever M is compact the dimension of $H^*(M)$ is finite;
  - $H^*(M)$ and its algebra structure are topologically invariant. That is, if M_1 and M_2 are homeomorphic, then $H^*(M_1)$ and $H^*(M_2)$ are isomorphic as algebras.
- The duality which appears in algebraic topology between homology and cohomology groups of a space extends to a duality of homology groups and de Rham groups via integration and Stokes's Theorem.

Mappings and de Rham Groups

Lemma

A C^∞ mapping $F : M_1 \to M_2$ induces an algebra homomorphism

$$F^* : H^*(M_2) \to H^*(M_1)$$

which carries $H^k(M_2)$ (linearly) into $H^k(M_1)$, for all k.
If F is the identity mapping on M, then

$$F^* : H^*(M) \to H^*(M)$$

is the identity isomorphism.
Under composition of mappings we have

$$(G \circ F)^* = F^* \circ G^*.$$

Mappings and de Rahm Groups (Cont'd)

- It is a property of differential forms that a C^∞ mapping $F : M_1 \to M_2$ defines a corresponding homomorphism

$$F^* : \bigwedge(M_2) \to \bigwedge(M_1).$$

We have $F^* d = dF^*$.

It follows that

$$F^*(Z^k(M_2)) \subseteq Z^k(M_1) \quad \text{and} \quad F^*(B^k(M_2)) \subseteq B^k(M_1).$$

Therefore, F^* induces a homomorphism, which we also denote by F^*,

$$F^* : H^k(M_2) \to H^k(M_1).$$

Now F^* is an algebra homomorphism on forms.

So $F^* : H^*(M_2) \to H^*(M_1)$ is also an algebra homomorphism.

Diffeomorphisms and Isomorphisms

Corollary

If M_1 and M_2 are diffeomorphic manifolds, then $H^*(M_1)$ and $H^*(M_2)$ are isomorphic rings.

- Let $F : M_1 \to M_2$ be a diffeomorphism and F^{-1} its inverse.
  Then

  $$F^{-1*} \circ F^* = (F \circ F^{-1})^* \quad \text{and} \quad F^* \circ F^{-1*} = (F^{-1} \circ F)^*$$

  are both the identity isomorphism.
  Hence F^* is an isomorphism with inverse F^{-1*}.

The de Rham Group of Dimension Zero

Theorem

Let M be a C^∞ manifold with a finite number r of components. Then $H^0(M) = \boldsymbol{V}^r$, a vector space over $\mathbb{R}$ of dimension r.

- $\bigwedge^0(M)$ consists of C^∞-functions on M.
  $Z^0(M)$ consists of those functions f for which $df = 0$.
  There are no forms of dimension less than zero.
  So $B^0(M) = \{0\}$ and $H^0(M) = Z^0(M)$.
  We have seen previously that
  $$df = 0 \quad \text{iff} \quad f \text{ is constant on each component } M_1, \ldots, M_r.$$
  Thus,
  $$H^0(M) \cong \{(a_1, \ldots, a_r) : a_i \in \mathbb{R}\},$$
  where $(a_1, \ldots, a_r)$ corresponds to the function taking the constant value a_i on M_i, $i = 1, \ldots, r$.

Remark

- Let $\{p\}$ be a zero-dimensional manifold.
- By the theorem,
$$H^0(\{p\}) \cong \mathbb{R}.$$
- This determines the de Rham groups of a point space.
- Since $\bigwedge^k(\{p\}) = 0$,
$$H^k(\{p\}) = 0, \quad \text{for } k > 0.$$

The First de Rham Group

Theorem

If a compact manifold M, or manifold with boundary, is simply connected, then

$$H^1(M) = \{0\}.$$

- Suppose ω is a closed one-form on M, that is,

$$d\omega = 0.$$

Then, there exists a function f on M, such that

$$df = \omega.$$

Thus, ω is exact.

Since every closed one-form is exact, $H^1(M) = \{0\}$.

The n-th de Rham Group

Theorem

Let M be a compact orientable manifold of dimension n, with $\partial M = \emptyset$. Then $H^n(M) \neq \{0\}$.

- Let Ω be a volume element.
  It is an n-form, which:
  - Is never zero at any point;
  - Gives the orientation of M.

  By a previous theorem, $\int_M \Omega > 0$.
  Suppose $\Omega = d\omega$, for some $(n-1)$-form ω.
  By Stokes's Theorem, since $\partial M = \emptyset$,
  $$\int_M \Omega = \int_M d\omega = \int_{\partial M} \omega = 0.$$
  On the other hand $d\Omega = 0$, since all $(n+1)$-forms vanish on M.
  Thus, Ω determines a nonzero class in $H^n(M)$.

Handling of Boundary Points

- Let $A \subseteq \mathbb{R}^n$ be either an open set or the closure of an open set.
- In the latter case we have in mind regular domains, cubes, simplices, and so on.
- Note that for either choice of A, $I \times A$ is the closure of an open set, its own interior, in $\mathbb{R} \times \mathbb{R}^n = \mathbb{R}^{n+1}$.
- By definition of differentiability of functions (in this instance its components) on A, when A is not open, a C^∞ k-form ω on A is the restriction to A of a k-form $\widetilde{\omega}$ on an open set U, with $A \subseteq U$.

Handling of Boundary Points (Cont'd)

- Our restrictions on A ensure that all derivatives of any C^∞ function f on A are defined at every $p \in A$ independently of the open set U and extension $\widetilde{f}$ which may be needed to define them at boundary points.
- This is a consequence of:
  - The continuity of all derivatives of $\widetilde{f}$ on U;
  - The fact that every $p \in A$ is either an interior point - where the derivatives are already defined without any $\widetilde{f}$ - or the limit of interior points.
- It follows that for a C^∞ form ω on A, $d\omega$ is defined, even at boundary points.

The Homotopy Operator

Definition

The **homotopy operator** $\mathscr{I}$ is defined to be an $\mathbb{R}$-linear operator from

$$\bigwedge^{k+1}(I \times A) \to \bigwedge^{k}(A).$$

On monomials $\mathscr{I}$ is defined as follows:

- If $\omega = \alpha(t,x)dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}$, we set $\mathscr{I}\omega = 0$;
- If $\omega = \alpha(t,x)dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$, we define $\mathscr{I}\omega$ by

$$\mathscr{I}\omega = \left(\int_0^1 \alpha(t,x)dt\right) dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$

Having been thus defined for monomials, we extend $\mathscr{I}$ to be $\mathbb{R}$-linear on $\bigwedge^{k+1}(I \times A)$ with values in $\bigwedge^{k}(A)$.

Remarks

- We will denote by $i_y : A \to I \times A$ the natural injection
$$i_t(x) = (t, x).$$
- Then
$$\omega_t = i^* \omega.$$
- In particular,
$$\omega_0 = i_0^* \omega \quad \text{and} \quad \omega_1 = i_1^* \omega.$$

Properties of the Homotopy Operator

Lemma

The homotopy operator $\mathscr{I} : \bigwedge^{k+1}(I \times A) \to \bigwedge^k(A)$ in addition to being $\mathbb{R}$-linear has the following properties:

(i) It commutes with C^∞ functions which are independent of t;

(ii) For all $\omega \in \bigwedge^{k+1}(I \times A)$ it satisfies the relation

$$\mathscr{I} d\omega + d\mathscr{I}\omega = \omega_1 - \omega_0.$$

- Suppose f is independent of t.

  Then we may consider it both as a function on $I \times A$ and on A.

  Moreover, independence of t, allows f to be moved through the integral sign in the definition of $\mathscr{I}$.

  Thus, $\mathscr{I} f\omega = f\mathscr{I}\omega$.

Properties of the Homotopy Operator (Cont'd)

- For the second property we must verify the equation directly.

  All of $d, \mathscr{I}, i_0^*$ and i_1^* are $\mathbb{R}$-linear.

  So it is enough to verify the equation for monomials.

  First we consider the case where ω does not involve dt,

$$\omega = \alpha(t,x) dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}.$$

  Then $\mathscr{I}\omega = 0$. So $d\mathscr{I}\omega = 0$.

  Also $\mathscr{I} d\omega$ is given by

$$\begin{aligned} \mathscr{I} d\omega &= \left(\int_0^1 \frac{\partial \alpha}{\partial t} dt\right) dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}} \\ &= (\alpha(1,x) - \alpha(0,x)) dx^{i_1} \wedge \cdots \wedge dx^{i_{k+1}}. \end{aligned}$$

  But the right side is then exactly $i_1^*\omega - i_0^*\omega = \omega_1 - \omega_0$.

  This establishes the equality for this case.

Properties of the Homotopy Operator (Cont'd)

- Now suppose that $\omega = \alpha(t,x)dt \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}$.
  Computing $\mathscr{I} d\omega$, we see that
  $$\mathscr{I} d\omega = -\sum_{j=1}^{n} \left(\int_0^1 \frac{\partial \alpha}{\partial x^j} dt \right) dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}.$$
  On the other hand using the Leibniz rule to differentiate under the integral sign, we may compute $d\mathscr{I}\omega$:
  $$\begin{array}{rcl} d\mathscr{I}\omega & = & d(\int_0^1 \alpha(t,x)dt)dx^{i_1} \wedge \cdots \wedge dx^{i_k} \\ & = & \sum_{j=1}^n (\int_0^1 \frac{\partial \alpha}{\partial x^j} dt) dx^j \wedge dx^{i_1} \wedge \cdots \wedge dx^{i_k}. \end{array}$$
  Adding these expressions, we see that $\mathscr{I} d\omega + d\mathscr{I}\omega = 0$.
  On the other hand since $i_1^* dt = 0 = i_0^* dt$, we have
  $0 = i_1^*\omega - i_0^*\omega = \omega_1 - \omega_0$.
  Thus, in all cases, the identity in Part (ii) holds.

Poincaré's Lemma

Lemma (Poincaré's Lemma)

Let A be a subset of $\mathbb{R}^n$ which is either open or is the closure of an open set. If A is star-shaped, then

$$H^k(A) = \{0\}, \quad \text{for all } k \geq 1.$$

Hence, $H^*(A)$ is isomorphic to the cohomology ring of a point.

- We recall that A is star-shaped if it contains a point 0, such that, for any $p \in A$, the segment $\overline{0p}$ lies entirely in A.

  By suitable choice of coordinates we may suppose that 0 is the origin.

  We define $H : I \times A \to A$ as

  $$H(t, x_1, \ldots, x_n) = (tx^1, \ldots, tx^n).$$

  If ω is a k-form on A, then $H^*\omega$ is a k-form on $I \times A$.

Poincaré's Lemma (Cont'd)

- By definition of $\mathscr{I}$, $i_0 : x \to (0, x)$ and $i_1 : x \to (1, x)$.
  Therefore, $H \circ i_0 : A \to \{0\}$ and $H \circ i_1 : A \to A$ is the identity.
  We apply $\mathscr{I}$ to $\bigwedge^k(I \times A)$, using the fact that $\bigwedge^k(\{0\})$, a point space, is trivial, for $k \geq 1$.
  We get
  $$d\mathscr{I}(H^*\omega) + \mathscr{I}d(H^*\omega) = i_1^*(H^*\omega) - i_0^*(H^*\omega).$$
  Suppose $d\omega = 0$. Then $dH^*\omega = 0$.
  So we have
  $$d\mathscr{I}H^*\omega = (H \circ i_1)^*\omega - (H \circ i_0)^*\omega = \omega.$$
  Therefore, every closed k-form ω on A is exact, if $k \geq 1$.
  If $k = 0$, then we may use the fact that A is connected to see that $H^0(A) \cong \mathbb{R}$.

Homotopic Maps and de Rham Homomorphisms

Theorem

Let M and N be compact manifolds and assume $\partial M = \emptyset$.
Let F and G be C^∞ mappings of M into N which are C^∞ homotopic.
Then the corresponding homomorphisms

$$F^*, G^* : H^*(M) \to H^*(N)$$

are equal.

- We use our previously defined operator $\mathscr{I}$.
  We construct a similar operator $\mathscr{I} : \bigwedge^{k+1}(I \times M) \to \bigwedge^k(M)$.
  First we note that M may be covered by a finite collection of coordinate neighborhoods, U_i, φ_i with

$$\varphi_i(U_i) = B_1^n(0), \quad n = \dim M,\ i = 1, \ldots, r,$$

  with a subordinate C^∞ partition of unity $\{f_i\}$, $\text{supp} f_i \subseteq U_i$.

Homotopic Maps and de Rham Homomorphisms (Cont'd)

- Then any $(k+1)$-form ω on $I \times M$ can be written as a sum of forms, with support in $I \times U_i$,

$$\omega = \sum_{i=1}^{r} \omega_i, \quad \omega_i = f_i \omega.$$

We may consider f_i, or any functions on M, as being also functions on $I \times M$, which are independent of t.

We define $\mathscr{I}$ to be additive so that

$$\mathscr{I}\omega = \sum \mathscr{I}\omega_i.$$

This leaves only the problem of defining $\mathscr{I}$ on forms with support in one of the neighborhoods $I \times U_i$.

Homotopic Maps and de Rham Homomorphisms (Cont'd)

- When ω has support in a neighborhood $I \times U$, where U, φ is a coordinate neighborhood with $\varphi(U) = B_1^n(0)$, we proceed as follows.

  Let $\widetilde{\varphi} : I \times U \to I \times B_1^n(0)$ be defined by

$$\widetilde{\varphi}(t, p) = (t, \varphi(p)).$$

  Then define $\mathscr{I}\omega$ on $I \times U$, using our previous definition of $\mathscr{I}$ for $I \times B_1^n(0)$, by

$$\mathscr{I}\omega|_U = \widetilde{\varphi}^*(\mathscr{I}(\widetilde{\varphi}^{-1*}\omega)),$$

  the $\mathscr{I}$ on the right side being the operator defined earlier.

  Further, let $\mathscr{I}\omega = 0$ on $M - U$.

  This defines a C^∞ k-form on M, the image of a $(k+1)$-form on $I \times M$.

Homotopic Maps and de Rham Homomorphisms (Cont'd)

- By a previous lemma for this form ω we have the relation

$$\mathscr{I} d\omega + d\mathscr{I}\omega = \omega_1 - \omega_0.$$

Now $\mathscr{I} d + d\mathscr{I}$ is an additive operator.
So, for an arbitrary $\omega \in \bigwedge^{k+1}(I \times M)$, we may apply the decomposition $\omega = \sum \omega_i$ to obtain

$$\begin{array}{rcl} \mathscr{I} d\omega + d\mathscr{I}\omega & = & \mathscr{I} d \sum \omega_i + d\mathscr{I} \sum \omega_I \\ & = & \sum \mathscr{I} d\omega_i + \sum d\mathscr{I}\omega_i \\ & = & \sum \mathscr{I} d\omega_I + \sum d\mathscr{I}\omega_i \\ & = & \sum((\omega_i)_1 - (\omega_i)_0) \\ & = & \omega_1 - \omega_0. \end{array}$$

Homotopic Maps and de Rham Homomorphisms (Cont'd)

- Finally, to complete the proof, we let ω be any closed k-form on N.

  We must show that $G^*\omega - F^*\omega$ is exact.

  Now let $H : M \times I \to M$ be the homotopy connecting F and G.

  Then, letting $i_t(p) = (t, p)$, as before, we have:

$$\begin{aligned} F(p) &= H(p,0) = H \circ i_0; \\ G(p) &= H(p,1) = H \circ i_1. \end{aligned}$$

  We know that $dH^*\omega = H^*d\omega = 0$.

  So we have

$$d\mathscr{I}H^*\omega = i_1^*H^*\omega - i_0^*H^*\omega = G^*\omega - F^*\omega.$$

Incontractibility of Compact Orientable Manifolds

- Intuition tells us that we cannot contract a sphere, or torus, over itself to a single point.

Corollary

Let M be a compact orientable C^∞ manifold ($\dim M > 0$), with $\partial M = \emptyset$. Then M is not contractible.

- By the previous theorem, with $M = N$, if i is homotopic to the constant map $F : M \to \{p_0\}$, then

$$i^* = F^*$$

as homomorphisms on the groups $H^k(M)$.

i^* is the identity isomorphism.

F^* is a homomorphism $H^k(M) \to H^k(\{p_0\})$ which is $\{0\}$, for $k \geq 1$.

This contradicts a previous theorem, if $\dim M > 0$.

Subsection 8

Further Applications of de Rham Groups

Maps from the Closed Ball to its Boundary

- Let D^n denote $\overline{B}_1^n(0)$, the closed unit ball in $\mathbb{R}^n$.
- D^n is a manifold with boundary, $\partial D^n = S^{n-1}$.

Lemma

There is no C^∞ map $F : D^n \to \partial D^n$ which leaves ∂D^n pointwise fixed.

- Suppose that there exists such a map F.

  Let G denote the identity map of $\partial D^n \to D^n$.

  Then $F \circ G = I$, the identity map of $\partial D^n \to \partial D^n$.

  This implies that $G^* \circ F^* = (F \circ G)^*$ induces the identity isomorphism on $H^*(\partial D^n)$.

Maps from the Closed Ball to its Boundary (Cont'd)

- Therefore, the homomorphism

$$F^* : H^{n-1}(\partial D^n) \to H^{n-1}(D^n)$$

must be injective.

That is, $\ker F^* = \{0\}$.

By Poincaré's Lemma, $H^{n-1}(D^n) = \{0\}$.

Hence, $\ker F^* = H^{n-1}(\partial D^n)$.

Therefore, $H^{n-1}(\partial D^n) = \{0\}$.

However, $\partial D^n = S^{n-1}$ is an orientable and compact manifold without boundary.

So we know that

$$H^{n-1}(\partial D^n) = H^{n-1}(S^{n-1}) \neq \{0\}.$$

This contradiction implies that no such map F exists.

The Brouwer Fixed Point Theorem

Theorem (Brouwer)

Let X be a topological space homeomorphic to D^n. Then any continuous map $F : X \to X$ has a fixed point. That is, for each F, there is at least one $x_0 \in X$, such that

$$F(x_0) = x_0.$$

- As a first step we note that it is enough to prove the theorem for D^n.

  Let $H : D^n \to X$ be a homeomorphism.

  Let $F : X \to X$ be any continuous mapping.

  Suppose $H^{-1} \circ F \circ H : D^n \to D^n$ has a fixed point y_0.

  Then $x_0 = H(y_0)$ is fixed by F.

The Brouwer Fixed Point Theorem (Cont'd)

- Moreover, even in the case of D^n, it is enough to establish the property for C^∞ maps $F : D^n \to D^n$.

  To see this, suppose every such C^∞ map has a fixed point.

  Assume there exists continuous $G : D^n \to D^n$ with no fixed point.

  Then $\|G(x) - x\|$ is bounded away from zero on the compact D^n.

  We may find an $\varepsilon > 0$, such that

  $$\|G(x) - x\| > 3\varepsilon.$$

  Using the Weierstraß Approximation Theorem, we approximate G to within ε by a C^∞ mapping G_1,

  $$\|G(x) - G_1(x)\| < \varepsilon, \quad \text{for all } x \in D^n.$$

The Brouwer Fixed Point Theorem (Cont'd)

- However, the values $G_1(x)$ are not necessarily in D^n, for every $x \in D^n$. So we replace G_1 by
$$F(x) = (1+\varepsilon)^{-1} G_1(x).$$
Clearly, $F(x)$ is defined and C^∞ on D^n. Moreover, $F(D^n) \subseteq D^n$. Since $\|G(x)\| \leq 1$, it follows that, for all $x \in D^n$:
  - $\|G_1(x)\| < 1+\varepsilon$;
  - $\|F(x)\| \leq 1$.

  Thus F, maps D^n into D^n and is C^∞.
  For $x \in D^n$,
$$\begin{array}{rcl} \|G(x) - F(x)\| & = & \|G(x) - (1+\varepsilon)^{-1} G_1(x)\| \\ & = & (1+\varepsilon)^{-1} \|\varepsilon G(x) + G(x) - G_1(x)\| \\ & \leq & \varepsilon \|G(x)\| + \|G(x) - G_1(x)\| \\ & = & 2\varepsilon. \end{array}$$

The Brouwer Fixed Point Theorem (Cont'd)

- From these inequalities we obtain a contradiction to the assumption that every C^∞ map $F : D^n \to D^n$ leaves some point fixed.

  Namely, for every $x \in D^n$ we have

$$\begin{aligned} \|F(x) - x\| &= \|(G(x) - x) - (G(x) - F(x))\| \\ &\geq \|G(x) - x\| - \|G(x) - F(x)\| \\ &\geq 3\varepsilon - 2\varepsilon \\ &= \varepsilon. \end{aligned}$$

  This contradiction shows that if every C^∞ map of D^n to D^n has a fixed point, then so must every continuous one.

  The proof of the theorem is then completed by the following lemma.

C^∞ Maps of the Closed Unit Ball and Fixed Points

Lemma

If $F : D^n \to D^n$ is a C^∞ map, then F has a fixed point.

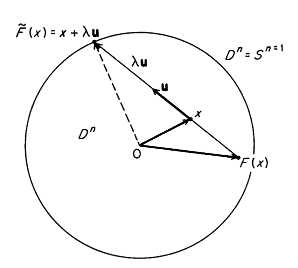
- Suppose $F : D^n \to D^n$ is C^∞ and has no fixed point.

  We use F to construct a C^∞ map $\widetilde{F} : D^n \to \partial D^n$ which leaves ∂D^n pointwise fixed.

  Given $x \in D^n$, let $\widetilde{F}(x)$ be the boundary point obtained by extending the segment $\overline{F(x)x}$ past x to the boundary of D^n.

  Note, if $x \in \partial D^n$, then $\widetilde{F}(x) = x$.

  In any case, $\widetilde{F}(D^n) \subseteq \partial D^n$.



  To see that $\widetilde{F}$ is C^∞, we express $\widetilde{F}$ explicitly using vectors in $\mathbb{R}^n$.

C^∞ Maps of the Closed Unit (Cont'd)

- Namely, we have
$$\widetilde{F}(x) = x + \lambda \boldsymbol{u},$$
where:
  - x denotes the vector from $(0, \ldots, 0)$ to $x = (x^1, \ldots, x^n)$;
  - $\boldsymbol{u}$ is the unit vector directed from $F(x)$ to x and lying on this segment, more precisely,
$$\boldsymbol{u} = \frac{x - F(x)}{\|x - F(x)\|};$$
  - $\lambda = -(x, \boldsymbol{u}) + [1 - (x, x) + (x, \boldsymbol{u})^2]^{1/2}$ denotes the length of the vector on $\boldsymbol{u}$ with initial point x and terminal point $\widetilde{F}(x)$ on ∂D^n.

Since F is C^∞, it is easy to check that $\widetilde{F}$ is C^∞.
The scalar λ is the unique nonnegative number such that
$$\|x + \lambda \boldsymbol{u}\| = 1.$$
Since F is C^∞, $\boldsymbol{u}$ is C^∞.
So wherever $1 - (x, x) + (x, \boldsymbol{u})^2 > 0$, then $\widetilde{F}(x)$ is also C^∞.

C^∞ Maps of the Closed Unit (Cont'd)

- However, $1-(x,x) \geq 0$, with equality only if $x \in S^{n-1}$.

  Moreover, $(x,\boldsymbol{u})^2 \geq 0$, with equality only when $\boldsymbol{u}$ is orthogonal to x.

  That is, when $x - F(x)$ is orthogonal to x.

  However, $(x,\boldsymbol{u}) = 0$ cannot occur when $(x,x) = 1$, that is, on a point of S^{n-1}, since in this case $F(x)$ would be exterior to D^n.

  Thus, $1-(x,x)+(x,\boldsymbol{u})^2 > 0$ on D^n and $\widetilde{F}$ is C^∞.

  The existence of $\widetilde{F}$ contradicts a previous lemma.

  So F has a fixed point.

The Antipodal Map of S^{n-1}

Theorem

If n is odd, then there is no C^∞ homotopy between the antipodal map $A: S^{n-1} \to S^{n-1}$ and the identity map of S^{n-1}.

- The sphere is an orientable manifold.

  In fact we may define the oriented orthonormal frames of $T_x(S^{n-1})$ at each $x \in S^{n-1}$ in the following fashion.

  Each $x \in S^{n-1}$ determines a unit vector $\boldsymbol{x} = \overline{0x}$.

  The elements of $T_x(S^{n-1})$ correspond to the vectors in the orthogonal complement of $\boldsymbol{x}$.

  Let $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}$ be an orthonormal frame of $T_x(S^{n-1})$ in the induced metric of $\mathbb{R}^n$.

  Then $\boldsymbol{x}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}$ is an orthonormal frame of $\mathbb{R}^n$.

  We use the natural parallelism to identify vectors at distinct points of $\mathbb{R}^n$.

The Antipodal Map of S^{n-1} (Cont'd)

- Two frames, $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}$ and $\boldsymbol{e}'_1, \ldots, \boldsymbol{e}'_n$ at x will be said to have the *same* orientation if the corresponding frames $\boldsymbol{x}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}$ and $\boldsymbol{x}, \boldsymbol{e}'_1, \ldots, \boldsymbol{e}'_{n-1}$ do.

  From the canonical orientation of $\mathbb{R}^n$ we obtain an orientation of S^{n-1} by choosing as oriented that class of frames for which $\boldsymbol{x}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}$ is an oriented frame of $\mathbb{R}^n$.

  Let Ω be the unique $(n-1)$-form on S^{n-1} which takes the value $+1$ on all oriented orthonormal frames $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}$.

The Antipodal Map of S^{n-1} (Cont'd)

- $A : S^{n-1} \to S^{n-1}$ is the restriction to S^{n-1} of a linear, in fact an orthogonal, map of $\mathbb{R}^n$.

  So its Jacobian is constant and just the map A itself.

  Thus, under A, the frame $\boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}$ at x goes to the frame $-\boldsymbol{e}_1, \ldots, -\boldsymbol{e}_{n-1}$ at $-x$.

  It is clear that this will be oriented according to our orientation of S^{n-1} if and only if n is even.

  In that case, $\boldsymbol{x}, \boldsymbol{e}_1, \ldots, \boldsymbol{e}_{n-1}$ and $-\boldsymbol{x}, -\boldsymbol{e}_1, \ldots, -\boldsymbol{e}_{n-1}$ are coherently oriented frames of $\mathbb{R}^n$.

  Therefore, $A^*\Omega = (-1)^n\Omega$ and, when n is odd, $\Omega = -A^*\Omega$.

The Antipodal Map of S^{n-1} (Cont'd)

- Suppose there is a C^∞ homotopy connecting A and the identity.

  Then $\Omega - A^*\Omega$ must be exact by a previous theorem.

  But, by Stokes's theorem, the integral over S^{n-1} of an exact form is zero.

  This means that, when n is odd,

$$2\int_{S^{n-1}} \Omega = \int_{S^{n-1}} (\Omega - A^*\Omega) = 0.$$

  However, the volume element is positive.

  So $\int_{S^{n-1}} \Omega = 0$ is impossible.

Non-Orientability of $P^n(\mathbb{R})$

Corollary

Real projective space $P^n(\mathbb{R})$ is not orientable when n is even.

- Suppose that $P^n(\mathbb{R})$ is orientable.
  We know that S^n is a (two-sheeted) covering manifold of $P^n(\mathbb{R})$.
  So $P^n(\mathbb{R})$ can be obtained from S^n as the orbit space of the group of two elements $\mathbb{Z}_2$ acting on S^n.
  This action is obtained by letting the generator of $\mathbb{Z}_2$ correspond to the antipodal map A.
  Suppose Ω is a nowhere vanishing n-norm on $P^n(\mathbb{R})$.
  Let $F : S^n \to P^n(\mathbb{R})$ be the covering map.
  Then $F^*\Omega = \Omega^*$ is a nowhere vanishing n-form on S^n.
  Moreover, since $F \circ A = F$, we see that $A^*\Omega^* = \Omega^*$.
  But this, as we have seen above, is not possible if $n + 1$ is odd.
  Thus, $P^n(\mathbb{R})$ is not orientable when n is even.

C^∞ Vector Fields on S^n

Theorem

If n is even, then there does not exist a C^∞-vector field X on S^n which is not zero at some point.

- We suppose that such a vector field exists.

  We show that this implies that the antipodal map A and the identity map I on S^n are C^∞ homotopic.

  Let X be a C^∞-vector field on S^n such that X is never zero.

  Then $\frac{X}{\|X\|}$ is a C^∞-vector field of unit vectors.

  So we may suppose to begin with that $\|X\| = 1$ on S^n.

  If x is a point of S^n, let X_x be the corresponding vector of the field.

  Treat $\mathbb{R}^{n+1}$ as a vector space and think of x as a radius vector.

  Then we have $(x, X_x) = 0$ for every x.

C^∞ Vector Fields on S^n (Cont'd)

- We define the homotopy $H : S^n \times I \to S^n$ by

$$H(x,t) = (\cos \pi t)x + (\sin \pi t)X_x.$$

  Then $H(x,t)$ is C^∞.

  Moreover, $\|H(x,t)\| \equiv 1$.

  So $H(x,t)$ defines a map of $S^n \to S^n$, for each t.

  Thus, $H(x,0) \equiv x$ and $H(x,1) \equiv -x$, as claimed.

  However, the existence of such a homotopy when n is even contradicts the previous proposition.

  Therefore, in this case no such vector field exists.

Remark

- Consider the case when n is odd.
- Consider the vector field X_x assigning to

$$x = (x^1, x^2, \ldots, x^n, x^{n+1}) \in S^n$$

the unit vector

$$X_x = x^2 \frac{\partial}{\partial x^1} - x^1 \frac{\partial}{\partial x^2} + \cdots + x^{n+1} \frac{\partial}{\partial x^n} - x^n \frac{\partial}{\partial x^{n+1}}$$

orthogonal to x.

- We have noted previously that X defines a nowhere vanishing field of tangent vectors to S^n.
- It follows that, in this case, A is homotopic to the identity.

Invariant k-Forms

- Suppose that G is a compact connected Lie group, e.g., $SO(n)$.
- Let $\theta : G \times M \to M$ be an action of G on a compact manifold M.
- θ_g denotes the diffeomorphism of M defined by

$$\theta_g(p) = \theta(g, p).$$

- A covariant tensor φ on M, in particular an exterior differential form, is said to be **invariant** if

$$\theta_g^* \varphi = \varphi, \quad \text{for each } g \in G.$$

- We know that, for every form φ,

$$d(\theta_g^* \varphi) = \theta_g^*(d\varphi).$$

- So if φ is invariant, $d\varphi$ is also.

Invariant k-Forms (Cont'd)

- Let $\widetilde{\bigwedge}^k(M)$ denote the subspace of $\bigwedge^k(M)$ which consists of all invariant k-forms.
- Then, as we have just seen,

$$d\left(\widetilde{\bigwedge}^k(M)\right) \subseteq \widetilde{\bigwedge}^{k+1}(M).$$

- We define the set of closed invariant forms of degree k

$$\widetilde{Z}^k(M) = \left\{\varphi \in \widetilde{\bigwedge}^k(M) : d\varphi = 0\right\}.$$

- We also define the set of "invariantly exact" forms of degree k

$$\widetilde{B}^k(M) = d\left(\widetilde{\bigwedge}^{k-1}(M)\right) \subseteq \widetilde{Z}^k(M).$$

Invariant de Rham Groups

Definition

The invariant de Rham groups of M, denoted by $\widetilde{H}^k(M)$, are defined by

$$\widetilde{H}^k(M) = \widetilde{Z}^k(M)/\widetilde{B}^k(M).$$

- We note that the natural inclusion i of $\widetilde{\bigwedge}^k(M)$ in $\bigwedge^k(M)$ takes:
  - $\widetilde{Z}^k(M)$ into $Z^k(M)$;
  - $\widetilde{B}^k(M)$ into $B^k(M)$.
- Hence, i induces a homomorphism

$$i_* : \widetilde{H}^k(M) \to H^k(M).$$

The Linear Operator $\mathscr{P}$

- In order to study the homomorphism

$$i_* : \widetilde{H}^k(M) \to H^k(M),$$

we define an $\mathbb{R}$-linear operator

$$\mathscr{P} : \bigwedge^k(M) \to \widetilde{\bigwedge}^k(M).$$

- Let

$$\varphi \in \bigwedge^k(M).$$

- Let Ω denote the bi-invariant volume element for which $\mathrm{vol}(G) = 1$.
- Define $\mathscr{P}\varphi$ by

$$\mathscr{P}\varphi(X_1, \ldots, X_k) = \int_G \theta_g^* \varphi(X_1, \ldots, X_k)\Omega.$$

Properties of $\mathscr{P}$

Lemma

$\mathscr{P}$ takes a k-form to an invariant k-form, that is,

$$\mathscr{P}\left(\bigwedge^k(M)\right) \subseteq \widetilde{\bigwedge}^k(M).$$

Moreover:

(i) If $\varphi \in \widetilde{\bigwedge}^k(M)$, then $\mathscr{P}\varphi = \varphi$;

(ii) $d\mathscr{P} = \mathscr{P}d$.

Properties of $\mathscr{P}$ (Cont'd)

- It is easy to check that $\mathscr{P}\varphi \in \bigwedge^k(M)$ and in fact is G-invariant.

$$\begin{aligned}
\theta_a^*\mathscr{P}\varphi(X_1,\ldots,X_k) &= \mathscr{P}\varphi(\theta_{a*}X_1,\ldots,\theta_{a*}X_k)\\
&= \int_G \theta_g^*\varphi(\theta_{a*}X_1,\ldots,\theta_{a*}X_k)\Omega\\
&= \int_G \theta_a^*[\theta_g^*\varphi(X_1,\ldots,X_k)]\Omega\\
&= \int_G \theta_{ga}^*\varphi(X_1,\ldots,X_k)\Omega\\
&= \int_G \theta_g^*\varphi(X_1,\ldots,X_k)\Omega.
\end{aligned}$$

The fact that $\mathscr{P}\varphi$ is C^∞ and Property (ii) are consequences of the Leibniz rule for differentiating under the integral sign.

Properties of $\mathscr{P}$ (Cont'd)

- If φ is G-invariant, then

$$\theta_g^* \varphi = \varphi, \quad \text{for all } g \in G.$$

More precisely at each $p \in M$,

$$\theta_g^* \varphi_{\theta(g,p)}(X_{1p}, \ldots, X_{kp}) = \varphi_p(X_{1p}, \ldots, X_{kp}).$$

From this it follows that

$$\begin{aligned} \mathscr{P}\varphi(X_1, \ldots, X_k) &= \int_G \theta_g^* \varphi(X_1, \ldots, X_k)\Omega \\ &= \varphi(X_1, \ldots, X_k) \int_G \Omega. \end{aligned}$$

But we have $\int_G \Omega = 1$.

So $\mathscr{P}\varphi = \varphi$ and Property (i) is established.

Property of i_*

Theorem

The homomorphism $i_* : \widetilde{H}^k(M) \to H^k(M)$ is an isomorphism into for each $k = 0, 1, \ldots, \dim M$.

- Suppose that $[\widetilde{\varphi}]$ is an element of $\widetilde{H}^k(M)$ and that $\widetilde{\varphi}$ is a closed invariant form on M belonging to the class $[\widetilde{\varphi}]$.
  To see that i_* is one-to-one, we show that, if $\widetilde{\varphi} = d\sigma$, $\sigma \in \bigwedge^{k-1}(M)$, then $\widetilde{\varphi}$ is the image under d of an element of $\widetilde{\bigwedge}^{k-1}(M)$.
  That is, that, if $\widetilde{\varphi}$ is exact, then it is "invariantly exact".
  This follows from the preceding lemma since $\mathscr{P}\sigma \in \widetilde{\bigwedge}^{k-1}(M)$ and
  $$\widetilde{\varphi} = \mathscr{P}\widetilde{\varphi} = \mathscr{P}d\sigma = d(\mathscr{P}\sigma).$$

  Remark: It is also true, but somewhat harder to prove directly, that i_* is onto, that is, $\widetilde{H}^k(M)$ is isomorphic to $H^k(M)$, for all k.

Bi-Invariant Tensors on Connected Lie Groups

Lemma

Let Φ_e be a covariant tensor of order r on $T_e(G)$, where G is a connected Lie group. If $\mathrm{Ad}g^*\Phi_e = \Phi_e$, that is, if Φ_e determines a bi-invariant tensor on G, then for any $X_1, \ldots, X_r, Z \in \mathfrak{g}$, we have

$$\sum_{i=1}^{r} \Phi(X_1, \ldots, [Z, X_i], \ldots, X_r) = 0.$$

- Let Φ be the bi-invariant covariant tensor on G determined by Φ_e.
  Suppose $Z \in \mathfrak{g}$ is a left-invariant vector field on G.
  We have seen that:
  - Z is complete;
  - The one-parameter group action $\theta : R \times G \to G$ which it determines is given by right translations by the elements of a uniquely determined one-parameter subgroup $g(t) = \exp tZ$ by the formula $\theta_t = R_{g(t)}$.

Bi-Invariant Tensors on Connected Lie Groups (Cont'd)

- We have previously established the following formula for C^∞-vector fields on a manifold,

$$[Z, X]_p = \lim_{t \to 0} \frac{1}{t}[\theta_{-t*} X_{\theta_t(p)} - X_p].$$

Suppose that $p = e$ and that X is a left-invariant vector field.

Then $[Z, X]$ is just the product in the Lie algebra $\mathfrak{g}$.

Identifying $\mathfrak{g}$ with $T_e(G)$, we may write

$$[Z, X] = \lim_{t \to 0} \frac{1}{t}[R_{g(-t)*} X_{g(t)} - X_e].$$

Bi-Invariant Tensors on Connected Lie Groups (Cont'd)

- By hypothesis, Φ is bi-invariant.

  So
  $$R^*_{g(-t)}\Phi - \Phi = 0.$$
  Thus, for any $X_1, \ldots, X_r \in \mathfrak{g}$,
  $$\Phi(R^*_{g(-t)}X_1, \ldots, R^*_{g(-t)}X_r) - \Phi(X_1, \ldots, X_r) = 0.$$
  Now we do the following:
  - Add and subtract
    $$\Phi(X_1, \ldots, X_{i-1}, R^*_{g(-t)}X_i, \ldots, R^*_{g(-t)}X_r), \quad i = 1, \ldots, r;$$
  - Then multiply by $\frac{1}{t}$;
  - Finally, let $t \to 0$.

  The outcome is the formula of the lemma.

Closedness of Bi-Invariant Forms on Lie Groups

Corollary

Every bi-invariant exterior form on a Lie group G is closed.

- Let ω be an exterior differential r-form.

  Suppose ω is left-invariant and $X_0, X_1, \ldots, X_r$ are left-invariant.

  Then

$$d\omega(X_0, \ldots, X_r) = \sum_{i=1}^{r} \omega(X_0, \ldots, [X_{i-1}, X_i], \ldots, X_r).$$

  We previously established this formula for $r = 2$.

  The method of proof in the general case is the same.

  The corollary is an immediate consequence.

Bi-Invariant r-Forms on Compact, Connected, Lie Groups

- Suppose that G acts on itself by both left and right translations.
- Let $G = M$ and $K = G \times G$, the direct product of Lie groups.
- Define $\theta : K \times M \to M$, for all $x \in M = G$ and $k = (g_1, g_2) \in K$, by

$$\theta(k, x) = g_1 x g_2 (= R_{g_2} \circ L_{g_1}(x)).$$

- Then the K-invariant forms $\widetilde{\varphi}$ on G are exactly the bi-invariant forms.

Bi-Invariant r-Forms and de Rham Groups

Corollary

Each bi-invariant r-form on a compact, connected, Lie group G determines a nonzero element of $H^r(G)$.

- By the corollary, each $\widetilde{\varphi} \in \widetilde{H}^r(G)$, that is, each bi-invariant r-form, is closed.

  We know that if it is exact, then it must be of the form $d\widetilde{\sigma}$, with $\widetilde{\sigma}$ bi-invariant.

  But then it is zero, by the corollary again, since $d\widetilde{\sigma} = 0$.

Example

- Consider any compact, connected, non-Abelian Lie group G.
- For example, $SO(n)$, the orthogonal matrix group (with elements of determinant $+1$), for $n \geq 3$.
- We claim that $H^3(G) \neq \{0\}$.
- We consider that the exterior three-form

$$\varphi(X, Y, Z) = ([X, Y], Z)$$

  on G, where (X, Y) denotes the bi-invariant inner product.
- We have:
  - $X, Y \in \mathfrak{g}$ implies that $[X, Y]$ is left-invariant;
  - $\mathrm{Ad}(g)$ is an automorphism of g.
- It follows readily that φ is bi-invariant.
- Further, we have:
  - $[X, Y] = -[Y, X]$;
  - (X, Y) is symmetric.
- These yield the alternating property of φ.

Example (Cont'd)

- By the preceding corollary, φ is closed and, if it is not zero, it determines an element of $H^3(G)$.
- Suppose that $\varphi = 0$.
- Then for all $X, Y, Z \in \mathfrak{g}$, we have

$$\varphi(X, Y, Z) = ([X, Y], Z) = 0.$$

- In particular, we have $([X, Y], [X, Y]) = 0$.
- It follows that

$$[X, Y] = 0, \quad \text{for all } X, Y \in \mathfrak{g}.$$

Example (Cont'd)

- This means, according to a previous section, that the one-parameter groups of G commute.
- It follows that there is a neighborhood U of e which consists of commuting elements.
- By the connectedness of G, the elements of U generate G.
- So G is commutative, contrary to assumption.
- This means that φ determines a nonvanishing element $[\varphi]$ of $H^3(G)$.

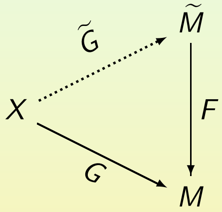
Subsection 9

Covering Spaces and the Fundamental Group

Covering Maps

- Suppose that M is a manifold.
- Let $\widetilde{M}$ be a covering manifold.
- Denote by $F : \widetilde{M} \to M$ the (C^∞) covering mapping.
- If X is a topological space and $G : X \to M$ a continuous mapping, then a continuous mapping $\widetilde{G} : X \to \widetilde{M}$ is said to **cover** G if

$$F \circ \widetilde{G} = G.$$



- We also say $\widetilde{G}$ is a **lift** of G.

  Example: If $f : I \to M$ is a path or loop, then $\widetilde{f} : I \to \widetilde{M}$ is a path which covers it, if $F \circ \widetilde{f}(t) = f(t)$, for $0 \leq t \leq 1$.
- If a covering $\widetilde{f}$ of a given path f exists at all, then it is uniquely determined by its value on a single point, say by $f(0)$.

Coverings of a Continuous Mapping

Lemma

If $F : \widetilde{M} \to M$ is a covering and X is a connected space, then two (continuous) mappings

$$\widetilde{G}_1, \widetilde{G}_2 : X \to \widetilde{M}$$

covering a continuous mapping $G : X \to M$ agree if they have the same value at a single point $x_0 \in X$.

- Let
$$A = \{x \in X : \widetilde{G}_1(x) = \widetilde{G}_2(x)\}.$$
Then A is closed by continuity of $\widetilde{G}_1$ and $\widetilde{G}_2$.
We show that A is also open.

Coverings of a Continuous Mapping (Cont'd)

- Let $x \in A$.

  Let U be a neighborhood of $\widetilde{G}_1(x) = \widetilde{G}_2(x)$, such that $F|_U$ is a diffeomorphism of U to M.

  Then G_1 and G_2 must agree on the open set

  $$V = \widetilde{G}_1^{-1}(U) \cap \widetilde{G}_2^{-1}(U).$$

  In fact, if $y \in V$, then, by hypothesis,

  $$F \circ \widetilde{G}_1(y) = F \circ \widetilde{G}_2(y).$$

  But $\widetilde{G}_1(y)$ and $\widetilde{G}_2(y)$ are in U.

  Moreover, on U, F is one-to-one.

  So $\widetilde{G}_1(y) = \widetilde{G}_2(y)$.

  Finally, since A is not empty and X is connected, $A = X$.

Coverings of Paths

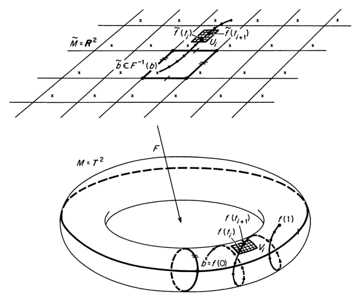
Theorem

Let $f : I \to M$ be a path in M with initial point $b = f(0)$.
Let $F : \widetilde{M} \to M$ be a covering and $\widetilde{b} \in F^{-1}(b)$.
Then there is a unique path $\widetilde{f}$ in $\widetilde{M}$ with initial point $\widetilde{f}(0) = \widetilde{b}$.

- Uniqueness is a consequence of the previous proposition.
  To prove existence, suppose

$$0 = t_0 < t_1 < \cdots < t_n = 1$$

is any partition of I such that for each i, $f([t_i, t_{i+1}])$ lies in an admissible neighborhood V_i with respect to the covering. The existence of such a partition follows from the compactness of I and the continuity of f.



Coverings of Paths (Cont'd)

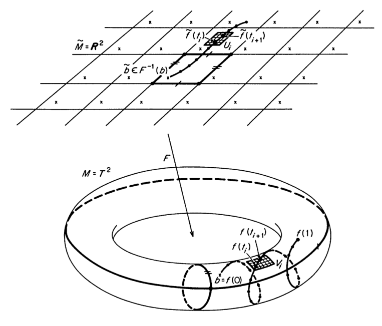
- We let $f(0) = b$ and let $\widetilde{b} \in \widetilde{M}$ denote a point over b, that is,

$$F(\widetilde{b}) = b.$$

Let U_1 be the unique connected component of $F^{-1}(V_1)$ containing $\widetilde{b}$.

We define $f(t)$, $0 \leq t \leq t_1$, by

$$\widetilde{f}(t) = (F|U_1)^{-1}(f(t)).$$



Then $\widetilde{f}(t_1) \in U_1 \cap U_2$, where U_2 is the unique component of $F^{-1}(V_2)$ containing $\widetilde{f}(t_1)$.

This allows us to define $f(t) = (F|_{U_2})^{-1}(f(t))$, for $t_1 \leq t \leq t_2$.

So we can determine $\widetilde{f}$ on $[t_0, t_2]$.

We can continue in this fashion to define $\widetilde{f}$ on all of I.

Lifting Homotopy Paths

Theorem

Let $f, g : I \to M$ be paths and $H : I \times I \to M$ a (relative) homotopy of f to g leaving endpoints fixed. Suppose $\widetilde{f}, \widetilde{g} : I \to \widetilde{M}$ cover f, g and have the same initial point. Then they have the same endpoint and there exists a unique homotopy $\widetilde{H} : I \times I \to \widetilde{M}$ of $\widetilde{f}$ to $\widetilde{g}$ covering H. Endpoints remain fixed for $\widetilde{H}$ also.

- We define $\widetilde{H} : I \times I \to \widetilde{M}$ using the previous theorem.
  For each fixed t,
  $$H_t(s) = H(s,t), \quad 0 \leq s \leq 1,$$
  is a path on M.
  It lifts to a unique path $\widetilde{H}_t(s)$ on $\widetilde{M}$ with
  $$\widetilde{H}_t(0) = \widetilde{f}(0) = \widetilde{g}(0),$$
  the common initial point of $\widetilde{f}$ and $\widetilde{g}$.

Lifting Homotopy Paths (Cont'd)

- We let
$$\widetilde{H}(s,t) = \widetilde{H}_t(s).$$
This defines $\widetilde{H} : I \times I \to \widetilde{M}$, with the property that $H = F \circ \widetilde{H}$.
But it is necessary to show that $\widetilde{H}$ is continuous.
Let $t_0 \in I$ be chosen.
Take a partition of the line $I \times \{t_0\}$ in $I \times I$ by
$$0 = s_0 < s_1 < \cdots < s_n = 1,$$
such that each interval $\{(s,t_0) : s_i \leq s \leq s_{i+1}\}$ is carried by H into an admissible neighborhood V_i on M.
Suppose $\widetilde{H}_i(s_i,t_0)$ have been defined at some stage.
This point of $\widetilde{M}$ determines unambiguously a component U_i of $F^{-1}(V_i)$ covering V_i and necessarily
$$\widetilde{H}_i(s,t_0) = (F|_{U_i})^{-1}(H(s,t_0)), \quad s_i \leq s \leq s_{i+1}.$$

Lifting Homotopy Paths (Cont'd)

- However, by the continuity of H, there exists $\delta > 0$, such that, for each $i = 0, 1, 2, \ldots, n-1$, the image $H(Q_i) \subseteq M$ of the cube $Q_i = \{(s,t) : s_i \leq s \leq s_{i+1}, t_0 - \delta \leq t \leq t_0 + \delta\}$ lies in V_i also.

  Hence, on all of Q_i,

  $$\widetilde{H}_t(s) = \widetilde{H}(s,t) = (\pi|_{U_i})^{-1}(H(s,t)).$$

  This shows that $\widetilde{H}$ is continuous on Q_i.

  This holds for each $i = 0, \ldots, n-1$.

  So $\widetilde{H}$ is continuous on a δ-strip $\{(s,t) : |t - t_0| < \delta\}$ around the segment $I \times \{t_0\} \subseteq I \times I$.

  But t_0 was arbitrarily chosen.

  Hence, $\widetilde{H}$ is continuous on $I \times I$.

Lifting Homotopy Paths (Cont'd)

- To complete the proof we notice that $\widetilde{H}$, being continuous, takes $\{1\} \times I$ into a connected set.

  Namely, the set of terminal points of $\widetilde{H}_t(1)$, $0 \leq t \leq 1$.

  We have
  $$F(\widetilde{H}(1,t)) = H(1,t) = f(1) = g(1).$$

  As this is a single point, the connected set lies in the discrete set $\pi^{-1}(f(1))$.

  It is, therefore, a single point, as claimed.

  We constructed $\widetilde{H}$ so that the initial points $\widetilde{H}_t(0)$, $0 \leq t \leq 1$, are all $\widetilde{f}(0)$.

  The existence (as constructed) and uniqueness of $\widetilde{H}$ show that this was the only possibility.

Lifting Homotopy Paths (Cont'd)

Corollary

If $\widetilde{b} \in \widetilde{M}$ lies over $b \in M$, then

$$F_* : \pi_1(\widetilde{M}, \widetilde{b}) \to \pi_1(M, b)$$

is an injective isomorphism.

- We know F_* is a homomorphism.

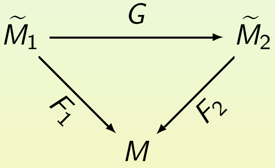
  Using the previous theorem with $\widetilde{f}, \widetilde{g}$ loops at $\widetilde{b}$, we see that

  $$F \circ \widetilde{f} \sim F \circ \widetilde{g} \quad \text{implies} \quad \widetilde{f} \sim \widetilde{g}.$$

  This is equivalent to F_* being injective.

Covering Isomorphisms

- Let $\widetilde{M}_1$ and $\widetilde{M}_2$ be coverings of a manifold M.
- Let the covering maps be $F_1 : \widetilde{M}_1 \to M$ and $F_2 : \widetilde{M}_2 \to M$.
- Then a homeomorphism $G : \widetilde{M}_1 \to \widetilde{M}_2$ such that $F_1 = F_2 \circ G$ and $F_2 = F_1 \circ G^{-1}$ is called an **isomorphism** of the coverings.



- In particular, an automorphism, that is, isomorphism, $G : \widetilde{M} \to \widetilde{M}$ is exactly a covering transformation, as given previously.
- Using admissible neighborhoods, it is apparent that the differentiability of F_1 and F_2 implies that of G and G^{-1}.
- We show that in a sense isomorphism classes of coverings of M are in one-to-one correspondence with subgroups of the fundamental group.

Subgroups and Covering Isomorphisms

Theorem

Let $F_1 : \widetilde{M}_1 \to M$ and $F_2 : \widetilde{M}_2 \to M$ be coverings of the same manifold M. Suppose that, for $b \in M$, $\widetilde{b}_1 \in \widetilde{M}_1$, $\widetilde{b}_2 \in \widetilde{M}_2$, with $F_1(\widetilde{b}_1) = b = F_2(\widetilde{b}_2)$, we have

$$F_{1*}\pi_1(\widetilde{M}_1, \widetilde{b}_1) = F_{2*}\pi_2(\widetilde{M}_2, \widetilde{b}_2).$$

Then there is exactly one isomorphism $G : \widetilde{M}_1 \to \widetilde{M}_2$ taking $\widetilde{b}_1$ to $\widetilde{b}_2$.

- Let $\widetilde{p} \in \widetilde{M}_1$.

  We define $G(\widetilde{p})$ as follows.

  Let $\widetilde{f}_1$ be a path such that $\widetilde{f}_1(0) = \widetilde{b}_1$ and $\widetilde{f}_1(1) = \widetilde{p}$.

  Then the path $f = F_1 \circ \widetilde{f}_1$ on M has a unique lifting to a path $\widetilde{f}_2$ on $\widetilde{M}_2$ covering f and with initial point $\widetilde{f}_2(0) = \widetilde{b}_2$.

  We define $G(\widetilde{p}) = \widetilde{f}_2(1)$.

Subgroups and Covering Isomorphisms (Cont'd)

- Of course we must show that:
  - The definition is independent of the path $\widetilde{f}_1$ chosen;
  - G is continuous.

  On the other hand, once these facts are proved, then, immediately from the definition, we get that:
  - $F_1 = F_2 \circ G$;
  - $G(\widetilde{b}_1) = \widetilde{b}_2$;
  - G is unique.

  This definition is natural.

  Let G have the properties required in the theorem.

  Then it must take $\widetilde{f}_1$ to a path $\widetilde{f}_2 \circ G$ on $\widetilde{M}_2$, such that:
  - $\widetilde{f}_2 \circ G$ covers $f = F_1 \circ \widetilde{f}_1$;
  - $\widetilde{f}_2 \circ G$ runs from $\widetilde{b}_2$ to $G(\widetilde{p})$.

Subgroups and Covering Isomorphisms (Cont'd)

- Now suppose that $\widetilde{f}_1$ and $\widetilde{g}_1$ are distinct paths on $\widetilde{M}_1$ from $\widetilde{b}_1$ to $\widetilde{p}$.
  Let $f = F_1 \circ \widetilde{f}_1$ and $g = F_1 \circ \widetilde{g}_1$.
  Consider the loop $f * g^{-1}$ with
  $$g^{-1}(s) = g(1-s), \quad 0 \leq s \leq 1.$$
  This loop determines an element $[f * g^{-1}]$ of $F_{1*}\pi_1(\widetilde{M}_1, \widetilde{b}_1)$.
  Hence, also the (same) element of $F_{2*}\pi_2(\widetilde{M}_2, \widetilde{b}_2)$.
  In view of the preceding corollary, if we lift this to a path from $\widetilde{b}_2$, its terminal point will necessarily be $\widetilde{b}_2$.
  So the lifted paths $\widetilde{f}_2$ and $\widetilde{g}_2$ on $\widetilde{M}_2$ beginning at $\widetilde{b}_2$ both end at the same point, that is,
  $$\widetilde{f}_2(1) = \widetilde{g}_2(1).$$
  It follows that, by using either $\widetilde{f}_1$ or $\widetilde{g}_1$, we obtain the same value for $G(\widetilde{p})$.

Subgroups and Covering Isomorphisms (Cont'd)

- By the preceding argument, there is a one-to-one correspondence between points of $\widetilde{M}_i$, $i = 1, 2$, and equivalence classes (under relative homotopy with endpoints fixed) of paths f on M issuing from b.

  Let $p \in M$.

  Let $[f]$ a homotopy class of paths from b to p.

  $[f]$ determines a point $\widetilde{p}_{[f]}$ of $\widetilde{M}_1$ which lies over p.

  Indeed, the class $[f]$ lifts to a class $[\widetilde{f}]$.

  All curves of $[\widetilde{f}]$ issue from the point $\widetilde{b}_1$.

  We have just seen that they all have as terminal point $\widetilde{p}_{[f]}$.

Subgroups and Covering Isomorphisms (Cont'd)

- Suppose we make this identification.

  So we may let $[f]$ denote $\widetilde{p}_{[f]}$.

  Then F_1 projects the class of paths $[f]$ to the common terminal point of its elements, that is, $F_1([f]) = f(1)$.

  Similarly for F_2, $\widetilde{M}_2$.

  The classes of loops at b correspond to the points over b.

  That is, the elements of $\pi_1(M, b)$ are in one-to-one correspondence with the points over b.

Subgroups and Covering Isomorphisms (Cont'd)

- It is clear that G is one-to-one onto.

  Moreover, G^{-1} is described in a symmetrical way to G.

  So G^{-1} is C^∞.

  Now let $\widetilde{p}_2 = G(\widetilde{p}_1) \in \widetilde{M}_2$.

  Let V, ψ be an admissible coordinate neighborhood of $p = F_i(\widetilde{p}_i)$ on M, $i = 1, 2$, such that:
  - $\psi(p) = 0$;
  - $\psi(V) = B_1^n(0) \subseteq \mathbb{R}^n$.

  Suppose f is a path from b to p on M which lifts to paths $\widetilde{f}_i$ joining $\widetilde{b}_i$ to $\widetilde{p}_i$ on $\widetilde{M}_i$, $i = 1, 2$.

  Then we see that this path may be used in the definition of G as described above.

Subgroups and Covering Isomorphisms (Cont'd)

- Let q be an arbitrary point in V.

  We have a radial path (in the local coordinates), say g_q, from p to q.

  Moreover, $f_q = f * g_q$ lifts to paths from $\widetilde{b}_i$ to $\widetilde{q}_i$ in the component $\widetilde{U}_i$ of $F_i^{-1}(V)$ containing $\widetilde{b}_i$, $i = 1, 2$.

  Thus, $G(q_1) = q_2$.

  This description is unique and valid for every $q \in V$.

  So $G : \widetilde{U}_1 \to \widetilde{U}_2$ is one-to-one and onto.

  In fact G may be described by

  $$G|_{\widetilde{U}_1} = (F_2|_{\widetilde{U}_2})^{-1} \circ (F_1|_{\widetilde{U}_1}).$$

  Thus, $G|_{\widetilde{U}_1}$ is a diffeomorphism.

  Since $\widetilde{M}_1$ is covered by open sets of this type, G is differentiable.

  This completes the proof.

Simply Connected Coverings

Corollary

If $F : \widetilde{M} \to M$ is a covering and $\widetilde{M}$ is simply connected, then the covering transformations are simply transitive on each set $F^{-1}(p)$.
If we fix $\widetilde{b} \in \widetilde{M}$ and $b \in M$ with $F(\widetilde{b}) = b$, then these choices determine a natural isomorphism

$$\Phi : \pi_1(M, b) \to \widetilde{\Gamma}$$

of the fundamental group of M onto the group of covering transformations.

- Suppose that $q_1, q_2 \in F^{-1}(p)$.
  We apply the preceding theorem with $M_1 = \widetilde{M}$, $M_2 = \widetilde{M}$.
  Note that because $\widetilde{M}$ is simply connected, $\pi_1(\widetilde{M}, q_i) = \{1\}$, $i = 1, 2$.
  Hence, $F_*(\pi_1(\widetilde{M}, q_1)) = \{1\} = F_*(\pi_1(\widetilde{M}, q_2))$.
  We get a covering transformation $G : \widetilde{M} \to \widetilde{M}$, with $G(q_1) = q_2$.
  By a previous theorem, the group $\widetilde{\Gamma}$ of covering transformations must be simply transitive on $F^{-1}(p)$, for each $p \in M$.

Simply Connected Coverings (Cont'd)

- We have fixed $b \in M$ and $\widetilde{b} \in \pi^{-1}(b)$.
  We may establish an isomorphism of $\pi_1(M, b)$ and $\widetilde{\Gamma}$ as follows.
  Let $[g] \in \pi_1(M, b)$.
  Let $\widetilde{g}$ be the lift of $g \in [f]$ to $\widetilde{M}$ determined by $\widetilde{g}(0) = \widetilde{b}$.
  We have seen earlier that any two curves $\widetilde{g}_1, \widetilde{g}_2$ which are lifts of curves of homotopic curves, in particular two loops of $[g]$, with $\widetilde{g}_1(0) = \widetilde{b} = \widetilde{g}_2(0)$, must have the same terminal point $\widetilde{b}_1$ and must be homotopic (with endpoints fixed).
  Since g is a loop, $F(\widetilde{b}) = b = F(\widetilde{b}_1)$.
  We let $\Phi[g] \in \widetilde{\Gamma}$ be the covering transformation
  $$\widetilde{b} \mapsto \widetilde{b}_1 = \widetilde{g}(1).$$
  This defines $\Phi : \pi_1(M, b) \to \widetilde{\Gamma}$.
  We can check that Φ is a homomorphism using the arguments of the preceding theorem.

Simply Connected Coverings (Cont'd)

- We show that Φ is one-to-one.

  If $\Phi[g] = 1$, then $\widetilde{g}(0) = \widetilde{b} = \widetilde{g}(1)$.

  So $\widetilde{g}$ determines an element of $\pi_1(\widetilde{M}, \widetilde{b})$.

  This group contains only the identity.

  So $\widetilde{g} \sim e_{\widetilde{b}}$ by a homotopy $\widetilde{H}$.

  Then $H = F \circ \widetilde{H}$ is a homotopy of g to e_b.

  It follows that $[g] = 1$.

  Hence, Φ is one-to-one.

Simply Connected Coverings (Cont'd)

- We show that Φ is onto.
  Let $G_1 \in \widetilde{\Gamma}$.
  Let $\widetilde{b}_1 = G_1(\widetilde{b})$.
  There is a path $\widetilde{g}$ from $\widetilde{b}$ to $\widetilde{b}_1$.
  We have $F(\widetilde{b}) = F[G_1(\widetilde{b})]$.
  So, by definition of covering transformation, $g = F \circ \widetilde{g}$ is a loop at b.
  It determines $[g] \in \pi_1(M, b)$.
  But the covering transformation $G = \Phi([f])$ agrees with G_1 on $\widetilde{b}$,
  $$G_1(\widetilde{b}) = \widetilde{b}_1 = G(\widetilde{b}).$$
  So we must have $G = G_1$, by a previous lemma.

Subgroups of Fundamental Group and Coverings

Theorem

Let M be a connected manifold and b a fixed point of M.
Then, corresponding to each subgroup $H \subseteq \pi_1(M, b)$, there is a covering $F : \widetilde{M} \to M$, such that, for some $\widetilde{b} \in F^{-1}(b)$, we have

$$F_* \pi_1(M, b) = H.$$

F and M are unique to within isomorphism.

- The uniqueness is just the previous theorem.

  Its proof also indicates how the space must be constructed.

  The points of $\widetilde{M}$ will consist of equivalence classes of paths from b.

  Two such paths f, g are equivalent if and only if:
  - $f(1) = g(1)$;
  - $[f * g^{-1}] \in H$, where g^{-1} denotes the path $g^{-1}(s) = g(1-s)$, $0 \le s \le 1$.

Subgroups of Fundamental Group and Coverings (Cont'd)

- Since H is a subgroup, the preceding relation is an equivalence.
  We denote it by $f \approx g$.
  We denote by $\{f\}$ the equivalence class of f (or point of $\widetilde{M}$).
  The projection map $F : \widetilde{M} \to M$ is defined by

$$F(\{f\}) = f(1), \quad \text{for any } f \in \{f\}.$$

  Let $\{f\} \in \widetilde{M}$ and $p = f(1)$.
  Let V, ψ be a coordinate neighborhood of p on M, with:
  - $\psi(p) = 0$;
  - $\psi(V) = B_1^n(0)$, the open n-ball.

  For each $q \in V$, there is a unique path g_q from p to q corresponding to a radial line in $\psi(V)$.
  Then $q \to \{f * g_q\}$ defines a map $\theta_f : V \to \widetilde{M}$.
  For all q in V,

$$F \circ \theta_f(q) = F\{f \circ g_q\} = f \circ g_q(1) = q.$$

Subgroups of Fundamental Group and Coverings (Cont'd)

- Suppose h is a path from b to q also.

  Assume that $h \not\approx f$, that is, $\{h \circ f^{-1}\} \notin H$.

  Then it is easy to see that

$$\theta_f(V) \cap \theta_h(V) = \emptyset.$$

  Indeed, assume, for some $q \in V$, we have $\{f * g_q\} = \{h * g_q\}$.

  But hen $[f * g_q * (h * g_q)^{-1}] = [f * h^{-1}]$ is an element of H.

  This contradicts the assumption.

  We may now check that the sets $\theta_f(V)$, with coordinate maps $\psi \circ F$, define a manifold structure on $\widetilde{M}$.

  Moreover, his structure makes $F : \widetilde{M} \to M$ a covering, with $\{V, \psi\}$ as admissible neighborhoods.

Subgroups of Fundamental Group and Coverings (Cont'd)

- Finally, we must establish that $F_*(\pi_1(\widetilde{M}, \widetilde{b})) = H$, where $\widetilde{b} = \{e_b\}$, the point of $\widetilde{M}$ determined by the constant path at b.

  Suppose that f(t), $0 \leq t \leq 1$, is a loop at b with $[f] \in H$.

  Then $f(0) = f(1) = b$.

  We define a one-parameter family f_t of paths from b by

  $$f_t(s) = f(st), \quad 0 \leq s, t \leq 1.$$

  Let

  $$\widetilde{f}(t) = \{f_t(s)\}.$$

Subgroups of Fundamental Group and Coverings (Cont'd)

- Then

$$\widetilde{f}(t), \quad 0 \leq t \leq 1,$$

is a path on $\widetilde{M}$, with

$$F(\widetilde{f}(t)) = f_t(1) = f(t).$$

Hence, $\widetilde{f}$ covers f and is a loop at $\widetilde{b}$.

We can check, using methods similar to those used above, that this actually determines an isomorphism F_* of $\pi_1(\widetilde{M}, \widetilde{b})$ onto H.

This completes the proof.

Connected Manifold As Orbit Space of Fundamental Group

- If we take $H = \{1\}$ we have a very important corollary.

Corollary

Every connected manifold M has a simply connected covering which is unique to within isomorphism.
Choice of $\widetilde{b} \in F^{-1}(b)$, for $b \in M$, determines an isomorphism of $\pi_1(M, b)$ onto $\widetilde{\Gamma}$ the group of covering transformations.
Then $\widetilde{M}/\widetilde{\Gamma}$ is diffeomorphic to M, that is, M is the orbit space of its fundamental group acting properly discontinuously on its universal covering $\widetilde{M}$.