Introduction to Differential Geometry

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Subsection 1

The Geometry of Surfaces in E^3

Local Coordinates

- Suppose that M is an imbedded surface.
- We consider only a portion of M covered by a single coordinate neighborhood U, φ.
- Moreover, we assume that $W = \varphi(U)$ is a connected open subset of $\mathbb{R}^2$, the uv-plane.
- Thus, $p \in U \subseteq M$ has coordinates $(u(p), v(p)) = \varphi(p)$.
- Take the Euclidean three-dimensional space with a fixed Cartesian coordinate system, i.e., identify $\boldsymbol{E}^3$ with $\mathbb{R}^3$.
- The imbedding or parameter mapping $\varphi^{-1} : W \to U \subseteq \mathbb{R}^3$ is given by

$$x^i = f^i(u, v), \quad i = 1, 2, 3.$$

- Let the coordinate frames be

$$E_1 = \varphi_*^{-1}\left(\frac{\partial}{\partial u}\right) \quad \text{and} \quad E_2 = \varphi_*^{-1}\left(\frac{\partial}{\partial v}\right).$$

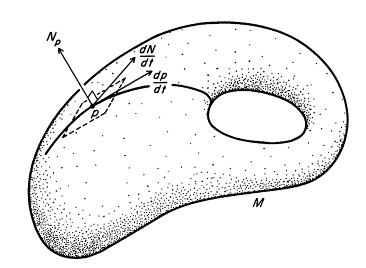
Unit Normal Vector

- Suppose that M is orientable and oriented with U, φ giving the orientation.
- Orientation is an important condition on M, since we are then able to define, without ambiguity, the unit normal vector field N to M.
- It is the unique unit vector at each $p \in M$ which is:
  - Orthogonal to $T_p(M) \subseteq T_p(\mathbb{R}^3)$;
  - So chosen that E_1, E_2, N form a frame at p with the same orientation as $\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}$, the standard orthonormal frame of $\mathbb{R}^3$.
- Length and orthogonality are defined in terms of the inner product (X, Y) of Euclidean space.
- The inner product induces a Riemannian metric on M by restriction.

The Derivative of the Normal Vector

- Let $p(t)$ be any differentiable curve on M with:
  - $p(0) = p$;
  - $\dot{p}(0) = X_p \in T_p(M)$.
- Restricting N to $p(t)$ gives a vector field $N(t) = N_{p(t)}$ along $p(t)$.
- This may be differentiated in $\mathbb{R}^3$ as a vector field along a space curve, giving a derivative $\frac{dN}{dt}$, which is itself a vector field along $p(t)$.
- Applying the inner product rule and using $(N, N) = 1$, we have

$$0 = \frac{d}{dt}(N, N) = 2\left(\frac{dN}{dt}, N\right).$$



- This means that $\frac{dN}{dt}$ is orthogonal to $N(t)$ at each point $p(t)$.
- Hence, $\frac{dN}{dt}$ is tangent to M, i.e., $\frac{dN}{dt} \in T_{p(t)}(M)$.

Independence on Curve

Theorem

The vector $\frac{dN}{dt}|_{t=0}$ depends only on X_p and not on the curve $p(t)$ chosen. Let

$$S(X_p) = -\frac{dN}{dt}|_{t=0}.$$

Then $X_p \to S(X_p)$ is a linear map of $T_p(M) \to T_p(M)$.

- Consider an arbitrary element of $I_p(M)$

$$X_p = aE_{1p} + bE_{2p}.$$

It is written as a linear combination of the coordinate frame E_{1p}, E_{2p} of the coordinate neighborhood U, φ containing p.

Let

$$p(t) = (f^1(u(t), v(t)), f^2(u(t), v(t)), f^3(u(t), v(t)))$$

be any differentiable curve with $p(0) = p$ and $\dot{p}(0) = X_p$.

Independence on Curve (Cont'd)

- Suppose $p(0)$ has coordinates $u_0 = u(0)$ and $v_0 = v(0)$.

  Since $\dot{p}(0) = X_p$, we have $\dot{p}(0) = aE_{1p} + bE_{2p}$, that is:
  - $\dot{u}(0) = a$;
  - $\dot{v}(0) = b$.

  We denote by $n^i(u, v)$ the components of N on U relative to the standard frames in $\mathbb{R}^3$,

$$N = n^1(u, v)\frac{\partial}{\partial x^1} + n^2(u, v)\frac{\partial}{\partial x^2} + n^3(u, v)\frac{\partial}{\partial x^3}.$$

  Then, along the curve

$$N(t) = \sum_{i=1}^{3} n^i(u(t), v(t))\frac{\partial}{\partial x^i}.$$

Independence on Curve (Cont'd)

- Moreover,

$$\begin{aligned}(\frac{dN}{dt})_0 &= \sum_{i=1}^3 \left[(\frac{\partial n^i}{\partial u})_{\varphi(p)}\dot{u}(0) + (\frac{\partial n^i}{\partial v})_{\varphi(p)}\dot{v}(0)\right]\frac{\partial}{\partial x^i} \\ &= a\left(\sum_{i=1}^3 (\frac{\partial n^i}{\partial u})_{\varphi(p)}\frac{\partial}{\partial x^i}\right) + b\left(\sum_{i=1}^3 (\frac{\partial n^i}{\partial v})_{\varphi(p)}\frac{\partial}{\partial x^i}\right).\end{aligned}$$

  This shows that $S(X_p)$ depends linearly on the components of X_p.

  Now $\frac{dN}{dt}|_{t=0}$ lies in $T_p(M)$.

  So $S : T_p(M) \to T_p(M)$ is a linear map.

  Moreover the only values that appear in the formula are:
  - $(u(0), v(0))$, the coordinates of p;
  - $\dot{u}(0), \dot{v}(0)$, the components of $\dot{p}(0) = X_p$.

  Thus, $(\frac{dN}{dt})_0$ depends on p and X_p and not on the curve used in the calculation.

Remark

- The linear map $S : T_p(M) \to T_p(M)$, given at each $p \in M$, is independent of:
  - The choice of coordinate system U, φ on M;
  - The Cartesian coordinate system used in Euclidean space.
- This is because N is defined using only the orientations of M and Euclidean space and the inner product of the Euclidean space.
- The differentiation depends only on the existence of parallel orthonormal frames in Euclidean space.
- Thus N, $\frac{dN}{dt}$ and S are independent of coordinates and involve only the geometry of M as an imbedded surface in Euclidean space.
- The operator S has been called the **shape operator**.

Example

- Suppose M is the xy-plane.

  Then $N = E_3$, a constant vector.

  So $S(X_p) = 0$.
- Suppose M is a sphere of radius R.

  The unit normal N at $(x^1, x^2, x^3) \in M$ is given by

  $$N = \frac{x^1}{R}\frac{\partial}{\partial x^1} + \frac{x^2}{R}\frac{\partial}{\partial x^2} + \frac{x^3}{R}\frac{\partial}{\partial x^3}.$$

  Suppose we move in any direction tangent to the sphere along a great circle curve, parametrized by arclength so that $\|X_p\| = 1$.

  Then

  $$S(X_p) = -\frac{dN}{ds} = \frac{1}{R}X_p.$$

Bilinear Forms

- Suppose M is a C^∞ submanifold.
- Recall the linear map $S : T_p(M) \to T_p(M)$, more accurately S_p, which we have determined at each $p \in M$.
- We may use S to define a C^∞ covariant tensor field on M.
- Let $S : \boldsymbol{V} \to \boldsymbol{V}$ be a linear operator on a vector space $\boldsymbol{V}$ with inner product (X, Y).
- Then the formula

  $$\Psi(X, Y) = (S(X), Y)$$

  defines a bilinear form, or covariant tensor of order 2, on $\boldsymbol{V}$.
- The form Ψ is symmetric if and only if

  $$(S(X), Y) = (X, S(Y))$$

  holds for all $X, Y \in \boldsymbol{V}$.
- If Ψ is symmetric, S is called **symmetric** or **self-adjoint**.

Properties of S

Theorem

$S(X)$ is a symmetric operator on the tangent space $T_p(M)$ for each $p \in M$ and $\Psi(X,Y)$ is a symmetric covariant tensor of order 2. The components of S and Ψ are C^∞ if M is a C^∞ submanifold.

- To prove the statements we compute the components of $\Psi(X,Y)$. Let U, φ be a coordinate neighborhood. Let $\varphi^{-1}: W \to U \subseteq M$ be the corresponding parametrization. Below we compute the components of $\Psi(X,Y)$ relative to the coordinate frames

$$E_1 = \varphi_*^{-1}\left(\frac{\partial}{\partial u}\right) \quad \text{and} \quad E_2 = \varphi_*^{-1}\left(\frac{\partial}{\partial v}\right).$$

We use $\frac{\partial N}{\partial u}$ and $\frac{\partial N}{\partial v}$ to denote the derivatives of N along the coordinate curves on M obtained by holding one coordinate fixed and allowing the other to vary (as parameter along the curve).

Properties of S (Cont'd)

- We have
$$\Psi(E_1,E_2)=(S(E_1),E_1)=-(\tfrac{\partial N}{\partial u},E_1),$$
$$\Psi(E_1,E_2)=(S(E_1),E_2)=-(\tfrac{\partial N}{\partial u},E_2),$$
$$\Psi(E_2,E_1)=(S(E_2),E_1)=-(\tfrac{\partial N}{\partial v},E_1),$$
$$\Psi(E_2,E_2)=(S(E_2),E_2)=-(\tfrac{\partial N}{\partial v},E_2).$$
Denote by $X=X(u,v)$ the position vector from 0 to $\varphi^{-1}(u,v)$,
$$X=f^1(u,v)\frac{\partial}{\partial x^1}+f^2(u,v)\frac{\partial}{\partial x^2}+f^3(u,v)\frac{\partial}{\partial x^3}.$$
Then $X_u=E_1$ and $X_v=E_2$ are just the vectors whose components are the corresponding derivatives of the components of X with respect to u and v.
That is,
$$X_u=\frac{\partial X}{\partial u}=E_1 \quad \text{and} \quad X_v=\frac{\partial X}{\partial v}=E_2.$$

Properties of S (Cont'd)

- Recall that $(N, X_u) = 0 = (N, X_v)$.

  Differentiate to obtain

$$\begin{array}{rcl} -(\frac{\partial N}{\partial u}, X_u) & = & (N, X_{uu}) = \sum n_i \frac{\partial^2 f^i}{\partial u^2}, \\ -(\frac{\partial N}{\partial v}, X_u) & = & (N, X_{vu}) = \sum n_i \frac{\partial^2 f^i}{\partial v \partial u} = (N, X_{uv}) = -(\frac{\partial N}{\partial u}, X_v), \\ -(\frac{\partial N}{\partial v}, X_v) & = & (N, X_{vv}) = \sum n_i \frac{\partial^2 f^i}{\partial v^2}. \end{array}$$

  So the components of Ψ, and hence of S, are C^∞ if M is.

  The second relation shows that $\Psi(X, Y) = \Psi(Y, X)$.

  So the tensor Ψ is symmetric.

Second Fundamental Form

- Consider the 2×2 matrix of the components of the symmetric tensor Ψ,
$$(\ell_{ij}) = (\Psi(E_i, E_j)).$$
- It will often be written
$$\begin{pmatrix} \ell & m \\ m & n \end{pmatrix},$$
where:
  - $\ell = (N, X_{uu}) = \ell_{11}$;
  - $m = (N, X_{uv}) = \ell_{12} = \ell_{21}$;
  - $n = (N, X_{vv}) = \ell_{22}$.
- The bilinear form $\Psi(X, Y)$ is called the **second fundamental form** of the surface M.

First Fundamental Form

- The inner product (X, Y) is called the **first fundamental form**.
- Recall that, in the general Riemannian case, the components of the Riemannian metric (X, Y) are denoted by g_{ij}.
- However, in the classical case of a surface M in Euclidean space, one often uses E, F, G.
- Thus,

$$\begin{array}{rcl} g_{11} & = & E = (X_u, X_u), \\ g_{12} & = & F = (X_u, X_v) = (X_v, X_u) = F = g_{21}, \\ g_{22} & = & G = (X_v, X_v). \end{array}$$

Remark: It is a classical theorem of differential geometry (which we shall not prove) that two surfaces M_1 and M_2 in $\mathbb{R}^3$ are congruent if and only if they correspond in such fashion that, at corresponding points, both fundamental forms agree.

Characteristic Values of S

Theorem

At each $p \in M$, the characteristic values of the linear transformation S are real numbers k_1 and k_2, $k_1 \geq k_2$.

- If $k_1 \neq k_2$, then the characteristic vectors belonging to them are orthogonal.
- If $k_1 = k_2 = k$ at p, then $S(X_p) = kX_p$, for every X_p in $T_p(M)$.

The numbers k_1 and k_2 are the maximum and minimum values of

$$\Psi(X_p, X_p) = (S(X_p), X_p),$$

over all unit vectors $X_p \in T_p(M)$.

- These statements are taken directly from theorems of linear algebra.
- Here we only sketch a proof for the case $k_1 \neq k_2$.

Characteristic Values of S (Cont'd)

- All vectors are elements of $T_p(M)$, p fixed.

  Suppose $k_1 > k_2$ are the characteristic values.

  They are real, since S is self-adjoint.

  Let F_1, F_2 be unit characteristic vectors corresponding to k_1, k_2.

  We have

$$k_1(F_1, F_2) = (S(F_1), F_2) = (F_1, S(F_2)) = k_2(F_1, F_2).$$

  This implies that, when $k_1 \neq k_2$,

$$(F_1, F_2) = 0.$$

  Replacing F_2 by $-F_2$ if necessary, we may suppose F_1, F_2 is an orthonormal basis with the same orientation as $T_p(M)$.

Characteristic Values of S (Cont'd)

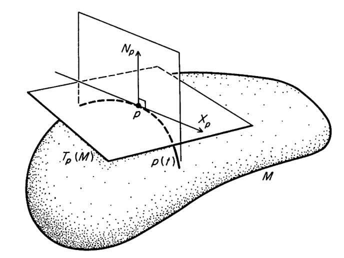
- Next we show that k_1 and k_2 are the maximum and minimum values of $(S(X_p), X_p)$, for unit vectors X_p.
  Any unit vector $X_p \in T_p(M)$ may be written
  $$X_p = \cos\theta \widetilde{F}_1 + \sin\theta \widetilde{F}_2.$$
  Let $k(\theta)$ denote $(S(X_p), X_p) = \Psi(X_p, X_p)$.
  F_1, F_2 is an oriented, orthonormal frame.
  So we have
  $$k(\theta) = k_1 \cos^2\theta + k_2 \sin^2\theta.$$
  Differentiating gives
  $$\frac{dk}{d\theta} = 2(k_2 - k_1)\sin\theta\cos\theta.$$
  Hence, the extrema of $k(\theta)$ occur when $\theta = 0$, $\frac{1}{2}\pi$, π or $\frac{3}{2}\pi$.
  In other words, when $X_p = \pm F_1$ or $\pm F_2$.
  So k_1 and k_2 are maximum and minimum values of $(S(X_p), X_p)$.

Umbilical and Planar Points

- The values k_1 and k_2 are the maximum and minimum of the expression
$$\frac{\Psi(X_p, X_p)}{(X_p, X_p)},$$
over all $X_p \neq 0$ in $T_p(M)$.
- The points p at which $k_1 = k_2$ are called:
  - **Umbilical points** of M, if $k_1 \neq 0$;
  - **Planar points** of M, otherwise.
- Note that a sphere of radius R consists entirely of umbilical points with
$$k_1 = \frac{1}{R} = k_2.$$
- Similarly, if M is a plane, every point is planar with $k_1 = 0 = k_2$.

Geometrical Interpretation

- We shall now interpret $k(\theta) = \Psi(X_p, X_p)$ geometrically.
- Let p be a point of M and X_p a unit tangent vector at p.
- X_p and N_p determine a plane on which we may take:
  - p as origin;
  - X_p, N_p as unit vectors along the axes (in that order).
- This gives a coordinate system and orientation on the plane.



- The plane intersects M along a curve which, of course, lies on M and on the plane, and passes through p.
- It is called the **normal section** at p determined by X_p.
- There is clearly such a curve for each X_p.

Geometrical Interpretation (Cont'd)

- The vector N_p is the normal to the curve at p.
- Moreover, X_p is the unit tangent vector to the curve at p.
- Write the curve as $p(t)$, with $p(0) = p$ and arclength as parameter.
- We have $\dot{p}(t) = \frac{dp}{dt}$, a unit vector for every t.
- So we get $\dot{p}(0) = X_p$.
- Differentiate $(N, \frac{dp}{dt}) = 0$ along the curve.
- We find that

$$\left(\frac{dN}{dt}, \frac{dp}{dt}\right) = -\left(N, \frac{d^2p}{dt^2}\right) = -\widetilde{k},$$

  the curvature of the plane curve $p(t)$, as defined previously.
- In particular, at $p = p(0)$,

$$\left(\frac{dN}{dt}, X_p\right) = -(S(X_p), X_p).$$

Normal Curvature, Principal Curvatures and Directions

- Let again, as above,

$$X_p = \cos\theta F_1 + \sin\theta F_2.$$

- We find that $k(\theta) = \widetilde{k}$ is the curvature of the normal section determined by X_p.
- For this reason $k(\theta)$ is called the **normal curvature** (of the section determined by X_p).
- k_1 and k_2, the maximum and minimum of $k(\theta)$, are called **principal curvatures** at p.
- The corresponding unit vectors F_{1p}, F_{2p} (chosen to conform to the orientation) are called **principal directions** at p.

Using Coordinates

- To study the surface at p we choose an xyz-coordinate system in Euclidean space so that:
  - The origin is at p;
  - $T_p(M)$ is the xy-plane;
  - The principal directions F_{1p}, F_{2p} and unit normal N_p at p are $\frac{\partial}{\partial x}$, $\frac{\partial}{\partial y}$, $\frac{\partial}{\partial z}$, unit vectors on the x-, y-, z-axes, respectively.
- Let $x = u$, $y = v$ and
$$z = f(u, v)$$
be the (parametric) equation of the surface.
- Then we may identify the xy- and uv-planes.
- Moreover, we may assume that the parameter mapping φ^{-1} takes some open set W on the xy-plane onto an open set U on M.
- The conditions then imply:
  - $f(0, 0) = 0$;
  - $f_x(0, 0) = 0 = f_y(0, 0)$.

Using Coordinates (Cont'd)

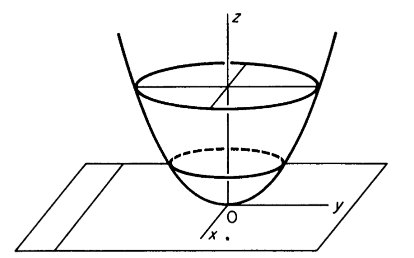
- If we compute the components of the first fundamental form at p, we obtain $E = 1 = G$ and $F = 0$.
- For the second fundamental form, recall that
$$\varphi^{-1} : (x, y) \to (x, y, f(x, y))$$
is the parametric representation of M.
- Thus, at p,
$$\begin{array}{rcl} \ell & = & (\frac{\partial}{\partial z}, f_{xx}\frac{\partial}{\partial z}) = f_{xx}, \\ m & = & (\frac{\partial}{\partial z}, f_{xy}\frac{\partial}{\partial z}) = f_{xy}, \\ n & = & (\frac{\partial}{\partial z}, f_{yy}\frac{\partial}{\partial z}) = f_{yy}. \end{array}$$
- Now the fact that we have chosen coordinate axes so that $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ are principal directions tells us that $m = 0$ and $\ell = k_1$, $n = k_2$.
- Thus, at $x = 0$, $y = 0$, we have
$$k(\theta) = f_{xx} \cos^2 \theta + f_{yy} \sin^2 \theta.$$

Using Coordinates (Cont'd)

- Let $f(x, y)$ be expanded in Taylor series at $(0, 0)$.
- Then
$$z = f(x, y) = f_{xx}(0, 0)x^2 + f_{yy}(0, 0)y^2 + R_2,$$
where R_2 contains terms of higher order.
- Let $f_{xx}(0, 0) = a$ and $f_{yy}(0, 0) = b$.
- Then we see that the normal sections of $z = ax^2 + by^2$ have the same sectional curvatures at p as does the given surface.
- Therefore the quadric surfaces must give typical examples.

Example

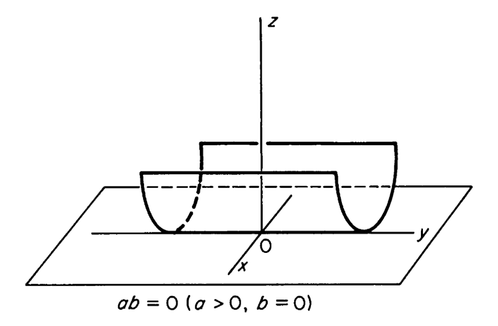
- $z = ax^2 + by^2$, $ab > 0$.



- This is an elliptic paraboloid.
- The principal curvatures are a and b.
  - If both are positive, it lies above the xy-plane;
  - If both are negative, it lies below.
- In either case when k_1 and k_2 have the same sign, the surface is (locally) on one side of $T_p(M)$.

Example

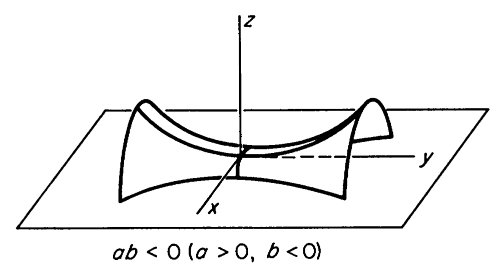
- $z = ax^2 + by^2$, $ab = 0$.



- If both are zero, we have the xy-plane as our surface;
- If one, say $b = 0$, then we have a parabolic cylinder which is above the xy-plane, if $a > 0$.

Example

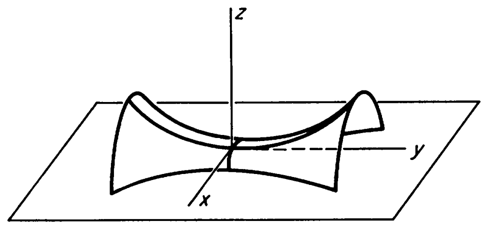
- $z = ax^2 + by^2$, $ab < 0$.



- In this case we have a hyperbolic paraboloid or saddle surface with the *xy*-plane tangent at the saddle.

Example

- E.g., consider $a = 1$ and $b = -1$.



$ab < 0$ $(a > 0,\ b < 0)$

- Then

$$k(\theta) = \cos^2\theta - \sin^2\theta.$$

- Hence $k(\theta)$ varies from $+1$ to -1 and is zero at $\pm\frac{\pi}{4}$, $\pm\frac{3\pi}{4}$.
- When $k_1 > 0$ and $k_2 < 0$, then the surface must have points (locally) on both sides of $T_p(M)$.

Subsection 2

The Gaussian and Mean Curvatures of a Surface

Gaussian and Mean Curvature

- The negative of the trace and determinant of any matrix of the linear transformation S are the coefficients of the characteristic polynomial of S and are important invariants.
- The determinant is the product of the characteristic values,

$$K = k_1 k_2.$$

- It is called the **Gaussian curvature** of the surface.
- The trace is the sum of the characteristic values $k_1 + k_2$.
- The quantity

$$H = \frac{1}{2}(k_1 + k_2)$$

  is called the **mean curvature** of the surface.
- We will compute these quantities directly from the-components of the fundamental forms, using any parametrization of the surface.

Computing the Gaussian and Mean Curvatures

Theorem

We have

$$K = \frac{\ell n - m^2}{EG - F^2} \quad \text{and} \quad H = \frac{1}{2}\frac{G\ell - 2Fm + En}{EG - F^2}.$$

- Consider the parametrization of M near p, i.e., on the coordinate neighborhood U, φ.

  Let $E_1 = X_u$ and $E_2 = X_v$ be the corresponding coordinate frames.

  Suppose the components of the operator S, in terms of E_1, E_2, are

  $$S(X_u) = aX_u + bX_v \quad \text{and} \quad S(X_v) = cX_u + dX_v.$$

  We may write

  $$K = \begin{vmatrix} a & b \\ c & d \end{vmatrix} \quad \text{and} \quad 2H = a + d.$$

Computing the Gaussian and Mean Curvatures (Cont'd)

- Let $\times$ be the cross product of vectors in 3-dimensional Euclidean space.

  In terms of X_u, X_v we have

$$\begin{aligned} KN &= K(X_u \times X_v) = S(X_u) \times S(X_v); \\ 2HN &= 2H(X_u \times X_v) = S(X_u) \times X_v + X_u \times S(X_v). \end{aligned}$$

  Note that

$$(X_u \times X_v, X_u \times X_v) = \|X_u \times X_v\|^2 = EG - F^2.$$

  For any vectors X, Y, U, V of $\mathbb{R}^3$, we have the Lagrange identities

$$((X \times Y), (U \times V)) = \begin{vmatrix} (X, U) & (X, V) \\ (Y, U) & (Y, V) \end{vmatrix}.$$

Computing the Gaussian and Mean Curvatures (Cont'd)

- We obtain the formula for K by taking inner products on both sides of the first equation with $X_u \times X_v$.

$$\begin{aligned} K(X_u \times X_v, X_u \times X_u) &= (S(X_u) \times S(X_v), X_u \times X_v) \\ K(EG - F^2) &= \begin{vmatrix} (S(X_u), X_u) & (S(X_u), X_v) \\ (S(X_v), X_u) & (S(X_v), X_v) \end{vmatrix} \\ K(EG - F^2) &= \ell n - m^2 \\ K &= \frac{\ell n - m^2}{EG - F^2}. \end{aligned}$$

Computing the Gaussian and Mean Curvatures (Cont'd)

- We obtain the formula for H by taking inner products on both sides of the second equation with $X_u \times X_v$.

$$\begin{aligned}
2H(X_u \times X_v, X_u \times X_v) &= (S(X_u) \times X_v, X_u \times X_v) \\
&\quad + (X_u \times S(X_v), X_u \times X_v) \\
2H(EG - F^2) &= \begin{vmatrix} (S(X_u), X_u) & (S(X_u), X_v) \\ (X_v, X_u) & (X_v, X_v) \end{vmatrix} \\
&\quad + \begin{vmatrix} (X_u, X_u) & (X_u, X_v) \\ (S(X_v), X_u) & (S(X_v), X_v) \end{vmatrix} \\
2H(EG - F^2) &= \ell G - Fm + nE - mF \\
H &= \frac{\ell G - 2Fm + En}{2(EG - F^2)}.
\end{aligned}$$

The Case $K > 0$

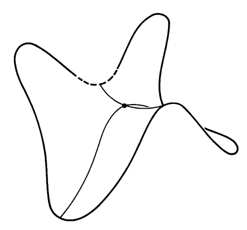
- The Gaussian curvature K is the product of the principal curvatures k_1 and k_2.
- Thus, $K > 0$ at p, if both k_1 and k_2 are different from zero and have the same sign.
  - If $k_1 > 0$ and $k_2 > 0$, the curve of each normal section curves toward the normal.
    So the surface lies entirely on the same side of the tangent plane as the normal N_p sufficiently near the point p.
  - If $k_1 < 0$ and $k_2 < 0$, each curve goes away from the normal.
    So the surface (near p) lies entirely on the opposite side to N_p.
- Equivalently, introducing local coordinates in $\mathbb{R}^3$, $K > 0$ if and only if the function $z = f(x, y)$ has a strict relative extremum at the point.

The Case $K < 0$

- Suppose $K < 0$.
- Then k_1 and k_2 are different from zero and have opposite signs.
- This means that the surface is like a saddle surface.
- Some normal sections are concave toward the normal N and some concave away from it.

The Case $K = 0$

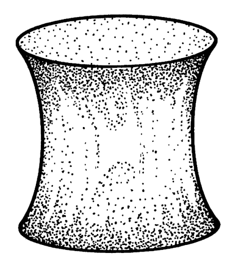
- If $k = 0$, one of the principal curvatures must be zero and then little can be said.
- In addition to the plane, we have:
  - $z = (x^2 + y^2)^2$, obtained by revolving $z = x^4$ around the z-axis.
  - $z = x(x^2 - 3y^2)$, the so-called **monkey saddle**.



This is similar to the usual saddle surface except that there are three valleys running down from the pass.
Two for the monkey's legs and one for its tail.

The Case of Mean Curvature

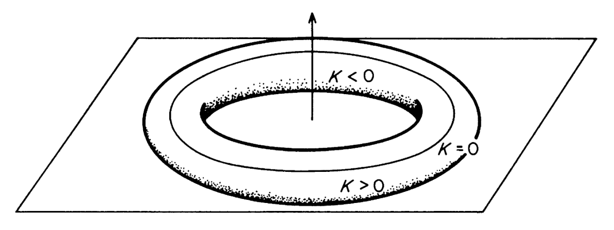
- Surfaces for which the mean curvature vanishes are of special interest.
- They are **minimal surfaces**, like the surfaces formed by a soap film stretched over a wire frame.
- They have the defining property of being surfaces of minimal area among all surfaces with a given boundary (the wire frame).



- Thus, in a sense, they generalize the geodesics-curves of minimal length joining two fixed points.
- Like the equation of geodesics, the vanishing of the mean curvature guarantees the property of minimality only in a local sense.

Example

- Consider a torus.
- Look at the two circles running around the torus which are the points of contact with the two parallel tangent planes orthogonal to its axis.



- We intuitively we can see that they divide the torus into:
  - An inner portion on which $K < 0$;
  - An outer portion at which $K > 0$.
- Along the two circles $K = 0$, since along these circles the normal vector remains parallel to the z-axis.

Example

- Consider a parametrization of the saddle surface $z = xy$,
$$(u, v) \to (u, v, uv).$$
- Then
$$X_u = \frac{\partial}{\partial x^1} + v\frac{\partial}{\partial x^3} \quad \text{and} \quad X_v = \frac{\partial}{\partial x^2} + u\frac{\partial}{\partial x^3}.$$
- So we get
$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1+v^2 & uv \\ uv & 1+u^2 \end{pmatrix}.$$
- A normal to the curve is given by
$$\lambda N = (-v, -u, 1),$$
where the normalizing factor is $\lambda = (1 + u^2 + v^2)^{1/2}$.
- Moreover, we have
$$X_{uu} = 0 = X_{vv} \quad \text{and} \quad X_{vu} = \frac{\partial}{\partial x^3}.$$

Example (Cont'd)

- So we obtain

$$\begin{aligned} \ell &= (N, X_{uu}) = (N, 0) = 0; \\ m &= (N, X_{vu}) = (N, \tfrac{\partial}{\partial x^3}) = \tfrac{1}{\lambda}; \\ n &= (N, X_{vv}) = (N, 0) = 0. \end{aligned}$$

- It follows that

$$\begin{pmatrix} \ell & m \\ m & n \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{\lambda} \\ \frac{1}{\lambda} & 0 \end{pmatrix}.$$

- Therefore, using the formulas, we compute

$$\begin{aligned} K &= \tfrac{\ell n - m^2}{EG - F^2} = \tfrac{0 - \frac{1}{\lambda^2}}{(1+v^2)(1+u^2)-(uv)^2} = \tfrac{-\frac{1}{\lambda^2}}{\lambda^2} = -\tfrac{1}{\lambda^4}; \\ H &= \tfrac{1}{2}\tfrac{G\ell - 2Fm + En}{EG - F^2} = \tfrac{1}{2}\tfrac{0 - 2uv\frac{1}{\lambda} + 0}{(1+v^2)(1+u^2)-(uv)^2} = \tfrac{1}{2}\tfrac{-2uv}{(1+u^2+v^2)\lambda} = -\tfrac{uv}{\lambda^3}. \end{aligned}$$

The Theorema Egregium of Gauss

- The entire subject of differential geometry was influenced by a very profound discovery of Gauss which may be stated as follows.

Theorem (Gauss)

Let M_1 and M_2 be two surfaces in Euclidean space.
Suppose that

$$F : M_1 \to M_2$$

is a diffeomorphism between them which is also an isometry.
Then the Gaussian curvature K is the same at corresponding points.

- To see the meaning of this theorem we consider some examples.

Example

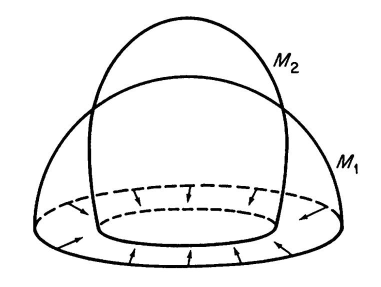
- Let M_1 be a plane.
- Let M_2 a right circular cylinder of radius R in Euclidean space $\mathbb{R}^3$.
- Suppose we roll the cylinder over the plane.
- Then we obtain a correspondence which does not change the length of curves or the angle between intersecting curves.
- Hence, it is an isometry.
- We know that $K = 0$ for the plane.
- According to the theorem the same must be true of the cylinder.
- Note that they do not have the same second fundamental form.
- That is, ℓ, m and n do not vanish identically for the cylinder.
- In fact curvatures of the normal sections vary from zero to $\frac{1}{R}$.
- This depends on the imbedded shape of the surface.
- By contrast, K depends only on the Riemannian metric induced on M.

Example

- Let M_1 be any open subset of the sphere of radius R.
- Let M_2 be a plane.
- We know that $K_1 \equiv \frac{1}{R^2} \neq 0$ and $K_2 \equiv 0$.
- The theorem implies that there exists no diffeomorphism of M_1 into M_2 that is an isometry.
- For example, any plane map of a portion of the globe must distort some metric properties (distance or length of curves, angles, areas, and so on).

Example

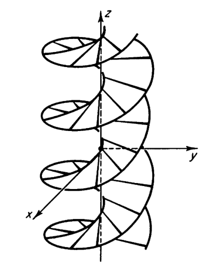
- There do exist surfaces isometric to, but not congruent to, say, the upper hemisphere.
- Suppose this hemisphere to be made of a thin sheet of brass.
- It is intuitively clear that we may bend it by squeezing at the edge without introducing any creases.
- This will give a surface isometric to the original since length of curves is unchanged.

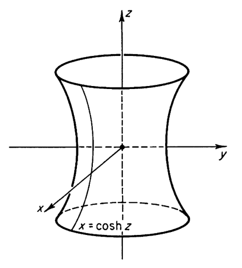


- It follows that K is the same at corresponding points.
- However, the surfaces are not congruent.

Example

- Among the more interesting examples of (locally) isometric surfaces are the **helicoid** and the **catenoid**.





- The first surface is given parametrically by

$$(u, v) \to (u \cos v, u \sin v, v), \;\; u > 0, \;\; -\infty < v < \infty.$$

  It is similar in shape to a spiral staircase.

- The catenoid is obtained by revolving the catenary $x = \cosh z$ around the z-axis. We may parametrize it as

$$(z, \theta) \to (\cos\theta \cosh z, \sin\theta \cosh z, z), \;\; -\infty < z < \infty, \;\; 0 < \theta < 2\pi.$$

- The isometry between these surfaces is given by $v = \theta, u = \sinh z$.

Proof of Gauss' Theorem

- Recall that, at a point $p \in M$, the value of the Gaussian curvature K is given by

$$K = \frac{\ell n - m^2}{EG - F^2},$$

where E, F, G and ℓ, m, n are the components of the first and second fundamental forms, respectively, relative to a system of local coordinates u, v in a neighborhood U of p.

The value of the ratio K is independent of the coordinates chosen although E, F, G and ℓ, m, n are not.

Suppose the surface in $\mathbb{R}^3$ is given by

$$X = X(u, v).$$

Then

$$E_1 = X_u \quad \text{and} \quad E_2 = X_v.$$

Proof of Gauss' Theorem (Cont'd)

- We have seen that

$$\ell n - m^2 = \left(\frac{\partial N}{\partial u}, E_1\right)\left(\frac{\partial N}{\partial v}, E_2\right) - \left(\frac{\partial N}{\partial u}, E_2\right)\left(\frac{\partial N}{\partial v}, E_1\right).$$

We also have

$$E = (E_1, E_1),\ \ F = (E_1, E_2),\ \ G = (E_2, E_2).$$

Thus, we obtain

$$EG - F^2 = (E_1, E_1)(E_2, E_2) - (E_1, E_2)^2.$$

But E, F, G are the coefficients of the Riemannian metric.

So it is enough to show that

$$\ell n - m^2 = K(EG - F^2)$$

depends only on the Riemannian metric.

Proof of Gauss' Theorem (Cont'd)

- We shall show that

$$\ell n - m^2 = R(E_1, E_2, E_2, E_1),$$

where $R(X, Y, Z, W)$ is the covariant tensor of order 4 defined previously.

Then K is given by

$$K = \frac{R(E_1, E_2, E_2, E_1)}{EG - F^2} = \frac{R(E_1, E_2, E_2, E_1)}{(E_1, E_1)(E_2, E_2) - (E_1, E_2)^2}.$$

The left side is independent of local coordinates.

Thus, the right side is also.

In fact, it can be shown that replacing E_1, E_2 at a point by any pair of vectors F_1, F_2, spanning the same plane, leaves unchanged the expression on the right.

We shall prove that expression gives K.

Proof of Gauss' Theorem (Cont'd)

- This implies that the expression, defined at each point of an imbedded surface M, is independent of local coordinates on M, and, moreover, it depends only on the Riemannian metric.
  Clearly this is true of the denominator.
  We recall that, by definition,
  $$(R(E_1, E_2) \cdot E_2, E_1) = (\nabla_{E_1} \nabla_{E_2} E_2 - \nabla_{E_2} \nabla_{E_1} E_2 - \nabla_{[E_1, E_2]} E_2, E_1).$$
  This depends only on the Riemannian metric by the Fundamental Theorem of Riemannian Geometry.
  In the present case, E_1 and E_2 denote coordinate frames of local coordinates u, v and we know that $[E_1, E_2] = 0$.
  So we must show only that
  $$\ell n - m^2 = (\nabla_{E_1} \nabla_{E_2} E_2 - \nabla_{E_2} \nabla_{E_1} E_2, E_1).$$
  We may compute the right-hand side using the definition of $\nabla_{E_1} Z$, $i = 1, 2$ (for any tangent vector field Z).

Proof of Gauss' Theorem (Cont'd)

- Take $\frac{\partial Z}{\partial u}$ and $\frac{\partial Z}{\partial v}$.
  Project them to the tangent plane at each point of the surface to obtain $\frac{DZ}{\partial u} = \nabla_{E_1} Z$ and $\frac{DZ}{\partial v} = \nabla_{E_2} Z$.
  If N denotes the unit normal, and $E_1 = X_u$ and $E_2 = X_v$, then we get

$$\nabla_{E_1} E_2 = X_{uv} - (N, X_{uv})N, \quad \nabla_{E_2} E_2 = X_{vv} - (N, X_{vv})N.$$

  Differentiate again and project onto the tangent plane (by subtracting the normal component of the derivative).
  This gives

$$\begin{array}{rcl}\nabla_{E_2}(\nabla_{E_1} E_2) & = & X_{vuv} - (N, X_{uv})N_v - c_1 N; \\ \nabla_{E_1}(\nabla_{E_2} E_2) & = & X_{uvv} - (N, X_{vv})N_u - c_2 N.\end{array}$$

  We next take an inner product of each term above with E_1.
  As $(N, E_1) = 0$, the terms involving c_1 and c_2 multiplying N vanish.
  So there is not need to compute c_1 or c_2.

Proof of Gauss' Theorem (Cont'd)

- For $R(E_1, E_2, E_2, E_1)$ we obtain

$$\begin{aligned}(\nabla_{E_1}\nabla_{E_2}E_2 - \nabla_{E_2}\nabla_{E_1}E_2, E_1) &= (X_{uvv}, X_u) - (N, X_{vv})(N_u, N_u) \\ &\quad -(X_{vuv}, X_u) + (N, X_{uv})(N_v, X_u).\end{aligned}$$

  This must be seen to be equal to the earlier evaluation of $\ell n - m^2$ above, namely,

$$\ell n - m^2 = (N_u, X_u)(N_v, X_v) - (N_u, X_v)(N_v, X_u).$$

  The proof is finished by noting that:

  - $X_{vuv} = X_{uvv}$;
  - Since $(N, X_u) = 0 = (N, X_v)$, we have

$$(N, X_{vv}) = -(N_v, X_v) \quad \text{and} \quad (N, X_{uv}) = -(N_u, X_v).$$

Subsection 3

Basic Properties of the Riemann Curvature Tensor

Review of Curvature of Riemannian Manifold

- We have defined previously the curvature tensor $R(X, Y, Z, W)$ of a Riemannian manifold M.
- Recall that it is a covariant tensor field of order 4 whose value at any point $p \in M$ is determined as follows.
- Let X, Y, Z, W be vector fields whose values at p are given, say X_p, Y_p, Z_p, W_p.
- Then

$$R(X_p, Y_p, Z_p, W_p) = (\nabla_{X_p} \nabla_Y Z - \nabla_{Y_p} \nabla_X Z - \nabla_{[X,Y]_p} Z, W_p).$$

- We have shown that this is independent of the vector fields chosen.
- Moreover, it defines a C^∞ covariant tensor field.

The Curvature Operator

- Similarly, the vector fields X, Y define at each $p \in M$ a linear operator, the curvature operator, $R(X_p, Y_p)$ on $T_p(M)$ by the prescription

$$R(X_p, Y_p) \cdot Z_p = \nabla_{X_p} \nabla_Y Z - \nabla_{Y_p} \nabla_X Z - \nabla_{[X,Y]_p} Z_p.$$

- It is, like the curvature tensor, linear in X, Y, Z in the sense of a $C^\infty(M)$ module.
- That is, if $f \in C^\infty(M)$, then

$$fR(X, Y) \cdot Z = R(fX, Y) \cdot Z = R(X, fY) \cdot Z = R(X, Y) \cdot fZ.$$

- Obviously the curvature tensor and the curvature operator are related by the equality

$$R(X, Y, Z, W) = (R(X, Y) \cdot Z, W).$$

Symmetry Relations

Theorem

The following symmetry relations hold for the curvature tensor and curvature operator at each point, and hence for all vector fields.

(i) $R(X,Y)\cdot Z + R(Y,X)\cdot Z = 0$;

(ii) $R(X,Y)\cdot Z + R(Y,Z)\cdot X + R(Z,X)\cdot Y = 0$;

(iii) $(R(X,Y)\cdot Z, W) + (R(X,Y)\cdot W, Z) = 0$;

(iv) $(R(X,Y)\cdot Z, W) = (R(Z,W)\cdot X, Y)$.

(i) We have

$$\begin{aligned} & R(X,Y)\cdot Z + R(Y,X)\cdot Z \\ &= \nabla_{X_p}\nabla_Y Z - \nabla_{Y_p}\nabla_X Z - \nabla_{[X,Y]_p} Z_p \\ &\qquad + \nabla_{Y_p}\nabla_X Z - \nabla_{X_p}\nabla_Y Z - \nabla_{[Y,X]_p} Z_p \\ &= -\nabla_{[X,Y]_p} Z_p + \nabla_{[X,Y]_p} Z_p = 0. \end{aligned}$$

Symmetry Relations (Cont'd)

(ii) $R(X, Y, Z, W)$ is a tensor.

So it is linear with respect to C^∞ functions.

This implies that it suffices to prove the statements for the vectors of a field of coordinate frames, say $E_1, \ldots, E_n$.

For these vector fields the Lie products $[E_i, E_j] = 0$.

So if X, Y, Z are chosen from among $E_1, \ldots, E_n$, then proving Property (ii) reduces to showing that

$$\begin{aligned}\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) + \nabla_Y(\nabla_Z X) - \nabla_Z(\nabla_Y X) \\ + \nabla_Z(\nabla_X Y) - \nabla_X(\nabla_Z Y) = 0.\end{aligned}$$

By definition of Riemannian connection,

$$\nabla_X Y - \nabla_Y X = [X, Y] = 0.$$

Using this, we find that the terms on the left cancel two by two.

Symmetry Relations (Cont'd)

(iii) Note that, for all X, Y, X, W,

$$\begin{aligned}(R(X,Y)\cdot(Z+W), Z+W) \\ &= (R(X,Y)\cdot Z, Z) + (R(X,Y)\cdot Z, W) \\ &+ (R(X,Y)\cdot W, Z) + (R(X,Y)\cdot W, W).\end{aligned}$$

So, Property (iii) is equivalent to the statement that, for all X, Y, Z,

$$(R(X,Y)\cdot Z, Z) = 0.$$

As before, it is enough to prove this for X, Y, Z chosen from among the vectors of the coordinate frames so that $[X, Y] = 0$.

Applying the definitions, we see that

$$(R(X,Y)\cdot Z, Z) = (\nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z), Z) = 0$$

if and only if $(\nabla_X(\nabla_Y Z), Z)$ is symmetric in X, Y.

Symmetry Relations (Cont'd)

- Differentiating the inner product (Z, Z) with respect to X and Y, we get

$$Y(X(Z,Z)) = 2Y(\nabla_X Z, Z) = 2(\nabla_Y(\nabla_X Z), Z) + 2(\nabla_X Z, \nabla_Y Z).$$

It now follows that

$$(\nabla_Y(\nabla_X Z), Z) = \frac{1}{2} YX(Z,Z) - (\nabla_X Z, \nabla_Y Z).$$

But $[X, Y] = 0$.

So $(XY - YX)f \equiv 0$, for any function f.

Taking $f = (Z, Z)$, we see that the right side is symmetric in X, Y.

Therefore, so is the left side.

Symmetry Relations (Cont'd)

(iv) Property (iv) derived from the first three properties.
By Property (ii), we have

$$(R(X,Y)\cdot Z,W)+(R(Y,Z)\cdot X,W)+(R(Z,X)\cdot Y,W)=0.$$

Then, using Properties (i)-(iii) we obtain the relation

$$(R(X,Y)\cdot Z,W)+(R(Y,W)\cdot Z,X)+(R(X,W)\cdot Y,Z)=0.$$

E.g., applying Property (ii), we get

$$(R(X,Y)\cdot W,Z)+(R(Y,W)\cdot X,Z)+(R(W,X)\cdot Y,Z)=0.$$

Then, multiplying by -1 and using Property (i), we get

$$-(R(X,Y)\cdot W,Z)-(R(Y,W)\cdot X,Z)+(R(W,X)\cdot Y,Z)=0.$$

Finally, using Property (iii), we get

$$(R(X,Y)\cdot Z,W)+(R(Y,W)\cdot Z,X)+(R(X,W)\cdot Y,Z)=0.$$

Symmetry Relations (Cont'd)

- We got the equations

$$(R(X,Y)\cdot Z,W)+(R(Y,Z)\cdot X,W)+(R(Z,X)\cdot Y,W)=0,$$
$$(R(X,Y)\cdot Z,W)+(R(Y,W)\cdot Z,X)+(R(X,W)\cdot Y,Z)=0.$$

In a similar way, we obtain two more equations

$$(R(Y,Z)\cdot X,W)+(R(Y,W)\cdot Z,X)+(R(Z,W)\cdot X,Y)=0,$$
$$(R(Z,W)\cdot X,Y)+(R(Z,X)\cdot Y,W)+(R(X,W)\cdot Y,Z)=0.$$

Now add the first two and subtract the last two to get

$$2(R(X,Y)\cdot Z,W)-2(R(Z,W)\cdot X,Y)=0.$$

This finally gives

$$(R(X,Y)\cdot Z,W)=(R(Z,W)\cdot X,Y).$$

Component Functions

- In any coordinate neighborhood U, φ we have coordinate frames $E_1, \ldots, E_n$.
- We may introduce n^4 functions of the coordinates $R^j_{ik\ell}$, $1 \leq i, j, k, \ell \leq n$ by the equations

$$R(E_k, E_\ell) \cdot E_i = \sum_j R^j_{ik\ell} E_j.$$

- Similarly we may define the components $R_{ijk\ell}$ of the Riemannian curvature tensor by the equations

$$R_{ijk\ell} = (R(E_k, E_\ell) \cdot E_i, E_j) = \sum_h R^h_{ik\ell} g_{hj},$$

  where $g_{ij} = (E_i, E_j)$ are the components of the Riemannian metric.
- By linearity both $R(X, Y) \cdot Z$ and $(R(X, Y) \cdot Z, W)$ are determined on U by these locally defined functions.

Using Components

- The preceding theorem may be written in terms of components.

Corollary

For all $1 \leq i, j, k, \ell \leq n$ we have:

(i) $R^j_{ik\ell} + R^j_{i\ell k} = 0$;

(ii) $R^j_{ik\ell} + R^j_{k\ell i} + R^i_{\ell jk} = 0$;

(iii) $R_{ijk\ell} + R_{jik\ell} = 0$;

(iv) $R_{ijk\ell} = R_{k\ell ij}$;

(v) $R_{ijk\ell} + R_{ik\ell j} + R_{i\ell jk} = 0$.

- We remark that Property (v) is an immediate consequence of $R_{ijk\ell} = \sum_h R^h_{ik\ell} g_{hj}$, the symmetry of g_{ij} and Properties (ii) and (iii).

Sectional Curvature

- The Riemann curvature tensor $(R(X,Y)\cdot Z, W)$ is used to define the sectional curvature, which plays an important role in the geometry of Riemannian manifolds.
- At any $p \in M$ we denote by π a **plane section**, that is, a two-dimensional subspace of $T_p(M)$.
- Such a section is determined by any pair of mutually orthogonal unit vectors X, Y at p.

Definition

The **sectional curvature** $K(\pi)$ of the section π with orthonormal basis X, Y is defined as

$$K(\pi) = -R(X,Y,X,Y) = -(R(X,Y)\cdot X, Y).$$

Changing Coordinate Vectors

- Symmetry and linearity yield the following property.
- Suppose X, Y are replaced by any pair of vectors X', Y', with
$$X = \alpha X' + \beta Y' \quad \text{and} \quad Y = \gamma X' + \delta Y'.$$
- Then, we get
$$\frac{1}{\Delta^2}(R(X', Y') \cdot X', Y') = (R(X, Y) \cdot X, Y),$$
where $\Delta = \alpha\delta - \beta\gamma$ is the determinant of coefficients.
- If X', Y' is also an orthonormal pair, then $\Delta = \pm 1$.
- So the definition of $K(\pi)$ is independent of the pair used.
- If it is just any arbitrary linearly independent pair, then using $\Delta^2 = (X', X')(Y', Y') - (X', Y')^2$, we have
$$K(\pi) = -\frac{(R(X', Y') \cdot X', Y')}{(X', X')(Y', Y') - (X', Y')^2}.$$

Changing Coordinate Vectors (Cont'd)

- Consider local coordinates.
- We saw that

$$K(\pi) = -\frac{(R(X', Y') \cdot X', Y')}{(X', X')(Y', Y') - (X', Y')^2}.$$

- Assume that $X' = \sum_i \alpha^i E_i$, $Y' = \sum_j \beta^j E_j$.
- Use $(E_i, E_j) = g_{ij}$.
- Then, with the notation above, concerning $R_{ijk\ell}$, we obtain

$$K(\pi) = -\frac{\sum R_{ijk\ell} \alpha^i \beta^j \alpha^k \beta^\ell}{\sum (g_{ik} g_{j\ell} - g_{i\ell} g_{jk}) \alpha^i \beta^j \alpha^k \beta^\ell},$$

where summation is over i, j, k, ℓ.

Curvature from Sectional Curvatures

Theorem

If $\dim M \geq 3$ and the sectional curvature is known on all sections of $T_p(M)$, then the Riemann curvature tensor is uniquely determined at p.

- Let $R(X, Y, Z, W)$ and $\widetilde{R}(X, Y, Z, W)$ be two tensors with the symmetry properties of the preceding theorem.

  Let $A(X, Y, Z, W)$ be their difference.

  It is also be a tensor with these symmetry properties.

  Our assumption is that for all X, Y, $R(X, Y, X, Y) = \widetilde{R}(X, Y, X, Y)$.

  Equivalently, for all X, Y, $A(X, Y, X, Y) = 0$.

  We must show that this implies that $A = 0$, i.e., that, for all X, Y, Z, W,

$$A(X, Y, Z, W) = 0.$$

Curvature from Sectional Curvatures (Cont'd)

- Let $p \in M$ and $F_1, \ldots, F_n$ be a frame or basis of $T_p(M)$.

  We denote by $A_{ijk\ell}$ the components of A.

  Let α^i, β^j be the components of vectors X, Y relative to this basis.

  Then by hypothesis, for any $\alpha^1, \ldots, \alpha^n$ and $\beta^1, \ldots, \beta^n$,

$$\sum_{i,j,k,\ell} A_{ijk\ell}\alpha^i\beta^j\alpha^k\beta^\ell = 0.$$

  We make specific choices for the α^i and β^j.

  Let δ_{ij} denote the Kronecker δ, that is,

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Curvature from Sectional Curvatures (Cont'd)

- First, set $\alpha^i = \delta_{i_0 i}$ and $\beta^j = \delta_{j_0 j}$.
  The equation above gives
  $$A_{i_0 j_0 i_0 j_0} = 0, \quad \text{for all } 1 \leq i_0, j_0 \leq n.$$
  Next, set $\alpha^i = \delta_{i_0 i}$ and $\beta^{j_0} = \beta^{k_0} = 1$ and $\beta^j = 0$, for all other j.
  Then by Property (iv) of the corollary we have
  $$A_{i_0 j_0 i_0 k_0} = 0.$$
  Finally, let both α^i and β^j vanish except at two values of i and two of j at which it has the value 1.
  Then, using Property (ii) and the results just established, we obtain
  $$0 = A_{ijk\ell} + A_{kj\ell i} + A_{i\ell kj} + A_{k\ell ij} = 2A_{ijk\ell} + 2A_{i\ell kj} = -2A_{ik\ell j}.$$
  Thus, $A_{ijk\ell} = 0$ for all $1 \leq i, j, k, \ell \leq m$.

Isotropic Manifolds

- Let M be a Riemannian manifold.
- Let p be a point in M.
- We say M is **isotropic at** p if the curvature is the same constant K_p on every section at p.
- M is called **isotropic** if it is isotropic at every point.
- A two-dimensional Riemannian manifold is (trivially) isotropic.

Components of Curvature Tensor for Isotropic Manifolds

Corollary

Let M be a Riemannian manifold. Suppose p is an isotropic point of M. Let U, φ be a coordinate neighborhood with:

- Coordinate frames $E_1, \ldots, E_n$;
- Riemannian metric $g_{ij} = (E_i, E_j)$.

Then, at the point p,

$$R_{ijk\ell} = -K_p(g_{ik}g_{j\ell} - g_{i\ell}g_{jk}).$$

- One may check that the right side defines a tensor of order 4 on $T_p(M)$ with the same symmetry properties as $R(X, Y, Z, W)$ and with constant value on all sections.

  The corollary then follows from the uniqueness theorem.

Manifolds of Constant Curvature

Definition

An isotropic Riemannian manifold is called a manifold of **constant curvature** if K_p is the same at every point.

- An example is Euclidean space where $K_p \equiv 0$.

The Ricci Curvature

- Let M be a Riemannian manifold.
- Let $R(X, Y, Z, W)$ denote the curvature tensor on M.
- We use this curvature tensor to define:
  - A (covariant) tensor field $S(X, Y)$ of order 2;
  - A (scalar) function on M.
- Let $p \in M$ and let $F_{1p}, \ldots, F_{np}$ be an orthonormal basis at p.
- Consider the operator

$$S_p(X_p, Y_p) = \sum_{i=1}^{n} R(F_{ip}, X_p, Y_p, F_{ip}) = \sum_{i=1}^{n} (R(F_{ip}, X_p) \cdot Y_p, F_{ip}).$$

- We may verify that S_p:
  - Is independent of the choice of orthonormal basis;
  - Defines a symmetric, C^∞ covariant tensor field S on M.

The Ricci Curvature (Cont'd)

Definition

The tensor field $S(X,Y)$ is called the **Ricci curvature** of M.
M is called an **Einstein manifold** if there is a constant c, such that

$$S(X,Y) = c(X,Y),$$

that is, $S(X,Y)$ is a constant multiple of the Riemannian metric on M.
The function r on M, defined by

$$r(p) = \sum_{i,j=1}^{n} R(F_{ip}, F_{jp}, F_{jp}, F_{ip}) = \sum_{j=1}^{n} S(F_{jp}, F_{jp})$$

is called the **scalar curvature** of M.

- Spaces of constant curvature are examples of Einstein manifolds.

Sectional Curvature in Lie Groups

Theorem

Let G be a compact Lie group with a bi-invariant Riemannian metric. On G, the sectional curvatures at e (hence everywhere) are given by

$$K(\pi_e) = -R(X_e, Y_e, X_e, Y_e) = +\frac{1}{4}([X, Y], [X, Y]),$$

where X, Y are an orthonormal pair of left-invariant vector fields spanning the section π_e at e. The curvature operator is similarly given at e, hence at all points by

$$R(X, Y) \cdot Z = -\frac{1}{4}[[X, Y], Z]$$

with X, Y, Z left-invariant vector fields.

- We have seen that for left-invariant vector fields X, Y, the connection of a bi-invariant metric on G given by $\nabla_X Y = \frac{1}{2}[X, Y]$.

Sectional Curvature in Lie Groups (Cont'd)

- Applying first the definition and then the Jacobi identity, we obtain

$$\begin{array}{rcl} R(X,Y)\cdot Z & = & \nabla_X(\nabla_Y Z) - \nabla_Y(\nabla_X Z) - \nabla_{[X,Y]}Z \\ & = & \frac{1}{4}[X,[Y,Z]] - \frac{1}{4}[Y,[X,Z]] - \frac{1}{2}[[X,Y],Z] \\ & = & \frac{1}{4}[Z,[X,Y]] \\ & = & -\frac{1}{4}[[X,Y],Z]. \end{array}$$

We also know that, for left-invariant vector fields U, V, W on G,

$$([U,V],W) = (U,[V,W]).$$

Thus, if X, Y are left-invariant and are an orthonormal basis at e of π, a plane section, the sectional curvature is

$$K(\pi) = -R(X,Y,X,Y) = \frac{1}{4}([[X,Y],X],Y) = \frac{1}{4}([X,Y],[X,Y]).$$

Ricci Tensor Formula

Corollary

Let G be a compact Lie group with a bi-invariant Riemannian metric.
Let X, Y, Z be left-invariant vector fields.
Then the Ricci tensor $S(X, Y)$ is given by the formula

$$S(X, Y) = -\frac{1}{4}\mathrm{tr}(\mathrm{ad}X \circ \mathrm{ad}Y).$$

Moreover, it is positive semi-definite and bi-invariant on G.
Each compact semisimple G is an Einstein manifold relative to any bi-invariant Riemannian metric.

- By the formula, the linear operator $Z \to R(Z, Y) \cdot X$ on G is defined at e for the left-invariant vector field by

$$R(Z, Y) \cdot X = -\frac{1}{4}(\mathrm{ad}X)(\mathrm{ad}Y) \cdot Z.$$

Ricci Tensor Formula (Cont'd)

- It can be shown that an alternative definition of $S(X, Y)$ is that it is the trace of the linear operator

$$\begin{array}{rcl} T_p(M) & \to & R_p(M) \\ Z_p & \mapsto & R(Z_p, X_p) \cdot Y_p \end{array}$$

on the tangent space at each point.

We also have

$$S(X, Y) = S(Y, X).$$

Now, for all Z,

$$R(Z, Y) \cdot X = -\frac{1}{4}[X, [Y, Z]].$$

So we get

$$S(X, Y) = -\frac{1}{4}(\mathrm{ad} X)(\mathrm{ad} Y) \cdot Z.$$

Ricci Tensor Formula (Cont'd)

- On the other hand, suppose $F_1, \ldots, F_n$ is an orthonormal basis of left-invariant vector fields.

  Then we have

  $$(\mathrm{ad}X \cdot F_i, F_j) = ([X, F_i], F_j) = (F_i, [X, F_j]) = (F_i, \mathrm{ad}X \cdot F_j).$$

  So the matrix (a_{ij}) of $\mathrm{ad}X$, relative to this basis, is skew symmetric.

  Hence,

  $$\mathrm{tr}\,\mathrm{ad}X\,\mathrm{ad}X = \sum_{i,j} a_{ij}a_{ji} = -\sum_{i,j} a_{ij}^2.$$

  It follows that

  $$S(X, X) = -\mathrm{tr}\,\mathrm{ad}X\,\mathrm{ad}X = \sum a_{ij}^2 \geq 0.$$

  Equality holds only when $\mathrm{ad}X = 0$.

  Hence, $S(X, Y)$ is positive semidefinite.

Ricci Tensor Formula (Cont'd)

- Moreover, if G is semisimple, it is positive definite.

  Now, if X, Y, Z are left-invariant, so is $R(Z, Y) \cdot X$.

  The same holds for its trace $S(X, Y)$.

  This means that $S(X, Y)$ is a bi-invariant Riemannian metric on a semisimple G.

  However two bi-invariant metrics can differ only by a scalar multiple.

  It follows that, with a bi-invariant metric, G is Einstein.

Subsection 4

The Curvature Forms and the Equations of Structure

Coframes

- Let U be a neighborhood on the Riemannian manifold M.
- Suppose on U is defined a C^∞ family of coframes
$$\theta^1, \ldots, \theta^n.$$
- Thus, automatically, we also have a dual C^∞ family of frames
$$E_1, \ldots, E_n.$$
- They may or may not be coordinate frames of a coordinate neighborhood U, φ.
- The components of the Riemann metric on U are still denoted by
$$g_{ij} = (E_i, E_j).$$

Properties of Coframes

- According to a previous theorem, there exist uniquely determined one-forms θ_i^j on U satisfying:
  (i) $d\theta^i = \sum_j \theta^j \wedge \theta_j^i$, $1 \leq i \leq n$;
  (ii) $dg_{ij} = \sum_k \theta_i^k g_{kj} + \sum_k g_{ik}\theta_j^k$, $1 \leq i,j \leq n$.
- Define
$$\theta_{ij} = \sum_k \theta_i^k g_{kj}.$$
- Then Equations (ii) assume the simpler form
$$dg_{ij} = \theta_{ij} + \theta_{ji}.$$
- In the special case where the frames are orthonormal, that is, $gij = \delta_{ij}$, we will use ω^i, ω_i^j instead of θ^i, θ_i^j.
- Then Equations (ii) become
$$0 = \omega_i^j + \omega_j^i, \quad 1 \leq i,j \leq n.$$

Connection Forms

- The forms θ_i^j determine, and are determined by the Riemannian connection.
- Thus if $\nabla_{E_i} E_j = \sum_k \Gamma_{ij}^k$, then
$$\theta_i^j = \sum_k \Gamma_{ki}^j \theta^k.$$
- Equivalently,
$$\nabla_X E^j = \sum_k \theta_j^k(X) E_k.$$
- The one-forms θ_j^k, $1 \leq j, k \leq n$, are called the **connection forms**.
- We have that $\Gamma_{ij}^k = \Gamma_{ji}^k$ only if $E_1, \ldots, E_n$ satisfy $[E_i, E_j] = 0$, as is the case for *coordinate frames*.
- This symmetry was derived from $\nabla_{E_i} E_j - \nabla_{E_j} E_i = [E_i, E_j]$, which we have made part of the definition of Riemannian connection.
- $\nabla_{E_i} E_j - \nabla_{E_j} E_i = [E_i, E_j]$ is equivalent to Equations (i).

Curvature Forms

- Now suppose that $R^j_{ik\ell}$, $1 \leq i,j,k,\ell \leq n$, are the components of the curvature (as an endomorphism) relative to the given frames, i.e.,

$$R(E_k, E_\ell) \cdot E_i = \sum_j R^j_{ik\ell} E_j.$$

- Then we define n^2 two-forms Ω^j_i, $1 \leq i,j \leq n$, by

$$\Omega^j_i = \sum_{1 \leq k < \ell \leq n} R^j_{ik\ell} \theta^k \wedge \theta^\ell = \frac{1}{2} \sum_{k,\ell=1}^n R^j_{ik\ell} \theta^k \wedge \theta^\ell.$$

- It follows that

$$\sum_{j=1}^n \Omega^j_i (E_k, E_\ell) E_j = \sum_{j=1}^n R^j_{ik\ell} E_j = R(E_k, E_\ell) \cdot E_i.$$

Curvature Forms (Cont'd)

- By linearity this extends to any vector fields X, Y so that
$$R(X,Y)\cdot E_i = \sum_j \Omega_i^j(X,Y)E_j.$$
- Thus, $(\Omega_i^j(X,Y))$ is the matrix of the curvature operator relative to the basis $E_1,\ldots,E_n$.
- Note that the properties of $R(X,Y)\cdot Z$ imply that $\Omega_i^j(X,Y)$ at p depend only on the values of X and Y at p, not on the vector fields.
- Obviously, $\Omega_i^j(X,Y) = -\Omega_i^j(Y,X)$.
- These n^2 forms Ω_i^j on U_j are called the **curvature forms**.
- They depend on the Riemannian metric and on the particular frame-field we use on U.

Curvature Forms and Connection Forms

Theorem

Using the notation above, the forms Ω_i^j on U are defined by the equations

$$\Omega_i^j = d\theta_i^j - \sum_{k=1}^{n} \theta_i^k \wedge \theta_k^j, \quad 1 \leq i, j \leq n.$$

- It is sufficient to verify that, on any vector fields X, Y on U, the value of the two-forms on each side of the equation is the same.

  This is equivalent to showing that

$$R(X, Y) \cdot E_i = \sum_j \left(\left(d\theta_i^j - \sum_k \theta_i^k \wedge \theta_k^j \right) (X, Y) \right) E_j, \quad i = 1, \ldots, n.$$

Curvature Forms and Connection Forms (Cont'd)

- By definition,

$$R(X,Y)\cdot E_i = \nabla_X(\nabla_Y E_i) - \nabla_Y(\nabla_X E_i) - \nabla_{[X,Y]}E_i.$$

This may be rewritten

$$R(X,Y)\cdot E_i = \nabla_X\left(\sum_j \theta_i^j(Y)E_j\right) - \nabla_Y\left(\sum_j \theta_i^j(X)E_j\right) - \sum_j \theta_i^j([X,Y])E_j.$$

Since $\theta_i^j(Y)$ and $\theta_i^j(X)$ are functions, the right-hand side is equal to

$$\sum_j (X(\theta_i^j(Y)) - Y(\theta_i^j(X)) - \theta_i^j([X,Y]))E_j + \sum_{j,k}\theta_i^j(Y)\theta_j^k(X)E_k - \sum_{j,k}\theta_i^j(X)\theta_j^k(Y)E_k.$$

Curvature Forms and Connection Forms (Cont'd)

We got

$$\begin{aligned} R(X,Y)\cdot E_i &= \textstyle\sum_j (X(\theta_i^j(Y)) - Y(\theta_i^j(X)) - \theta_i^j([X,Y]))E_j \\ &\quad + \textstyle\sum_{j,k} \theta_i^j(Y)\theta_j^k(X)E_k - \sum_{j,k} \theta_i^j(X)\theta_j^k(Y)E_k. \end{aligned}$$

Applying a previous lemma, we get that the right side equals

$$\sum_j \left\{ d\theta_i^j(X,Y) - \sum_k \left[\theta_i^k(X)\theta_k^j(Y) - \theta_i^k(Y)\theta_k^j(X) \right] \right\} E_j.$$

This proves that

$$R(X,Y)\cdot E_j = \sum_j \left(d\theta_i^j - \sum_k \theta_i^k \wedge \theta_k^j \right)(X,Y)E_j.$$

Summary: Equations of Structure

- Let U be any open subset of a Riemannian manifold M on which is defined a field of coframes $\theta^1, \ldots, \theta^n$.
- Let $E_1, \ldots, E_n$ denote the uniquely determined dual frame-field.
- Let $g_{ij} = (E_i, E_j)$ on U.
- Then there exist n^2 uniquely determined one-forms θ_i^j on U satisfying Equations (i) and (ii):
  (i) $d\theta^i = \sum_j \theta^j \wedge \theta_j^i$, $1 \leq i \leq n$;
  (ii) $dg_{ij} = \sum_k \theta_i^k g_{kj} + \sum_k g_{ik} \theta_j^k$, $1 \leq i, j \leq n$.
- They determine the two-forms Ω_i^j, and hence the curvature on U, by

$$\Omega_i^j = d\theta_i^j - \sum_{k=1}^{n} \theta_i^k \wedge \theta_k^j, \quad 1 \leq i, j \leq n.$$

- Equations (i), (ii) and the displayed one are known as Cartan's **equations of structure**.

Summary: Equations of Structure (Cont'd)

- As noted, it is often convenient to write $\theta_{ij} = \sum_s \theta_i^s g_{sj}$ so that (ii) takes a simpler form.
- We may define, similarly,

$$\Omega_{ij} = \sum_s \Omega_i^s g_{sj}.$$

- Then

$$\Omega_{ij} = \frac{1}{2} \sum_{k,\ell} R_{ijk\ell} \theta^k \wedge \theta^\ell,$$

  since we have previously seen that $R_{ijk\ell} = \sum_s g_{js} R_{ik\ell}^s$, where $R_{ijk\ell} = R(F_k, F_\ell, F_i, F_j)$.
- The symmetry properties imply that $\Omega_{ij} = -\Omega_{ji}$.

Summary: The Orthonormal Case

- Suppose the frame-field is orthonormal.
- That is, it consists of vectors $E_1, \ldots, E_n$, with

$$(E_i, E_j) = \delta_{ij}.$$

- As noted above, Equations (i) and (ii) simplify:
  (i) $d\omega^i = \sum_j \omega^j \wedge \omega_j^i$, $1 \leq i \leq n$;
  (ii) $0 = \omega_i^j + \omega_j^i$, $1 \leq i, j \leq n$.
- Moreover,

$$\Omega_{ij} = \Omega_i^j, \quad R_{ijk\ell} = R_{ik\ell}^j, \quad \omega_i^j = \omega_{ij}.$$

- These enable us to formulate a restatement.

The Orthonormal Case (Cont'd)

Corollary

The forms $\omega^1, \ldots, \omega^n$, dual to a field of orthonormal frames, determine uniquely a set of one-forms ω_i^j, $1 \leq i, j \leq n$, satisfying:

(i) $d\omega^i = \sum \omega_k^i \wedge \omega^k$;

(ii) $\omega_i^j + \omega_j^i = 0$;

And we also have:

(iii) $d\omega_i^j - \sum_k \omega_i^k \wedge \omega_k^j = \sum_{k<\ell} \omega^k \wedge \omega^\ell = \Omega_i^j = \Omega_{ij}$.

Relative to these frames the matrix

$$(\Omega_{ij}(X, Y))$$

of the curvature operator $R(X, Y)$ is a skew-symmetric matrix.

Components of Curvature and Connection Forms

Corollary

Let Γ_{ij}^k denote the coefficients of the connection forms relative to coordinate frames $E_1, \ldots, E_n$ of a coordinate neighborhood U, φ. That is, with $\theta^1, \ldots, \theta^n$ being dual to $E_1, \ldots, E_n$,

$$\theta_j^k = \sum_{\ell} \Gamma_{\ell j}^k \theta^\ell.$$

Then $\Gamma_{ij}^k = \Gamma_{ji}^k$ and

$$R_{ik\ell}^j = \frac{\partial \Gamma_{i\ell}^j}{\partial x^k} - \frac{\partial \Gamma_{ik}^j}{\partial x^\ell} + \sum_h (\Gamma_{ik}^h \Gamma_{h\ell}^j - \Gamma_{i\ell}^h \Gamma_{hi}^j).$$

Components of Curvature and Connection Forms (Cont'd)

- According to the theorem $\Omega_i^j = d\theta_i^j - \sum_h \theta_i^h \wedge \theta_h^j$.
  Hence
  $$\Omega_i^j = \sum_\ell (d\Gamma_{\ell i}^j \wedge \theta^\ell + \Gamma_{\ell i}^j d\theta^\ell) - \sum_{k,\ell} \sum_h \Gamma_{ki}^h \Gamma_{\ell h}^j \theta^k \wedge \theta^\ell.$$
  Now $\Gamma_{ij}^k = \Gamma_{ji}^k$, since $[E_i, E_j] = 0$ for coordinate frames.
  Since $\theta^j \wedge \theta^i = -\theta^i \wedge \theta^j$, it follows that
  $$d\theta^i = \sum_j \theta^j \wedge \theta_j^\ell = \sum_{i,j} \Gamma_{ij}^\ell \theta^j \wedge \theta^i = 0.$$
  Therefore, the second equation above may be written as
  $$\begin{aligned} \frac{1}{2} \sum_{k,\ell=1}^n R_{ik\ell}^j \theta^k \wedge \theta^\ell &= \sum_{k,\ell} \frac{1}{2} \left(\frac{\partial \Gamma_{\ell i}^j}{\partial x^k} - \frac{\partial \Gamma_{ki}^j}{\partial x^\ell} \right) \theta^k \wedge \theta^\ell \\ &\quad - \frac{1}{2} \sum_{k,\ell} \sum_h (\Gamma_{ki}^h \Gamma_{\ell h}^j - \Gamma_{\ell i}^h \Gamma_{kh}^j) \theta^k \wedge \theta^\ell. \end{aligned}$$

Components of Curvature and Connection Forms (Cont'd)

- Now the coefficients on both left and right are skew-symmetric in the indices k, ℓ.

  So these equations imply equality of coefficients.

  To obtain the (standard) formula of the corollary, one uses:
  - The symmetry of Γ_{ij}^k in i, j;
  - The fact that $\theta^k \wedge \theta^\ell = -\theta^\ell \wedge \theta^k$;
  - Change of index of summation where necessary.

Manifolds of Dimension 2

Corollary

If $\dim M = 2$, then

$$d\omega_1^2 = \Omega_1^2 = +K\omega^1 \wedge \omega^2,$$

where K is the Gaussian curvature of M.

- In proving Gauss's Theorema Egregium we saw that if E_1, E_2 are orthonormal unit vectors, then

$$K = -R(E_1, E_2, E_1, E_2) = -(R(E_1, E_2) \cdot E_1, E_2) = -R_{1212}.$$

On the other hand since $g_{ij} = (E_i, E_j) = \delta_{ij}$ we have

$$\Omega_1^2 = \Omega_{12} = -R_{1212}\omega^1 \wedge \omega^2.$$

Manifolds of Dimension 2 (Cont'd)

- Now $\omega_i^j + \omega_j^i = 0$.
  So we get
  $$\omega_1^1 = 0 = \omega_2^2.$$
  Thus, by the preceding corollary,
  $$\sum_{k=1}^{2} \omega_1^k \wedge \omega_k^2 = 0 \quad \text{and} \quad d\omega_1^2 = \Omega_1^2.$$
- Note that these equations are independent of the particular orthonormal frame field on $U \subseteq M$.

Geometric Interpretation of Sectional Curvature

- Let M be a Riemannian manifold.
- Let π be a plane section at a point p of M.
- Let N_p be an open, two-dimensional submanifold of M:
  - Consisting of geodesic arcs through p;
  - Tangent at p to the section π.

Theorem

If we use on N_p the Riemannian metric induced by that of M, then the sectional curvature $K(\pi)$ is equal to the Gaussian curvature of N_p at p.

- Consider a normal neighborhood of p

$$U = \exp_p B_\varepsilon.$$

That is, we choose $\varepsilon > 0$ such that

$$B_\varepsilon = \{X_p \in T_p(M) : \|X_p\| < \varepsilon\}$$

is mapped diffeomorphically onto an open set $U \subseteq M$.

Geometric Interpretation of Sectional Curvature (Cont'd)

- The plane section π corresponds to a two-dimensional subspace $V_\pi \subseteq T_p(M)$.

  We may suppose that N_p is the image of $V_\pi \cap B_\varepsilon$.

  Since U is a normal neighborhood, it is covered simply by the geodesics of length ε issuing from p.

  They are given by

  $$\exp_p tX_p, \quad 0 \leq t \leq \varepsilon,$$

  for each X_p with $\|X_p\| = 1$.

Geometric Interpretation of Sectional Curvature (Cont'd)

- Now choose an orthonormal basis $E_{1p}, \ldots, E_{np}$ of $T_p(M)$, with E_{1p}, E_{2p} a basis of V_π.

  Then

  $$(x^1, \ldots, x^n) \to \exp_p\left(\sum x^i E_{ip}\right)$$

  establishes a system of normal coordinates on U.

  Moreover, the coordinate map φ is the inverse of the above.

  Thus, N_p is described by

  $$x^3 = \cdots = x^n = 0.$$

  Additionally, $U \cap N_p, \varphi$ is a coordinate system on N_p, with x^1, x^2 as coordinates.

Geometric Interpretation of Sectional Curvature (Cont'd)

- Let $E_1, \ldots, E_n$ denote the coordinate frames.

  They agree at p with the given frame.

  Moreover, E_1, E_2 are tangent to N_p everywhere on N_p.

  We denote the dual coframes by $\theta^1, \ldots, \theta^n$, with connection forms

$$\theta_j^k = \sum_i \Gamma_{ij}^k \theta^i.$$

  Note that $\Gamma_{ij}^k(0) = 0$.

  That is, $\theta_j^k = 0$ at $p \in U$.

  From those frames, by the Gram-Schmidt process we obtain a family of orthonormal frames $F_1, \ldots, F_n$ in U with the property that F_1, F_2 are a linear combination of E_1, E_2.

  So F_1, F_2 are tangent to N_p at each of its points.

Geometric Interpretation of Sectional Curvature (Cont'd)

- We denote by $\omega^1, \ldots, \omega^n$ the dual coframes to $F_1, \ldots, F_n$.

  We let ω_i^j be the corresponding connection forms.

  They satisfy the equations

$$\omega_i^j + \omega_j^i = 0 \quad \text{and} \quad d\omega^i = \sum_k \omega_k^i \wedge \omega^k.$$

  We shall see that for $j > 2$, $\omega_1^j = \omega_2^j = 0$ at p.

  First recall that at p,

$$\nabla_{X_p} E_i = \sum_j \theta_i^j(X_p) E_j = 0 \quad \text{and} \quad \nabla_{X_p} F_i = \sum \omega_i^j(X_p) F_j.$$

Geometric Interpretation of Sectional Curvature (Cont'd)

- Now, for $i = 1, 2$,

$$F_i = a_i^1 E_1 + a_i^2 E_2.$$

So

$$\nabla_{X_p} F_i = (X_p a_i^1) E_1 + (X_p a_i^2) E_2 + a_i^1 \nabla_{X_p} E_1 + a_i^2 \nabla_{X_p} E_2.$$

Since $\Gamma_{ij}^k(0) = 0$, the last two terms vanish.

So, for $i = 1, 2$, $\nabla_{X_p} F_i$ is a linear combination of E_1 and E_2.

Hence, $\nabla_{X_p} F_i$ is a linear combination of F_1 and F_2.

Thus, for $i = 1, 2$,

$$\nabla_{X_p} F_i = \omega_i^1(X_p) F_1 + \omega_i^2(X_p) F_2.$$

Moreover, for $i = 1, 2$ and $j > 2$, $\omega_i^j(X_p) = 0$.

Geometric Interpretation of Sectional Curvature (Cont'd)

- Denote by $I : N_p \to M$ the imbedding.
  Let $\widetilde{\omega}^i = I^* \omega^i$, $\widetilde{\omega}_i^j = I^* \omega_i^j$.
  I^* is a homomorphism of $\bigwedge(M) \to \bigwedge(N_p)$ and commutes with d.
  So
  $$d\widetilde{\omega}^i = \sum_k \widetilde{\omega}_k^i \wedge \widetilde{\omega}^k \quad \text{and} \quad \widetilde{\omega}_i^j + \widetilde{\omega}_j^i = 0.$$
  F_1, F_2 span the tangent space to N_p.
  Moreover, if $j = 1$ or $j = 2$ and $i > j$,
  $$\widetilde{\omega}^i(F_j) = (I^* \omega^i)(F_j) = \omega^i(I_* F_j) = \omega^i(F_j) = 0.$$
  Therefore, for $i > 2$, $\widetilde{\omega}^i = 0$.
  Thus, $\widetilde{\omega}^1, \widetilde{\omega}^2$ are dual to F_1, F_2 restricted to N_p.
  Moreover, together with $\widetilde{\omega}^1 = \widetilde{\omega}^2$, they satisfy Equations (i) and (ii), which determine the connection forms uniquely.

Geometric Interpretation of Sectional Curvature (Cont'd)

- It follows from the preceding corollary that

$$d\widetilde{\omega}_1^2 = K\widetilde{\omega}^1 \wedge \widetilde{\omega}^2.$$

On the other hand, we have on M

$$d\omega_1^2 = \sum_k \omega_1^k \wedge \omega_k^2 + \sum_{k<\ell} R_{12k\ell}\omega^k \wedge \omega^\ell.$$

Apply I^* to both sides and evaluate at p.
We get the equality (at p)

$$d\widetilde{\omega}_1^2 = R_{1212}\widetilde{\omega}^1 \wedge \widetilde{\omega}^2.$$

It follows that the sectional curvature

$$K(\pi) = -R_{1212} = K_p,$$

the Gaussian curvature at p of the surface N_p.

The Curvature of an n-Sphere

Corollary

Let M be an n-sphere of radius a in $\mathbb{R}^{n+1}$ with the Riemannian metric induced from $\mathbb{R}^{n+1}$. Then M has constant sectional curvature $\frac{1}{a^2}$.

- Let p be a point of M.

  Then the geodesics through p tangent to a plane π in $T_p(M)$ are great circles.

  They form a 2-sphere of radius a.

  We have seen that the Gaussian curvature of such a 2-sphere is $\frac{1}{a^2}$.

  So the corollary follows from the theorem.

Isotropic Manifolds and Constant Curvature

Theorem

If M is a connected, isotropic Riemannian manifold and $\dim M > 3$, then M has constant curvature.

- Let K_p be the value of the sectional curvature at p.
  This i constant on all sections by hypothesis.
  We must show that this function on M is constant.
  That is, w must show $dK = 0$.
  Let U be a neighborhood of $p \in M$ with an orthonormal frame field.
  Let $\omega^1, \ldots, \omega^n$ be the dual coframe field.
  We use the expression for $R_{ijk\ell}$ in a previous corollary, which now becomes
  $$R_{ijk\ell} = K(\delta_{ik}\delta_{j\ell} - \delta_{i\ell}\delta_{jk}).$$
  We obtain $\Omega_i^j = \Omega_{ij} = K\omega^i \wedge \omega^j$, in which K depends only on p, not on the (orthonormal) frames used.

Isotropic Manifolds and Constant Curvature (Cont'd)

- Take the exterior derivative of the structure equation

$$d\omega_i^j = \sum \omega_i^k \wedge \omega_k^j + \Omega_i^j.$$

We obtain

$$\begin{array}{rcl} 0 & = & \sum(d\omega_i^k \wedge \omega_k^j - \omega_i^k \wedge d\omega_k^j) \\ & & +dK \wedge \omega^i \wedge \omega^j + Kd\omega^i \wedge \omega^j - K\omega^i \wedge d\omega^j. \end{array}$$

We substitute for $d\omega_i^k$, $d\omega^i$, and so on, from a previous corollary.
After simplifying, we get, for all $i, j = 1, \ldots, n$,

$$dK \wedge \omega^i \wedge \omega^j = 0.$$

Now $dK = K_1\omega^1 + \cdots + K_n\omega^n$, a linear combination of $\omega^1, \ldots, \omega^n$.
Moreover, $\omega^\ell \wedge \omega^i \wedge \omega^j \neq 0$, if ℓ, i, j are distinct.
So the displayed equation can only hold if $dK = 0$ on U.
But U is a neighborhood of p and p is arbitrary.
Therefore, $dK = 0$ and K is constant.

Examples

- According to the preceding corollary, the sphere of radius a with the Riemannian metric induced by the Euclidean space with contains it has constant positive curvature.
- Euclidean space itself with its standard Riemannian metric has curvature identically zero, since with the usual coordinates $\Gamma_{ij}^k = 0$ and $R_{ijk\ell} = 0$.
- An example of a manifold of constant negative curvature of arbitrary dimension will be given later.

Subsection 5

Differentiation of Covariant Tensor Fields

Translation Along a Curve

- Let M be a Riemannian manifold.
- Consider a covariant tensor field Φ of order r on M, $\Phi \in \mathscr{T}^r(M)$.
- Suppose given a curve

$$p(t), \quad a \leq t \leq b,$$

  on M of differentiability class C^1 at least.
- Let $\Phi_{p(t)}$ denote the restriction of Φ to $p(t)$.
- Then $\Phi_{p(t)} \in \mathscr{T}^r(T_{p(t)}(M))$, that is, $\Phi_{p(t)}$ is a tensor field along $p(t)$.
- Using previous results, we denote by τ_t parallel translation along $p(t)$ from a fixed point $p(t_0)$ of the curve,

$$\tau_t : T_{p(t_0)}(M) \to T_{p(t)}(M).$$

- This is an isomorphism of these tangent spaces.
- It is uniquely determined by $p(t)$ and the Riemannian structure.

Derivative of Tensor Along a Curve

Definition

With the preceding notation, the **derivative** $\frac{D\Phi}{dt}$ **of the tensor** Φ **along the curve** is defined at the point $p(t_0)$ by

$$\left(\frac{D\Phi}{dt}\right)_{t_0} = \lim_{t \to t_0} \frac{1}{t - t_0} \left(\tau_t^* \Phi_{p(t)} - \Phi_{p(0)}\right).$$

- As thus defined $(\frac{D\Phi}{dt})_{t_0}$ is a covariant tensor of order r on the vector space $T_{p(t_0)}(M)$.

Derivative of Tensor Along a Curve (Cont'd)

- Consider any set of r vectors $X^1_{p(t_0)}, \ldots, X^r_{p(t_0)} \in T_{p(t_0)}(M)$.
- Then $\frac{D\Phi}{dt}$ at $p(t_0)$ is the limit as $t \to t_0$ of the expression

$$\frac{1}{t - t_0}(\tau_t^* \Phi_{p(t)}(X^1_{p(t_0)}, \ldots, X^r_{p(t_0)}) - \Phi_{p(t_0)}(X^1_{p(t_0)}, \ldots, X^r_{p(t_0)})).$$

- For each value of t near t_0, this is a multiple by $\frac{1}{t-t_0}$ of the difference of two tensors $\tau_t^* \Phi_{p(t)}$ and $\Phi_{p(t)}$ on $T_{p(t_0)}(M)$.
- Both are covariant r tensors on the same vector space.
- It follows that the limit is also such a tensor.
- We repeat this procedure at each t_0 on the interval (a, b).
- The process gives a covariant tensor field $\frac{D\Phi}{dt}$ along $p(t)$, provided that suitable differentiability conditions are satisfied.

Differentiability Conditions

- Satisfying "suitable" differentiability conditions means that, for any C^k family of vector fields
$$X_t^i = X_{p(t)}^i, \quad i = 1, \ldots, r,$$
defined along the C^k curve $p(t)$, the value of $\frac{D\Phi}{dt}$ on them,
$$\frac{D\Phi}{dt}(X_t^1, \ldots, X_t^r), \quad a < t < b,$$
should be a function of class C^{k-1} (C^∞ when $k = \infty$) of t.
- This should be true in the most frequent situation where:
  - $X^1, \ldots, X^r$ are C^∞-vector fields on M;
  - $X_t^1, \ldots, X_t^r$ are their restrictions to the curve $p(t)$.
- In the next result, we show that this is indeed a consequence of our definition and derive computational formulas.
- For convenience, we suppose Φ is C^∞.

A Formula for the Derivative

Lemma

Let Φ be a C^∞-covariant tensor field of order r on M.
Let $p(t)$, $a < t < b$, be a curve of class C^k, $k \geq 1$, on M.
Let $X_t^1, \ldots, X_t^r \in T_{p(t)}(M)$ be vector fields of class C^k along the curve.
Then, for each t_0 on the interval (a, b), we have

$$\left(\frac{D\Phi}{dt}\right)_{t_0}(X_{t_0}^1, \ldots, X_{t_0}^r) = \left(\frac{d}{dt}[\Phi_{p(t)}(X_t^1, \ldots, X_t^r)]\right)_{t=t_0} - \sum_{i=1}^r \Phi_{p(t_0)}\left(X_{t_0}^1, \ldots, \left(\frac{DX^i}{dt}\right)_{t_0}, \ldots, X_{t_0}^r\right).$$

- The lemma will establish the fact that $\frac{D\Phi}{dt}$ evaluated on C^k-vector fields along the curve is differentiable of class C^{k-1} at least.
- If $k = \infty$, then $\frac{D\Phi}{dt}$ will be a C^∞-tensor field along the curve.
- That is, its value on C^∞-vector fields will be a C^∞ function of t.
- For lower differentiability classes, the class of $\frac{D\Phi}{dt}$ will also be lower.

Proof of the Formula

- By definition we have

$$\begin{array}{rcl}\left(\frac{D\Phi}{dt}\right)_{t_0} & = & \lim_{t\to t_0}\frac{1}{t-t_0}(\tau_t^*\Phi_{p(t)}(X_{t_0}^1,\ldots,X_{t_0}^r)-\Phi_{p(t_0)}(X_{t_0}^1,\ldots,X_{t_0}^r)) \\ & = & \lim_{t\to t_0}\frac{1}{t-t_0}(\Phi_{p(t)}(\tau_t(X_{t_0}^1),\ldots,\tau_t(X_{t_0}^r)) \\ & & \qquad -\Phi_{p(t_0)}(X_{t_0}^1,\ldots,X_{t_0}^r)).\end{array}$$

Then for each $i=1,\ldots,r$, in turn, we subtract and add

$$\Phi_{p(t)}\left(X_t^1,X_t^2,\ldots,X_t^i,\tau_t(X_{t_0}^{i+1}),\ldots,\tau_t(X_{t_0}^r)\right).$$

Rearranging and collecting terms, and using both linearity at $p(t)$ and the continuity of the tensor Φ, we may rewrite the defining equation

$$\begin{array}{rcl}\left(\frac{D\Phi}{dt}\right)_{t_0} & = & \sum_{i=1}^r\Phi_{p(t)}(X_t^1,\ldots,\lim_{t\to t_0}\frac{1}{t-t_0}(\tau_t(X_{t_0}^i)-X_t^i), \\ & & \qquad \tau_t(X_{t_0}^{i+1}),\ldots,\tau_t(X_{t_0}^r)) \\ & & +\lim_{t\to t_0}\frac{1}{t-t_0}(\Phi_{p(t)}(X_t^1,\ldots,X_t^r)-\Phi_{p(t_0)}(X_{t_0}^1,\ldots,X_{t_0}^r)).\end{array}$$

Proof of the Formula (Cont'd)

- We now use the fact that for any C^k-vector field X_t along $p(t)$,

$$\begin{aligned}\lim_{t\to t_0}\frac{\tau_t(X_{t_0})-X_t}{t-t_0} &= -\lim_{t\to t_0}\tau_t\left(\frac{\tau_{-t}(X_t)-X_{t_0}}{t-t_0}\right)\\ &= -\tau_0\left(\frac{DX_t}{dt}\right)_{t_0}\\ &= -\left(\frac{DX_t}{dt}\right)_{t_0}.\end{aligned}$$

  Therefore passing to the limit in the expression for $(\frac{D\Phi}{dt})_{t_0}$ completes the proof of the lemma.
- We can verify from the formula itself that $(\frac{D\Phi}{dt})_{t_0}$ depends $\mathbb{R}$-linearly on the values of the vector fields $X_t^1,\ldots,X_t^r$ at $p(t_0)$.
- So the formula does define an $\mathbb{R}$-linear function, that is, a covariant tensor of order r on the vector space $T_{p(t_0)}(M)$.

The Case of Parallel Vector Fields

Corollary

Let $X_0^1, \ldots, X_0^r \in T_{p(t_0)}(M)$ be given and suppose that $X_t^1, \ldots, X_t^r$ are the uniquely determined parallel vector fields along $p(t)$, $a < t < b$, which take these values at $p(t_0)$. Then the formula of the preceding lemma becomes

$$\left(\frac{D\Phi}{dt}\right)_{t_0}(X_{t_0}^1, \ldots, X_{t_0}^r) = \left(\frac{d}{dt}\Phi_{p(t)}(X_t^1, \ldots, X_t^r)\right)_{t_0}.$$

- By definition of X_t^i we have $\frac{DX_t^i}{dt} \equiv 0$, $i = 1, \ldots, r$.

  So the conclusion follows from the formula of the preceding lemma.
- This corollary makes it clear that $(\frac{D\Phi}{dt})_{t_0}$ depends only on the tensor field Φ and on the curve $p(t)$, $a < t < b$.

Independence of Choice of Curve

Lemma

Let Φ be a C^∞-covariant tensor field of order r on M and $p \in M$.
Let $X^1, \ldots, X^r$ are C^∞-vector fields on some neighborhood U of p.
Let $X_p^1, \ldots, X_p^r$ denote their value at p.
Consider two C^1 curves on M, $F(t)$, $-\varepsilon < t < \varepsilon$, and $G(s)$, $-\delta < s < \delta$, such that:

- $F(0) = p = G(0)$;
- $\dot{F}(0) = Y_p = \dot{G}(0)$ is their common tangent vector at p.

Then

$$\left(\frac{D\Phi}{dt}\right)_0 (X_p^1, \ldots, X_p^r) = \left(\frac{D\Phi}{ds}\right)_0 (X_p^1, \ldots, X_p^r).$$

That is, the two tensors on $T_p(M)$ defined by differentiating Φ along each of the curves are the same.

Independence of Choice of Curve (Cont'd)

- Suppose that f is a C^∞ function on U.

  Then $f(F(t))$ is its restriction to the curve $F(t)$.

  Moreover,

$$\left(\frac{d}{dt}f(F(t))\right)_{t=0} = F_*\left(\frac{d}{dt}\right)f = Y_p f.$$

  Similarly, restricting f to $G(s)$, differentiating with respect to s and evaluating at $s = 0$ gives

$$\left(\frac{d}{ds}f(G(s))\right)_{t=0} = G_*\left(\frac{d}{dt}\right)f = Y_p f.$$

Independence of Choice of Curve (Cont'd)

- We apply the preceding to the function
$$f(q) = \Phi_q(X_q^1, \ldots, X_q^r).$$
We see that in the formula of the lemma, the first term in case of either curve (and derivative of Φ) is the same, namely
$$Y_p(\Phi(X^1, \ldots, X^r)).$$
On the other hand, by our original definition of $\nabla_{Y_p} X$ for a vector field X, we have
$$\nabla_{Y_p} X = \left(\frac{DX^i_{p(t)}}{dt} \right)_0 = \left(\frac{DX^i_{p(s)}}{ds} \right)_0 .$$
Hence, the remaining terms in the formula agree also.

The Covariant Derivative

- We denote the covariant tensor of order r on $T_p(M)$, which we have defined by differentiation of Φ along curves through p with Y_p as tangent at p by

$$\nabla_{Y_p}\Phi = \left(\frac{D\Phi}{dt}\right)_0 (X_p^1, \ldots, X_p^r).$$

Definition

The covariant r-tensor on $T_p(M)$ just defined from differentiation of Φ along curves through p, with Y_p as tangent at p, is denoted

$$\nabla_{Y_p}\Phi \in \mathscr{T}^r(T_p(M)).$$

It is called the **covariant derivative of Φ at p in the direction Y_p**.

Comments

- According to the facts in the proof above, the covariant derivative is given by the formula

$$\begin{array}{rcl}\nabla_{Y_p}\Phi(X^1,\ldots,X^r) & = & Y_p(\Phi(X^1,\ldots,X^r) \\ & & -\sum_{i=1}^r \Phi_p(X^1,\ldots,\nabla_{Y_p}X^i,\ldots,X_p^r)),\end{array}$$

  where $X^1,\ldots,X^r$ are vector fields on a neighborhood of p.
- Only the values of $X^1,\ldots,X^r$ at p affect the value of $\nabla_{Y_p}\Phi$ on $T_p(M)$.

The Covariant $r+1$ Tensor Field Ψ

Theorem

Let Φ be a C^∞-covariant tensor field of order r on M, $\Phi \in \mathscr{T}^r(M)$. Then we may define on M a C^∞-covariant tensor field Ψ of order $r+1$ by the formula

$$\Psi_p(X_p^1, \ldots, X_p^r; Y_p) = (\nabla_{Y_p}\Phi)(X_p^1, \ldots, X_p^r).$$

- By preceding work, it is only necessary to prove two more facts.
  - For each $p \in M$, Ψ_p is linear in the last variable, with the others fixed;
  - For any C^∞-vector fields $X^1, \ldots, X^r, Y$ the formula above defines a C^∞ function of p.

The Covariant $r+1$ Tensor Field Ψ (Cont'd)

- Note that each term of the formula is linear in Y_p as a real-valued function on $T_p(M)$.

  Consequently, if we fix the vector fields $X^1, \ldots, X^r$, then the mapping $T_p(M) \to \mathbb{R}$ defined by that formula

  $$Y_p \to (\nabla_{Y_p}\Phi)(X_p^1, \ldots, X_p^r)$$

  is linear.

  On the other hand, it is clear that for C^∞-vector fields $X^1, \ldots, X^r; Y$ the function

  $$\Psi(X^1, \ldots, X^r; Y) = (\nabla_Y \Phi)(X_1, \ldots, X_r)$$

  is C^∞.

Components in Local Coordinates

- Let U, φ be a local coordinate system with:
  - Local coordinates $x^1, \ldots, x^n$;
  - Coordinate frames $E_1, \ldots, E_n$, such that

$$\nabla_{E_i} E_j = \sum_k \Gamma_{ij}^k E_k.$$

- Let Φ be a C^∞-covariant tensor field of order r on M.
- Let its components be

$$\Phi_{i_1 \ldots i_r} = \Phi(E_{i_1}, \ldots, E_{i_r}).$$

Formulas in Local Coordinates

Corollary

Let Φ be a C^∞-covariant tensor field of order r on M.
Consider the C^∞-covariant tensor field Ψ of order $r+1$ given by

$$\Psi_p(X_p^1,\ldots,X_p^r;Y_p)=(\nabla_{Y_p}\Phi).$$

The components

$$\Psi_{j_1,\ldots,j_{r+1}}=\Psi(E_{j_1},\ldots,E_{j_{r+1}})$$

of Ψ on U are given by the formulas

$$\Psi_{j_1,\ldots,j_{r+1}}=\frac{\partial}{\partial x_{j_{r+1}}}\Phi_{j_1\cdots j_r}-\sum_{k,i}\Gamma^k_{j_{r+1}j_i}\Phi_{j_1\cdots k\cdots j_r},$$

$$k=1,\ldots,n,\ i=1,\ldots,r.$$

Parallel Tensor Fields

Definition

A tensor field $\Phi \in \mathscr{T}^r(M)$ is said to be **parallel along a curve** $p(t)$ if

$$\frac{D\Phi}{dt} \equiv 0$$

along the curve. It is said to be **parallel** if

$$\frac{D\Phi}{dt} = 0$$

along every curve on M.

Remarks

- If, for every $X_p \in T_p(M)$ and all $p \in M$,

$$\nabla_{X_p}\Phi = 0,$$

then Φ is parallel.
- So if it is parallel along geodesics, for example, then it will be parallel.
- This follows from the preceding lemma and the fact that there is a geodesic tangent to any given vector X_p.
- Suppose, also, that

$$p(t), \quad a \leq t \leq b,$$

is a curve of class C^1, say.
- Then Φ is parallel along $p(t)$ if and only if it satisfies

$$\frac{d}{dt}(\Phi(X_t^1, \ldots, X_t^r)) \equiv 0,$$

for every set $X_t^1, \ldots, X_t^r$ of parallel vector fields along the curve $p(t)$.

Parallel Sections and Constant Curvature

- Let M be a Riemannian manifold of constant curvature K.
- Then, by definition, for any orthonormal pair of vectors X_p, Y_p the sectional curvature $R(X_p, Y_p, X_p, Y_p) = -K$.
- Suppose $p(t)$ is any curve through p with, say, $p(0) = p$.
- Let $X_{p(t)}, Y_{p(t)}$ be the uniquely determined parallel fields such that $X_p = X_{p(0)}$ and $Y_p = Y_{p(0)}$.
- Then $X_{p(t)}, Y_{p(t)}$ is orthonormal at each $p(t)$.

Parallel Sections and Constant Curvature (Cont'd)

- Moreover,
$$R(X_{p(t)}, Y_{p(t)}, X_{p(t)}, Y_{p(t)}) = -K,$$
a constant independent of t.
- It follows that, for any parallel vector fields along $p(t)$, say
$$X_t^i, \quad i = 1, 2, 3, 4,$$
we have
$$\frac{d}{dt} R(X_t^1, X_t^2, X_t^3, X_t^4) \equiv 0.$$
- Indeed the values of all of the sectional curvatures uniquely determine the curvature.
- Thus the curvature is parallel if it is constant on parallel sections π_t along any curve $p(t)$.

Symmetric Spaces and Parallel Curvature Tensors

Theorem (Cartan)

If M is a Riemannian symmetric space, then the curvature tensor is parallel.

- Any isometry of a Riemannian manifold preserves parallelism.

  It carries parallel vector fields, sections, and so on, along a curve to parallel vector fields, sections, and so on, along the image.

  Moreover, isometries preserve the curvature,

  $$R_p(X_p, Y_p, Z_p, W_p) = R_{F(p)}(X_{F(p)}, Y_{F(p)}, Z_{F(p)}, W_{F(p)}).$$

  Finally isometries carry geodesics to geodesics.

  This is because parallelism, curvature and geodesics are all defined in terms of the Riemannian metric.

Symmetric Spaces and Parallel Curvature Tensors (Cont'd)

- Now to show that the curvature is parallel, it is enough to show that it is constant on parallel vector fields along geodesics.

  Suppose $p(t)$ is a geodesic.

  Then, according to a previous theorem, the vectors

  $$X_{p(t)}, Y_{p(t)}, Z_{p(t)}, W_{p(t)}$$

  of the parallel vector field determined by $X_{p(0)}, Y_{p(0)}, \ldots$ are given by isometries τ_c of M .

  Therefore, the curvature is constant on parallel fields along the geodesic $p(t)$.

Remarks

- This is more general than constant curvature.
- We have seen an example of a symmetric space - a compact semisimple Lie group G with bi-invariant metric - in which the curvatures on various sections π_e at the identity vary between 0 (if there is an Abelian subgroup of dimension two) and a positive maximum value.
- Thus G is not isotropic.
- Hence, it is not of constant curvature in this metric.
- However, it does have parallel curvature.
- This raises the interesting question of how those Riemannian manifolds with parallel curvature may be otherwise characterized or described.
- The answer to this is given by the following two theorems which are stated without proof.

Manifolds With Parallel Curvature

Theorem (Cartan)

Let M be a Riemannian manifold with parallel curvature. Then M is locally symmetric. That is, each point $p \in M$ has a neighborhood U, such that, there is an involutive isometry $\sigma_p : U \to U$, with p as its only fixed point.

- Of course, a manifold may be locally symmetric without being globally symmetric, that is, symmetric in the sense of our original definition of symmetric space.
- For example, Euclidean space or a sphere, with its usual Riemannian metric, is no longer a symmetric space if a single point is removed, since we have seen that a symmetric space is necessarily complete.
- But it is still locally symmetric.
- Even if completeness is assumed, together with parallel curvature, we still cannot be quite sure that the space is symmetric.

Manifolds With Parallel Curvature

- However, if the Riemannian manifold is complete and has parallel curvature, then we may be sure that its universal covering (with the naturally induced Riemannian metric) is a symmetric space.

Theorem (Cartan-Ambrose)

Let M and N be complete, connected Riemannian manifolds of the same dimension, each with parallel curvature, and suppose further that M is simply connected.
Let $p \in M$ and $q \in N$ and

$$\varphi : T_p(M) \to T_q(N)$$

a linear mapping which preserves the inner product and the curvature.

Manifolds With Parallel Curvature (Cont'd)

Theorem (Cartan-Ambrose Cont'd)

That is, for arbitrary $X_p, Y_p, Z_p, W_p \in T_p(M)$, we have

$$\begin{aligned} (\varphi(X_p), \varphi(Y_p))_q &= (X_p, Y_p)_p, \\ R_q(\varphi(X_p), \varphi(Y_p), \varphi(Z_p), \varphi(W_p)) &= R_p(X_p, Y_p, Z_p, W_p). \end{aligned}$$

Then there is a unique C^∞ mapping $F : M \to N$ with the properties:

(i) $F(p) = q$;

(ii) $F_* : T_p(M) \to T_q(N)$ is the same as φ;

(iii) F is a Riemannian covering (that is, it is a covering such that F_* is an isometry on each tangent space and, thus, a local isometry).

Subsection 6

Manifolds of Constant Curvature

Curvature Forms

- Let M be a Riemannian manifold.
- Let $E_1, \ldots, E_n$ be an orthonormal frame field on an open set $U \subseteq M$.
- Let ω^i, $1 \leq i \leq n$, denote the field of coframes dual to $E_1, \ldots, E_n$.
- Let ω_i^j, $1 \leq i, j \leq n$, denote the corresponding connection forms.
- Based on preceding results, we have

Lemma

Let M have constant curvature K. Then the curvature forms $\Omega_i^j = d\omega_i^j + \sum_k \omega_i^k \wedge \omega_k^j$ are given by

$$\Omega_i^j = K\omega^i \wedge \omega^j.$$

Assume, conversely, that on a neighborhood U of each point of M there is an orthonormal frame field $E_1, \ldots, E_n$ for which the uniquely determined ω^i, ω^j satisfy this equation. Then M has constant curvature K.

Curvature Range

- Recall that Euclidean space with its standard Riemannian metric is a space of zero curvature.
- Also, the n-sphere of radius a in $\mathbb{R}^{n+1}$ with the induced Riemannian metric has constant curvature $K = \frac{1}{a^2}$.
- Thus for every nonnegative real number K, we have already found an example of Riemannian manifold of arbitrary dimension n with constant curvature K.
- We now give an example of an n-dimensional Riemannian manifold of constant curvature $K = -1$.
- A slight variation can produce an example for any $K < 0$.

Example: Hyperbolic Space

- Let M be the open upper half-space of $\mathbb{R}^n$ defined by

$$M = \{x \in \mathbb{R}^n : x^n \geq 0\}.$$

- The Riemannian metric given by the line element

$$ds^2 = \frac{(dx^1)^2 + \cdots + (dx^n)^2}{(x^n)^2}.$$

- More precisely, we note that, as a manifold, M is covered by a single coordinate system with:
  - Local coordinates $x^1, \ldots, x^n$;
  - Coordinate frames $\frac{\partial}{\partial x^1}, \ldots, \frac{\partial}{\partial x^n}$.
- This is because, as a manifold, M corresponds to an open subset of $\mathbb{R}^n$.

Example: Hyperbolic Space (Cont'd)

- In these local coordinates, the components of the Riemannian metric Φ are given by
$$g_{ij} = \Phi\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right) = \frac{\delta_{ij}}{(x^n)^2}.$$
- We use the preceding lemma to see that this manifold has constant curvature $K = -1$.
- Let
$$E_i = x^n \frac{\partial}{\partial x^i}, \quad i = 1, \ldots, n.$$
- These define an orthonormal frame field on all of M.
- We denote by $\omega^1, \ldots, \omega^n$ the dual coframes.
- They are given by
$$\omega^i = \frac{1}{x^n} dx^i, \quad i = 1, \ldots, n.$$

Example: Hyperbolic Space (Cont'd)

- Consider the forms
$$\omega_i^j = \delta_{nj}\omega^i - \delta_{ni}\omega^j.$$
- It is easy to verify that they satisfy the equations
$$d\omega^i = \sum_{j=1}^n \omega^j \wedge \omega_j^i \quad \text{and} \quad \omega_i^j + \omega_j^i = 0.$$
- Hence, they must be the connection forms, since these are uniquely determined by these conditions.
- Finally, taking the exterior derivative of ω_i^j, we obtain
$$\Omega_i^j = d\omega_i^j - \sum \omega_i^k \wedge \omega_k^j = -\omega^i \wedge \omega^j.$$
- Then, by the preceding lemma, M has constant curvature $K = -1$.
- We call this **hyperbolic space**.
- It is denoted by H^n (for its underlying space, the "half-plane").

Simple Connectedness and Completeness

- Now we have examples of spaces of positive, zero, and negative constant curvature.
- Note that all three examples are simply connected.
  - When $K > 0$, our example was the compact manifold S^n;
  - When $K = 0$ or $K = -1$, the corresponding manifolds $\boldsymbol{E}^n$ and H^n are diffeomorphic to $\mathbb{R}^n$.
- Since S^n is compact, it is complete.
- We also know $\boldsymbol{E}^n$ to be a complete Riemannian manifold.
- We shall prove later that H^n is complete.

Complete Simply Connected of Constant Curvature

Theorem

Every complete, simply connected Riemannian manifold M of constant curvature $K = +1, 0$ or -1 is isometric to one of the three examples above:

- To S^n, if $K = +1$;
- To $\boldsymbol{E}^n$, if $K = 0$;
- To H^n, if $K = -1$.

Manifolds of Constant Curvature (Cont'd)

Theorem (Cont'd)

More precisely, let $p \in M$, and q in either $S^n, \boldsymbol{E}^n$ or H^n according to whether $K = +1, 0$ or -1. Assume, also, given a prescribed linear map of $T_p(M)$ onto the tangent space at q which preserves the inner product. Then there is exactly one isometry F of M to the corresponding space of constant curvature:

- Taking p to q;
- Such that F_* corresponds to the given linear mapping on $T_p(M)$.

- This is an immediate consequence of the Cartan-Ambrose Theorem once we know that H^n is complete (proved later).

Isometries

Corollary

Let M be S^n, $\boldsymbol{E}^n$ or H^n and let $E_{1p}, \ldots, E_{np}$, $E_{1q}, \ldots, E_{nq}$ be orthonormal frames at two arbitrary points p, q of M. Then there is a unique isometry of M, that takes:

- p to q;
- E_{ip} to E_{iq}, $i = 1, \ldots, n$.

- This shows that the group of isometries is transitive on M.
- So it is plausible that in each of these cases this is a Lie group.
- We already know this, however, for:
  - S^n, whose group of isometries is $O(n+1)$;
  - $\boldsymbol{E}^n$, whose group of isometries consists of rotations and translations and their products.
- We will study the group of all isometries of H^n only for $n = 2$.

Riemannian Coverings

- Let M be a Riemannian manifold.
- Let $\widetilde{M}$ a covering manifold, with covering map $F : \widetilde{M} \to M$
- Then there is a unique Riemannian metric on $\widetilde{M}$, such that F is a local isometry.
- When M has this metric, the covering will be called a **Riemannian covering**.

Properties of Riemannian Coverings

- The following facts are quite easily verified from the definitions.
  (i) F carries geodesics to geodesics and each geodesic on M is covered by a unique geodesic on $\widetilde{M}$;
  (ii) If M is complete, then $\widetilde{M}$ is also complete (convergence of Cauchy sequences is a local phenomenon);
  (iii) The covering transformations are isometries of $\widetilde{M}$.
- With the aid of these facts one may take a step towards reducing the determination of manifolds of constant curvature to a group theoretic problem.

Universal Covering Manifolds

Theorem

Let M be a complete manifold of constant curvature $K = +1, 0$ or -1. Then the universal covering manifold $\widetilde{M}$ is isometric to S^n, $\boldsymbol{E}^n$ or H^n, respectively. Moreover, M is the orbit space of a subgroup Γ of the group of isometries of $\widetilde{M}$ which acts freely and properly discontinuously on $\widetilde{M}$.

- The theorem follows from the fact that $\widetilde{M}$ is complete, simply connected, and (since the covering mapping is a local isometry) has the same constant curvature as M.
- By the theory of covering spaces, we know that:
  - $M = \widetilde{M}/\Gamma$;
  - The covering transformations Γ act freely and properly discontinuously (as a group of isometries).
- We give some indication of how this may be used by considering some examples.

In Search of Spaces of Positive Curvature

- We look for Riemannian manifolds of constant curvature $K = +1$.
- We must find subgroups Γ of the group of isometries of S^n, the unit sphere, which act freely and properly discontinuously on S^n.
- The isometries of S^n are contained in $O(n+1)$, which acts in the usual way on the unit sphere in $\mathbb{R}^{n+1}$.
- It follows that $\Gamma \subseteq O(n+1)$.
- The assumption that Γ acts freely means that no element of Γ, except the identity, leaves a point of S^n fixed.
- Let $A \in \Gamma$ and $A \neq I$.
- Then A cannot have $+1$ as a characteristic value.

In Search of Spaces of Positive Curvature (Cont'd)

- Moreover, Γ must be a group of finite order.
- Otherwise, there must be an $x \in S^n$, such that

$$\Gamma x = \{Ax : A \in r\}$$

  has a limit point.
- This would contradict proper discontinuity.
- Thus, we must find finite subgroups of $O(n+1)$ no element of which (except the identity) leaves a vector x fixed.
- This is a necessary condition for Γ.
- However, it can be shown that it is also sufficient.

Example

- The simplest example of a subgroup Γ of $O(n+1)$ of the type described is the group consisting of two elements, $\Gamma = \{\pm I\}$.
- The orbit space S^n/Γ is the collection of all antipodal pairs of points on S^n.
- As we have seen earlier, this is just the projective space $P^n(\mathbb{R})$.
- Thus, for every n, we have at least two inequivalent spaces of constant curvature:
  - The real projective space;
  - Its universal (Riemannian) covering space S^n.

The Case of Even Dimension

Fact

If n is even, then S^n and $P^n(\mathbb{R})$ are the only complete manifolds of constant curvature $K = +1$.

- Let Γ be a properly discontinuous group of isometries acting freely on S^n.

  Then $\Gamma \subseteq O(n+1)$.

  So each $A \in \Gamma$ is an $(n+1) \times (n+1)$ orthogonal matrix.

  The degree of its characteristic polynomial is an odd number $n+1$.

  Therefore, A must have a real characteristic value.

  But the characteristic values of an orthogonal matrix are of absolute value one.

  Thus, A has ± 1 as a characteristic value.

The Case of Even Dimension (Cont'd)

- We have seen that only the identity on Γ can have $+1$ as a characteristic value.

  Hence -1 is a characteristic value of A.

  This implies that A^2 has $+1$ as characteristic value.

  So $A^2 = I$.

  Hence, each of the characteristic values of A is either $+1$ or -1.

  So, one of the following holds:
  - All are $+1$ and $A = I$;
  - All are -1 and $A = -I$.

  This completes the proof when combined with the preceding example.

Example

- When n is odd, other possibilities can occur.
- As an indication, we will show that, in the case of S^3, there exist many examples of finite subgroups $\Gamma \subseteq O(4)$, which act freely on S^3 and, thus, give manifolds S^3/Γ of constant positive curvature.
- The examples are based on the algebra $\boldsymbol{K}$ of quaternions.
- That is, on the real linear combinations

$$\boldsymbol{q} = x + y\boldsymbol{i} + z\boldsymbol{j} + w\boldsymbol{k}$$

of the four symbols $1, \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ with:

  - The usual rules of multiplication;
  - Componentwise addition.

Example (Cont'd)

- We denote by $\overline{\boldsymbol{q}}$, the conjugate of $\boldsymbol{q}$,

$$\overline{\boldsymbol{q}} = x - y\boldsymbol{i} - z\boldsymbol{j} - w\boldsymbol{k}.$$

- We denote by $\|\boldsymbol{q}\|$ the usual norm

$$\|\boldsymbol{q}\| = (\boldsymbol{q}\overline{\boldsymbol{q}})^{1/2}.$$

- Then $\boldsymbol{K}$ is in obvious one-to-one linear correspondence with $\mathbb{R}^4$.
- This norm corresponds to the standard norm in $\mathbb{R}^4$.
- Consider the set of quaternions of norm one

$$\boldsymbol{K}_1 = \{\boldsymbol{q} : \|\boldsymbol{q}\| = 1\}.$$

- They correspond to $S^3 \subseteq \mathbb{R}^4$.

Example (Cont'd)

- As usual, we identify:
  - $\boldsymbol{K}$ and $\mathbb{R}^4$ as vector spaces and as manifolds;
  - $\boldsymbol{K}_1$ and S^3 as manifolds.
- For all $\boldsymbol{q}_1, \boldsymbol{q}_2 \in \boldsymbol{K}$,
$$\|\boldsymbol{q}_1\boldsymbol{q}_2\| = \|\boldsymbol{q}_1\|\|\boldsymbol{q}_2\|.$$
- So $\boldsymbol{K}_1$ is a group with respect to quaternion multiplication.
- For $\boldsymbol{q} \in \boldsymbol{K}_1$, consider then left translation $L_{\boldsymbol{q}} : \boldsymbol{K} \to \boldsymbol{K}$, defined by
$$L_{\boldsymbol{q}}(\boldsymbol{x}) = \boldsymbol{q}\boldsymbol{x}.$$
- It is an $\mathbb{R}$-linear mapping of $\boldsymbol{K}$ onto $\boldsymbol{K}$.
- Moreover, it preserves the norm of $\boldsymbol{x}$,
$$\|L_{\boldsymbol{q}}(\boldsymbol{x})\| = \|\boldsymbol{x}\|.$$

Example (Cont'd)

- So $L_{\boldsymbol{q}}$ is an orthogonal linear transformation on $\boldsymbol{K} = \mathbb{R}^4$.
- In brief, $S^3 = \boldsymbol{K}_1$ is a group space and left translations are orthogonal transformations, in fact isometries, of S^3, with its usual Riemannian structure.
- But no left translation, except the identity, can have a fixed point.
- So we need only find examples of finite subgroups Γ of $\boldsymbol{K}_1$.
- Each such example determines a three-dimensional manifold of constant positive curvature.
- Further, they are all determined in this way.

Example (Cont'd)

- To find finite subgroups of $\boldsymbol{K}_1$ one uses the following fact.
- There is a natural homomorphism $\pi : \boldsymbol{K}_1 \to SO(3)$ which is onto and has kernel $+1$ ($+1$ is the unit quaternion).
- We now describe this homomorphism.
- Let $\mathbb{R}^3$ be identified with the subspace of $\boldsymbol{K}$ of all quaternions of the form $\boldsymbol{q} = y\boldsymbol{i} + z\boldsymbol{j} + w\boldsymbol{k}$, with real part $x = 0$.
- Then to each $\boldsymbol{q}' \in \boldsymbol{K}_1$ we let correspond the rotation $\pi(\boldsymbol{q}')$ of $\mathbb{R}^3$ given by
$$\pi(\boldsymbol{q}') : \boldsymbol{q} \mapsto \boldsymbol{q}'\boldsymbol{q}(\boldsymbol{q}')^{-1}.$$
- Now, if $\Gamma_1 \subseteq SO(3)$ is a finite subgroup, then $\Gamma = \pi^{-1}(\Gamma_1)$ is a finite subgroup of $\boldsymbol{K}_1$.
- Such subgroups of $SO(3)$ are easy to find - the group of symmetries of any regular solid (omitting those of determinant -1) give examples.

Spaces of Zero Curvature

- Now consider the Riemannian manifolds which have Euclidean space of the same dimension as their universal Riemannian covering space.
- They are the (complete) spaces of zero curvature.
- Thus they are of the form $M = \boldsymbol{E}^n/\Gamma$, the orbit space of a subgroup Γ of the group of isometries (rigid motions) of $\boldsymbol{E}^n$.
- Suppose we identify $\boldsymbol{E}^n$ with $\mathbb{R}^n$ and use vector space notation.
- Then each isometry is of the form

$$x \to Ax + b,$$

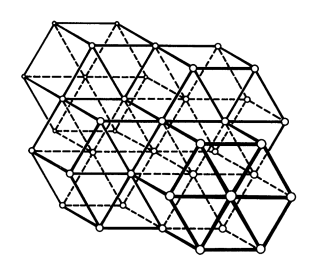
  where:
  - $A \in O(n)$ and determines a rotation of the space;
  - $b = (b^1, \ldots, b^n)$ and determines a translation of the space.
- Locally, the geometry of any such M is just that of Euclidean space.
- So these spaces might seem to lack interest, but this is not the case.

Spaces of Zero Curvature (Examples)

- The global behavior between $\boldsymbol{E}^n$ and $M = \boldsymbol{E}^n/\Gamma$ may be different.
- A particular example is given by the global behavior of geodesics in such spaces.
- We have already noted this in the case of two examples.
  - The cylinder, which is just $\boldsymbol{E}^2/\Gamma$ with
  $$\Gamma = \{x \to x + ne_1 : e_1 = (1, 0), n \in \mathbb{Z}\};$$
  - The torus T^2, which is obtained as the orbit space of the group of translations
  $$\{x \to x + ne_1 + me_2 : n, m \in \mathbb{Z}, e_1 = (1, 0), e_2 = (0, 1)\}.$$

Crystal Structures

- Historically, the study of these spaces is linked to that of the study of crystal structures on the plane $\boldsymbol{E}^2$ and in Euclidean space $\boldsymbol{E}^3$.
- That is, to uniform coverings of the plane by congruent polygons and of $\boldsymbol{E}^3$ by congruent polyhedra.
- It is fairly easy to convince ourselves that the symmetries of such crystalline structures - rigid motions carrying the structures onto themselves - form a subgroup Γ of the group of rigid motions which acts properly discontinuously.



- However, elements of such groups may well have fixed points.
- So these groups are somewhat more general than those which generate examples of manifolds of zero curvature.

Crystallographic Groups

- It was proved in the 19th century that there were only a finite number of crystal structures on $\boldsymbol{E}^3$.
- In his address of 1900, Hilbert asked whether the number of possible isomorphism classes of properly discontinuous groups of motions Γ of $\boldsymbol{E}^n$ for which the orbit space $\boldsymbol{E}^n/\Gamma$ is compact is finite, for every n.
- These are called **crystallographic groups**.
- Hilbert's question was answered affirmatively by Bieberbach in 1911.
- This implies, in particular, that, for every dimension n, there exist finitely many compact Riemannian manifolds of curvature zero.
- Among these, of course, is the torus T^n.
- It is a consequence of Bieberbach's work that every such manifold has the torus as covering space.

The Hyperbolic Space of Dimension 2

- Consider H^2 as given in a preceding example.
  - We write (x, y) for (x^1, x^2);
  - We identify H^2 with the upper half-plane of the complex numbers $\mathbb{C}$ by the correspondence
$$(x, y) \leftrightarrow z = x + iy.$$
- Then H^2 is the open subset of $\mathbb{C}$, consisting of all complex numbers z with positive imaginary part $\text{Im}z > 0$.

The Hyperbolic Space of Dimension 2 (Cont'd)

- We may then write the Riemannian metric, or line element

$$ds^2 = \sum_{i,j=1}^{2} g_{ij} dx^i dx^j,$$

in the complex or real form

$$ds^2 = \frac{dz d\overline{z}}{(\mathrm{Im} z)^2} = \frac{dx^2 + dy^2}{y^2}.$$

- We have considered this Riemannian manifold and its isometries.
- The reason for passing to complex coordinates is that it makes it much simpler to define and work with the group of isometries.
- Of course, other representations of H^2 and its group of isometries are often used, some of which extend to H^n for all n.

Linear Fractional Transformations

- Recall that mappings on $\mathbb{C}$ of the form

$$z \mapsto w = \frac{az+b}{cz+d},$$

  wih $a, b, c, d \in \mathbb{C}$ such that $ad - bc \neq 0$, are isometries of H^2.
- In analytic function theory they are called **linear fractional transformations**.

Theorem

The group G of linear fractional transformations, such that a, b, c, d are real numbers and $ad - bc = +1$, is exactly the group of isometries of H^2, identified with the upper halfplane of $\mathbb{C}$.

Linear Fractional Transformations (Cont'd)

Theorem (Cont'd)

The mapping $F : Sl(2,\mathbb{R}) \to G$ defined by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \left(z \mapsto w = \frac{az+b}{cz+d} \right)$$

is a homomorphism of $Sl(2,\mathbb{R})$ onto G, with kernel $\pm I$.

- Almost all statements were proved in a previous example.
- It remains to show that this group contains all of the isometries.

Linear Fractional Transformations (Cont'd)

- Note that the last statement is verified by a straightforward computation.

  We show, next, that the first statement is correct.

  Let w be the image of $z \in H^2$ by a transformation of G.

  Then
  $$\text{Im} w = \frac{\text{Im} z}{|cz + d|^2} > 0.$$

  So the upper half-plane maps onto itself.

  If we compute dw, we find that
  $$dw = \frac{dz}{(cz + d)^2}.$$

Linear Fractional Transformations (Cont'd)

- From $dw = \frac{dz}{(cz+d)^2}$ it follows that

$$\frac{dwd\overline{w}}{(\mathrm{Im}w)^2} = \frac{dzd\overline{z}}{(\mathrm{Im}z)^2}.$$

  So ds^2 is preserved.

  This is a shorthand way of seeing that the components of g_{ij} transform as they should for an isometry.

- This mapping could be given in terms of real and imaginary parts.

  That is, one could compute the functions $u(x,y)$ and $v(x,y)$, such that

$$w = u(x,y) + iv(x,y).$$

  Then the mapping could be written without use of complex variables.

  However, the computations become more difficult.

Linear Fractional Transformations (Cont'd)

- We next see that this group G contains all isometries.

  Recall, first, that it acts transitively on the upper half-plane.

  Recall, also, that it is transitive on directions.

  Indeed, it has been shown that the orbit of $i = \sqrt{-1}$ is all of H^2.

  This implies transitivity.

  It also implies that the isotropy subgroup of i consists of elements of G corresponding to matrices in $Sl(2, \mathbb{R})$ of the form

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}.$$

  This subgroup of G is transitive on directions at i.

  In fact, it acts as $SO(2)$ on the tangent space to H^2 at i.

  These facts, together with a previous corollary, prove the assertion.

Fractional Transformations and Geodesics

- We note that angles on H^2 in terms of the given Riemannian metric are the same as angles on $\mathbb{R}^2$.
- From complex function theory we have the following facts.
  - Linear fractional transformations are analytic mappings on the complex plane.
  - As such, they are conformal, that is, they preserve angles between curves.
  - Linear fractional transformations carry circles and straight lines of $\mathbb{C}$ into circles and straight lines.
- It follows that any circle which is orthogonal to the real axis will be carried by any element of G into a circle orthogonal to the real axis or a vertical straight line.
- We can show that vertical straight lines are geodesics of H^2.
- It follows that any circle orthogonal to the real axis is also a geodesic.

Fractional Transformations and Geodesics (Cont'd)

- A little Euclidean geometry shows that, through a given $z_0 \in H^2$, there is exactly one circle (or vertical line) tangent to each direction at z_0 and orthogonal to the real axis.
- Now isometries take geodesics to geodesics.
- So this gives every geodesic through z_0.
- One important consequence is that every geodesic can be extended to infinite length so that H^2 is seen to be a complete metric space.
- It is sufficient to check this for just one geodesic, namely,

$$x = 0, \quad y = t, \quad 0 < t < \infty.$$

- The length of this geodesic from $t = a$ to $t = b$ is

$$\int_a^b \frac{dt}{t}.$$

- So it is unbounded in both directions, i.e., as $a \to 0$ or $b \to \infty$.
- This shows it is indefinitely extendable.

Fractional Transformations and Geodesics (Cont'd)

- We also saw that H^2 is an example of a symmetric space, which means that it must be complete.
- We have previously noted that:
  - H^2 is the space of non-Euclidean geometry;
  - It is easy to see from this description of geodesics that Euclid's postulate of parallels is not satisfied (although all the other postulates of Euclid are!).
- This behavior of geodesics should be contrasted with that on S^2 and $P^2(\mathbb{R})$, spaces of constant positive curvature.
- On those, every pair of geodesics intersect, twice on S^2 and once $P^2(\mathbb{R})$.

Completeness of H^n

- Note that any translation of H^n in a direction parallel to the plane $x^n = 0$ is an isometry.
- The same holds for a rotation of the underlying $\mathbb{R}^n$ leaving x^n fixed.
- That is, a linear transformation of the variables $x^1, \ldots, x^{n-1}$, with orthogonal matrix, is an isometry.
- Thus any 2-plane determined by a point $x \in H^n$ and unit vector X_x at x can be carried to the submanifold

$$H^2 = \{x \in H^n : x^1 = \cdots = x^{n-1} = 0\}$$

by an isometry of H^n.

Completeness of H^n (Cont'd)

- We can verify that geodesics of H^2 are geodesics of H^n.
- So, from the facts concerning H^2 and known properties of geodesics, every geodesic of H^n can be extended to infinite length.
- This means that H^n is complete.
- It also means that the geodesics of H^n are exactly the semicircles whose center lies on the $(n-1)$-plane $x^n = 0$ and whose plane is perpendicular to it.