Introduction to the Theory of Distributions

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 500

Subsection 1

The Space of Test Functions $\mathscr{D}$

The Space $C_0^\infty(\Omega)$

- Recall Ω will denote a nonempty open subset of $\mathbb{R}^n$.
- Recall, also, that, for every compact subset K of Ω, there is defined a linear space $C_K^\infty(\Omega)$ and a topology which makes it into a Fréchet space.
- The union of the spaces $C_K^\infty(\Omega)$ as K ranges over all compact subsets of Ω, is denoted by $C_0^\infty(\Omega)$.
- Every function in $C_0^\infty(\Omega)$ is infinitely differentiable on Ω and its support is a compact subset of Ω.
- The topology of $C_K^\infty(\Omega)$ as a closed subspace of $C^\infty(\Omega)$ was defined by the seminorms

$$p_m(\phi) = \sup\{|\partial^\alpha \phi(x)| : x \in K, |\alpha| \le m\}, \quad m \in \mathbb{N}_0,$$

with the sets $B_m(r) = \{\phi \in C_K^\infty(\Omega) : p_m(\phi) < r\}$ as a local base.

The Topology on $C_0^\infty(\Omega)$

- $C_K^\infty(\Omega)$ is also a closed subspace of $C_0^\infty(\Omega)$;
- We define the topology of

$$C_0^\infty(\Omega) = \bigcup_{K \subseteq \Omega} C_K^\infty(\Omega)$$

  to be the finest locally convex topology for which the identity map $C_K^\infty(\Omega) \to C_0^\infty(\Omega)$ is continuous, for every $K \subseteq \Omega$.
- Thus, a convex, balanced set $U \subseteq C_0^\infty(\Omega)$ is a neighborhood of 0 in $C_0^\infty(\Omega)$ if and only if $U \cap C_K^\infty(\Omega)$ is a neighborhood of 0 in $C_K^\infty(\Omega)$, for every $K \subseteq \Omega$.
- The collection of all such neighborhoods U constitutes a local base for the topology we have defined on $C_0^\infty(\Omega)$.
- $C_0^\infty(\Omega)$ is known as the **inductive limit** of the topologies on $C_K^\infty(\Omega)$.

Properties of the Topology on $C_0^\infty(\Omega)$

- $C_0^\infty(\Omega)$, with the inductive limit topology, is a locally convex space.
- The original topology on $C_K^\infty(\Omega)$, for any $K \subseteq \Omega$, is clearly the topology that $C_K^\infty(\Omega)$ inherits as a subspace of $C_0^\infty(\Omega)$.
- If Ω_1 is an open subset of Ω, then $C_0^\infty(\Omega_1)$ is a subspace of $C_0^\infty(\Omega)$.
  This is because every function in $C_0^\infty(\Omega_1)$ may be extended as a C_0^∞ function into Ω by defining it to be 0 on $\Omega - \Omega_1$.

Continuity of Linear Functionals on $C_0^\infty(\Omega)$

Theorem

A linear functional on $C_0^\infty(\Omega)$ is continuous if and only if its restriction to $C_K^\infty(\Omega)$ is continuous, for every compact subset K of Ω.

- Let T be a linear functional on $C_0^\infty(\Omega)$. By a previous theorem, T is continuous if and only if it is continuous at $0 \in C_0^\infty(\Omega)$.

  Let K be a compact set in Ω, and T_K the restriction of T to $C_K^\infty(\Omega)$.

  If V is any neighborhood of $0 \in \mathbb{C}$, and T is continuous at 0, then $T^{-1}(V)$ is a neighborhood of 0 in $C_0^\infty(\Omega)$.

  So $T^{-1}(V) \cap C_K^\infty(\Omega) = T_K^{-1}(V)$ is a neighborhood of 0 in $C_K^\infty(\Omega)$.

  Conversely, suppose T_K is continuous at 0, for every K.

  Then $T_K^{-1}(V) = T^{-1}(V) \cap C_K^\infty(\Omega)$ is a neighborhood of 0 in $C_K^\infty(\Omega)$, for every $K \subseteq \Omega$. Consequently, $T^{-1}(V)$ is a neighborhood of 0 in $C_0^\infty(\Omega)$.

The Space of Test Functions

- The locally convex space $C_0^\infty(\Omega)$, endowed with the inductive limit topology, is called the **space of test functions**.
- It is denoted by $\mathscr{D}(\Omega)$, in accordance with Schwartz's notation.
- We use $\mathscr{D}_K$ to denote the locally convex space $C_K^\infty(\Omega)$, where K is a compact subset of Ω.
- For any $\phi \in \mathscr{D}(\Omega)$, we define the norms

$$|\phi|_m = \sup\{|\partial^\alpha \phi(x)| : x \in \Omega, |\alpha| \le m\}, \quad m \in \mathbb{N}_0.$$

- When ϕ is in $\mathscr{D}_K$, $|\phi|_m$ coincides with the seminorm $p_m(\phi)$.

Bounded Subsets of $\mathscr{D}(\Omega)$

Theorem

E is a bounded subset of $\mathscr{D}(\Omega)$ if and only if the following two conditions are satisfied:

(i) $E \subseteq \mathscr{D}_K$, for some $K \subseteq \Omega$.

(ii) E is bounded in $\mathscr{D}_K$, in the sense that, for every nonnegative integer m, there is a finite constant M_m, such that $|\phi|_m \leq M_m$, for all $\phi \in E$.

- The sufficiency of (i) and (ii) is clear.

  For necessity, let E be a subset of $\mathscr{D}(\Omega)$, which lies in no $\mathscr{D}_K$.

  Then, there is a sequence of functions $\phi_k \in E$ and a sequence of points $x_k \in \Omega$, with no cluster point in Ω, such that $\phi_k(x_k) \neq 0$, $k \in \mathbb{N}$.

  Let

$$U = \left\{ \phi \in \mathscr{D}(\Omega) : |\phi(x_k)| < \frac{1}{k} |\phi_k(x_k)|, k \in \mathbb{N} \right\}.$$

Bounded Subsets of $\mathscr{D}(\Omega)$ (Cont'd)

- We defined

$$U = \left\{ \phi \in \mathscr{D}(\Omega) : |\phi(x_k)| < \frac{1}{k} |\phi_k(x_k)|, k \in \mathbb{N} \right\}.$$

Note that each K contains only a finite number of points of (x_k).

So the intersection $\mathscr{D}_K \cap U$ is a neighborhood of 0 in $\mathscr{D}_K$, for every K.

Hence, U is a neighborhood of 0 in $\mathscr{D}(\Omega)$.

But $\phi_k \notin kU$, for any k. So no multiple of U contains E.

Thus, E is unbounded.

Hence, if E is bounded in $\mathscr{D}(\Omega)$, then condition (i) must hold.

Condition (ii) follows from the fact that the topology of $\mathscr{D}_K$ is the topology it inherits as a subspace of $\mathscr{D}(\Omega)$.

Convergence in $\mathscr{D}(\Omega)$

Theorem

A sequence of (ϕ_k) in $\mathscr{D}(\Omega)$ converges to 0 if and only if the following two conditions are satisfied:

(i) There is a compact subset K of Ω, such that $\text{supp}\phi_k \subseteq K$, for all k.

(ii) $\partial^\alpha \phi_k \to 0$ uniformly on K, for all α.

($\Leftarrow$) Conditions (i) and (ii) imply that $\phi_k \to 0$ in $\mathscr{D}_K$. Since the identity map from $\mathscr{D}_K$ to $\mathscr{D}(\Omega)$ is continuous, $\phi_k \to 0$ in $\mathscr{D}(\Omega)$.

($\Rightarrow$) Conversely, if $\phi_k \to 0$ in $\mathscr{D}(\Omega)$, then (ϕ_k) is a bounded sequence in $\mathscr{D}(\Omega)$, as seen previously. From the preceding theorem, (ϕ_k) lies in $\mathscr{D}_K$, for some $K \subseteq \Omega$. Condition (i) now follows. But then $\phi_k \to 0$ in the subspace topology of $\mathscr{D}_K$. So Condition (ii) also follows.

Non-Metrizability of the Topology of $C_0^\infty(\Omega)$

Claim: The topology defined on $C_0^\infty(\Omega)$ is not metrizable.

Assume that d is a metric which defines the topology of $\mathscr{D}(\Omega)$.

Let $\Omega = \bigcup K_n$, with K_n compact and $K_n \subseteq K_{n+1}^\circ$, for all n.

Choose $\phi_n \in \mathscr{D}(\Omega)$, such that $\text{supp}\phi_n \nsubseteq K_n$.

Multiplication by a constant is a continuous mapping from $\mathscr{D}(\Omega)$ into $\mathscr{D}(\Omega)$.

So we can find $\lambda_n > 0$ small enough so that $d(0, \lambda_n \phi_n) < \frac{1}{n}$, for every n.

This means that the sequence $\lambda_n \phi_n \to 0$ in $\mathscr{D}(\Omega)$.

This, however, is not possible, since $\text{supp}(\lambda_n \phi_n)$ cannot be contained in a single compact subset of Ω.

Topological Completeness of $\mathscr{D}(\Omega)$

- Claim: $\mathscr{D}(\Omega)$ is complete in the topological sense.

  We know $\mathscr{D}_K$ is complete in the topological sense.

  If (ϕ_k) is a Cauchy sequence in $\mathscr{D}(\Omega)$, it is bounded.

  By a previous theorem, (ϕ_k) lies in $\mathscr{D}_K$, for some $K \subseteq \Omega$.

  Since $\mathscr{D}_K$ is complete, (ϕ_k) converges in $\mathscr{D}_K$.

  Consequently, it converges in $\mathscr{D}(\Omega)$.

The Space $\mathscr{D}^m(\Omega)$

- We can also define the topology of

$$C_0^m(\Omega) = \bigcup_{K \subseteq \Omega} C_K^m(\Omega)$$

  to be the finest locally convex topology in which the identity map from $C_K^m(\Omega)$ to $C_0^m(\Omega)$ is continuous for every compact set $K \subseteq \Omega$.
- The resulting topological vector space will be denoted by $\mathscr{D}^m(\Omega)$.

Corollary

A sequence (ϕ_k) in $\mathscr{D}^m(\Omega)$ converges to 0 if and only if:

(i) There is a compact set $K \subseteq \Omega$, such that $\text{supp}\phi_k \subseteq K$, for all k;

(ii) $\partial^\alpha \phi_k \to 0$ uniformly on K, for all $|\alpha| \leq m$.

Example

- Consider the function $\alpha : \mathbb{R}^n \to \mathbb{R}$ defined by

$$\alpha(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}}, & \text{on } |x| < 1 \\ 0, & \text{on } |x| \geq 1 \end{cases}.$$

It has as support the closed unit ball $\overline{B}(0,1)$ in $\mathbb{R}^n$.
Clearly α lies in $\mathscr{D}(\mathbb{R}^n)$.

- The sequence $\alpha_k = \frac{1}{k}\alpha$ satisfies the conditions of the theorem.
  So it converges to 0 in $\mathscr{D}(\mathbb{R}^n)$.
- Consider the sequence $\alpha_k = \frac{1}{k}\alpha \circ \frac{1}{k}$ defined by $\alpha_k(x) = \frac{1}{k}\alpha(\frac{x}{k})$, $x \in \mathbb{R}^n$.
  α_k does not converge in $\mathscr{D}(\mathbb{R}^n)$, because $\text{supp}\alpha_k = \overline{B}(0,k)$ does not satisfy Condition (i).
- The sequence $\alpha_k = \frac{1}{k}\alpha \circ k$ has a sequence of shrinking supports $\overline{B}(0,\frac{1}{k})$.
  However, the partial derivatives of α_k do not converge to 0 on any neighborhood of the origin.
  So Condition (ii) of the theorem is violated and the sequence diverges.

Subsection 2

Distributions

Distributions on Ω

Definition

A **distribution on** Ω is a continuous linear functional on $\mathscr{D}(\Omega)$.

- We denote the linear space of all distributions on Ω by $\mathscr{D}'(\Omega)$, the topological dual of $\mathscr{D}(\Omega)$.

Theorem

A linear functional T on $\mathscr{D}(\Omega)$ is a distribution if and only if, for every compact set $K \subseteq \Omega$, there exists a nonnegative integer m and a finite constant M, such that $|T(\phi)| \leq M|\phi|_m$, for all $\phi \in \mathscr{D}_K$.

- T is in $\mathscr{D}'(\Omega)$ iff T is continuous in $\mathscr{D}(\Omega)$ iff, by a previous theorem, T_k is continuous in $\mathscr{D}_K(\Omega)$, for every compact $K \subseteq \Omega$, iff, by a previous theorem, T_k is bounded on $\mathscr{D}_K$, for every compact $K \subseteq \Omega$, iff, by the topology of $\mathscr{D}_K$, the given condition holds.

Lebesgue Integrable Functions

- Denote the Lebesgue integral of the measurable function f over the measurable set $E \subseteq \mathbb{R}^n$ by
$$\int_E f(x)dx.$$
- It will sometimes be abbreviated to $\int_E f dx$ or $\int_E f$, when the measure function is clear from the context.
- In this convention, E is often dropped when $E = \mathbb{R}^n$.
- $L^1(\Omega)$ denotes the linear space of complex Lebesgue integrable functions on Ω, i.e., all functions $f : \Omega \to \mathbb{C}$ whose integral $\int_\Omega |f(x)|dx$ is finite.

Locally Lebesgue Integrable Functions

- The function f is **locally integrable on** Ω if $\int_E |f(x)|dx$ is finite on every compact subset E of Ω.
- $L^1_{\text{loc}}(\Omega)$ denotes the space of locally integrable functions on Ω.
- All continuous functions on $\mathbb{R}^n$, for example, are locally integrable, although some of them, such as polynomials, are not integrable on $\mathbb{R}^n$.
- Clearly $L^1(\Omega) \subseteq L^1_{\text{loc}}(\Omega)$.

Distributions Defined by Locally Integrable Functions

- If $f \in L^1_{\text{loc}}(\Omega)$, then the linear functional T_f, defined on $\mathscr{D}(\Omega)$ by
$$T_f(\phi) = \int_\Omega f(x)\phi(x)dx, \quad \phi \in \mathscr{D}(\Omega),$$
is bounded.
Let $K = \text{supp}\phi$. Then
$$|T_f(\phi)| \leq \sup_{x \in \Omega} |\phi(x)| \int_K |f(x)|x = |\phi|_0 \int_K |f(x)|dx.$$
Therefore, $T_f \in \mathscr{D}'(\Omega)$.
- Sometimes we denote the distribution T_f simply by f and write
$$T_f(\phi) = \langle f, \phi \rangle = \int_\Omega f(x)\phi(x)dx, \quad \phi \in \mathscr{D}(\Omega).$$
- Continuous functions on Ω are locally integrable.
- So every $f \in C^0(\Omega)$ defines a distribution T_f as above.

The Order of a Distribution

- Compare with the framework developed in the preceding theorem, i.e., with
$$|T(\phi)| \leq M|\phi|_m.$$
- Here $M = \int_K |f|$ clearly depends on K, but the integer $m = 0$ works for all K.
- T_f is then said to be of **order** 0.
- The **order** of the distribution T is the smallest m for which the inequality holds for all K.
- If no such m exists, T is of **infinite order**.

Example

- Let $f \in L^1_{\text{loc}}(\mathbb{R} - \{0\})$ satisfy $|f(x)| \leq \frac{c}{|x|^m}$ on $|x| \leq 1$, for some positive integer m and a positive constant c.

  Claim: There is a distribution $T \in \mathscr{D}'(\mathbb{R})$ of order $\leq m$, such that $T = T_f$ on $\mathscr{D}(\mathbb{R} - \{0\})$.

  Let $\phi \in \mathscr{D}(\mathbb{R})$ be arbitrary.

  Then, there is a number $a > 1$, such that $\phi(x) = 0$ on $|x| > a$.

  For any x, we can use Taylor's formula to write, for some $t \in (0,1)$,

$$\phi(x) = \phi(0) + x\phi'(0) + \cdots + \frac{x^{m-1}}{(m-1)!}\phi^{(m-1)}(0) + \frac{x^m}{m!}\phi^{(m)}(tx).$$

  Now we define

$$\begin{array}{rcl} T(\phi) & = & \int_{|x|>1} f(x)\phi(x)dx + \int_{|x|\leq 1} f(x)[\phi(x) - \sum_{k=0}^{m-1} \frac{x^k}{k!}\phi^{(k)}(0)]dx \\ & = & \int_{|x|>1} f(x)\phi(x)dx + \int_{|x|\leq 1} f(x)\frac{x^m}{m!}\phi^{(m)}(tx)dx \end{array}$$

Example (Cont'd)

- We obtain, for some positive constants A and B,

$$\begin{aligned} |T(\phi)| &= \int_{|x|>1} f(x)\phi(x)dx + \int_{|x|\le 1} f(x)\frac{x^m}{m!}\phi^{(m)}(tx)dx \\ &\le |\phi|_0 \int_{1<|x|<a} |f(x)|dx + \int_{|x|\le 1} \frac{c}{m!}|\phi^{(m)}(tx)|dx \\ &\le A|\phi|_0 + B|\phi|_m. \end{aligned}$$

Hence T is a distribution on $\mathbb{R}$ of order $\le m$.

We show that T is represented by f on $\mathbb{R} - \{0\}$.

Let $\phi \in \mathscr{D}(\mathbb{R})$ with $\text{supp}\phi \subseteq \mathbb{R} - \{0\}$. Then $\phi^{(k)}(0) = 0$, for all k.

Therefore,

$$T(\phi) = \int_{\mathbb{R}} f(x)\phi(x)dx = \langle f, \phi \rangle.$$

Regular versus Singular Distributions

- A distribution T is said to be **regular** if there is a locally integrable function f on Ω, such that

$$T(\phi) = \langle f, \phi \rangle = \int_\Omega f(x)\phi(x)dx, \quad \phi \in \mathscr{D}(\Omega).$$

- Otherwise, it is **singular**.

  Example: The distribution corresponding to $f(x) = \frac{1}{x^m}$, $m \geq 1$, $x \neq 0$, is singular on $\mathbb{R}$, since f is not integrable on a neighborhood of 0.

The Dirac Distribution

- For any fixed point $\xi \in \Omega$, we define

$$T(\phi) = \phi(\xi), \quad \phi \in \mathscr{D}(\Omega).$$

  T is clearly a linear functional on $\mathscr{D}(\Omega)$.

  T is continuous, since $\phi \to 0$ in $\mathscr{D}(\Omega)$ implies that $\phi(\xi) \to 0$ in $\mathbb{C}$.

  T is known as the **Dirac distribution** and is denoted by δ_ξ.

  δ_0 usually abbreviated to δ.

  Thus, $\delta(\phi) = \phi(0)$, for all $\phi \in \mathscr{D}(\Omega)$.

  This distribution obviously has zero order.

The Dirac Distribution: Singularity

- We show that δ_ξ is not regular.

  Let, for $\varepsilon > 0$ and every $x \in \mathbb{R}$,

$$\phi_\varepsilon(x) = \alpha\left(\frac{x}{\varepsilon}\right) = \begin{cases} e^{-\frac{\varepsilon^2}{\varepsilon^2 - |x|^2}}, & |x| < \varepsilon \\ 0, & |x| \geq \varepsilon \end{cases}.$$

  ϕ_ε is clearly in $\mathscr{D}(\mathbb{R})$ and $|\phi_\varepsilon(x)| \leq \phi_\varepsilon(0) = \frac{1}{e}$.

  Suppose δ were regular. Then, for some $f \in L^1_{\text{loc}}(\mathbb{R})$,

$$\delta(\phi_\varepsilon) = \int f(x)\phi_\varepsilon(x)dx = \int_{|x|\leq\varepsilon} f(x)\phi_\varepsilon(x)dx.$$

  Consequently,

$$\frac{1}{e} = \phi_\varepsilon(0) = \delta(\phi_\varepsilon) \leq \frac{1}{e}\int_{|x|\leq\varepsilon} |f(x)|dx \xrightarrow{\varepsilon\to 0} 0.$$

  But this is impossible. Hence δ, and therefore δ_ξ, is singular.

The Dirac Distribution: Notation and Generalization

- Even though δ_ξ is singular, we write

$$\langle \delta_\xi, \phi \rangle := \delta_\xi(\phi) = \phi(\xi).$$

- In other words, the use of the bracket notation is not restricted to regular distributions.
- If Σ is a hypersurface in $\mathbb{R}^n$ of dimension less than n, then for any locally integrable function f on Σ, we can define the distribution

$$T_f(\phi) = \int_\Sigma f \phi d\sigma, \quad \phi \in \mathscr{D}(\mathbb{R}^n).$$

- This is clearly a generalization of the Dirac distribution from the point 0 to the hypersurface Σ.
- T_f may be interpreted as a measure on $\mathbb{R}^n$ supported by Σ with density f.

Distributions Generated by Borel Measures

- The Riesz Representation Theorem asserts that to each continuous linear functional T on $C_0^0(\Omega)$, there corresponds a unique complex, locally finite, regular Borel measure μ on Ω, such that

$$T(\phi) = \int_\Omega \phi d\mu, \quad \phi \in C_0^0(\Omega).$$

- Such a measure defines a continuous linear functional on $C_0^0(\Omega)$.
- So the correspondence between T and μ is bijective.
- The measure function corresponding to the regular distribution T_f is given by $\mu(E) = \int_E f$, for any measurable set $E \subseteq \mathbb{R}^n$.
- The Dirac distribution δ_ξ which is defined on $\mathscr{D}(\Omega)$ by $\langle \delta_\xi, \phi \rangle = \phi(\xi)$ is also continuous on $C_0^0(\Omega)$ and corresponds to the measure function

$$\mu(E) = \begin{cases} 1, & \text{if } \xi \in E \\ 0, & \text{if } \xi \notin E \end{cases}.$$

Non-Borel Measurable Distribution

- The mapping $T(\phi) = \phi'(0)$ defines a continuous linear functional on $\mathscr{D}(\mathbb{R})$. In fact, on $C_0^m(\mathbb{R})$, for $m \geq 1$, but not on $C_0^0(\mathbb{R})$.
- Thus, T is a distribution which is not a measure.
- In higher dimensions, the functional

$$T(\phi) = \partial_k \phi(0), \quad \phi \in \mathscr{D}(\mathbb{R}^n),$$

  where $1 \leq k \leq n$, is a (singular) distribution in $\mathbb{R}^n$ of order 1.
- More generally, the functional $\phi \mapsto \partial^\alpha \phi(0)$, for any $\alpha \in \mathbb{N}_0^n$, is a distribution in $\mathbb{R}^n$ of order $|\alpha|$.

Example: The function $\begin{cases} \frac{1}{x}, & \text{if } x \in (0,\infty) \\ 0, & \text{otherwise} \end{cases}$ is not integrable on any neighborhood of 0, and does not define a distribution on $\mathbb{R}$.

Its restriction to $(0,\infty)$, on the other hand, is continuous and therefore defines a regular distribution in $(0,\infty)$.

Subsection 3

Differentiation of Distributions

Derivative of a Distribution

- When $f \in C^1(\mathbb{R})$, it defines a distribution and has a derivative f' which is also a distribution.
- Viewing a distribution as a generalization of a function, it is desirable to define the distributional derivative of f so that it agrees with f'.
- Integration by parts gives the following result, where $\phi \in \mathscr{D}(\mathbb{R})$,

$$\begin{array}{rcl} \langle f', \phi \rangle & = & \int_{-\infty}^{\infty} f'(x)\phi(x)dx \\ & = & f(x)\phi(x)\Big|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} f(x)\phi'(x)dx \\ & = & -\langle f, \phi' \rangle. \end{array}$$

Definition

For any $T \in \mathscr{D}'(\Omega)$, we define

$$\partial_k T(\phi) = -T(\partial_k \phi), \quad \phi \in \mathscr{D}(\Omega).$$

Higher Derivatives of Distributions

- By using induction, we obtain the more general formula

$$\partial^\alpha T(\phi) = (-1)^{|\alpha|} T(\partial^\alpha \phi), \quad \phi \in \mathscr{D}(\Omega),\ \alpha \in \mathbb{N}_0^n$$

  or $\langle \partial^\alpha T, \phi \rangle = (-1)^{|\alpha|} \langle T, \partial^\alpha \phi \rangle$.
- The right-hand side is well defined for any multi-index α, because $\phi \in \mathscr{D}(\Omega)$, and represents a continuous linear functional on $\mathscr{D}(\Omega)$.
- Thus a distribution has derivatives, in the sense of the above definition, of all orders.
- Furthermore, $\partial^\alpha \partial^\beta T = \partial^\beta \partial^\alpha T$, for any $T \in \mathscr{D}'(\Omega)$.

$$\begin{aligned} \partial^\alpha \partial^\beta T(\phi) &= (-1)^{|\alpha|} \partial^\beta T(\partial^\alpha \phi) \\ &= (-1)^{|\alpha|+|\beta|} T(\partial^\beta \partial^\alpha \phi) \\ &= (-1)^{|\alpha|+|\beta|} T(\partial^\alpha \partial^\beta \phi) \\ &= \partial^\beta \partial^\alpha T(\phi). \end{aligned}$$

Distributional vs. Ordinary Derivatives

- If $f \in C^m(\Omega)$, then the formula for integration by parts can be used to show that the distributional derivative of f coincides with its conventional, or classical, derivative in the sense that

$$\partial^\alpha T_f = T_{\partial^\alpha f}, \text{ for all } |\alpha| \leq m.$$

- In general this relation does not hold, as may be seen from some of the following examples.

Example (Distributional vs. Ordinary Derivatives)

- Define $x_+ = \begin{cases} x, & x > 0 \\ 0, & x \le 0 \end{cases}.$

  As a function x_+ is not differentiable at $x = 0$ in the classical sense.

  As a distribution, it can be differentiated by the preceding formula.

  Define the **Heaviside function** $H(x) = \begin{cases} 1, & x > 0 \\ 0, & x < 0 \end{cases}.$

  Then we have

$$\begin{array}{rcl} \langle x_+', \phi\rangle & = & -\langle x_+, \phi'\rangle, \quad \phi \in \mathscr{D}(\mathbb{R}) \\ & = & -\int_0^\infty x\phi'(x)dx \\ & = & -x\phi(x)\big|_0^\infty + \int_0^\infty \phi(x)dx \\ & = & 0 + \int_{-\infty}^\infty H(x)\phi(x)dx \\ & = & \langle H, \phi\rangle. \end{array}$$

  Similarly,

$$\langle x_+'', \phi\rangle = \langle H', \phi\rangle = -\langle H, \phi'\rangle = -\int_0^\infty \phi'(x)dx = \phi(0).$$

  Therefore, $x_+'' = H' = \delta$.

Example (Cont'd)

- We can go further:

$$\begin{array}{rcl} \langle x_+''', \phi \rangle & = & \langle \delta', \phi \rangle = -\langle \delta, \phi' \rangle = -\phi'(0); \\ & \vdots & \\ \langle x_+^{(k+2)}, \phi \rangle & = & \langle \delta^{(k)}, \phi \rangle = (-1)^k \phi^{(k)}(0). \end{array}$$

  It is important to note in this example that x_+ and H are differentiated as distributions and not as functions.
  - In the case of x_+ it makes no difference, since $x_+' = H$ almost everywhere (a.e.) in the classical sense as well.
  - But the classical derivative of H is 0 a.e..
    When we write $H' = \delta$ we really mean $T_H' = \delta$.

Derivative of Non-Continuously Differentiable Function

- As in the case of x_+, the distributional and the classical derivatives of a function may coincide even when the function is not continuously differentiable.

  Example: Let f be a differentiable function on $I = (a, b)$.

  Suppose its (classical) derivative f' is integrable on I.

  Such a function can be expressed as the integral of its derivative

  $$f(x) = \int_c^x f'(t)dt + f(c), \quad x, c \in I.$$

  The, for all $\phi \in \mathscr{D}(I)$,
  - $(f\phi)' = f'\phi + f\phi'$;
  - $\int_I (f\phi)' = 0$, because $f\phi$ vanishes outside a closed subinterval of I.

  Hence,

  $$\int_I f'\phi + \int_I f\phi' = 0.$$

Example (Cont'd)

- Let T'_f be the distributional derivative of T_f.
  Then
  $$T'_f(\phi) = -T_f(\phi') = -\int_I f\phi' = \int_I f'\phi = T_{f'}(\phi).$$
  Therefore, $T'_f = T_{f'}$.
- More generally, suppose f is absolutely continuous on I.
  Then:
  - f' exists almost everywhere;
  - f' is integrable on I;
  - $f(x) = \int_c^x f'(t)dt + f(c)$, $x, c \in I$.

  The equality $T'_f = T_{f'}$, then follows by the same argument.

Example (Punctured Intervals)

- Let $c \in (a,b) = I$ and $f \in C^1(I - \{c\})$.

  Suppose the left- and right-hand limits at c,

$$f(c^-) = \lim_{\substack{x \to c \\ x < c}} f(x) \quad \text{and} \quad f(c^+) = \lim_{\substack{x \to c \\ x > c}} f(x)$$

  are finite and f' is bounded in a neighborhood of c.

  Then the distributions $T_{f'}$ and T_f' in $\mathscr{D}'(I)$ are related by

$$T_f' = T_{f'} + [f(c^+) - f(c^-)]\delta_c.$$

  We show this in the next slide.

Example (Cont'd)

- Suppose ϕ is any function in $\mathscr{D}(I)$.

$$\begin{aligned}
T_f'(\phi) &= -T_f(\phi') = -\int_a^b f(x)\phi'(x)dx \\
&= -\int_a^c f(x)\phi'(x)dx - \int_c^b f(x)\phi'(x)dx \\
&= -\lim_{\varepsilon_1\to 0}\int_a^{c-\varepsilon_1} f(x)\phi'(x)dx - \lim_{\varepsilon_2\to 0}\int_{c+\varepsilon_2}^b f(x)\phi'(x)dx \\
&= -\lim_{\varepsilon_1\to 0}[f(x)\phi(x)\big|_a^{c-\varepsilon_1} - \int_a^{c-\varepsilon_1} f'(x)\phi(x)dx] \\
&\qquad - \lim_{\varepsilon_2\to 0}[f(x)\phi(x)\big|_{c+\varepsilon_2}^b - \int_{c+\varepsilon_2}^b f'(x)\phi(x)dx] \\
&= -f(c^-)\phi(c) + f(c^+)\phi(c) + \int_a^b f'(x)\phi(x)dx \\
&= \langle f',\phi\rangle + [f(c^+) - f(c^-)]\langle\delta_c,\phi\rangle.
\end{aligned}$$

In particular, when f is the Heaviside function, $H' = 0$ on $I - \{0\}$ and we obtain the expected result $T_H' = \delta$.

Notational Clarifications

- x_+ and H are functions defined on $\mathbb{R}$ which represent distributions on $\mathscr{D}(\mathbb{R})$, since each is locally integrable.
- The classical derivative of H is the function which is 0 almost everywhere, and represents the zero distribution.
- But the distributional derivative of H is δ, which is not a function.
- In the sequel, derivatives will always be taken in the distributional sense.
- The pointwise notation $H'(x)$ is meaningful only when it applies to the classical derivative, since we have no way of evaluating a distribution at a point.
- If it is interpreted properly, this notation can be useful when we wish to keep track of the point variable.
- It is convenient at times to write $H' = \delta$, $H'(x) = \delta(x)$, or $H'_x = \delta_x$, on $\mathbb{R}$, rather than the more accurate $T'_H = \delta$ on $\mathscr{D}(\mathbb{R})$.

Example (Characteristic Functions)

- For any subset E of $\mathbb{R}^n$, we define its **characteristic function** by

$$I_E(x) = \begin{cases} 1, & x \in E \\ 0, & x \in \mathbb{R}^n - E \end{cases}.$$

If E is a bounded open subset of $\mathbb{R}^n$, with a smooth boundary ∂E, then, using the Divergence Theorem,

$$\langle \partial_k I_E, \phi \rangle = -\langle I_E, \partial_k \phi \rangle = -\int_E \partial_k \phi(x) dx = -\int_{\partial E} \phi(x) \cos\theta_k d\sigma,$$

where:
  - θ_k is the angle between the x_k-axis in $\mathbb{R}^n$ and the outward normal to ∂E;
  - $d\sigma$ is the Euclidean measure on ∂E.

Thus, $\partial_k I_E$ is a measure of density $-\cos\theta_k$ on ∂E.
For the special case when $n = 1$ and $E = (a, b)$, we have:
  - $I_E(x) = H(x-a) - H(x-b)$;
  - $I'_E(x) = \delta(x-a) - \delta(x-b) = \delta_a - \delta_b$.

Example (Regularization)

- $\log|x|$ is locally integrable on $\mathbb{R}$.

  So it defines a distribution in $\mathscr{D}(\mathbb{R})$.

  Its classical derivative $\frac{d}{dx}\log|x| = \frac{1}{x}$, $x \neq 0$, does not define a distribution as pointed out previously.

  We explore the relation between the distributional derivative of $\log|x|$ and $\frac{1}{x}$.

$$\left\langle \frac{d}{dx}\log|x|, \phi \right\rangle = \langle \log|x|, \phi' \rangle = -\int_{-\infty}^{\infty} \log|x| \phi'(x) dx.$$

  Now, with $\log|x|\phi'(x)$ integrable in the neighborhood of 0,

$$\left\langle \frac{d}{dx}\log|x|, \phi \right\rangle = -\lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon} \log|x|\phi'(x) dx.$$

Example (Regularization Cont'd)

- Since ϕ has compact support and is differentiable at $x=0$,

$$\begin{aligned}\langle \tfrac{d}{dx}\log|x|,\phi\rangle &= -\lim_{\varepsilon\to 0}\Big[\log|x|\phi(x)\Big|_{\varepsilon}^{-\varepsilon} - \int_{|x|\geq\varepsilon}\tfrac{1}{x}\phi(x)dx\Big] \\ &= \lim_{\varepsilon\to 0}\Big[2\varepsilon\log\varepsilon\tfrac{\phi(\varepsilon)-\phi(-\varepsilon)}{2\varepsilon} + \int_{|x|\geq\varepsilon}\tfrac{1}{x}\phi(x)dx\Big] \\ &= \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\tfrac{1}{x}\phi(x)dx.\end{aligned}$$

- $\lim\limits_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{1}{x}\phi(x)dx$ is called the **Cauchy principal value** of the divergent integral $\int_{-\infty}^{\infty}\frac{1}{x}\phi(x)dx$ and is denoted by $\mathrm{pv}\int_{-\infty}^{\infty}\frac{1}{x}\phi(x)dx$.
- Thus, the distributional derivative of $\log|x|$, which is not a function, denoted by $\mathrm{pv}\frac{1}{x}$, is obtained from the divergent integral $\int_{-\infty}^{\infty}\frac{1}{x}\phi(x)dx$ by taking its principal value.
- This process is known as **regularizing the integral**.

Regularization of $T_{\partial^\alpha f}$

- If the function f is locally integrable but $\partial^\alpha f$ is not, then $\partial^\alpha T_f$ is called a **regularization** of $T_{\partial^\alpha f}$.

  By the same token, ϕ' being differentiable at $x = 0$,

$$\begin{array}{rcl}\langle \frac{d}{dx}\mathrm{pv}\frac{1}{x}, \phi\rangle & = & -\langle \mathrm{pv}\frac{1}{x}, \phi'\rangle \\ & = & -\lim\limits_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{1}{x}\phi'(x)dx \\ & = & -\lim\limits_{\varepsilon\to 0}[\log|x|\phi'(x)\big|_{\varepsilon}^{-\varepsilon} - \int_{|x|\geq\varepsilon}\log|x|\phi''(x)dx] \\ & = & \lim\limits_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\log|x|\phi''(x)dx.\end{array}$$

  The last integral is well defined, since $\log|x|\phi''$ is integrable on $\mathbb{R}$, and represents the action of the distribution $\frac{d}{dx}\mathrm{pv}\frac{1}{x}$ on ϕ.

Example

- Consider the differential operator $L = \frac{d^2}{dx^2} - 3\frac{d}{dx} + 2$ in $\mathbb{R}$.
  Let
  $$h(x) = \begin{cases} e^x, & x \leq 0 \\ e^{2x}, & x > 0 \end{cases}.$$
  Let T_h be the distribution defined by the continuous function h.
  Claim: $LT_h = \delta$.
  For any $\phi \in \mathscr{D}(\mathbb{R})$, we have
  $$LT_h(\phi) = \langle h'' - 3h' + 2h, \phi \rangle = \langle h, \phi'' \rangle + 3\langle h, \phi' \rangle + 2\langle h, \phi \rangle.$$
  Now
  $$\begin{array}{rcl} \langle h, \phi'' \rangle & = & \int_{-\infty}^0 e^x \phi''(x) dx + \int_0^\infty e^{2x} \phi''(x) dx \\ & = & [\phi'(0) - \int_{-\infty}^0 e^x \phi'(x) dx] + [-\phi'(0) - 2\int_0^\infty e^{2x} \phi'(x) dx] \\ & = & -[\phi(0) - \int_{-\infty}^0 e^x \phi(x) dx] - 2[-\phi(0) - 2\int_0^\infty e^{2x} \phi(x) dx] \\ & = & \phi(0) + \int_{-\infty}^0 e^x \phi(x) dx + 4\int_0^\infty e^{2x} \phi(x) dx. \end{array}$$

Example (Cont'd)

- We also have

$$\begin{array}{rcl}\langle h,\phi'\rangle & = & \int_{-\infty}^{0} e^{x}\phi'(x)dx+\int_{0}^{\infty} e^{2x}\phi'(x)dx \\ & = & -\int_{-\infty}^{0} e^{x}\phi(x)dx-2\int_{0}^{\infty} e^{2x}\phi(x)dx; \\ \langle h,\phi\rangle & = & \int_{-\infty}^{0} e^{x}\phi(x)dx+\int_{0}^{\infty} e^{2x}\phi(x)dx.\end{array}$$

Hence, $LT_h(\phi)=\phi(0)$, for every $\phi\in\mathscr{D}(\mathbb{R})$. So $LT_h=\delta$.

Note that the function h, though continuous, has a jump discontinuity in its derivative at $x=0$ given by

$$h'(0^+)-h'(0^-)=2e^0-e^0=1.$$

This accounts for the δ distribution when h is differentiated a second time.

On $\mathbb{R}-\{0\}$, the function h is twice differentiable and satisfies $Lh=0$.

Generalizing the Differential Operator

- Let
$$L = \frac{d^2}{dx^2} + a\frac{d}{dx} + b,$$
with $a, b \in \mathbb{R}$, be a differential operator in $\mathbb{R}$.
Suppose that f_1 and f_2 are two C^2 solutions in $\mathbb{R}$ of $Lf = 0$, satisfying
$$f_1(0) = f_2(0), \quad f_2'(0) - f_1'(0) = 1.$$
Let h be the continuous function defined by
$$h(x) = \begin{cases} f_1(x), & x \le 0 \\ f_2(x), & x > 0 \end{cases}.$$
Let T_h be the distribution defined by h.
We can verify that $LT_h = \delta$.
The solution x_+ of $T'' = \delta$ is in accordance with this construction.

The Laplacian Operator

- In $\mathbb{R}^n$ the partial differential operator
$$\sum_{k=1}^{n} \partial_k^2$$
is known as the **Laplacian operator**, and will be denoted by Δ.

Example: The function $\log|x|$ is locally integrable in $\mathbb{R}^2$.

We obtain its (distributional) Laplacian derivative
$$\Delta \log|x| = (\partial_1^2 + \partial_2^2) \log|x|.$$

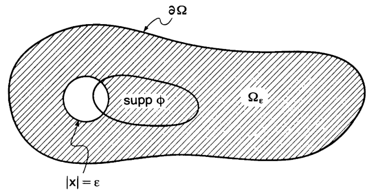
By the differentiation formula, for all $\phi \in \mathscr{D}(\mathbb{R}^2)$,
$$\begin{array}{rcl} \langle \Delta \log|x|, \phi \rangle & = & \langle \log|x|, \Delta\phi \rangle \\ & = & \int_{\mathbb{R}^2} \log|x| \Delta\phi(x) dx \\ & = & \lim\limits_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \log|x| \Delta\phi(x) dx. \end{array}$$

Intermission: Green's First and Second Formulas

- Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set with sufficiently smooth boundary.
- Let $u, v \in C^2(\overline{\Omega})$ be pair of functions.
- By the Divergence Theorem, we get **Green's First Formula**
$$\int_\Omega \left[u\Delta v + \sum_{k=1}^n (\partial_k u)(\partial_k v) \right] = \int_{\partial\Omega} u\partial_\eta v,$$
where ∂_η is the differential operator with respect to the outward normal η on $\partial\Omega$.
- By interchanging u and v, we get
$$\int_\Omega \left[v\Delta u + \sum_{k=1}^n (\partial_k u)(\partial_k v) \right] = \int_{\partial\Omega} v\partial_\eta u.$$
- By subtracting, we obtain **Green's Second Formula**
$$\int_\Omega (u\Delta v - v\Delta u) = \int_{\partial\Omega} (u\partial_\eta v - v\partial_\eta u).$$

The Laplacian Operator (Cont'd)

- Choose Ω so that it contains:
  - The support $\text{supp}\phi$;
  - The closed ball $\overline{B}(0,\varepsilon)$, for some $\varepsilon > 0$.



By Green's Second Formula on $\Omega_\varepsilon = \Omega - \overline{B}(0,\varepsilon) = \{x \in \Omega : |x| > \varepsilon\}$, we obtain

$$\begin{aligned}\int_{\Omega_\varepsilon} \log|x| \Delta\phi(x) dx &= \int_{\Omega_\varepsilon} \phi(x) \Delta \log|x| dx \\ &+ \int_{\partial\Omega_\varepsilon} [\log|x| \partial_\eta \phi(x) - \phi(x) \partial_\eta \log|x|] d\sigma,\end{aligned}$$

where η is the outward normal on $\partial\Omega_\varepsilon$.

Since ϕ and $\partial_\eta\phi$ vanish on the boundary $\partial\Omega$, we have

$$\begin{aligned}\int_{|x|\geq\varepsilon} \log|x| \Delta\phi(x) dx &= \int_{|x|\geq\varepsilon} \phi(x) \Delta \log|x| dx \\ &+ \int_{|x|=\varepsilon} [\log|x| \partial_\eta \phi(x) - \phi(x) \partial_\eta \log|x|] d\sigma.\end{aligned}$$

The Laplacian Operator (Cont'd)

- With $|x| = (x_1^2 + x_2^2)^{1/2} = r$, we have $\partial_\eta = -\partial_r$ on the circle $|x| = \varepsilon$. Moreover, for all $x \neq 0$, we also have

$$\begin{array}{rcl} \Delta \log|x| & = & \partial_1\left(\frac{1}{|x|}\partial_1|x|\right) + \partial_2\left(\frac{1}{|x|}\partial_2|x|\right) \\ & = & \partial_1\left(\frac{x_1}{|x|^2}\right) + \partial_2\left(\frac{x_2}{|x|^2}\right) \\ & = & \frac{x_2^2 - x_1^2}{|x|^4} + \frac{x_1^2 - x_2^2}{|x|^4} = 0. \end{array}$$

Thus, the first integral on the right side drops out, and we have

$$\begin{array}{rcl} \int_{|x|\geq\varepsilon} \log|x| \Delta\phi(x) dx & = & \int_{|x|=\varepsilon} \left[\log\varepsilon \partial_\eta \phi(x) - \phi(x) \frac{x_1^2 + x_2^2}{|x|^2}\right] d\sigma \\ & = & \int_{|x|=\varepsilon} \left[\frac{1}{\varepsilon}\phi(x) - \log\varepsilon \partial_r \phi(x)\right] d\sigma. \end{array}$$

The Laplacian Operator (Conclusion)

- Now ϕ is in $C_0^\infty(\mathbb{R}^2)$.

  So its derivative $\partial_r\phi$ is bounded on $\mathbb{R}^2$ by some constant, say M.

  Hence
  $$|\log\varepsilon\int_{|x|=\varepsilon}\partial_r\phi(x)d\sigma|\le 2\pi\varepsilon|\log\varepsilon|M\overset{\varepsilon\to 0}{\longrightarrow}0.$$

  Moreover,
  $$\frac{1}{\varepsilon}\int_{|x|=\varepsilon}\phi(x)d\sigma=\frac{1}{\varepsilon}\int_{|x|=\varepsilon}[\phi(x)-\phi(0)]d\sigma+\frac{1}{\varepsilon}\phi(0)\int_{|x|=\varepsilon}d\sigma.$$

  ϕ is continuous at $x=0$. So $\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{|x|=\varepsilon}[\phi(x)-\phi(0)]d\sigma=0$.

  Therefore,
  $$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\int_{|x|=\varepsilon}\phi(x)d\sigma=0+2\pi\phi(0).$$

  Thus, $\langle\Delta\log|x|,\phi\rangle=2\pi\phi(0)$, for all $\phi\in\mathscr{D}(\mathbb{R}^2)$, i.e., $\Delta\log|x|=2\pi\delta$.

Example

- We determine $\Delta\left(\frac{1}{|x|}\right)$ in $\mathbb{R}^3$.

  In $\mathbb{R}^3$, $\frac{1}{|x|}$ is integrable in the neighborhood of 0. We have

$$\left\langle \Delta\frac{1}{|x|},\phi\right\rangle = \left\langle \frac{1}{|x|},\Delta\phi\right\rangle = \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\frac{1}{|x|}\Delta\phi(x)dx.$$

  We also have

$$\begin{aligned}\int_{|x\geq\varepsilon}\frac{1}{|x|}\Delta\phi(x)dx &= \int_{|x|\geq\varepsilon}\phi(x)\Delta\left(\frac{1}{|x|}\right)dx \\ &+\int_{|x|=\varepsilon}\left[\frac{1}{|x|}\partial_\eta\phi(x)-\phi(x)\partial_\eta\left(\frac{1}{|x|}\right)\right]d\sigma.\end{aligned}$$

  Note that $\Delta\left(\frac{1}{|x|}\right)=(\partial_1^2+\partial_2^2+\partial_3^2)(x_1^2+x_2^2+x_3^2)^{-1/2}=0$, when $x\neq 0$.
  So the first integral on the right-hand side vanishes.
  Therefore, with $\partial_\eta=-\partial_r$,

$$\int_{|x|\geq\varepsilon}\frac{1}{|x|}\Delta\phi(x)dx=-\frac{1}{\varepsilon}\int_{|x|=\varepsilon}\partial_r\phi(x)d\sigma-\frac{1}{\varepsilon^2}\int_{|x|=\varepsilon}\phi(x)d\sigma.$$

Example (Cont'd)

- Now $\partial_r \phi$ is a bounded function in $\mathbb{R}^3$.

  So there is a positive M such that $|\partial_r \phi(x)| \le M$, for all $x \in \mathbb{R}^3$.

  Hence,

$$\left| \frac{1}{\varepsilon} \int_{|x|=\varepsilon} \partial_r \phi(x) d\sigma \right| \le \frac{M}{\varepsilon} \int_{|x|=\varepsilon} d\sigma = 4\pi\varepsilon M \xrightarrow{\varepsilon\to 0} 0.$$

  We are left with

$$\frac{1}{\varepsilon^2} \int_{|x|=\varepsilon} \phi(x) dx = \frac{1}{\varepsilon^2} \int_{|x|=\varepsilon} [\phi(x) - \phi(0)] dx + \frac{1}{\varepsilon^2} \int_{|x|=\varepsilon} \phi(0) dx.$$

  The first integral on the right-hand side tends to 0 as $\varepsilon \to 0$.

  The second is just $4\pi\phi(0)$.

  Thus, $\left\langle \Delta \frac{1}{|x|}, \phi \right\rangle = -4\pi\phi(0)$, for every $\phi \in \mathscr{D}(\mathbb{R}^3)$.

  Therefore, $\Delta \frac{1}{|x|} = -4\pi\delta$.

Subsection 4

Convergence of Distributions

Weak Topology and Weak Convergence of Distributions

- On the vector space $\mathscr{D}'(\Omega)$, the **weak topology** is the locally convex topology defined by the family of seminorms

$$p_\phi(T) = |T(\phi)|, \quad \phi \in \mathscr{D}(\Omega), \ T \in \mathscr{D}'(\Omega).$$

- This leads to the following definition of (weak) convergence in $\mathscr{D}'(\Omega)$.

Definition (Weak Convergence in $\mathscr{D}'(\Omega)$)

The sequence (T_k) in $\mathscr{D}'(\Omega)$ **converges** to 0 if and only if, for every $\phi \in \mathscr{D}(\Omega)$, the sequence $(T_k(\phi))$ converges to 0 in $\mathbb{C}$.

- This is "pointwise" convergence on $\mathscr{D}(\Omega)$.
- We write $T_k \to T$ in $\mathscr{D}'(\Omega)$ if the sequence $(T_k - T)$ converges to 0.

Strong Convergence of Distributions

- In the **strong** or **uniform convergence** in $D'(\Omega)$, $T_k \to 0$ is equivalent to $T_k(\phi) \to 0$ uniformly on every bounded subset of $\mathscr{D}(\Omega)$.
- Strong, or uniform, convergence implies weak convergence.
- Convergence in $\mathscr{D}'(\Omega)$ will be taken in the weak sense unless otherwise qualified.

Sequential Completeness

Theorem

The space of distributions $\mathscr{D}'(\Omega)$ is (sequentially) complete.

- Suppose (T_k) is a Cauchy sequence in $\mathscr{D}'(\Omega)$.

  Then it is bounded, i.e., there is a neighborhood U of 0 in $\mathscr{D}(\Omega)$ and a positive number M, such that $|T_k(\phi)| \leq M$, for all $\phi \in U$ and $k \in \mathbb{N}$.

  Also, $(T_k(\phi))$ is a Cauchy sequence in $\mathbb{C}$, for every $\phi \in \mathscr{D}(\Omega)$.

  Therefore its limit exists. Let T be defined by

  $$T(\phi) = \lim T_k(\phi), \quad \phi \in \mathscr{D}(\Omega).$$

  T is clearly linear.

  For all $\phi \in U$, $|T(\phi)| = \lim |T_k(\phi)| \leq M$. So T is bounded on U.

  Therefore, T is continuous on $\mathscr{D}(\Omega)$.

Limits and Derivatives

Corollary

If $T_k \in \mathscr{D}'(\Omega)$, for every $k \in \mathbb{N}$, and $\lim T_k = T$, then $\lim \partial^\alpha T_k = \partial^\alpha T$, for every multi-index $\alpha \in \mathbb{N}_0^n$.

- For any $\phi \in \mathscr{D}(\Omega)$, we have

$$\begin{aligned} \lim(\partial^\alpha T_k)(\phi) &= (-1)^{|\alpha|} \lim T_k(\partial^\alpha \phi) \\ &= (-1)^{|\alpha|} T(\partial^\alpha \phi) \\ &= \partial^\alpha T(\phi). \end{aligned}$$

Almost Everywhere Convergence vs Convergence in $\mathscr{D}'(\Omega)$

Theorem

If f_k is a sequence of functions in $L^1_{\text{loc}}(\Omega)$ which converges to f a.e. in Ω, and $|f_k| \leq g$, for some $g \in L^1_{\text{loc}}(\Omega)$, then $f_k \to f$ in $\mathscr{D}'(\Omega)$.

- For every $\phi \in \mathscr{D}(\Omega)$, we have

$$T_{f_k}(\phi) = \langle f_k, \phi \rangle = \int_\Omega f_k \phi \stackrel{k\to\infty}{\longrightarrow} \int_\Omega f \phi,$$

by the Lebesgue Dominated Convergence Theorem.

But, we have $\int_\Omega f\phi = T_f(\phi)$.

Therefore, $T_{f_k} \to T_f$.

Convergence a.e. vs. Convergence in $\mathscr{D}'$

- Convergence a.e. for a sequence of locally integrable functions does not imply its convergence in $\mathscr{D}'$.

  Example: Consider the sequence

$$f_k(x) = \begin{cases} k^2, & |x| < \frac{1}{k} \\ 0, & |x| \geq \frac{1}{k} \end{cases}.$$

  It converges to 0 a.e.

  Let $\phi \in \mathscr{D}(\mathbb{R})$ be such that $\phi = 1$ in $(-1,1)$.

  Then $\langle f_k, \phi \rangle = 2k$.

  This does not converge.

Convergence in $\mathscr{D}'$ vs. Pointwise Convergence

- Distributional convergence does not imply pointwise convergence.

  Example: Consider
  $$(\sin kx).$$
  Then, we have, for every $\phi \in \mathscr{D}(\mathbb{R})$,
  $$\begin{array}{rcl}\langle \sin kx, \phi\rangle & = & \int_{-\infty}^{\infty} \sin kx \phi(x) dx \\ & = & \int_{-\infty}^{\infty} (-\frac{1}{k}\cos kx)' \phi(x) dx \\ & = & -\frac{1}{k}\cos kx \phi(x) \mid_{-\infty}^{\infty} + \frac{1}{k}\int_{-\infty}^{\infty} \cos kx \phi'(x) dx \\ & = & \frac{1}{k}\int_{-\infty}^{\infty} \cos kx \phi'(x) dx \stackrel{k\to\infty}{\longrightarrow} 0.\end{array}$$
  Clearly, $(\sin kx)$ does not converge pointwise.

Example (Convergence in $\mathscr{D}'(\mathbb{R})$)

- Let
$$T_n = n\delta - \left(\sum_1^n \frac{1}{k}\right)\delta' - \left(\sum_1^n \delta_{1/k}\right).$$
To show that T_n converges in $\mathscr{D}'(\mathbb{R})$ we must show that $\lim T_n(\phi)$ exists, for every $\phi \in \mathscr{D}(\mathbb{R})$.
We have
$$T_n(\phi) = n\phi(0) + \left(\sum_1^n \frac{1}{k}\right)\phi'(0) - \sum_1^n \phi\left(\frac{1}{k}\right).$$
By Taylor's Formula, we can write
$$\phi(x) = \phi(0) + x\phi'(0) + x^2\psi(x),$$
where ψ is a C^∞ function which is bounded by some constant, say M.

Example (Convergence in $\mathscr{D}'(\mathbb{R})$ Cont'd)

- Now we obtain

$$\begin{aligned} T_n(\phi) &= n\phi(0) + \left(\sum_1^n \frac{1}{k}\right)\phi'(0) - \sum_1^n \phi\left(\frac{1}{k}\right) \\ &= n\phi(0) + \left(\sum_1^n \frac{1}{k}\right)\phi'(0) - \sum_1^n \left[\phi(0) + \frac{1}{k}\phi'(0) + \frac{1}{k^2}\psi\left(\frac{1}{k}\right)\right] \\ &= -\sum_1^n \frac{1}{k^2}\psi\left(\frac{1}{k}\right). \end{aligned}$$

Therefore, for $m < n$,

$$|T_n(\phi) - T_m(\phi)| \leq M \sum_m^n \frac{1}{k^2}.$$

So $(T_n(\phi))$ is a Cauchy sequence in C.

So its limit exists.

Example: Delta-Convergent Sequences

- Even when the sequence of functions f_k converges a.e. and in $\mathscr{D}'$, the two limits may not be equal.

  Example: Consider

$$f_k(x) = \begin{cases} k, & \text{if } |x| < \frac{1}{2k} \\ 0, & \text{if } |x| \geq \frac{1}{2k} \end{cases}.$$

  Clearly, $\int f_k(x)dx = 1$. Moreover, $f_k \to 0$ a.e. on $\mathbb{R}$.

  For any function ϕ in $\mathscr{D}(\mathbb{R})$, by the continuity of ϕ at 0,

$$\langle f_k, \phi \rangle = \phi(0) + k \int_{-1/2k}^{1/2k} [\phi(x) - \phi(0)] dx \stackrel{k \to \infty}{\longrightarrow} \phi(0).$$

  Hence, $\lim f_k = \delta$.

- A sequence of functions, such as (f_k), which converges to δ in $\mathscr{D}'(\Omega)$ is called a **delta-convergent sequence**.

Construction of Delta-Convergent Sequences

Theorem

Let f be a nonnegative integrable function on $\mathbb{R}^n$ with $\int f(x)dx = 1$ and

$$f_\lambda(x) = \frac{1}{\lambda^n} f\left(\frac{x}{\lambda}\right) = \frac{1}{\lambda^n} f\left(\frac{x_1}{\lambda}, \ldots, \frac{x_n}{\lambda}\right), \quad \lambda > 0.$$

Then $f_\lambda \to \delta$ in $\mathscr{D}'(\mathbb{R}^n)$ as $\lambda \to 0$.

- Note that

$$\int f_\lambda(x)dx = \int f\left(\frac{x}{\lambda}\right)\frac{1}{\lambda^n}dx = \int f(\xi)d\xi = 1.$$

Therefore,

$$\begin{aligned} \lim_{\lambda\to 0}\langle f_\lambda, \phi\rangle &= \lim_{\lambda\to 0}\int f_\lambda(x)\phi(x)dx \\ &= \phi(0) + \lim_{\lambda\to 0}\int f_\lambda(x)[\phi(x) - \phi(0)]dx. \end{aligned}$$

Construction of Delta-Convergent Sequences (Cont'd)

- We also have

$$\begin{array}{rcl} \left|\int f_\lambda(x)[\phi(x)-\phi(0)]dx\right| & \leq & \int_{|x|\leq r}\left|f_\lambda(x)[\phi(x)-\phi(0)]\right|dx \\ & & +\int_{|x|\geq r}\left|f_\lambda(x)[\phi(x)-\phi(0)]\right|dx \\ & \leq & \sup_{|x|\leq r}|\phi(x)-\phi(0)|\int_{|x|\leq r}f_\lambda(x)dx \\ & & +\sup_{|x|\geq r}|\phi(x)-\phi(0)|\int_{|x|\geq r}f_\lambda(x)dx \\ & \leq & \sup_{|x|\leq r}|\phi(x)-\phi(0)|+M\int_{|\xi|\geq r/\lambda}f(\xi)d\xi, \end{array}$$

where M is the max of $|\phi(x)-\phi(0)|$ on $\mathbb{R}^n$.

Let $\varepsilon>0$ be arbitrary.

Because ϕ is continuous at 0, we can make the first term less than $\frac{1}{2}\varepsilon$ by choosing r small enough.

Because f is integrable on $\mathbb{R}^n$, we can choose λ small enough so that the second term is less than $\frac{1}{2}\varepsilon$.

Example

- Recall the equality $\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \pi$.

  Define

  $$f(x) = \frac{1}{\pi(1+x^2)}.$$

  Then

  $$\int_{-\infty}^{\infty} \frac{1}{\pi(1+x^2)} = 1.$$

  Let

  $$f_\lambda(x) = \frac{1}{\lambda} f\left(\frac{x}{\lambda}\right) = \frac{1}{\lambda} \frac{1}{\pi[1+(\frac{x}{\lambda})^2]} = \frac{\lambda}{\pi(x^2+\lambda^2)}.$$

  By the previous theorem, in $\mathscr{D}'(\mathbb{R})$,

  $$f_\lambda \xrightarrow{\lambda \to 0} \delta.$$

Example

- Recall the equality $\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$.

  We obtain

$$\begin{array}{rcl} \int_{\mathbb{R}^n} e^{-|x|^2} dx & = & \int_{\mathbb{R}^n} \prod_{k=1}^n e^{-x_k^2} dx_k \\ & = & \prod_{k=1}^n \int_{-\infty}^{\infty} e^{-x_k^2} dx_k \\ & = & \sqrt{\pi}^n. \end{array}$$

  Replacing the parameter $\lambda > 0$ in the previous theorem by $\sqrt{\lambda}$, we obtain the following function defined on $\mathbb{R}^n$, for all positive values of λ,

$$f_\lambda(x) = \frac{1}{\sqrt{\pi\lambda}^n} e^{-|x|^2/\lambda}.$$

  By the theorem,

$$f_\lambda \xrightarrow{\lambda \to 0} \delta.$$

Subsection 5

Multiplication by Smooth Functions

The Product of a C^∞ Function with a Distribution

- For any $T \in \mathscr{D}'(\Omega)$ and $f \in C^\infty(\Omega)$, we define fT as the linear functional
$$(fT)(\phi) = T(f\phi), \quad \phi \in \mathscr{D}(\Omega).$$
- The product fT is well defined.

  Note that the product $f\phi$ is in $\mathscr{D}(\Omega)$.
- The product fT is in $\mathscr{D}'(\Omega)$.

  Suppose the sequence ϕ_k converges to 0 in $\mathscr{D}(\Omega)$.

  Then the sequence $f\phi_k$ also converges to 0 in $\mathscr{D}(\Omega)$.

  Therefore,
$$(fT)(\phi_k) = T(f\phi_k) \to 0.$$
  So fT is a continuous linear functional on $\mathscr{D}(\Omega)$.

Regularity of the Product

- Let $T \in \mathscr{D}'(\Omega)$ and $f \in C^\infty(\Omega)$.
- Suppose T is a regular distribution.
- Let g be a locally integrable function, such that, for all $\phi \in \mathscr{D}(\Omega)$,
$$T\phi = \langle g, \phi \rangle.$$
- Note that fg is also locally integrable.
- So we obtain
$$(fT_g)(\phi) = T_g(f\phi) = \langle g, f\phi \rangle = \int gf\phi = \langle fg, \phi \rangle.$$
- Thus, $fT_g = T_{fg}$ and fT is also regular.

Differentiation of a Product

- The ordinary rules of differentiating a product of two functions apply to fT when $f \in C^\infty(\Omega)$ and $T \in \mathscr{D}'(\Omega)$.

  Indeed we have, for all $\phi \in \mathscr{D}(\Omega)$

$$\begin{aligned}\partial_k(fT)(\phi) &= -fT(\partial_k\phi)\\ &= -T(f\partial_k\phi)\\ &= -T(\partial_k(f\phi)-(\partial_k f)\phi)\\ &= -T(\partial_k(f\phi))-T((\partial_k f)\phi)\\ &= \partial_k T(f\phi)-(\partial_k f)T(\phi)\\ &= f\partial_k T(\phi)+(\partial_k f)T(\phi).\end{aligned}$$

  Therefore,

$$\partial_k(fT)=(\partial_k f)T+f\partial_k T.$$

Differentiation of a Product: Leibniz Formula

- Let $f \in C^{\infty}(\Omega)$ and $T \in \mathscr{D}'(\Omega)$.
- We can use induction to show that Leibniz's formula

$$\partial^{\alpha}(fT) = \sum_{\beta=0}^{\alpha} \frac{\alpha!}{\beta!(\alpha-\beta)!}(\partial^{\beta} f)(\partial^{(\alpha-\beta)} T), \quad \alpha \in \mathbb{N}_0^n$$

  remains valid, were the summation is over the multi-indices from $(0,\ldots,0)$ to $(\alpha_1,\ldots,\alpha_n)$.

Example

- The product $\sin x\delta$ is the distribution defined on $\mathscr{D}(\mathbb{R})$ by

$$\langle \sin x\ \delta, \phi\rangle = \langle \delta, \sin x\ \phi\rangle = \sin 0\ \phi(0) = 0.$$

On the other hand, $\sin x\delta'$ is given by

$$\begin{aligned}\langle \sin x\ \delta', \phi\rangle &= \langle \delta', \sin x\ \phi\rangle \\ &= -\langle \delta, (\sin x\ \phi)'\rangle \\ &= -\langle \delta, \cos x\ \phi + \sin x\ \phi'\rangle \\ &= -(\cos 0\ \phi(0) + \sin 0\ \phi'(0)) \\ &= -\phi(0).\end{aligned}$$

Subsection 6

Local Properties of Distributions

Zero Distributions

- It does not make sense to assign a value to a distribution at a given point in Ω, but we can define what it means for a distribution to vanish on an open subset of Ω.

Definition

For any $T \in \mathscr{D}'(\Omega)$ and any open subset G of Ω, we say that $T = 0$ **on** G if $T(\phi) = 0$, for every $\phi \in \mathscr{D}(G)$.

- We can now say that $T \in \mathscr{D}'(\Omega)$ is **zero** if $T = 0$ on Ω.
- We also say that $T_1, T_2 \in \mathscr{D}'(\Omega)$ are **equal** if $T_1 - T_2 = 0$ on Ω.

Examples

- We saw that, for all $\phi \in \mathscr{D}(\mathbb{R})$,

$$\langle \sin x\ \delta, \phi \rangle = 0.$$

  We conclude that $\sin x\ \delta = 0$ on $\mathbb{R}$.

- We also saw that, for all $\phi \in \mathscr{D}(\mathbb{R})$,

$$\langle \sin x\ \delta', \phi \rangle = -\phi(0) = -\langle \delta, \phi \rangle.$$

  We conclude that $\sin x\ \delta' = -\delta$ on $\mathbb{R}$.

- Earlier on, we interpreted the equality $T = T_f$ on $\mathscr{D}(\mathbb{R} - \{0\})$ to mean that T is represented by f on $\mathbb{R} - \{0\}$, i.e. that $T = T_f$ on $\mathbb{R} - \{0\}$.

Example

- Let $I = (a, b)$ be any interval in $\mathbb{R}$, including $\mathbb{R}$ itself.

  Suppose $T \in \mathscr{D}'(I)$ is such that $T' = 0$.

  Then T must be a constant.

  By hypothesis, for all $\phi \in \mathscr{D}(I)$, $T(\phi') = -T'(\phi) = 0$.

  Thus, T vanishes at every test function which can be expressed as the derivative of some function in $\mathscr{D}(I)$.

  Let $\mathscr{D}_0(I)$ be the subspace of $\mathscr{D}(I)$ characterized by the condition that $\phi \in \mathscr{D}_0(I)$ if and only if there exists a $\psi \in \mathscr{D}(I)$, such that $\phi = \psi'$.

  Claim: $\phi \in \mathscr{D}_0(I)$ if and only if $\int_a^b \phi(x)dx = 0$.

  This condition is clearly necessary.

  Suppose the condition is satisfied.

  Define $\psi(x) = \int_a^x \phi(t)dt$.

  Then $\psi \in \mathscr{D}(I)$ and $\psi' = \phi$.

Example (Cont'd)

- $T(\phi) = 0$, by hypothesis, for every $\phi \in \mathscr{D}_0(I)$.

  Let ϕ_0 be a fixed function in $\mathscr{D}(I)$, such that $\int_a^b \phi_0(x)dx = 1$.

  Given any $\phi \in \mathscr{D}(I)$, the function $\phi - (\int_a^b \phi(x)dx)\phi_0$ lies in $\mathscr{D}_0(I)$.

  Therefore,

$$T\left(\phi - \phi_0 \int_a^b \phi(x)dx\right) = 0.$$

  This gives $T(\phi) = c\int_a^b \phi(x)dx$, where c is the constant $T(\phi_0)$.

  This equation implies that T is the constant function c.

- Suppose $T \in \mathscr{D}'(I)$ satisfies $T' = c_1$, for some constant c_1.

  Define $S \in \mathscr{D}'(I)$ by $S = c_1 x$. Then $(T - S)' = 0$.

  Therefore, $T = c_1 x + c_2$, for some constant c_2.

- If $T^{(m)} = 0$, we can use induction to show T is a polynomial of degree $\leq m - 1$.

The Convolution

Definition

For any $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and $\phi \in C^\infty_K(\mathbb{R}^n)$, where K is a compact subset of $\mathbb{R}^n$, we define the **convolution** of f and ϕ as the function

$$\int f(x-y)\phi(y)dy = \int f(y)\phi(x-y)dy$$

which will be denoted by $(f * \phi)(x)$.

- Note that $f * \phi$ is also defined if ϕ is merely continuous with compact support in $\mathbb{R}^n$.
- $f * \phi$ is not necessarily defined when $\text{supp}\phi$ is not compact, unless, of course, $\text{supp}f$ is compact.

The Distribution β_λ

- Consider the C^∞ function

$$\alpha(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}}, & \text{on } |x| < 1 \\ 0, & \text{on } |x| \geq 1 \end{cases}.$$

  - It has support in the closed unit ball $\overline{B}(0,1)$.
  - Its integral over $\mathbb{R}^n$ is a finite positive number.
- Consider the function $\beta(x) = \frac{\alpha(x)}{\int \alpha(x) dx}$.
  - It is another C^∞ function with support in $\overline{B}(0,1)$.
  - Moreover, it satisfies $\int \beta(x) dx = 1$.
- Let, for any positive number λ,

$$\beta_\lambda(x) = \frac{1}{\lambda^n} \beta\left(\frac{x}{\lambda}\right).$$

  - $\beta_\lambda \in \mathscr{D}(\mathbb{R}^n)$, with $\text{supp}(\beta_\lambda) = \overline{B}(0,\lambda)$.
  - Moreover, $\int \beta_\lambda(x) dx = \int \beta(x) dx = 1$.

Properties of β_λ

Theorem

(i) If $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, then $f * \beta_\lambda \in C^\infty(\mathbb{R}^n)$.

(ii) If $f \in L^1(\mathbb{R}^n)$ with compact support K, then $\text{supp}(f * \beta_\lambda)$ is contained in a neighborhood of K defined by $K_\lambda = \bigcup_{x \in K} \overline{B}(x,\lambda) = K + \overline{B}(0,\lambda)$.

(iii) If $f \in C^0(\mathbb{R}^n)$, then, $f * \beta_\lambda \xrightarrow{\lambda \to 0} f$ uniformly on every compact subset of $\mathbb{R}^n$.

(i) $(f * \beta_\lambda)(x) = \int f(y)\beta_\lambda(x-y)dy = \int_{B(x,\lambda)} f(y)\beta_\lambda(x-y)dy$.
But $B(x,\lambda)$ is bounded and β_λ is infinitely differentiable.
Hence, $f * \beta_\lambda \in C^\infty(\mathbb{R}^n)$.

(ii) Suppose $x \notin K_\lambda$. Then $d(x,K) = \inf_{y \in K} |x-y| > \lambda$. So $\beta_\lambda(x-y) = 0$, for all $y \in K$. Consequently,

$$(f * \beta_\lambda)(x) = \int_K f(y)\beta_\lambda(x-y)dy = 0.$$

Properties of β_λ Part (iii)

(iii) Since f is continuous on $\mathbb{R}^n$, it is uniformly continuous on any compact subset E of $\mathbb{R}^n$. So, given $\varepsilon > 0$, there is a $\delta > 0$, such that, for all $x \in E$ and all $y \in B(0,\delta)$,

$$|f(x-y) - f(x)| < \varepsilon.$$

Then, for all $x \in E$ and all $\lambda \leq \delta$,

$$\begin{array}{rcl} |(f * \beta_\lambda)(x) - f(x)| & = & |\int [f(x-y) - f(x)]\beta_\lambda(y)dy| \\ & \leq & \int_{B(0,\lambda)} |f(x-y) - f(x)|\beta_\lambda(y)dy \\ & < & \varepsilon. \end{array}$$

- In this proof the only properties of β_λ that were used are:

$$\beta_\lambda \in C_0^\infty(\mathbb{R}^n), \quad \text{supp}\beta_\lambda \subseteq \overline{B}(0,\lambda), \quad \int \beta_\lambda(x)dx = 1.$$

Hence β_λ may be replaced in the statement of the theorem by any function with these properties.

Regularizing Sequence or Regularization

- The theorem indicates that the convolution of f with β_λ smoothes out the discontinuities in f while preserving its general shape.
- For that reason the sequence of functions

$$f_k = f * \beta_{1/k}$$

is called a **regularizing sequence**, or a **regularization**, of f.

Property of Compact Subsets of Ω

Corollary

If K is a compact subset of $\Omega \subseteq \mathbb{R}^n$, then there is a $\phi \in \mathscr{D}(\Omega)$, such that $0 \le \phi \le 1$ and $\phi = 1$ on K.

- There is no loss of generality in taking Ω to be bounded.

  Let K_δ be the δ-neighborhood of K, where $\delta = \frac{1}{3} d(K, \partial\Omega)$.

  Let I_{K_δ} be the characteristic function of K_δ.

  Consider the C^∞ function

$$\phi(x) = (I_{K_\delta} * \beta_\delta)(x) = \int_{K_\delta} \beta_\delta(x-y) dy.$$

  - $\phi = 1$ on K;
  - $0 \le \phi \le 1$ on $K_{2\delta}$;
  - $\phi = 0$ outside $K_{2\delta}$.

Density of $\mathscr{D}(\mathbb{R}^n)$ in $C_0^0(\mathbb{R}^n)$

Corollary

$\mathscr{D}(\mathbb{R}^n)$ is a dense subspace of $C_0^0(\mathbb{R}^n)$ with the identity map from $\mathscr{D}(\mathbb{R}^n)$ to $C_0^0(\mathbb{R}^n)$ continuous.

- Suppose ϕ_k converges in $\mathscr{D}(\mathbb{R}^n)$ to ϕ.
  Then there is a compact set $K \subseteq \mathbb{R}^n$, such that:
  - $\text{supp}\phi_k \subseteq K$, for all k;
  - ϕ_k converges uniformly to ϕ on K.

  But that implies $\phi_k \to \phi$ in $C_0^0(\mathbb{R}^n)$.

  Hence, the identity map from $\mathscr{D}(\mathbb{R}^n)$ to C_0^0 is continuous.

  Next, let ϕ be any function in $C_0^0(\mathbb{R}^n)$, with $\text{supp}\phi = K$.

  Then the sequence $\phi_k = \phi * \beta_{1/k}$ is supported in $K + \overline{B}(0,1)$.

  By the theorem, ϕ_k converges uniformly to ϕ on $K + \overline{B}(0,1)$.

Open Cover and Partition of Unity

Theorem

If $\{G_\alpha : \alpha \in A\}$ is a collection of open subsets of Ω, and $T \in \mathscr{D}'(\Omega)$ is zero on every G_α, then T is zero on the union $\bigcup_{\alpha \in A} G_\alpha$.

- Let $G = \bigcup G_\alpha$ and ϕ be in $\mathscr{D}(G)$ with $\text{supp}\phi = K$.

  The collection $\{G_\alpha\}$ is an open covering of the compact set K.

  It contains a finite subcovering of K, say, after relabeling, $G_1, \ldots, G_m$.

  For every $k \in \{1, \ldots, m\}$, we choose:
  - A compact set $K_k \subseteq G_k$ so that $K \subseteq \bigcup_{k=1}^m K_k^\circ$;
  - $\phi_k \in \mathscr{D}(G_k)$ so that $0 \le \phi_k \le 1$ and $\phi_k = 1$ on K_k.

  Now let

  $$\psi_1 = \phi_1, \quad \psi_k = \phi_k(1 - \phi_1) \cdots (1 - \phi_{k-1}),\ k = 2, \ldots, m.$$

  For $k \in \{1, \ldots, m\}$, $\psi_k \in \mathscr{D}(G_k)$, $0 \le \psi_k \le 1$. Moreover, $\sum_{k=1}^m \psi_k = 1$ on a neighborhood of K. So $\phi = \sum_{k=1}^m \phi\psi_k$. But $\phi\psi_k \in \mathscr{D}(G_k)$ and $T = 0$ on G_k. So $T(\phi) = \sum_{k=1}^m T(\pi\psi_k) = 0$.

Partition of Unity Subordinate to an Open Cover

- Suppose $\{G_\alpha : \alpha \in A\}$ is a collection of open subsets of Ω.
- The set of functions $\{\psi_1, \ldots, \psi_m\}$, constructed in the theorem, is called a C^∞ **partition of unity** subordinate to the open cover $\{G_1, \ldots, G_m\}$ of K.

The Support of a Distribution

Definition

The **support** of $T \in \mathscr{D}'(\Omega)$ is the complement in Ω of the largest open subset of Ω where $T = 0$.

Example: Consider $\delta \in \mathscr{D}'(\Omega)$.

We know that $\langle \delta, \phi \rangle = 0$, for every ϕ in $\mathscr{D}(\Omega - \{0\})$.

So the support of δ is $\{0\}$.

- Note that, if T is a distribution and f is a C^∞ function which vanishes on $\operatorname{supp} T$, it does not necessarily follow that $fT = 0$.

  Example: We have seen that $x\delta' = -\delta$.

- On the other hand, if f vanishes on a neighborhood of $\operatorname{supp} T$, then we may conclude that $fT = 0$.

Compact Support and Finite Order

Theorem

Every distribution with compact support is of finite order.

- Suppose $T \in \mathscr{D}'(\Omega)$ and $\operatorname{supp} T$ is compact.

  There is $\psi \in \mathscr{D}(\Omega)$, with $\psi = 1$ on some open set containing $\operatorname{supp} T$.

  For any $\phi \in \mathscr{D}(\Omega)$, the support of $\phi - \psi\phi$ does not intersect $\operatorname{supp} T$: $\operatorname{supp}(\phi - \psi\phi) \subseteq \Omega - \operatorname{supp} T$. So $T(\phi - \psi\phi) = 0$. I.e., $T(\phi) = T(\psi\phi)$.

  Let $K = \operatorname{supp}\psi$. By a previous theorem, there is a nonnegative integer m and a constant M_1, such that $T(\phi) \le M_1 |\phi|_m$, for all $\phi \in \mathscr{D}_K$.

  By the Leibniz Formula for the derivative of $\psi\phi$, there is a constant M_2, such that $|\psi\phi|_m \le M_2 |\phi|_m$, for all $\phi \in \mathscr{D}(\Omega)$.

  For this choice of ψ and for every $\phi \in \mathscr{D}(\Omega)$, we have

$$|T(\phi)| = |T(\psi\phi)| \le M_1 |\psi\phi|_m \le M_1 M_2 |\phi|_m.$$

Finite Order and Compact Support

- A distribution of finite order does not necessarily have compact support.

  Example: Any locally integrable function defines a distribution of order 0.

  Example: Consider

  $$\sum_{k=0}^{\infty} \delta_k^{(k)}.$$

  This is an example of a distribution of infinite order.

Linear Combinations of Derivatives of Delta

- Consider a linear combination of derivatives of the Dirac measure on $\mathbb{R}^n$
$$T = \sum_{|\alpha| \leq m} c_\alpha \partial^\alpha \delta.$$
  - It has support $\{0\}$;
  - For all $\phi \in \mathscr{D}(\mathbb{R}^n)$, $T(\phi) = \sum_{|\alpha| \leq m} c_\alpha (-1)^{|\alpha|} \partial^\alpha \phi(0)$.

  Note that
$$\begin{aligned} |T(\phi)| &\leq \sum_{|\alpha| \leq m} |c_\alpha \partial^\alpha \phi(0)| \\ &\leq M_{mn} \max_{|\alpha| \leq m} |\partial^\alpha \phi(0)| \\ &\leq M_{mn} |\phi|_m. \end{aligned}$$
  Here M_{mn} is a positive constant which depends on m and n.
  This implies that the order of T is m.
- We will see later that every distribution with support $\{0\}$ is a finite linear combination of derivatives of δ.

$\mathscr{D}(\Omega)$ as a Subspace of $L^p(\Omega)$

- The space $L^p(I)$, where $I = (a, b)$ and $1 \leq p < \infty$, is the completion of $C_0^0(I)$ in the norm
$$f \mapsto \|f\|_p = \left[\int_I |f(x)|^p dx\right]^{1/p}.$$
- More generally, for any open set $\Omega \subseteq \mathbb{R}^n$, we can also define $L^p(\Omega)$ to be the completion of $C_0^0(\Omega)$ in the norm $\|\cdot\|_p$, with I replaced by Ω.
- It is a standard result of real analysis that this definition is equivalent to the usual definition of $L^p(\Omega)$ as the linear space of measurable functions on Ω with finite norm $\|\cdot\|_p$.
- Since convergence in $C_0^0(\Omega)$ implies convergence in $L^p(\Omega)$, and in view of a previous corollary, we have

Theorem

$\mathscr{D}(\Omega)$ is a dense subspace of $L^p(\Omega)$, for $1 \leq p < \infty$, with the identity map from $\mathscr{D}(\Omega)$ to $L^p(\Omega)$ continuous.

Approximation of an L^p Function in $\mathscr{D}$

- Next, we show, given an L^p function, how to construct the approximating sequence in $\mathscr{D}$.
- We use (γ_k) to denote the sequence $(\beta_{1/k})$.

  Let $u \in L^p(\mathbb{R}^n)$.

  We carry out the following steps:
  - First, we show that

$$\|u * \gamma_k\|_p \leq \|u\|_p, \quad 1 \leq p < \infty;$$

  - We, then, conclude that, in $L^p(\mathbb{R}^n)$, for $u \in L^p(\mathbb{R}^n)$,

$$u * \gamma_k \to u.$$

Approximation of an L^p Function in $\mathscr{D}$ (Part (i))

(i) Suppose, first, $1 < p < \infty$. Then

$$\|u * \gamma_k\|_p^p = \int \left| \int \gamma_k(y) u(x-y) dy \right|^p dx.$$

We can write $u\gamma_k = (u\gamma_k^{1/p})(\gamma_k^{1/q})$, where $\frac{1}{p} + \frac{1}{q} = 1$.
Now use Hölder's Inequality to obtain

$$\int \gamma_k(y)|u(x-y)|dy \leq \left[\int \gamma_k(y)|u(x-y)|^p dy\right]^{1/p} \left[\int \gamma_k(y) dy\right]^{1/q}.$$

Taking into account $\int \gamma_k(y) dy = 1$, we get

$$\begin{array}{rcll} \|u * \gamma_k\|_p^p & \leq & \int \int \gamma_k(y)|u(x-y)|^p dy dx & \\ & = & \int \gamma_k(y)[\int |u(x-y)|^p dx] dy & \text{(Fubini's Theorem)} \\ & = & \int \gamma_k(y) \|u\|_p^p dy & \\ & = & \|u\|_p^p. & \end{array}$$

Approximation of an L^p Function in $\mathscr{D}$ (Part (i) Cont'd)

- If $p = 1$,
$$\begin{array}{rcl} \|u * \gamma_k\|_1 & \leq & \int\int \gamma_k(y)|u(x-y)|dydx \\ & = & \int \gamma_k(y)[\int |u(x-y)|dx]dy \\ & = & \|u\|_1. \end{array}$$
Hence, for all $p \in [1, \infty)$,
$$\|u * \gamma_k\|_p \leq \|u\|_p.$$

Approximation of an L^p Function in $\mathscr{D}$ (Part (ii))

(ii) Let $u \in L^p(\mathbb{R}^n)$ and $\varepsilon > 0$ be arbitrary.
Since C_0^0 is dense in L^p, choose $\phi \in C_0^0(\mathbb{R}^n)$, such that $\|u - \phi\|_p < \varepsilon$.
Then, by Part (i),

$$\|u * \gamma_k - \phi * \gamma_k\|_p = \|(u - \phi) * \gamma_k\|_p \leq \|u - \phi\|_p < \varepsilon.$$

Now we take into account the fact that:

- $\phi * \gamma_k$ and ϕ are supported in the compact set $K = \text{supp}\phi + \overline{B}(0,1)$;
- $\phi * \gamma_k \to \phi$ uniformly on K.

So we can write, for k large enough,

$$\begin{aligned} \|\phi * \gamma_k - \phi\|_p &= [\textstyle\int_K |(\phi * \gamma_k)(x) - \phi(x)|^p dx]^{1/p} \\ &\leq \sup_{x \in K} |(\phi * \gamma_k)(x) - \phi(x)| [\textstyle\int_K dx]^{1/p} \\ &< \varepsilon. \end{aligned}$$

Thus,

$$\|u * \gamma_k - u\|_p \leq \|u * \gamma_k - \phi * \gamma_k\|_p + \|\phi * \gamma_k - \phi\|_p + \|\phi - u\|_p < 3\varepsilon.$$

Approximation of an L^p_{loc} Function in $\mathscr{D}$

- Let $u \in L^p_{\text{loc}}(\mathbb{R}^n)$ and K be any compact set in $\mathbb{R}^n$.
  Let I_K be the characteristic function of K.
  The function $v = uI_K$ lies in $L^p(\mathbb{R}^n)$.
  The sequence $v * \gamma_k$ converges to v in $L^p(\mathbb{R}^n)$.
  I.e., $u * \gamma_k \to u$ in the L^p norm on every compact subset of $\mathbb{R}^n$.
  With this convergence in $L^p_{\text{loc}}(\mathbb{R}^n)$, $\mathscr{D}(\mathbb{R}^n)$ is also dense in $L^p_{\text{loc}}(\mathbb{R}^n)$.

Zero Distributions and Functions Zero a.e.

- Recall that every locally integrable function f defines a distribution T_f.
- If $f = g$ a.e., then clearly $T_f = T_g$.
- We show that, conversely, if $T_f = T_g$, for two locally integrable functions f and g, then $f = g$ a.e..
- Suppose $f \in L^1_{\text{loc}}(\mathbb{R}^n)$, such that $T_f = 0$ in $\mathscr{D}'(\mathbb{R}^n)$.
  We prove that $f = 0$ a.e.
  (i) Suppose, first, that $f \in L^1(\mathbb{R}^n)$.
  Take into account that:
    - $\gamma_k(x-y)$ lies in $\mathscr{D}(\mathbb{R}^n)$, for every fixed x;
    - $T_f = 0$ on $\mathscr{D}(\mathbb{R}^n)$.

  So we have

$$(f * \gamma_k)(x) = \int f(y)\gamma_k(x-y)dy = 0.$$

  Hence, in $L^1(\mathbb{R}^n)$,

$$f = \lim(f * \gamma_k) = 0.$$

  This means that $f = 0$ a.e..

Zero Distributions and Functions Zero a.e. (Cont'd)

(ii) Now let $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ and K be a compact set in $\mathbb{R}^n$.
Choose $\psi \in \mathscr{D}(\mathbb{R}^n)$, such that $0 \le \psi \le 1$ and $\psi = 1$ on K.
This is always possible by a previous corollary.
Thus, $\psi f \in L^1(\mathbb{R}^n)$.
If $\phi \in \mathscr{D}(\mathbb{R}^n)$, then, by hypothesis,

$$T_{\psi f}(\phi) = T_f(\psi\phi) = 0.$$

By Part (i), we conclude that $\psi f = 0$ a.e. in $\mathbb{R}^n$.
This implies that $f = 0$ a.e. on K.
K being arbitrary, this means that $f = 0$ a.e..

- The proof depends essentially on $\mathscr{D}(\mathbb{R}^n)$ being dense in $L^1_{\text{loc}}(\mathbb{R}^n)$.

Subsection 7

Distributions of Finite Order

The Space $\mathscr{D}^{m\prime}(\Omega)$

- Recall that $\mathscr{D}^m(\Omega)$, $m \in \mathbb{N}_0$, is the linear space $C_0^m(\Omega)$ equipped with the inductive limit topology of $\{C_K^m(\Omega) : K \subseteq \Omega\}$.
- This is the locally convex topology in which a set is open if and only if its intersection with $C_K^m(\Omega)$ is open, for every compact $K \subseteq \Omega$.
- In turn, $C_K^m(\Omega)$ carries its natural locally convex topology defined by the seminorms

$$p_i(\phi) = \sup\{|\partial^\alpha \phi(x)| : x \in K, |\alpha| \le i\}, \quad 0 \le i \le m \le \infty.$$

- This topology on $\mathscr{D}^m(\Omega)$ is weaker than the topology of $\mathscr{D}(\Omega)$.
- Thus, the inclusion $\mathscr{D}(\Omega) \subseteq \mathscr{D}^m(\Omega)$ is in fact a continuous injection.
- Consequently, the dual space $\mathscr{D}^{m\prime}(\Omega)$ is a subspace of $\mathscr{D}'(\Omega)$.

Characterization of $\mathscr{D}^{m\prime}(\Omega)$ as a Subspace of $\mathscr{D}'(\Omega)$

Theorem

$\mathscr{D}^{m\prime}(\Omega)$ consists of all the distributions in $\mathscr{D}'(\Omega)$ of order $\leq m$.

- Suppose $T \in \mathscr{D}^{m\prime}(\Omega)$. Then, by definition, there is a constant M, such that $|T(\phi)| \leq M|\phi|_m$, for all $\phi \in \mathscr{D}^m(\Omega)$. The restriction of T to $\mathscr{D}(\Omega)$ is therefore a distribution of order $\leq m$.

  Conversely, suppose $T \in \mathscr{D}'(\Omega)$ is of order m.

  So there is a constant M, such that $|T(\phi)| \leq M|\phi|_m$, for all $\phi \in \mathscr{D}(\Omega)$.

  Now $\mathscr{D}(\Omega) \subseteq \mathscr{D}^m(\Omega) \subseteq \mathscr{D}^0(\Omega)$.

  Moreover, by a previous corollary, $\mathscr{D}(\Omega)$ is dense in $\mathscr{D}^0(\Omega) = C_0^0(\Omega)$.

  Hence, $\mathscr{D}(\Omega)$ is dense in $\mathscr{D}^m(\Omega)$.

  Thus, the continuous linear functional T may be extended by continuity to $\mathscr{D}^m(\Omega)$, with the inequality $|T(\phi)| \leq M|\phi|_m$ still valid.

  It follows that $T \in \mathscr{D}^{m\prime}(\Omega)$.

$\mathscr{D}_F(\Omega)$ and the Projective Limit Topology

- Let $\mathscr{D}_F(\Omega)$ be the set $\bigcap_{m=0}^{\infty} C_0^m(\Omega) = C_0^{\infty}(\Omega)$ equipped with the weakest topology in which the identity map $i_m : \mathscr{D}_F(\Omega) \to \mathscr{D}^m(\Omega)$ is continuous for every $m \in \mathbb{N}_0$.
- This is a locally convex topology which is induced by the topologies of $\mathscr{D}^m(\Omega)$ under the inverse maps i_m^{-1}.
- If $\mathscr{U}_m$ is a base of 0-neighborhoods in $\mathscr{D}^m(\Omega)$, the finite intersections of the sets $i_m^{-1}(U_m)$, where $U_m \in \mathscr{U}_m$ and $m \in \mathbb{N}_0$, form a base of 0-neighborhoods for the topology of $\mathscr{D}_F$.
- This topology on $\mathscr{D}_F$ is called the **projective limit** of the topologies of $\{\mathscr{D}^m(\Omega)\}$.
- This is a dual topology to the inductive limit.

Comparing $\mathscr{D}_F(\Omega)$ with $\mathscr{D}(\Omega)$

- Although $\mathscr{D}_F(\Omega)$ and $\mathscr{D}(\Omega)$ represent the same set, namely $C_0^\infty(\Omega)$, they are different topological spaces.

  Claim: The topology of $\mathscr{D}(\Omega)$ is stronger than that of $\mathscr{D}_F(\Omega)$.

  Consider any sequence ϕ_k in $\mathscr{D}(\Omega)$ which converges to ϕ.

  By a previous theorem, there is a compact set $K \subseteq \Omega$ which contains $\text{supp}\phi_k$, for all k, and $|\phi_k - \phi|_m \to 0$, for all m.

  This implies that $\phi_k \to \phi$ in $\mathscr{D}^m(\Omega)$, for every m.

  Hence, $\phi_k \to \phi$ in $\mathscr{D}_F(\Omega)$.
- Thus, the identity map from $\mathscr{D}(\Omega)$ to $\mathscr{D}_F(\Omega)$ is continuous.
- So the corresponding dual spaces $\mathscr{D}'_F(\Omega)$ and $\mathscr{D}'(\Omega)$ are related by the (proper) inclusion $\mathscr{D}'_F(\Omega) \subseteq \mathscr{D}'(\Omega)$.

Characterization of $\mathscr{D}'_F(\Omega)$

Theorem

$\mathscr{D}'_F(\Omega)$ consists of all the distributions in $\mathscr{D}'(\Omega)$ of finite order. In other words, $\mathscr{D}'_F(\Omega) = \bigcup_{m=0}^{\infty} \mathscr{D}^{m\prime}(\Omega)$.

- Suppose $T \in \mathscr{D}'(\Omega)$ is of finite order, say m.
  Then, by the preceding theorem, $T \in \mathscr{D}^{m\prime}(\Omega)$.
  Its restriction to $C_0^\infty(\Omega)$ is continuous in the topology of $\mathscr{D}'_F(\Omega)$.
  Hence, $T \in \mathscr{D}'_F(\Omega)$.
  Now let $T \in \mathscr{D}'_F(\Omega)$. Then, there is a neighborhood U of $0 \in \mathscr{D}_F(\Omega)$, such that, for all $\phi \in U$,
  $$|T(\phi)| \leq M.$$
  But U contains a neighborhood of the form $U_1 \cap \cdots \cap U_k \cap C_0^\infty(\Omega)$, where U_i is a neighborhood of $0 \in \mathscr{D}^{m_i}(\Omega)$, i.e., of form
  $$\{\phi \in C_0^\infty(\Omega) : |\phi|_{m_i} \leq \varepsilon_i\}.$$

Characterization of $\mathscr{D}'_F(\Omega)$ (Cont'd)

- Let $\varepsilon = \min\{\varepsilon_1, \ldots, \varepsilon_k\}$ and $m = \max\{m_1, \ldots, m_k\}$.

  Then

  $$\{\phi \in C_0^\infty(\Omega) : |\phi|_m \le \varepsilon\} \subseteq \{\phi \in C_0^\infty(\Omega) : |\phi|_{m_i} \le \varepsilon_i\} \subseteq U.$$

  Thus, for all $\phi \in C_0^\infty(\Omega)$, such that $|\phi|_m \le \varepsilon$, the linear functional T satisfies

  $$|T(\phi)| \le M.$$

  This means that T is a continuous linear functional on $C_0^\infty(\Omega)$ in the topology induced by $\mathscr{D}^m(\Omega)$.

  Therefore, T is a distribution of order m.

Projective and Inductive Limits and Dual Spaces

- With $\mathscr{D}'_F(\Omega) = \bigcup \mathscr{D}^{m\prime}(\Omega)$, we can also define a topology on $\mathscr{D}'_F(\Omega)$ through the inductive limit of the topologies of $\{\mathscr{D}^{m\prime}(\Omega)\}$.
- It turns out that this topology coincides with the one that we have already defined on $\mathscr{D}'_F(\Omega)$ as the dual of $\mathscr{D}_F(\Omega)$.
- Since the topology of $\mathscr{D}_F(\Omega)$ is the projective limit of the topologies of $\{\mathscr{D}^m(\Omega)\}$, we see that these two methods of defining a topology are naturally suited to dual spaces, in this case $\mathscr{D}_F(\Omega)$ and $\mathscr{D}'_F(\Omega)$.

Distributions and Measures on Open Sets

- A **Radon measure** on an open set $\Omega \subseteq \mathbb{R}^n$ is an element of $\mathscr{D}^{0\prime}(\Omega)$.
- That is, a Radon measure on an open set $\Omega \subseteq \mathbb{R}^n$ is a continuous linear functional on $\mathscr{D}^0(\Omega) = C_0^0(\Omega)$, or a distribution of order 0.
- As a continuous linear functional on $\mathscr{D}^0(\Omega)$, it is also represented, according to the Riesz Representation Theorem, by a regular Borel measure on Ω.

Positivity of a Real Linear Functional

Definition

A real linear functional T on a real linear space of functions F is said to be **positive** if $T(f) \geq 0$, whenever $f \in F$, $f \geq 0$.

- If T is positive on $C_0^0(\Omega)$, we show that T is continuous on $C_0^0(\Omega)$. Hence, it defines a (positive) Radon measure on Ω.
  By a previous corollary, it suffices to prove that:
  If $\phi_k \in C_0^0(\Omega)$, with $\text{supp}\phi_k$ contained in some compact set $K \subseteq \Omega$ and $|\phi_k|_0 = \sup_{x \in K} |\phi_k(x)| \xrightarrow{k \to \infty} 0$, then $T(\phi_k) \xrightarrow{k \to \infty} 0$.
  Choose $\psi \in C_0^0(\Omega)$, such that $0 \leq \psi \leq 1$ and $\psi = 1$ on K.
  Then $|\phi_k| \leq |\phi_k|_0 \psi$. Therefore, $-|\phi_k|_0 \psi \leq \phi_k \leq |\phi_k|_0 \psi$.
  Since T is positive, $-|\phi_k|_0 T(\psi) \leq T(\phi_k) \leq |\phi_k|_0 T(\psi)$.
  Hence, $\lim T(\phi_k) = 0$.
- Using the definition, we say $T \in \mathscr{D}'(\Omega)$ is **positive**, and write $T \geq 0$, if $T(\phi) \geq 0$, for all $\phi \geq 0$ in $\mathscr{D}(\Omega)$.

Example

- Let T be a positive distribution on Ω.

  To show that T is a Radon measure on Ω:
  - First extend T from $\mathscr{D}(\Omega)$ to $\mathscr{D}^0(\Omega)$;
  - Then prove that it is continuous as a linear functional on $\mathscr{D}^0(\Omega)$.

  Let $\phi \in \mathscr{D}^0(\Omega)$ be arbitrary. By a preceding corollary, there is a sequence $\phi_k \in \mathscr{D}(\Omega)$, such that $\phi_k \to \phi$ in $\mathscr{D}^0(\Omega)$.

  According to another corollary, suppϕ_k is contained in some compact set $K \subseteq \Omega$ and $|\phi_k - \phi|_0 \to 0$ on K.

  Choose $\psi \in \mathscr{D}(\Omega)$, such that $0 \le \psi \le 1$ in Ω and $\psi = 1$ on K.

  Now $|\phi_j(x) - \phi_k(x)| \le |\phi_j - \phi_k|_0 \psi(x)$.

  But $T \ge 0$ and $|\phi_j - \phi_k| \overset{j,k\to\infty}{\longrightarrow} 0$.

  Hence, $|T(\phi_j) - T(\phi_k)| \le |\phi_j - \phi_k|_0 T(\psi) \overset{j,k\to\infty}{\longrightarrow} 0$.

  Therefore, $\lim T(\phi_k)$ exists and we denote it by $T(\phi)$.

Example (Cont'd)

- If $\psi_k \in \mathscr{D}(\Omega)$ is another sequence which tends to ϕ in $\mathscr{D}^0(\Omega)$, the above argument implies that $T(\phi_k) - T(\psi_k) \stackrel{k\to\infty}{\longrightarrow} 0$.

  Therefore the limit $T(\phi)$ does not depend on the particular choice of the sequence ϕ_k, and we have shown that T has an extension to $\mathscr{D}^0(\Omega)$, which is clearly linear.

  To show that T is continuous on $\mathscr{D}^0(\Omega)$, it suffices to show (by work immediately preceding) that T is positive on $\mathscr{D}^0(\Omega)$.

  Let ϕ be any function in $C_0^0(\Omega)$ and $\phi \geq 0$.

  Then, for k large enough,

  - $\phi * \gamma_k \in \mathscr{D}(\Omega)$;
  - $\phi * \gamma_k \geq 0$.

  Hence, $T(\phi * \gamma_k) \geq 0$.

  Now $\phi * \gamma_k \stackrel{k\to\infty}{\longrightarrow} \phi$ in $\mathscr{D}^0(\Omega)$ and $T(\phi) = \lim T(\phi * \gamma_k) \geq 0$.

Subsection 8

Distributions Defined by Powers of x

Analyticity of Distributions

- Let $\lambda \mapsto T_\lambda$ be a mapping from $\mathbb{C}$ to $\mathscr{D}'(\Omega)$.
- We say T_λ is **analytic in** Λ if the function $\lambda \mapsto \langle T_\lambda, \phi\rangle$ is analytic in Λ, for every $\phi \in \mathscr{D}(\Omega)$.
- This definition extends the usual meaning of the analytic dependence of a function on a complex variable λ:

$$\lim_{\lambda\to\lambda_0} \frac{\langle T_\lambda, \phi\rangle - \langle T_{\lambda_0}, \phi\rangle}{\lambda - \lambda_0} = \left\langle \lim_{\lambda\to\lambda_0} \frac{T_\lambda - T_{\lambda_0}}{\lambda-\lambda_0}, \phi\right\rangle, \quad \phi \in \mathscr{D}(\Omega).$$

- Thus, when T_λ is a function of λ which is differentiable at λ_0, $\langle T_\lambda, \phi\rangle$ is differentiable at λ_0.
- When the limit in the equation exists, it defines a distribution which is denoted by $(\partial_\lambda T)_{\lambda_0}$.

Extending Analyticity in the Distributional Sense

- Let T_λ be the regular distribution defined by $|x|^\lambda = e^{\lambda \log |x|}$.
- The function $|x|^\lambda$ is locally integrable when $\text{Re}\lambda > -1$.
- So T_λ is analytic in λ on $\text{Re}\lambda > -1$.
- We will exploit the above definition of analyticity in the distributional sense to extend $|x|^\lambda$ as a distribution beyond $\text{Re}\lambda > -1$.
- This is done by continuing the function $\langle |x|^\lambda, \phi \rangle$ analytically, for every $\phi \in \mathscr{D}(\mathbb{R})$, to a larger connected subset of the complex λ-plane.

Example

- Consider the function

$$x_+^\lambda = \begin{cases} x^\lambda, & x > 0 \\ 0, & x \le 0 \end{cases}.$$

It is a locally integrable function for $\mathrm{Re}\lambda > -1$.

It determines the distribution

$$\langle x_+^\lambda, \phi \rangle = \int_0^\infty x^\lambda \phi(x) dx, \quad \phi \in \mathscr{D}(\mathbb{R}).$$

The right-hand side is analytic in $\mathrm{Re}\lambda > -1$, for every $\phi \in \mathscr{D}(\mathbb{R})$.

So the distribution x_+^λ is also analytic in $\mathrm{Re}\lambda > -1$.

Example

- If $\operatorname{Re}\lambda > -1$ and $\phi \in \mathscr{D}(\mathbb{R})$, we can write

$$\begin{aligned}
\int_0^\infty x^\lambda \phi(x)dx &= \int_0^\infty x^\lambda \phi(x)dx - \phi(0)\int_0^\infty x^\lambda H(1-x)dx + \phi(0)\int_0^1 x^\lambda dx \\
&= \int_0^\infty x^\lambda [\phi(x) - \phi(0)H(1-x)]dx + \phi(0)\int_0^1 x^\lambda dx \\
&= \int_0^1 x^\lambda [\phi(x) - \phi(0)]dx + \int_1^\infty x^\lambda \phi(x)dx + \frac{1}{\lambda+1}\phi(0).
\end{aligned}$$

  - The first integral on the right is convergent if $\operatorname{Re}\lambda > -2$.
    Note that ϕ is differentiable at 0.
    So $x^\lambda[\phi(x) - \phi(0)] = x^{1+\lambda}\frac{\phi(x)-\phi(0)}{x}$ is integrable on $[0,1]$.
  - The second integral is finite for all $\lambda \in \mathbb{C}$.
  - The third term is finite for all $\lambda \neq -1$.

  Therefore, x^λ can be continued analytically to

$$\Lambda = \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > -2, \lambda \neq -1\}.$$

Example (Cont'd)

- The subtraction of $\phi(0)H(1-x)$ from $\phi(x)$ is designed to reduce the order of the singularity of x^λ at $x=0$ while still preserving compact support for the integrand.

  This process can be repeated with higher order terms from the Taylor expansion of ϕ at $x=0$. In the m-th step,

$$\begin{aligned}
\langle x_+^\lambda,\phi\rangle &= \int_0^\infty x^\lambda\left[\phi(x)-\{\phi(0)+\cdots+\tfrac{x^{m-1}}{(m-1)!}\phi^{(m-1)}(0)\}H(1-x)\right]dx \\
&\quad+\sum_{k=1}^m \phi^{(k-1)}(0)\int_0^1 \tfrac{x^{\lambda+k-1}}{(k-1)!}dx \\
&= \int_0^1 x^\lambda\left[\phi(x)-\phi(0)-x\phi'(0)-\cdots-\tfrac{x^{m-1}}{(m-1)!}\phi^{(m-1)}(0)\right]dx \\
&\quad+\int_1^\infty x^\lambda\phi(x)dx+\sum_{k=1}^m \tfrac{1}{(\lambda+k)(k-1)!}\phi^{(k-1)}(0).
\end{aligned}$$

Example (Conclusion)

- The first integral on the right converges for $\text{Re}\lambda > -m-1$.
  $\phi(x) - \sum_{k=1}^{m} x^{k-1} \frac{\phi^{(k-1)}(0)}{(k-1)!}$ is of order x^m in the neighborhood of 0.
  When it is multiplied by x^λ, with $\text{Re}\lambda > -m-1$, the resulting function is integrable in the neighborhood of $x=0$.
- The third term on the right-hand side has simple poles at $\lambda = -1, -2, \ldots, -m$.

So the distribution x_+^λ may be continued analytically into $\text{Re}\lambda > -m-1, \lambda \neq -1, -2, \ldots, -m$.

Since m is arbitrary, x_+^λ is defined for all $\lambda \in \mathbb{C} - \mathbb{Z}^-$, where $\mathbb{Z}^-$ is the set of negative integers.

Example

- Consider the function

$$x_-^\lambda = \begin{cases} (-x)^\lambda, & x<0 \\ 0, & x \geq 0 \end{cases}.$$

It is locally integrable for $\mathrm{Re}\lambda > -1$.
It is also in $\mathscr{D}'(\mathbb{R})$ and analytic for $\mathrm{Re}\lambda > -1$.
It can be continued analytically into $\mathrm{Re}\lambda > -m-1$, $\lambda \neq -1, \ldots, -m$, by

$$\begin{array}{rcl}\langle x^\lambda, \phi\rangle & = & \int_{-\infty}^0 (-x)^\lambda \phi(x)dx \\ & = & \int_0^\infty x^\lambda \phi(-x)dx \\ & = & \displaystyle\int_0^1 x^\lambda \left[\phi(-x) - \sum_{k=1}^m \frac{(-x)^{k-1}}{(k-1)!}\phi^{(k-1)}(0)\right]dx \\ & & \displaystyle + \int_1^\infty x^\lambda \phi(-x)dx + \sum_{k=1}^m \frac{(-1)^{k-1}}{(\lambda+k)(k-1)!}\phi^{(k-1)}(0).\end{array}$$

Hence, the distribution x_-^λ is also defined for all $\lambda \in \mathbb{C} - \mathbb{Z}^-$.

Primitives

- Given a distribution $T \in \mathscr{D}'(\mathbb{R})$, any distribution S which satisfies $S'(\phi) = T(\phi)$, for every $\phi \in \mathscr{D}(R)$, is called a **primitive** of T.
- The extension of the function x_+^λ as a distribution outside $\text{Re}\lambda > -1$ should not be confused with the function x_+^λ which is well defined on $\mathbb{R} - \{0\}$ for all values of λ.
  - The distribution x_+^λ and the function x_+^λ are quite different when $\text{Re}\lambda < -1$.
  - The more we have to change the integral $\int x^\lambda \phi(x) dx$ to arrive at a definition of $\langle x^\lambda, \phi \rangle$, the more the resulting distribution will deviate from the function x^λ.
- Some books use the notation $[x_+^\lambda]$ or $\text{pf} x_+^\lambda$, the "pseudo-function" x_+^λ, to designate the distribution x_+^λ.