Introduction to the Theory of Distributions

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Subsection 1

Preliminary Concepts

Vector Spaces

- Denote by $\mathbb{R}$ and $\mathbb{C}$ the fields of real and complex numbers.
- We use Φ to denote either of these two fields.
- A **linear space**, or a **vector space**, over Φ is a nonempty set X on which two operations, **addition** and **scalar multiplication**, are defined such that:
  - (a) X is an abelian group under addition, i.e., to every pair $x, y \in X$, the sum $x + y$ is also in X, and we have for all $x, y, z \in X$:
    - (i) $x + y = y + x$;
    - (ii) $x + (y + z) + (x + y) + z$;
    - (iii) There is a zero element $0 \in X$, such that $x + 0 = x$, for all x;
    - (iv) For each $x \in X$, there is an element $-x \in X$, such that $x + (-x) = 0$.
  - (b) For every pair a, x with $a \in \Phi$ and $x \in X$, the scalar product $a \cdot x$ is an element in X, and we have for all $a, b \in \Phi$ and $x, y \in X$:
    - (i) $1 \cdot x = x$;
    - (ii) $a \cdot (b \cdot x) = (a \cdot b) \cdot x$;
    - (iii) $a \cdot (x + y) = a \cdot x + a \cdot y$;
    - (iv) $(a + b) \cdot x = a \cdot x + b \cdot x$.

Properties and Notation

- The zero element is unique.
- Every $x \in X$ has a unique additive inverse $-x$.
- $0 \cdot x = 0$ and $(-1) \cdot x = -x$, for every $x \in X$.
- $a \cdot 0 = 0$, for every $a \in \Phi$.
- The same symbol 0 is used to denote the zeros of both Φ and X.
- The dot symbol for the product is usually dropped.

Linear Independence, Basis and Dimension

- The elements of a vector space X are called **vectors**.
- The vectors $x_1,\ldots,x_n$ are **linearly independent** if the equation $a_1x_1+\cdots+a_nx_n=0$, with $a_k\in\Phi$, implies $a_k=0$, for all k.
- Otherwise, they are **linearly dependent**.
- The set $\{x_1,\ldots,x_n\}$ of vectors in X is said to **span** the space X if any $x\in X$ can be represented by a linear combination of the form $x=a_1x_1+\cdots+a_nx_n$, where $a_k\in\Phi$.
- Any (finite) set of linearly independent vectors $\{x_1,\ldots,x_n\}$ which spans X is called a **basis** of X.
- The **dimension** of X is then n, the number of elements in its basis.
- If no such basis exists, X is said to be **infinite dimensional**.
- In a linear space X with basis $\{x_1,\ldots,x_n\}$, any vector $x\in X$ has a unique representation of the form $x=a_1x_1+\cdots+a_nx_n$, in the sense that the scalar coefficients a_k are uniquely determined by x.

Subspaces

- A nonempty subset M of a linear space X is called a (**linear**) **subspace** of X if whenever $x, y \in M$ and $a \in \Phi$,
$$x + y \in M \quad \text{and} \quad ax \in M.$$
- In that case M is a linear space in its own right.
- $\{0\}$ is a subspace of every linear space.
- With $x \in X, \lambda \in \Phi$ and $A \subseteq X$, we use the notation
$$x + A = \{x + y : y \in A\}, \quad \lambda A = \{\lambda y : y \in A\}.$$
- The "sum" $A + B$ of two subsets of X denotes the set $\{x + y : x \in A, y \in B\}$;
- The "difference" $A - B$ will be used to denote the set $\{x \in A : x \notin B\}$, for any pair of sets A and B, i.e., the complement of B in A.

Subsets of a Vector Space

- We define three types of subsets of the linear space X:
  (1) $E \subseteq X$ is **convex** if, whenever $x, y \in E$ and $0 \leq \lambda \leq 1$, then
  $$\lambda x + (1 - \lambda) y \in E.$$
  Thus, a convex set contains the "line segment" joining x and y whenever it contains x and y.
  (2) $E \subseteq X$ is **balanced** if, whenever $x \in E$ and $|\lambda| \leq 1$, then $\lambda x \in E$.
  By choosing $\lambda = 0$, we see that every balanced set in X contains 0.
  (3) $E \subseteq X$ is **absorbing** if for every $x \in X$, there is a $\lambda > 0$, such that $x \in \lambda E$.
  Here again it is obvious that 0 is contained in every absorbing set.

n-Dimensional Euclidean Space

- The set $\mathbb{R}^n$ whose elements are the n-tuples $(x_1,\ldots,x_n)$, with $x_k \in \mathbb{R}$, $1 \le k \le n$, is an n-dimensional linear space over $\mathbb{R}$ under the operations

$$\begin{array}{rcl} (x_1,\ldots,x_n)+(y_1,\ldots,y_n) & = & (x_1+y_1,\ldots,x_n+y_n) \\ a(x_1,\ldots,x_n) & = & (ax_1,\ldots,ax_n). \end{array}$$

- If we define the distance between any two vectors (points) $x=(x_1,\ldots,x_n)$ and $y=(y_1,\ldots,y_n)$ in $\mathbb{R}^n$ by

$$|x-y| = \left[\sum_{k=1}^{n}(x_k-y_k)^2\right]^{1/2},$$

  then $\mathbb{R}^n$ is called ***n*-dimensional Euclidean space**.
- In this space the set of points x, such that $|x| \le r$, for some positive number r, defines a **ball of radius r and center** 0.
- Such balls are convex, balanced and absorbing.

Topological Spaces

- A **topological space** is a nonempty set X in which a collection τ of subsets is defined, such that:
  - τ contains X and the empty set $\emptyset$;
  - The intersection of any pair in τ is in τ;
  - The union of any collection in τ is in τ.
- The members of τ are known as **open sets**.
- τ is said to define a **topology** on X.
- Since different topologies may be defined on the same set X, the topological space should properly be denoted by the pair (X,τ), but we shall often use only X to denote the topological space.

Standard Terminology for Topological Spaces

- Let (X,τ) be a topological space.

(1) A **neighborhood** of $x \in X$ is any subset of X which contains an open set containing x.

(2) (X,τ) is a **Hausdorff space** if distinct points of X have disjoint neighborhoods.

(3) Let τ and σ be two topologies on X.

We say that τ is **stronger** (**finer**) than σ, or that σ is **weaker** (**coarser**) than τ, if $\sigma \subseteq \tau$, i.e., if every open set in (X,σ) is open in (X,τ).

(4) A collection $\sigma \subseteq \tau$ of open sets is a **base** for τ if every member of τ is a union of sets of σ.

Terminology on Topological Spaces II

(5) The **product topology** on $X \times Y$, where Y is another topological space, is the topology which has as a base the collection of all sets of the form $U \times V$, where U is an open set in X and V is an open set in Y.

(6) A collection σ of neighborhoods of $x \in X$ is a **local base** at x if every neighborhood of x contains a member of σ.

(7) A sequence $(x_n : n \in \mathbb{N})$ in the topological space X **converges** to a **limit** $x \in X$, written $\lim x_n = x$ or $x_n \to x$, if every neighborhood of x contains all but finitely many elements of the sequence.

A weaker requirement is to have an element of (x_n), different from x, in every neighborhood of x, in which case x is called a **cluster point** of (x_n).

In a Hausdorff space the limit of a sequence, if it exists, is unique.

Terminology on Topological Spaces III

(8) The **interior** of a set $E \subseteq X$ is the union of all open subsets of E.

It is denoted by E° and is clearly an open set.

(9) A subset E of X is **closed** if its complement in X, $E^c = X - E$ is open.

The **closure** of $E \subseteq X$ is the intersection of all closed sets which contain E.

The closure of E, denoted by $\overline{E}$, is always closed.

By De Morgan's law, its complement is the union of open subsets of E^c, and is therefore open.

We have

$$\begin{array}{rcl}(\overline{E})^c & = & (\bigcap\{F : E \subseteq F, F \text{ closed}\})^c \\ & = & \bigcup\{G : G \subseteq E^c, G \text{ open}\} \\ & = & (E^c)^\circ.\end{array}$$

Terminology on Topological Spaces IV

(10) A subset E of X is **dense** in X if $\overline{E} = X$.

Even when E is not a subset of X, we still say that E is **dense** in X if $E \cap X$ is dense in X.

(11) The **boundary** of $E \subseteq X$ is the set $\partial E = \overline{E} - E^\circ$.

It is closed since it is the intersection of the closed sets $\overline{E}$ and $(E^\circ)^c$.

(12) $E \subseteq X$ is **compact** if every collection of open sets of X whose union contains E has a finite subcollection whose union contains E.

(13) If $E \subseteq X$ and σ is the collection of sets $E \cap U$, where U runs through the open sets in τ, then σ is a topology on E.

With this **inherited topology** any subset of X becomes a topological space in its own right.

This topology is also referred to as the **subspace topology** of E in X.

Mappings I

- Consider a **mapping**, or a **map**, from a nonempty set X to a nonempty set Y, written $T : X \to Y$.
- When $Y = \Phi$, the mapping T is usually referred to as a **function** from X to Y.
  (i) The **image** of any $x \in X$ is denoted by $T(x) \in Y$.
  If $A \subseteq X$, the set $T(A) = \{T(x) : x \in A\} \subseteq Y$ is the **image**, under T, of A.
  If $B \subseteq Y$, the set $T^{-1}(B) = \{x \in X : T(x) \in B\} \subseteq X$ is the **preimage**, under T, of B.
  T is **injective** (or **one-to-one**) if $T(x_1) = T(x_2)$ implies $x_1 = x_2$, for any pair $x_1, x_2 \in X$.
  T is **surjective** (or **onto**) if $T(X) = Y$.
  When T is both injective and surjective it is called **bijective**.
  In this case, the inverse mapping $T^{-1} : Y \to X$ is defined by
  $$T^{-1}(y) = x \quad \text{if and only if} \quad T(x) = y.$$
  A bijective map from X to Y is also referred to as a **bijection** or a **one-to-one correspondence** from X to Y.

Linear Maps

(ii) If X and Y are linear spaces over Φ, then T is **linear** if

$$T(ax+by)=aT(x)+bT(y),$$

for every $a,b\in\Phi$ and $x,y\in X$.

- When T is linear it follows that:
  - $T(0)=0$;
  - $T(A)$ is a subspace of Y whenever A is a subspace of X;
  - $T^{-1}(B)$ is a subspace of X whenever B is a subspace of Y.
- In particular the subspace $T^{-1}(\{0\})\subseteq X$ is called the **null space** (or the **kernel**) of T and is denoted by $N(T)=\{x\in X:T(x)=0\}$.
- When X is a linear space over Φ and $Y=\Phi$ the linear function T is called a **linear functional**.

Continuous Maps

(iii) If X and Y are topological spaces, then T is **continuous** at $x \in X$ if, for every neighborhood V of $T(x)$, the set $T^{-1}(V)$ is a neighborhood of x.

T is **continuous** on X, or simply **continuous**, if it is continuous at every point in X.

Equivalently, T is continuous on X if and only if $T^{-1}(V)$ is an open set in X whenever V is an open set in Y.

By taking complements, T is continuous if and only if $T^{-1}(V)$ is a closed subset of X whenever V is a closed subset of Y.

Consequently, if $E \subseteq X$, then the identity mapping from E into X is continuous on E provided the topology of E is either the topology inherited from X or a stronger topology.

Homeomorphisms and Embeddings

- A **homeomorphism** from X to E is a continuous bijection from X to E whose inverse is continuous.
  Thus, when there is a homeomorphism from X to Y, the image of an open set in X is an open set in Y, and the inverse image of an open set in Y is an open set in X.
  The topologies on X and Y are therefore in a one-to-one correspondence, and the two spaces are said to be **homeomorphic**.

(iv) If X and Y are topological spaces and $T : X \to Y$ is an injective continuous mapping, then the mapping $S : X \to Z = T(X)$, defined by $S(x) = T(x)$, for all $x \in X$, is clearly bijective.
If S is a homeomorphism from X to Z, T is called a (**topological**) **embedding** of X in Y.
  - When $X \subseteq Y$ the identity mapping from X to Y is always an embedding whenever the topology of X coincides with its subspace topology in Y.
  - If X carries a stronger topology then we merely have a continuous injection of X into Y.

Metric Spaces

- A **metric space** X is a topological space in which the topology is generated by a metric, or distance, function $d : X \times X \to \mathbb{R}$ satisfying:
  (i) $0 \leq d(x,y) < \infty$;
  (ii) $d(x,y) = 0$ if and only if $x = y$;
  (iii) $d(x,y) = d(y,x)$;
  (iv) $d(x,y) = d(x,z) + d(z,y)$, for all $x,y,z \in X$.
- For any $x \in X$ and $r > 0$, the set $B(x,r) = \{y \in X : d(x,y) < r\}$ is called an **open ball with center at** x **and radius** r.
- By defining a subset of X to be open if and only if it is a (possibly empty) union of open balls, the axioms of open sets are satisfied and X becomes a topological space.
- Every metric space is Hausdorff.

  For any distinct pair $x,y \in X$, the open balls $B(x,r)$ and $B(y,r)$ are disjoint if $r < \frac{1}{2}d(x,y)$.

Sequences in Metric Spaces

- Every point of a metric space has a countable base of neighborhoods. One such choice is $\{B(x, \frac{1}{n}) : n \in \mathbb{N}\}$.
- Using this property it can be shown that if f maps the metric space X into the topological space Y, then f is continuous at $x \in X$ if and only if, for every sequence (x_n) in X which converges to x, the sequence $(f(x_n))$ converges to $f(x)$ in Y.
- If E is a subset of X and x is a cluster point of E, then there is a sequence in E which converges to x.
- The set of cluster points of E is contained in its closure $\overline{E}$.
- E will be dense in X if every $x \in X$ is the limit of a sequence (x_n) in E.

Cauchy Sequences and Complete Metric Spaces

- In a metric space X the sequence (x_n) is called a **Cauchy sequence** if, for every $\varepsilon > 0$, there is a positive integer N, such that

$$d(x_n, x_m) < \varepsilon, \text{ for all } n \geq N \text{ and } m \geq N.$$

- X is said to be (**sequentially**) **complete** if every Cauchy sequence in X converges to a point in X.

Metrizable Spaces and Isometries

- Not every topological space (X,τ) is a metric space because it is not always possible to define a metric on X, with the above properties, which will generate the topology τ.
- When this is possible, we say that the topological space is **metrizable**, and that its topology can be **induced** or **generated** by a metric.
- A homeomorphism h from a metric space (X,d_1) to another (Y,d_2) is called an **isometry** if it preserves distances, in the sense that

$$d_2(h(x_1),h(x_2)) = d_1(x_1,x_2), \quad \text{for all } x_1,x_2 \in X.$$

- Two metrics d_1 and d_2 on the same set X are said to be **equivalent** if the identity map from (X,d_1) onto (X,d_2) is a homeomorphism.

  This is equivalent to saying that a set is open with respect to one metric whenever it is open with respect to the other.

Subsection 2

Topological Vector Spaces

Topological Vector Spaces and Normed Spaces

- A **topological vector space** is a linear space X on which a topology τ is defined so that the operations of addition from $X \times X$ to X and scalar multiplication from $\Phi \times X$ to X are continuous.
- A linear space X is a **normed (linear) space** if to every $x \in X$ corresponds a real number $\|x\|$, called the **norm** of x, such that:
  - (i) $\|x\| \neq 0$, whenever $x \neq 0$;
  - (ii) $\|cx\| = |c|\|x\|$, for all $c \in \Phi$ and $x \in X$;
  - (iii) $\|x+y\| \leq \|x\| + \|y\|$, for all $x, y \in X$.
- Property (iii) yields $|\|x\| - \|y\|| \leq \|x-y\|$.
- Thus, the norm of a vector is never negative.
- Property (ii) implies that the norm of the zero vector is zero.

Banach Spaces

- A normed space is also a metric space if we define $d(x,y) = \|x - y\|$, as the distance from x to y.
- The normed space is called a **Banach space** if it is complete in this metric.
- The continuity of the operations $(x,y) \mapsto x + y$ and $(c,x) \mapsto cx$ is then a direct consequence of the above properties of the norm.

  Suppose $x_n \to x$ and $y_n \to y$ in X and $c_n \to c$ in Φ.

  Now we have

$$\begin{array}{rcl} \|(x_n + y_n) - (x + y)\| & \leq & \|x_n - x\| + \|y_n - y\|; \\ \|c_n x_n - cx\| & = & \|c_n(x_n - x) + (c_n - c)x\| \\ & \leq & |c_n| \|x_n - x\| + |c_n - c| \|x\|. \end{array}$$

  Therefore, $x_n + y_n \to x + y$ and $c_n x_n \to cx$ in X.

  Example: The n-dimensional Euclidean space $\mathbb{R}^n$, with the usual Euclidean distance, is a finite dimensional Banach space.

Boundedness and Translations

- We say that a subset E of a topological vector space X is **bounded** if, with a suitable contraction, it can be contained in any neighborhood of 0.
- More precisely, $E \subseteq X$ is **bounded** if, for every neighborhood U of $0 \in X$, there is a number $\lambda > 0$, such that $E \subseteq \lambda U$.
- For any point x_0 in the topological vector space X, consider the translation from X onto X defined by $x \mapsto x + x_0$.
  - It is injective.
  - By assumption, it is continuous.
  - Its inverse $x \mapsto x - x_0$ is also continuous.

  Therefore translation by x_0 is a homeomorphism in X.
- In particular, the set $U + x_0 = \{x + x_0 : x \in U\}$ is open whenever U is open.
- Consequently, τ is completely determined by any local base, which will always be taken at 0.

Scalability

- For any nonzero $\lambda \in \Phi$, the mapping from X onto itself defined by $x \mapsto \lambda x$ is a homeomorphism.
- We can use the continuity of the mapping $(\lambda, x) \mapsto \lambda x$ to conclude that every neighborhood of 0 is absorbing and contains a balanced neighborhood of 0.
  - Suppose U is a neighborhood of $0 \in X$. Let x is any (nonzero) point in X. The mapping $\lambda \mapsto \lambda x$ is continuous at $\lambda = 0$. So there is a neighborhood $\{\lambda \in \Phi : |\lambda| < \varepsilon\}$ of $0 \in \Phi$ which is mapped into U. Hence, $\lambda x \in U$, for all $|\lambda| < \varepsilon$. So $x \in \mu U$, for all $|\mu| > \frac{1}{\varepsilon}$.
  - The mapping $(\lambda, x) \mapsto \lambda x$ is continuous at $(\lambda, x) = (0, 0)$. So there is a neighborhood V of $0 \in X$ and a positive number ε, such that $\lambda V \subseteq U$, whenever $|\lambda| < \varepsilon$. Thus, the set $W = \bigcup_{|\lambda| < \varepsilon} \lambda V$ is a balanced neighborhood of 0 which is contained in U.
- We conclude that every topological vector space X has a balanced, absorbing local base.

Types of Topological Vector Spaces

- A topological vector space X is called a:
  (i) **locally convex space** if its topology has a local base whose members are convex sets;
  (ii) **locally bounded space** if 0 has a bounded neighborhood;
  (iii) **Frechet space** if it is locally convex, metrizable and complete;
  (iv) **normable space** if a norm can be defined on X which is compatible with the topology of X, in the sense that it generates the topology.

Cauchy Sequences and Completeness

- Let $\mathscr{B}$ is a local base for the topology of a topological vector space X.
- The sequence (x_n) in X is a **Cauchy sequence** if, to every $U \in \mathscr{B}$, corresponds an N, such that $x_n - x_m \in U$, for all $n \geq N$ and $m \geq N$.
- A topological vector space X is **complete** if every Cauchy sequence in X converges to a point in X.
- We remark that the notion of *completeness* of a topological vector space X is more general than sequential completeness, and is defined in terms of *Cauchy filters* instead of Cauchy sequences.

  But, for our purposes, it suffices to consider sequential completeness.

Completeness and Boundedness

- If (x_n) is a Cauchy sequence in the topological vector space X, the sequence (x_n) is bounded in the sense that the set $\{x_n\}$ is bounded.

  Let U be a neighborhood of 0.

  Then there is an N, such that $x_k - x_N \in U$, for all $k \geq N$.

  Thus, $\{x_k : k \geq N\} \subseteq x_N + U$.

  But $\lambda(x_N + U)$ may be contained in any neighborhood of 0 by a suitable choice of $\lambda > 0$.

  Hence, the sequence (x_n) is bounded.

Bounded and Continuous Maps

- Let X and Y be topological vector spaces over the same field Φ.
- Let T be a linear map from X to Y.
- T is said to be **bounded** if $T(A)$ is a bounded subset of Y for every bounded subset A of X.
- Since every bounded subset of X may be mapped homeomorphically into any neighborhood of $0 \in X$, T is bounded if and only if it is bounded on a neighborhood of 0.
- Similarly, T is continuous if and only if it is continuous at 0.

  Suppose T is continuous at 0. Then, for every neighborhood V of $0 \in Y$, there is a neighborhood U of $0 \in X$, such that $T(U) \subseteq V$.

  But then, for every $x_0 \in X$,

$$T(x_0 + U) = T(x_0) + T(U) \subseteq T(x_0) + V.$$

Algebraic and Topological Dual

- An important class of linear mappings consists of those for which $Y = \Phi$, i.e., the linear functional on X.
- This class is denoted by X^* and called **algebraic dual** of X.
- With the definition

$$(aT + bS)(x) = aT(x) + bS(x),$$

  for any $a, b \in \Phi$ and $x \in X$, X^* is a linear space over Φ.
- In the usual metric topology of Φ, the continuous linear functionals on X constitute a subspace X' of X^*.
- X' is called the **topological dual** of X, or simply the **dual** of X.

Characterization of Continuity

Theorem

If T is a linear functional on a topological vector space X, then the following statements are equivalent:

(i) T is continuous at 0.

(ii) T is continuous.

(iii) $T^{-1}(\{0\})$ is closed.

(iv) T is bounded.

(i)$\Leftrightarrow$(ii) This equivalence as already been proved.

(ii)$\Rightarrow$(iii) Suppose T is continuous. But $\{0\}$ is closed in Φ. So $T^{-1}(\{0\}) = N(T)$ is closed in X.

Characterization of Continuity

(iii)⇒(iv) Suppose $N(T)$ is closed.
If T is identically zero, then it is bounded.
Suppose T is not identically zero.
Then there is a point $x_0 \in X - N(T)$, with $T(x_0) = 1$.
Now $N(T)$ is closed. So its complement $X - N(T)$ is open.
Thus, there is a balanced neighborhood U of 0, such that
$x_0 + U \subseteq X - N(T)$. This implies that $(x_0 + U) \cap N(T) = \varnothing$.
Suppose $|T(x)| \geq 1$, for some $x \in U$.
Then $y = -\frac{x}{T(x)} \in U$ and $T(x_0 + y) = 1 - 1 = 0$. So $(x_0 + U) \cap N(T) \neq \varnothing$.
This gives a contradiction. So $|T(x)| < 1$, for all $x \in U$.
Thus, T is bounded.
(iv)⇒(i) Let T be bounded on some neighborhood U of 0.
Then there is a number $M > 0$, such that $|T(x)| < M$, for every $x \in U$.
For any $\varepsilon > 0$, we therefore have $|T(x)| < \varepsilon$, whenever x is in $\frac{\varepsilon}{M} U$.
But $\frac{\varepsilon}{M} U$ is also a neighborhood of 0. So T is continuous at 0.

Subsection 3

Seminorms and Locally Convex Spaces

Seminorms

- Let X be a linear space over Φ.
- A **seminorm** on X is a real-valued function p satisfying, for all $x, y \in X$ and $\lambda \in \Phi$:
  (i) $p(x+y) \leq p(x) + p(y)$ (**subadditivity**);
  (ii) $p(\lambda x) = |\lambda| p(x)$.
- For all $x, y \in X$,

$$p(x) = p(x - y + y) \leq p(x - y) + p(y).$$

Interchanging x and y and using property (ii), we obtain

$$p(y) \leq p(x - y) + p(x).$$

Therefore, we always have

$$|p(x) - p(y)| \leq p(x - y).$$

Properties of Seminorms

- In particular, $p(x) \geq 0$, for all $x \in X$.
- The equality $p(0) = 0$ follows directly from (ii).
- However, it may happen that $p(x) = 0$, for some $x \neq 0$.
- When $p(x) = 0$ implies $x = 0$, then p is a norm on X.
- For any linear functional T on X the function $p(x) = |T(x)|$ is an example of a seminorm on X.
- For $r > 0$, the set
$$B_p(r) := \{x \in X : p(x) < r\}$$
corresponds to the ball $B(0, r)$ in a metric space with center 0 and radius r.

Seminorms and Balls

Theorem

In a linear space X equipped with a seminorm p, the p-ball $B_p(r) = \{x \in X : p(x) < r\}$ is convex, balanced and absorbing.

- Let $x, y \in B_p(r)$ and $0 \le \lambda \le 1$.

  Then we have

$$p(\lambda x + (1-\lambda)y) \le \lambda p(x) + (1-\lambda)p(y) < r.$$

  So $B_p(r)$ is convex.

  Suppose $x \in B_p(r)$ and $|\lambda| \le 1$. Then $p(\lambda x) = |\lambda| p(x) < r$.

  Thus, $B_p(r)$ is balanced.

  Let $x \in X$ and $\lambda > p(x)$. Then $p(\frac{r}{\lambda}x) = \frac{p(x)}{\lambda} r < r$. Hence, $x \in \frac{\lambda}{r} B_p(r)$.

  So $B_p(r)$ is absorbing.

The Minkowski Functional

- Let E be an absorbing subset of the linear space X.
- Let x be any point of X.
- There is always a finite positive number λ, such that $\frac{1}{\lambda}x \in E$.
- The **Minkowski functional** μ_E of E is defined, for all $x \in X$, by

$$\mu_E(x) = \inf\left\{\lambda > 0 : \frac{1}{\lambda}x \in E\right\}.$$

Properties of the Minkowski Functional

- $\mu_E(0) = 0$.

  Every absorbing subset of X contains 0.
- $\mu_E : X \to [0, \infty)$.
- If, besides being absorbing, E is convex, then for each $x \in X$, the set

$$M_E(x) = \left\{\lambda > 0 : \frac{1}{\lambda} x \in E\right\} = \{\lambda > 0 : x \in \lambda E\}$$

  is convex and unbounded.

  $M_E(x)$ is the semi-infinite interval whose left endpoint is $\mu_E(x)$.

The Minkowski Functional as a Seminorm

Theorem

In a linear space X the Minkowski functional of a convex, balanced and absorbing set is a seminorm on X.

- Let E be a convex and absorbing subset of X. For any $x, y \in X$, we choose λ_1 and λ_2 so that $\mu_E(x) < \lambda_1$, $\mu_E(y) < \lambda_2$. Since E is convex, it then follows that $\frac{1}{\lambda_1}x \in E$, $\frac{1}{\lambda_2}y \in E$. Moreover,

$$\frac{1}{\lambda_1+\lambda_2}(x+y) = \frac{\lambda_1}{\lambda_1+\lambda_2}\frac{1}{\lambda_1}x + \frac{\lambda_2}{\lambda_1+\lambda_2}\frac{1}{\lambda_2}y$$

  is in E. Thus, $\mu_E(x+y) \leq \lambda_1+\lambda_2$. Since λ_1 and λ_2 can be taken arbitrarily close to $\mu_E(x)$ and $\mu_E(y)$, respectively, we conclude that $\mu_E(x+y) \leq \mu_E(x) + \mu_E(y)$.
  The relation $\mu_E(cx) = |c|\mu_E(x)$ is always true for $c > 0$.
  When E is balanced, it is also true for $|c| = 1$.
  Thus, $\mu_E(cx) = |c|\mu_E(x)$, for any $c \in \Phi$.

Continuity of the Minkowski Functional

- The definition of μ_E implies that

$$\{x \in X : \mu_E(x) < 1\} \subseteq E \subseteq \{x \in X : \mu_E(x) \le 1\}$$

  for any convex (and absorbing) subset E of the linear space X.

- If, moreover, X is a topological vector space, then E is a neighborhood of 0 if and only if μ_E is continuous.

  Suppose, first, that E is a neighborhood of 0. Then the inequality $|\mu_E(x) - \mu_E(y)| \le \mu_E(x - y)$, which follows from the subadditive property of μ_E, shows that it suffices to prove continuity at 0.

  But, by definition, for any $\varepsilon > 0$, if $x \in \varepsilon E$, then $\mu_E(x) \le \varepsilon$.

  Conversely, suppose μ_E is continuous. Then $\{x \in X : \mu_E(x) < 1\}$ is an open set which contains 0 and is contained in E. Indeed, in this case:
  - $\{x \in X : \mu_E(x) < 1\}$ is the interior E° of E;
  - $\{x \in X : \mu_E(x) \le 1\}$ is the closure $\overline{E}$ of E.

The Topology Induced by a Collection of Seminorms

Theorem

Given any set $\{p_i : i \in I\}$ of seminorms on a linear space X, there is a topology on X, compatible with its algebraic structure, in which every seminorm p_i, is continuous. Under this topology X is a locally convex topological space.

- Let $\mathscr{P} = \{p_i : i \in I\}$. For each $i \in I$, $r > 0$, $B_i(r) = \{x \in X : p_i(x) < r\}$ is convex, balanced and absorbing, according to a previous theorem.
  We take $B_i(r)$ to be an open neighborhood of 0.
  For any finite $I' \subseteq I$, let $\mathscr{P}' = \{p_i : i \in I'\}$ be the corresponding finite subset of $\mathscr{P}$. Define $B' = \bigcap_{i \in I'} B_i(1)$.
  Clearly B' is a convex, balanced and absorbing set.
  The collection $\mathscr{B} = \{rB' : \mathscr{P}' \subseteq \mathscr{P}, r > 0\}$, where $\mathscr{P}'$ runs through the finite subsets of $\mathscr{P}$, satisfies the properties of a base of neighborhoods of the origin. So $(X, \mathscr{B})$ is a locally convex space.

The Topology Induced by Seminorms (Cont'd)

- In this topology, every p_i is continuous because every $B_i(r)$ is a neighborhood of 0.
  It remains to show that the algebraic operations on X are also continuous. For any pair $x, y \in X$ and any B in $\mathscr{B}$, we have, using the convexity of B,
  $$\left(x + \frac{1}{2}B\right) + \left(y + \frac{1}{2}B\right) = (x+y) + B.$$
  So addition is continuous on $(X, \mathscr{B})$.
  For scalar multiplication, let $x \in X$, $\lambda \in \Phi$ and $B \in \mathscr{B}$.
  Note that $\mu y - \lambda x = \mu(y-x) + (\mu - \lambda)x$.
  So, if $\mu(y-x) \in \frac{1}{2}B$ and $(\mu-\lambda)x \in \frac{1}{2}B$, then $\mu y - \lambda x$ is in B.
  Pick ε small enough so that $\varepsilon x \in \frac{1}{2}B$.
  - The first condition is satisfied by choosing $y \in x + \frac{1}{2(|\lambda|+\varepsilon)}B$;
  - The second condition is satisfied by taking $|\mu - \lambda| < \varepsilon$.

The Separation Axiom

- The topology defined on X in this proof is the weakest topology in which every seminorm p, is continuous.

  It is referred to as the **topology generated by the family of seminorms** $\{p_i\}$.

- Even though $p(x) = 0$ does not guarantee that $x = 0$, if enough seminorms vanished at x, then, presumably, we may safely conclude that $x = 0$.

Definition

A family $\mathscr{P}$ of seminorms on the linear space X satisfies the **separation axiom**, or is **separating**, if, for every $x \neq 0$ in X, there is a seminorm $p \in \mathscr{P}$, such that $p(x) \neq 0$.

Separation and Locally Convex Hausdorff Spaces

Proposition

A linear space with a separating family of seminorms may be topologized to produce a locally convex Hausdorff space in which each seminorm is continuous. Conversely, any locally convex Hausdorff space X is a topological vector space in which the topolology is generated by a separating family of continuous seminorms defined by the Minkowski functionals of the convex local base of X.

- In any locally convex topological vector space X, let $\mathscr{B}$ be a convex and balanced local base in X. For each $B \in \mathscr{B}$, which is always absorbing, a previous theorem shows that the Minkowski functional μ_B is a seminorm on X. If X is Hausdorff then, for any nonzero vector x in X, there is $B \in \mathscr{B}$, such that $x \notin \mathscr{B}$. Consequently, $\mu_B(x) \geq 1$. Thus, $\{\mu_B : B \in \mathscr{B}\}$ is a separating family of seminorms in X.

Separation and Locally Convex Hausdorff Spaces (Cont'd)

- Conversely, suppose the seminorms $\{p_i\}$ on the linear space X are separating. Then the topology which they generate on X, according to the previous theorem is Hausdorff. This follows from the observation that if $x - y \neq 0$, then there is p_i, such that $p_i(x-y) = r > 0$. Then, the two neighborhoods $x + B_i(\frac{1}{2}r)$ and $y + B_i(\frac{1}{2}r)$ are disjoint.

  If not, there exists $z \in X$, with $p_i(z-x) < \frac{1}{2}r$ and $p_i(z-y) < \frac{1}{2}r$.

  Therefore,

$$p_i(x-y) \leq p_i(z-x) + p_i(z-y) < \frac{1}{2}r + \frac{1}{2}r = r.$$

  This gives a contradiction.

Metrizability of Locally Convex Spaces

Theorem

A locally convex space X is metrizable if and only if it is Hausdorff and has a countable local base.

- Suppose X is metrizable, with metric d. The balls $\{x \in X : d(0,x) < \frac{1}{n}, n \in \mathbb{N}\}$ are convex, balanced and absorbing and form a countable base at 0 for a Hausdorff topology.
  Suppose X is Hausdorff and has a countable local base $\mathscr{B} = \{B_i\}$. Its topology is generated by the countable, separating family of seminorms $\mathscr{P} = \{p_i\}$, where p_i, is the Minkowski functional of B_i.
  We define $d(x,y) = \sum_{i=1}^{\infty} \frac{2^{-i} p_i(x-y)}{1+p_i(x-y)}$, for all $x, y \in X$.
  In view of the inequality $\frac{a}{1+a} \le \frac{b}{1+b}$, $0 \le a \le b$, we have

$$\begin{aligned}\frac{p_i(x-y)}{1+p_i(x-y)} &= \frac{p_i[(x-z)+(z-y)]}{1+p_i[(x-z)+(z-y)]} \le \frac{p_i(x-z)+p_i(z-y)}{1+p_i(x-z)+p_i(z-y)} \\ &\le \frac{p_i(x-z)}{1+p_i(x-z)} + \frac{p_i(z-y)}{1+p_i(z-y)}.\end{aligned}$$

Metrizability of Locally Convex Spaces (Cont'd)

- Therefore d is subadditive.

  Moreover, $d(x,y)=0$ implies $x=y$ because $\mathscr{P}$ is separating.

  So d is clearly a metric on X.

  We have $d(x+z,y+z)=d(x,y)$, for any $x\in X$.

  So the sets $U_n=\{x\in X: d(0,x)<\frac{1}{2^n}\}$ form a base of neighborhoods at 0 for the topology of (X,d).

  Now the series which defines d converges uniformly on $X\times X$ and p_i is continuous on X. So d is continuous on $X\times X$ and U_n open in $(X,\mathscr{B})$.

  If $x\notin B_n$, then $p_n(x)\geq 1$. So $d(0,x)\geq 2^{-n}\frac{p_n(x)}{1+p_n(x)}\geq 2^{-n-1}$.

  Thus, $U_{n+1}\subseteq B_n$. So $\{U_n\}$ is a local base for the topology of $(X,\mathscr{B})$.

Corollary

A countable, separating family of seminorms on a linear space X generates a locally convex, metrizable topology on X.

Normable Locally Convex Hausdorff Spaces

Theorem

A locally convex, Hausdorff space X is normable if and only if its zero vector has a bounded neighborhood.

- Suppose X is normable. The open unit ball $\{x \in X : \|x\| < 1\}$ is a bounded neighborhood of 0. Suppose U is a bounded neighborhood of 0 in the locally convex space (X, τ). Then it contains a convex, balanced and absorbing open set U_0 which is also bounded.
  Let p_0 be the Minkowski functional of U_0. If $p_0(x) = 0$, then $x \in \lambda U_0$, for any $\lambda > 0$. But, since U_0 is bounded, every neighborhood of 0 contains λU_0, for some $\lambda > 0$. Hence, $x = 0$. So p_0 is a norm on X.
  The normed space (X, p_0) has a local base given by $\{\lambda U_0 : \lambda > 0\}$.
  But each λU_0 is an open set in τ on X. Moreover, every neighborhood of 0 in (X, τ) contains λU_0, for some $\lambda > 0$. Thus, p_0 generates τ.
- We assume from now on that all locally convex spaces are Hausdorff.

Remarks on Boundedness

- Suppose X is a topological vector space.
- Let E be a subset of X.
- Let us say that E is *topologically bounded* if it is absorbable by any neighborhood of 0.
- Let us say that E is *normally bounded* if X is normable, with norm $\|\cdot\|$, and there exists a positive constant M, such that $\|x\| \le M$, for all $x \in E$.
- When the topological vector space X is normable, then a subset E of X is topologically bounded if and only if it is normally bounded in X.
- In general, these two notions of boundedness are not equivalent.

Subsection 4

Examples of Locally Convex Spaces

Notation for Functions on $\mathbb{R}^n$

- In the calculus of n variables we use the n-tuple $\alpha = (\alpha_1, \ldots, \alpha_n)$ of nonnegative integers α_i as a multi-index and define:
  - $|\alpha| = \alpha_1 + \cdots + \alpha_n$;
  - $\alpha! = \alpha_1! \cdots \alpha_n!$.
- With $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we use the notation:
  - $x^\alpha = x_1^{\alpha_1} \cdots x_n^{\alpha_n}$;
  - $\partial = (\partial_1, \ldots, \partial_n) = (\frac{\partial}{\partial x_1}, \cdots, \frac{\partial}{\partial x_n})$;
  - $\partial^\alpha = \partial_1^{\alpha_1} \cdots \partial_n^{\alpha_n} = \frac{\partial^{|\alpha|}}{\partial^{\alpha_1} x_1 \cdots \partial^{\alpha_n} x_n}$.

Functions on $\mathbb{R}^n$ and Support

- Our functions will, in general, be complex-valued and defined on an open subset Ω of $\mathbb{R}^n$, with the usual Euclidean topology on $\mathbb{R}^n$.
- The **support** of a function $\phi : \Omega \to \mathbb{C}$, denoted by supp$\phi$, is defined to be the closure of the set $\{x \in \Omega : \phi(x) \neq 0\}$ in the topological space Ω, i.e., the smallest closed set containing $\{x \in \Omega : \phi(x) \neq 0\}$.

Examples of Function Spaces

(i) $C^m(\Omega)$ denotes the set of (complex-valued) functions defined on Ω with continuous derivatives of order m, where $m < \infty$, i.e., $\partial^\alpha \phi$ is continuous on Ω, for every α, with $|\alpha| \leq m$.

When $m = 0$, we have the set $C^0(\Omega)$ of continuous functions on Ω.

Clearly, $C^m(\Omega) \subseteq C^{m-1}(\Omega) \subseteq \cdots \subseteq C^0(\Omega)$.

(ii) $C^\infty(\Omega) = \bigcap_{m \geq 0} C^m(\Omega)$ is the set of functions on Ω with continuous derivatives of all orders.

(iii) $C_K^m(\Omega)$ is the set of functions in $C^m(\Omega)$ with support in K, where K will always denote a compact subset of Ω.

(iv) C_K^∞ is the set of functions in $C^\infty(\Omega)$ with support in K.

Comments on the Definitions

- Clearly $C^m(\Omega)$ is a linear space over $\mathbb{C}$, for $m \leq \infty$, by the usual definition of addition of functions and multiplication by complex numbers

$$(\phi + \psi)(x) = \phi(x) + \psi(x), \quad (c\phi)(x) = c\phi(x).$$

- $C_K^m(\Omega)$ is a subspace of $C^m(\Omega)$, for every m.
- A well-known example of a $C^\infty(\mathbb{R}^n)$ function of compact support is given by

$$\alpha(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}}, & \text{on } |x| < 1 \\ 0, & \text{on } |x| \geq 1 \end{cases}.$$

It has support in the unit ball $\{x \in \mathbb{R}^n : |x| \leq 1\}$.

Topology on $C^0(\Omega)$

- Since any open subset of $\mathbb{R}^n$ may be expressed as a countable union of compact sets in $\mathbb{R}^n$, we can write $\Omega = \bigcup K_i$, where K_i is a compact subset of $\mathbb{R}^n$ for all $i \in \mathbb{N}$.
  Without loss of generality, we may choose $K_1 \subseteq K_2 \subseteq K_3 \subseteq \cdots$.
  For any $\phi \in C^0(\Omega)$, we define the seminorm

  $$p_i(\phi) = \sup\{|\phi(x)| : x \in K_i\}, \quad i \in \mathbb{N}.$$

  Note that the increasing sequence (p_i) is clearly separating.
  The sets

  $$B_i(r) = \{\phi \in C^0(\Omega) : p_i(\phi) < r\}, \quad i \in \mathbb{N},\ r > 0,$$

  form a convex local topological base for $C^0(\Omega)$.
  The resulting topology is compatible with the metric

  $$d(\phi, \psi) = \sum_{i=1}^{\infty} 2^{-i} \frac{p_i(\phi - \psi)}{1 + p_i(\phi - \psi)}.$$

Topology on $C^0(\Omega)$ (Cont'd)

- Since convergence in this metric is uniform on every compact subset of Ω, the limit of every Cauchy sequence is always a continuous function on Ω.

  Thus the metric space $C^0(\Omega)$ is complete and is therefore a Fréchet space (locally convex, metrizable and complete).

  But $C^0(\Omega)$ is not normable because in every $B_i(r)$ we can always find a function ϕ for which $p_{i+1}(\phi)$ is as large as we please, so that no $B_i(r)$ can be bounded.

  Note, however, that every $B_i(r)$ is bounded in the metric d.

  In fact the whole space $C^0(\Omega)$ is bounded in this metric.

Topology on $C^m(\Omega)$

- Assume again that Ω is the union of a sequence of compact sets $K_1 \subseteq K_2 \subseteq \cdots$.

  For any $\phi \in C^m(\Omega)$, with $1 \le m < \infty$, we define the separating countable family of seminorms

$$p_{i,m}(\phi) = \sup\{|\partial^\alpha \phi(x)| : x \in K_i, |\alpha| \le m\}.$$

  The corresponding balls

$$B_{i,m}(r) = \{\phi \in C^m(\Omega) : p_{i,m}(\phi) < r\}$$

  provide a base for a topology on $C^m(\Omega)$ which makes it into a locally convex, metrizable space.

  The convergence of (ϕ_k) in $C^m(\Omega)$ is equivalent to the uniform convergence of $(\partial^\alpha \phi_k)$ on every compact subset of Ω, for all $|\alpha| \le m$.

Topology on $C^m(\Omega)$ (Cont'd)

- The topology of $C^m(\Omega)$ is the weakest in which the linear map $\partial^\alpha : C^m(\Omega) \to C^0(\Omega)$, $|\alpha| \leq m$, is continuous, where $C^0(\Omega)$ carries its natural topology of uniform convergence.

  So a sequence (ϕ_k) converges to ϕ in $C^m(\Omega)$ if and only if the sequence $(\partial^\alpha \phi_k)$ converges to $\partial^\alpha \phi$ in $C^0(\Omega)$, for all $|\alpha| \leq m$.

  This is equivalent to the uniform convergence of $\partial^\alpha \phi_k$ to $\partial^\alpha \phi$ on every compact subset of Ω.

  It implies the uniform convergence of (ϕ_k) to ϕ in $C^0(\Omega)$.

  So the topology of $C^m(\Omega)$ is stronger than its subspace topology in $C^0(\Omega)$.

  More generally, the topology of $C^\ell(\Omega)$ is stronger than its subspace topology in $C^m(\Omega)$ whenever $\ell \geq m \geq 0$.

  So the identity map from $C^\ell(\Omega)$ into $C^m(\Omega)$ is continuous.

Completeness of $C^m(\Omega)$

Theorem

The locally convex space $C^m(\Omega)$ is complete.

- Let (ϕ_k) be a Cauchy sequence in $C^m(\Omega)$. So it is a Cauchy sequence in $C^0(\Omega)$. Since $C^0(\Omega)$ is complete, $\phi_k \to \phi \in C^0(\Omega)$. The sequence $(\partial^\alpha \phi_k)$ is also a Cauchy sequence in $C^0(\Omega)$, for every α satisfying $|\alpha| \leq m$. Therefore, $\partial^\alpha \phi_k \to \phi_\alpha \in C^0(\Omega)$. But the operator $\partial_\alpha : C^m(\Omega) \to C^0(\Omega)$ is continuous. So $\partial^\alpha \phi = \partial^\alpha(\lim \phi_k) = \lim \partial^\alpha \phi_k = \phi_\alpha$ is in $C^0(\Omega)$. Hence, ϕ is in $C^m(\Omega)$.
- This theorem shows that $C^m(\Omega)$ is a Fréchet space.
- It is not normable because, as before, every neighborhood of 0 is unbounded.

Topology on $C^\infty(\Omega)$

- We write Ω as the union of an increasing sequence of compact sets (K_i).

  We define the seminorms

  $$p_i(\phi) = \sup\{|\partial^\alpha \phi(x)| : x \in K_i, |\alpha| \le i\}, \quad \phi \in C^\infty(\Omega).$$

  The balls $B_i(r) = \{\phi \in C^\infty(\Omega) : p_i(\phi) < r\}$ form a local base for the topology of $C^\infty(\Omega)$.

  The same argument as before shows that, with this topology, $C^\infty(\Omega)$ is a Fréchet space which is not normable.

  It is the weakest topology which makes, for all $m \ge 0$, the linear map $\partial^\alpha : C^\infty(\Omega) \to C^m(\Omega)$ continuous, where $C^m(\Omega)$ carries its natural topology.

Bounded Subsets of $C^\infty(\Omega)$

Theorem

A subset E of $C^\infty(\Omega)$ is bounded if and only if, for all $m \in \mathbb{N}_0 = \{0,1,2,\ldots\}$ and every compact set $K \subseteq \Omega$, there is a positive constant M, which depends on m and K, such that $|\partial^\alpha \phi(x)| \le M$, whenever $|\alpha| \le m$, $x \in K$ and $\phi \in E$.

- Suppose the elements of E satisfy the given inequality.
  Let U be a neighborhood of $0 \in C^\infty(\Omega)$. Any such neighborhood contains the balls $B_i = \{\phi \in C^\infty(\Omega) : p_i(\phi) < \frac{1}{i}\}$, for all values of i greater than some positive integer. Choose i large enough so that $i \ge m$ and $K_i \supseteq K$. If we now choose λ, such that $0 < \lambda < \frac{1}{M(i+1)}$, we obtain

$$\lambda E = \left\{\phi \in E : |\partial^\alpha \phi(x)| \le \frac{1}{i+1}, x \in K, |\alpha| \le m\right\} \subseteq B_{i+1} \subseteq U.$$

  This means that E is bounded.

Bounded Subsets of $C^\infty(\Omega)$ (Converse)

- Conversely, suppose E is bounded.

  By a suitable choice of the positive number λ, the set λE may be contained in any neighborhood of 0.

  In particular, for every B_i, there is a $\lambda_i > 0$, such that $\lambda_i E \subseteq B_i$.

  This means that

$$p_i(\phi) < \frac{1}{i\lambda_i}, \quad \phi \in E.$$

  Assume $m \in \mathbb{N}_0$ and $K \subseteq \Omega$ compact are given.

  We can choose i so that $i \geq m$ and $K_i \supseteq K$.

  Then the inequality of the hypothesis follows by choosing $M = \frac{1}{i\lambda_i}$.
- Note that, according to this theorem, no B_i can be bounded for any finite integer i.

More on the Topology of of $C^\infty(\Omega)$

- The system of seminorms that we have used to define the topology of $C^\infty(\Omega)$ is equivalently given by

$$p_{i,K}(\phi) = \sup\{|\partial^\alpha \phi(x)| : x \in K, |\alpha| \le i\},$$

as i runs through the nonnegative integers and K through the compact subsets of Ω.

The preceding theorem may then be restated as follows:

> $E \subseteq C^\infty(\Omega)$ is bounded if and only if, for every $m \in \mathbb{N}_0$ and every compact $K \subseteq \Omega$, there is an $M > 0$, such that $p_{m,K}(\phi) \le M$, for every $\phi \in E$.

Furthermore, the set $\{\phi \in C^\infty(\Omega) : p_{m,K}(\phi) < r\}$ is a neighborhood of 0, for every $m \in \mathbb{N}_0$, $K \subseteq \Omega$ and $r > 0$.

The convergence of (ϕ_k) in $C^\infty(\Omega)$ is equivalent to the uniform convergence of $(\partial^\alpha \phi_k)$, for every multi-index $\alpha = (\alpha_1, \ldots, \alpha_n)$, on every compact subset of Ω.

Topology on $C_K^m(\Omega)$, for $m \leq \infty$

Claim: $C_K^m(\Omega)$ is a closed subspace of $C^m(\Omega)$, for $m \leq \infty$.
Note that, for any $x \in \Omega$, the linear mapping T_x from $C^m(\Omega)$ to $\mathbb{C}$ defined by $T_x(\phi) = \phi(x)$ is continuous.
So its null space $N(T_x) = \{\phi \in C^m(\Omega) : \phi(x) = 0\}$ is closed, by a previous theorem, for every $x \in \Omega$.
But $C_K^m(\Omega) = \bigcap_{x \in \Omega - K} N(T_x)$. Hence, C_K^m is closed in $C^m(\Omega)$.
It follows that $C_K^m(\Omega)$ is also a Fréchet space.
For $m \leq \infty$, the seminorms on $C_K^m(\Omega)$ are given, for all $0 \leq i \leq m$, by

$$p_i(\phi) = \sup\{|\partial^\alpha \phi(x)| : x \in K, |\alpha| \leq i\}.$$

The local base they define is the collection of balls
$B_i(r) = \{\phi \in C_K^m(\Omega) : p_i(\phi) < r\}$.
These seminorms, in contrast to those of the previous examples, actually define norms on $C_K^m(\Omega)$, since $p_i(\phi) = 0$, for any $i \in \mathbb{N}_0$, $i \leq m$, implies $\phi = 0$.

Normability of $C_K^m(\Omega)$, for $m \leq \infty$

- If E is a bounded subset of $C_K^\infty(\Omega)$, then, by definition, for every B_i, there is a positive number λ_i, such that $E \subseteq \lambda_i B_i$.

  This is equivalent to saying that, for every nonnegative integer i, there is a constant M_i, such that

  $$\sup\{|\partial^\alpha \phi(x)| : \phi \in E, x \in K, |\alpha| \leq i\} \leq M_i.$$

  - Every B_i is therefore unbounded because it contains a ϕ for which $p_{i+1}(\phi)$ is arbitrarily large. Hence $C_K^\infty(\Omega)$ is not normable.
  - When $m < \infty$ the largest of the seminorms, i.e., p_m, which is actually a norm, makes $C_K^m(\Omega)$ into a Banach space.
- If $K_1 \subseteq K_2 \subseteq \Omega$, then $C_{K_1}^m(\Omega)$ is a closed subspace of $C_{K_2}^m(\Omega)$, $m \leq \infty$.

  The topology on $C_{K_1}^m(\Omega)$ is the topology it inherits as a subspace of $C_{K_2}^m(\Omega)$.