Introduction to the Theory of Distributions

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

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1. Distributions with Compact Support and Convolutions
   - The Dual Space of $C^\infty(\Omega)$
   - Tensor Product
   - Convolution
   - Regularization of Distributions
   - Local Structure of Distributions
   - Applications to Differential Equations

Subsection 1

The Dual Space of $C^\infty(\Omega)$

The Space $\mathscr{E}'(\Omega)$

- We use $\mathscr{E}(\Omega)$ to denote the Fréchet space $C^\infty(\Omega)$ topologized by the system of seminorms

$$p_{m,K}(\phi) = \sup\{|\partial^\alpha \phi(x)| : x \in K, |\alpha| \leq m\},$$

where $m \in \mathbb{N}_0$ and K runs through the compact subsets of Ω.
- We have seen that:
  - $\mathscr{D}_K$ is a closed subspace of $\mathscr{E}(\Omega)$, for every compact set $K \subseteq \Omega$;
  - The topology defined on $\mathscr{D}_K$ is the subspace topology inherited from $\mathscr{E}(\Omega)$.
- Therefore, the identity map from $\mathscr{D}_K$ to $\mathscr{E}(\Omega)$ is continuous.
- It follows that every continuous linear functional on $\mathscr{E}(\Omega)$ is also a continuous linear functional on $\mathscr{D}_K$.
- Since this is true for every $K \subseteq \Omega$, every continuous linear function on $\mathscr{E}(\Omega)$ is a continuous linear functional on $\mathscr{D}(\Omega)$.
- So every element in $\mathscr{E}'(\Omega)$, the dual space of $\mathscr{E}(\Omega)$, is a distribution.

Characterization of $\mathscr{E}'(\Omega)$

Theorem

For any open set Ω in $\mathbb{R}^n$, $\mathscr{E}'(\Omega)$ is the subspace of $\mathscr{D}'(\Omega)$ consisting of distributions with compact support.

- We saw that every element of $\mathscr{E}'(\Omega)$ defines a distribution in $\mathscr{D}'(\Omega)$.

  We now show that different elements in $\mathscr{E}'(\Omega)$ define different distributions by showing that $\mathscr{D}(\Omega)$ is a dense subspace of $\mathscr{E}(\Omega)$.

  Let (K_i) be an increasing sequence of compact subsets of Ω whose union is Ω. Let (ϕ_i) be a corresponding sequence in $\mathscr{D}(\Omega)$, such that $\phi_i = 1$ on a neighborhood of K_i. Let $\psi \in \mathscr{E}(\Omega)$. The function $\psi_i = \phi_i \psi$ is in $\mathscr{D}(\Omega)$. The function $\psi_i \to \psi$ in $\mathscr{E}(\Omega)$. Now, if $T = 0$ in $\mathscr{D}'(\Omega)$, then $T(\phi) = 0$, for all $\phi \in \mathscr{D}(\Omega)$. We obtain $T(\psi) = \lim T(\psi_i) = 0$. Hence, $T = 0$ in $\mathscr{E}'(\Omega)$.

Characterization of $\mathscr{E}'(\Omega)$ (Cont'd)

- Let $T \in \mathscr{E}'(\Omega)$. Then there is a bounded neighborhood of 0 in $\mathscr{E}(\Omega)$ which is mapped by T into the unit disc in $\mathbb{C}$.

  Thus, there is an integer $m \in \mathbb{N}_0$, a compact set $K \subseteq \Omega$ and a positive number r, such that the neighborhood of 0 in $\mathscr{E}(\Omega)$ defined by

  $$U = \{\phi \in \mathscr{E}(\Omega) : p_{m,K}(\phi) < r\}$$

  satisfies $|T(\phi)| \le 1$, for every $\phi \in U$.

  Suppose $\phi \in \mathscr{E}(\Omega)$ and $p_{m,K}(\phi) = 0$.

  Then $\lambda\phi \in U$, for every $\lambda > 0$. So $|T(\lambda\phi)| = \lambda|T(\phi)| \le 1$.

  Hence, $|T(\phi)| \le \frac{1}{\lambda}$, for every $\lambda > 0$. This means that $T(\phi) = 0$.

  But $p_{m,K}(\phi) = 0$, for every $\phi \in \mathscr{D}(\Omega - K)$. Hence $T = 0$ on $\Omega - K$.

  That is, $\operatorname{supp} T \subseteq K$.

Example

- The sequence $T_n = \sum_{k=1}^n a^k \delta_k$, with $a > 0$, converges in $\mathscr{D}'(\mathbb{R})$, but not in $\mathscr{E}'(\mathbb{R})$.

  Let $\phi \in \mathscr{D}(\mathbb{R})$. Then, there exists an integer m, such that $\phi = 0$ outside $[-m, m]$, and

  $$\langle T_n, \phi \rangle = \sum_{k=1}^m a^k \phi(k), \quad \text{if } n \geq m.$$

  Consequently, $\lim_{n\to\infty} \langle T_n, \phi \rangle = \sum_1^m a^k \phi(k)$ exists in $\mathscr{D}'(\mathbb{R})$.

  The sequence (T_n) also lies in $\mathscr{E}'(\mathbb{R})$.

  But it does not converge $\mathscr{E}'(\mathbb{R})$.

  Consider the test function $\phi(x) = a^{-x} \in \mathscr{E}(\mathbb{R})$. We get

  $$\langle T_n, \phi \rangle = \sum_1^n a^k a^{-k} = n \to \infty.$$

  Thus, the infinite sum $\sum_1^\infty a^k \delta_k$ lies in $\mathscr{D}'(\mathbb{R})$, but not in $\mathscr{E}'(\mathbb{R})$.

Zero Support and the Dirac Measure

Theorem

Every distribution whose support is $\{0\}$ may be represented by a unique finite linear combination of the derivatives of the Dirac measure δ.

- Suppose $T \in \mathscr{D}'(\Omega)$, $0 \in \Omega$ and $\operatorname{supp} T = \{0\}$.
  From a previous theorem, T has finite order, say m.
  By the preceding theorem, T lies in $\mathscr{E}'(\Omega)$.
  For any $\phi \in \mathscr{E}(\Omega)$, Taylor's Formula gives

  $$\phi(x) = \sum_{|\alpha| \le m} \frac{1}{\alpha!} \partial^\alpha \phi(0) x^\alpha + R_m(x),$$

  where $R_m \in \mathscr{E}(\Omega)$ and $\partial^\alpha R_m(0) = 0$, for all $|\alpha| \le m$.
  Since $\partial^\alpha R_m$ is continuous at 0 for every α, the derivatives $|\partial^\alpha R_m(x)|$, $|\alpha| \le m$ can be made arbitrarily small by taking $|x|$ small enough.
  Thus, for every $\varepsilon > 0$, there is $r > 0$, such that $|\partial^\alpha R_m(x)| < \varepsilon$, when $x \in B(0,r) = \{x : |x| < r\}$ and $|\alpha| \le m$.

Zero Support and the Dirac Measure (Cont'd)

- Using a previous result, we can choose $\psi_r \in \mathscr{D}(\Omega)$, such that $\operatorname{supp}\psi_r \subseteq B(0,r)$ and $\psi_r = 1$ on $\overline{B}(0,\frac{1}{2}r)$.
  The function $\phi_r = \psi_r R_m$ lies in $\mathscr{D}(\Omega)$.
  By Leibniz's formula, $\partial^\alpha \phi_r$ is a finite linear combination of products of the form $\partial^{\alpha-\beta}\psi_r \partial^\beta R_m$ with β running through $|\beta| \leq |\alpha| \leq m$.
  Now $\partial^{\alpha-\beta}\psi_r$ is bounded for all $|\alpha| \leq m$.
  So there is a constant M_1 (which depends on r and m), such that, for all $x \in \Omega$,
  $$\begin{array}{rcll} |\partial^\alpha \phi_r(x)| & \leq & M_1 |\partial^\beta R_m(x)| & \quad (|\beta| \leq |\alpha| \leq m) \\ & \leq & \varepsilon M_1. & \end{array}$$
  With $R_m = \phi_r$ on a neighborhood of $\operatorname{supp} T$, $|T(R_m)| = |T(\phi_r)|$ $\leq M_2 |\phi_r|_m$, for some constant M_2, since T is of order m.
  Thus, $|T(R_m)| \leq \varepsilon M_1 M_2$.
  Since $\varepsilon > 0$ was arbitrary, we conclude that $T(R_m) = 0$.

Zero Support and the Dirac Measure (Conclusion)

- Now we get back to

$$\phi(x) = \sum_{|\alpha|\le m} \frac{1}{\alpha!}\partial^\alpha\phi(0)x^\alpha + R_m(x).$$

We can write

$$T(\phi) = \sum_{|\alpha|\le m} \frac{1}{\alpha!} T(x^\alpha)\partial^\alpha\phi(0) = \sum_{|\alpha|\le m} c_\alpha\partial^\alpha\delta(\phi),$$

where $c_\alpha = (-1)^{|\alpha|}\frac{T(x^\alpha)}{\alpha!}$.
For uniqueness, assume $\sum_{|\alpha|\le m} c_\alpha\partial^\alpha\delta(\phi) = 0$.
Then,for every $\phi\in\mathscr{E}(\Omega)$, $\sum_{|\alpha|\le m}(-1)^{|\alpha|}c_\alpha\partial^\alpha\phi(0) = 0$.
Choose $\phi(x) = x^\beta$, $|\beta|\le m$, to obtain

$$0 = \sum_{|\alpha|\le m}(-1)^{|\alpha|}c_\alpha\partial^\alpha\phi(0) = (-1)^{|\beta|}c_\beta\beta!.$$

Thus, $c_\alpha = 0$, for all $|\alpha|\le m$.

Example

- Let m be a positive integer and T be a distribution on $\mathbb{R}$.

  Claim: $x^m T = 0$ if and only if T is a linear combination of $\delta, \delta', \ldots, \delta^{(m-1)}$ with constant coefficients.

  Suppose, first, that $T = \sum_{i<m} c_i \delta^{(i)}$.

  Then, for all $\phi \in \mathscr{D}(\mathbb{R})$,

$$\begin{aligned} x^m T(\phi) &= T(x^m \phi) \\ &= \sum_{i<m} c_i \langle \delta^{(i)}, x^m \phi \rangle \\ &= \sum_{i<m} (-1)^i c_i \langle \delta, \partial^i (x^m \phi) \rangle \\ &= 0. \end{aligned}$$

Example (Converse)

- Conversely, let $T(x^m\phi) = 0$, for every $\phi \in \mathscr{D}(\mathbb{R})$.

  Claim: $\operatorname{supp} T = \{0\}$.

  Let Ω be an open subset of $\mathbb{R} - \{0\}$ and ψ be in $\mathscr{D}(\Omega)$. Define

$$\phi(x) = \begin{cases} \frac{1}{x^m}\psi, & \text{on } \operatorname{supp}\psi \\ 0, & \text{on } \mathbb{R} - \operatorname{supp}\psi \end{cases}.$$

  $\phi(x)$ lies in $\mathscr{D}(\mathbb{R})$. Moreover, $T(\psi) = T(x^m\phi) = 0$.

  Hence, T vanishes on every open subset of $\mathbb{R} - \{0\}$.

  So it vanished on $\mathbb{R} - \{0\}$ itself. Therefore, $\operatorname{supp} T = \{0\}$.

  By the theorem, T may be represented by a finite sum of the form

$$T = \sum_{k=0}^{\ell} c_k \delta^{(k)}.$$

Example (Cont'd)

Claim: $c_k = 0$, for $k \geq m$.

We have the following properties:

- $\langle \delta^{(k)}, x^j \phi \rangle = 0$, when $k < j$;
- $\langle \delta^{(k)}, x^k \phi \rangle = (-1)^k k! \phi(0)$.

Suppose in $T = \sum_{k=0}^{\ell} c_k \delta^{(k)}$, $c_\ell \neq 0$, for $\ell \geq m$.

Then for $\phi \in \mathscr{D}(\mathbb{R})$, such that $\phi(0) \neq 0$, we have

$$\begin{aligned} 0 &= \langle x^m T, x^{\ell - m} \phi \rangle \\ &= \langle T, x^\ell \phi \rangle \\ &= \textstyle\sum_{k=0}^{\ell} \langle c_k \delta^{(k)}, x^\ell \phi \rangle \\ &= c_\ell (-1)^\ell \ell! \phi(0). \end{aligned}$$

This gives a contradiction.

Thus $c_k = 0$, for $k \geq m$.

Subsection 2

Tensor Product

Direct or Tensor Product

- Let Ω_1 be an open set in $\mathbb{R}^{n_1}$ and Ω_2 be an open set in $\mathbb{R}^{n_2}$.
- The product
$$\Omega_1 \times \Omega_2 = \{(x,y) : x \in \Omega_1, y \in \Omega_2\}$$
is an open set in the Euclidean space $\mathbb{R}^{n_1+n_2} = \mathbb{R}^{n_1} \times \mathbb{R}^{n_2}$.
- Let f be a function on Ω_1 and g a function on Ω_2.
- We define the **direct**, or **tensor**, **product** $f \otimes g$ on $\Omega_1 \times \Omega_2$ by
$$(f \otimes g)(x,y) = f(x)g(y).$$
- Clearly $(f \otimes g)(x,y) = (g \otimes f)(y,x)$, for every pair $(x,y) \in \Omega_1 \times \Omega_2$.
- $C_0^\infty(\Omega_1) \times C_0^\infty(\Omega_2)$ denotes the linear space of functions $\phi(x,y)$ that can be represented as finite sums of products of the form $\phi_1(x)\phi_2(y)$ with $\phi_i \in C_0^\infty(\Omega_i)$, $i = 1,2$.
- We show it is a dense subspace of the linear space $C_0^\infty(\Omega_1 \times \Omega_2)$.

Density of $C_0^\infty(\mathbb{R}) \times C_0^\infty(\mathbb{R})$ in $C_0^\infty(\mathbb{R}^2)$

Theorem

For $\phi(x,y) \in C_0^\infty(\mathbb{R}^{m+n})$, there are $\phi_{ij}(x) \in C_0^\infty(\mathbb{R}^n)$ and $\psi_{ij}(y) \in C_0^\infty(\mathbb{R}^m)$, such that $\phi_i(x,y) = \sum_{j=1}^{k_i} \phi_{ij}(x)\psi_{ij}(y)$ converges to ϕ in $\mathscr{D}(\mathbb{R}^{n+m})$.

- We present an outline of the proof for $n = m = 1$.
- Define

$$\Phi(x,y,t) = \begin{cases} \frac{1}{(2\sqrt{\pi t})^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(\xi,\eta) e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4t}} d\xi d\eta, & \text{if } t > 0 \\ \phi(x,y), & \text{if } t = 0 \end{cases}.$$

Changing variables $\xi_1 = \frac{\xi - x}{2\sqrt{t}}$, $\eta_1 = \frac{\eta - y}{2\sqrt{t}}$, we get

$$\Phi(x,y,t) = \frac{1}{(\sqrt{\pi})^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \phi(x + 2\xi_1\sqrt{t}, y + 2\eta_1\sqrt{t}) e^{-(xi_1^2+\eta_1^2)} d\xi_1 d\eta_1.$$

Proof (Cont'd)

- Recall that $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(\xi_1^2+\eta_1^2)} d\xi_1 d\eta_1 = \pi$.

  So we get

$$\begin{aligned}
&|\Phi(x,y,t)-\phi(x,y)| \\
&= \frac{1}{(\sqrt{\pi})^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |\phi(x+2\xi\sqrt{t}, y+2\eta\sqrt{t})-\phi(x,y)| e^{-(\xi^2+\eta^2)} d\xi d\eta \\
&= \frac{1}{\pi}\left\{\iint_{\xi^2+\eta^2\geq T^2} |\phi(x+2\xi\sqrt{t}, y+2\eta\sqrt{t})-\phi(x,y)| e^{-(\xi^2+\eta^2)} d\xi d\eta\right. \\
&\quad \left. + \iint_{\xi^2+\eta^2< T^2} |\phi(x+2\xi\sqrt{t}, y+2\eta\sqrt{t})-\phi(x,y)| e^{-(\xi^2+\eta^2)} d\xi d\eta\right\}.
\end{aligned}$$

  Now we can see that $\lim_{t\to 0^+}\Phi(x,y,t) = \phi(x,y)$ uniformly in (x,y):

  - ϕ is bounded and $e^{-(\xi^2+\eta^2)}$ is integrable in $\mathbb{R}^2$.
    So the first term in the sum approaches 0 as $T\to\infty$.
  - The second term, for fixed $T>0$, approaches 0 as $t\to 0^+$.

Proof (Cont'd)

- Now consider

$$\frac{\partial^{k+\ell}\Phi(x,y,t)}{\partial x^k \partial y^\ell} = \begin{cases} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{1}{(2\sqrt{\pi t})^2} \frac{\partial^{k+\ell}\phi(\xi,\eta)}{\partial \xi^k \partial \eta^\ell} e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4t}} d\xi d\eta, & \text{if } t>0 \\ \frac{\partial^{k+\ell}\phi(x,y)}{\partial x^k \partial y^\ell}, & \text{if } t=0 \end{cases}.$$

Reasoning as before, we obtain that, uniformly in (x,y),

$$\lim_{t\to 0^+} \frac{\partial^{k+\ell}\Phi(x,y,t)}{\partial x^k \partial y^\ell} = \frac{\partial^{k+\ell}\phi(x,y)}{\partial x^k \partial y^\ell}.$$

Now $\Phi(x,y,t)$, $t>0$, may be extended to a holomorphic function of complex x,y, for $|x|<\infty$ and $|y|<\infty$.

So, for all $\varepsilon>0$ and fixed $t>0$, $\Phi(x,y,t)$ may be expanded into a Taylor series

$$\Phi(x,y,t) = \sum_{k=0}^{\infty}\sum_{s=0}^{k} c_s(t) x^s y^{k-s}.$$

Proof (Cont'd)

- We expanded into Taylor series to obtain

$$\Phi(x,y,t) = \sum_{k=0}^{\infty} \sum_{s=0}^{k} c_s(t) x^s y^{k-s}.$$

This is absolutely and uniformly convergent if $|x| \le \varepsilon$ and $|y| \le \varepsilon$.
Differentiating term by term, we get

$$\frac{\partial^{k+\ell}\Phi(x,y,t)}{\partial x^k \partial y^\ell} = \sum_{k_1=0}^{\infty} \sum_{s=0}^{k_1} c_s(t) \frac{\partial^{k+\ell} x^s y^{k_1-s}}{\partial x^k \partial y^\ell}.$$

Take $\{t_i\}$, with $t_i \ge 0$, $t_i \to 0^+$.
Choose, for each i, a polynomial section $P_i(x,y)$ of the polynomial $\sum_{k=0}^{\infty} \sum_{s=0}^{k} c_s(t) x^s y^{k-s}$, such that $\lim_{i\to\infty} P_i(x,y) = \phi(x,y)$ in $\mathscr{E}(\mathbb{R}^2)$.
Thus, for every compact $K \subseteq \mathbb{R}^2$, $\lim_{i\to\infty} \partial^s P_i(x,y) = \partial^s \phi(x,y)$ uniformly on K, for all ∂^s. Adopt $u(x) \in C_0^\infty(\mathbb{R})$, $v(y) \in C_0^\infty(\mathbb{R})$, such that $u(x)v(y) = 1$ on $\mathrm{supp}(\phi(x,y))$. Then $\phi_i(x,y) = u(x)v(y)P_i(x,y)$ satisfy our requirements.

The Distributions T_1 and T_2

- T_i will denote a distribution in Ω_i.
- For a fixed $y \in \Omega_2$, the function $\phi(\cdot, y)$ belongs to $C_0^\infty(\Omega_1)$;
- So T_1 maps $\phi(\cdot, y)$ to the number $T_1(\phi(\cdot, y))$, denoted $T_1(\phi)(y)$.
- Thus, $T_1(\phi)$ is a function on Ω_2.
- Similarly, $T_2(\phi)$ is a function on Ω_1.
- The next theorem shows that $T_1(\phi)$ and $T_2(\phi)$ preserve all the smoothness properties of the test function space $\mathscr{D}$.

Derivatives in the Product Space

Theorem

If $\phi(x,y)\in\mathscr{D}(\Omega_1\times\Omega_2)$ and $T_1\in\mathscr{D}'(\Omega_1)$, then $T_1(\phi)\in\mathscr{D}(\Omega_2)$ and

$$\partial_y^\beta T_1(\phi)=T_1(\partial_y^\beta\phi),\ \text{for all } \beta\in\mathbb{N}_0^{n_2}.$$

- For any point $y\in\Omega_2$, let h be any nonzero real number such that $B(y,2|h|)\subseteq\Omega_2$. Let $h_k=(0,\ldots,h,\ldots,0)$ be the point in $\mathbb{R}^{n_2}$, with all coordinates 0 except the k-th.

  Let $\phi\in C_0^\infty(\Omega_1\times\Omega_2)$. But ϕ is differentiable with respect to y. So

  $$\phi(x,y+h_k)=\phi(x,y)+\partial_{y_k}\phi(x,y)h+R(x,y,h),$$

  where $\frac{1}{h}|R(x,y,h)|\to 0$ as $h\to 0$. Using the linearity and continuity of T_1, we see that $T_1(\phi(x,y))$ has a k-th partial derivative, as a function of y, and that $\partial_{y_k}T_1(\phi(\cdot,y))=T_1(\partial_{y_k}\phi(\cdot,y))$.

Derivatives in the Product Space (Cont'd)

- We saw that $\partial_{y_k} T_1(\phi(\cdot, y)) = T_1(\partial_{y_k}\phi(\cdot, y))$.

  The formula $\partial_y^\beta T_1(\phi) = T_1(\partial_y^\beta \phi)$ follows by induction.

  The assumption $\phi \in C_0^\infty(\Omega_1 \times \Omega_2)$ also implies that, for every x in a compact subset of Ω_1, the function $\partial_y^\beta \phi$ is continuous on Ω_2.

  Hence, by the continuity of T_1, so is $T_1(\partial_y^\beta \phi)$.

  But $\phi(x, y)$ has compact support in $\Omega_1 \times \Omega_2$.

  So the function $T_1(\phi(\cdot, y))$ has compact support in Ω_2.

Consequence for $\mathscr{E}$ and C^∞

Corollary

If $\phi(x,y) \in \mathscr{E}(\Omega_1 \times \Omega_2)$ and $T_1 \in \mathscr{E}'(\Omega_1)$, then $T_1(\phi) \in \mathscr{E}(\Omega_2)$ and

$$\partial_y^\beta T_1(\phi) = T_1(\partial_y^\beta \phi), \text{ for all } \beta \in \mathbb{N}_0^{n_2}.$$

- This result may be proved by replacing ϕ by $\psi\phi$, where $\psi \in C_0^\infty(\Omega)$ equals 1 on a neighborhood of $\operatorname{supp} T_1$ and using the theorem.

Corollary

If $\phi(x,y) \in C^\infty(\Omega_1 \times \Omega_2)$ has compact support as a function of x and y separately, then:

(a) $T_1(\phi)(y) \in C^\infty(\Omega_2)$, for every $T_1 \in \mathscr{D}'(\Omega_1)$;

(b) $T_2(\phi)(x) \in C^\infty(\Omega_1)$, for every $T_2 \in \mathscr{D}'(\Omega_2)$.

Example

- Let $\phi, \psi \in \mathscr{D}(\mathbb{R})$.
  (a) The tensor product $(\phi \otimes \psi)(x,y) = \phi(x)\psi(y)$ is in $\mathscr{D}(\mathbb{R}^2)$.
  For any $T \in \mathscr{D}'(\mathbb{R})$, $T(\phi \otimes \psi) = T(\phi)\psi(y)$ is a function in $\mathscr{D}(\mathbb{R})$.
  (b) The function $\phi(x+y)$ lies in $C^\infty(\mathbb{R}^2)$.
  However, it does not have compact support.
  E.g., $\phi \neq 0$ on the line $x+y=c$ in $\mathscr{R}^2$ whenever $\phi(c) \neq 0$.
  But, as a function of x and y separately, $\phi(x+y)$ has compact support.

We have

$$\langle 1_x, \phi(x,y) \rangle = \int \phi(x+y)dx = \int \phi(\xi)d\xi = \text{constant}.$$

This is a $C^\infty(\mathbb{R})$ function in agreement with the last corollary.

We also have $\langle \delta_x, \phi(x+y) \rangle = \phi(y)$. This lies in $C_0^\infty(\mathbb{R})$.

This would seem to suggest that if T_i in the corollary is taken in $\mathscr{E}'(\Omega_i)$, then $T_i(\phi)$ will have compact support.

Example (Cont'd)

- Now let $f \in L^1_{\text{loc}}(\Omega_1)$ and $g \in L^1_{\text{loc}}(\Omega_2)$.
  Then $f \otimes g$ is clearly in $L^1_{\text{loc}}(\Omega_1 \times \Omega_2)$.
  Let $\phi_i \in \mathscr{D}(\Omega_i)$, $i = 1, 2$.
  Then $\phi_1 \otimes \phi_2 \in \mathscr{D}(\Omega_1 \times \Omega_2)$ and we have

$$\begin{aligned} \langle f \otimes g, \phi_1 \otimes \phi_2 \rangle &= \int_{\Omega_1 \times \Omega_2} f(x) g(y) \phi_1(x) \phi_2(y) dx dy \\ &= \int_{\Omega_1} f(x) \phi_1(x) dx \int_{\Omega_2} g(y) \phi_2(y) dy \\ &= \langle f, \phi_1 \rangle \langle g, \phi_2 \rangle. \end{aligned}$$

- The next theorem generalizes this result.

The Tensor Product Distribution

Theorem

If $T_i \in \mathscr{D}'(\Omega_i)$, $i = 1, 2$, then there is a unique $T_1 \otimes T_2 \in \mathscr{D}'(\Omega_1 \times \Omega_2)$, defined by

$$(T_1 \otimes T_2)(\phi_1 \otimes \phi_2) = T_1(\phi_1) T_2(\phi_2),$$

for all tensor products $\phi_1 \otimes \phi_2$, where $\phi_i \in \mathscr{D}(\Omega_i)$, and such that

$$(T_1 \otimes T_2)(\phi) = T_1(T_2(\phi)) = T_2(T_1(\phi)), \quad \phi \in \mathscr{D}(\Omega_1 \times \Omega_2).$$

- Uniqueness follows from denseness of $\mathscr{D}(\Omega_1) \times \mathscr{D}(\Omega_2)$ in $\mathscr{D}(\Omega_1 \times \Omega_2)$.
  We show that $T_1 \otimes T_2$ is a distribution of $\Omega_1 \times \Omega_2$.
  Let K_i be a compact subset of Ω_i.
  By a previous theorem, there is a nonnegative integer m_i, and a nonnegative constant M_i, $i = 1, 2$, such that, for all $\phi_i \in \mathscr{D}_{K_i}$, $|T_i(\phi_i)| \leq M_i |\phi_i|_{m_i}$.

The Tensor Product Distribution (Cont'd)

- Let $\phi \in \mathscr{D}_K$, where $K = K_1 \times K_2$.
  The preceding theorem implies that $T_2(\phi)$ is in $\mathscr{D}_{K_1}$.
  Therefore, $T_1(T_2(\phi))$ is well defined.
  Moreover, it satisfies

$$|T_1(T_2(\phi))| \le M_1 |T_2(\phi)|_{m_1}.$$

  We also have $\partial_x^\alpha T_2(\phi(x,\cdot)) = T_2(\partial_x^\alpha \phi(x,\cdot))$.
  So we obtain

$$\begin{array}{rcl} |T_2(\phi(x,\cdot))|_{m_1} & = & \sup_{x \in K_1} \{\partial_x^\alpha T_2(\phi(x,\cdot))| : |\alpha| \le m_1\} \\ & = & \sup_{x \in K_1} \{T_2(\partial_x^\alpha \phi(x,\cdot))| : |\alpha| \le m_1\} \\ & \le & M_2 \sup_{\substack{|\alpha| \le m_1 \\ x \in K_1}} |\partial_x^\alpha \phi(x,\cdot)|_{m_2} \\ & \le & M_2 \sup_{x \in K_1, y \in K_2} \{|\partial_x^\alpha \partial_y^\beta \phi(x,y)| : |\alpha| \le m_1, |\beta| \le m_2\}. \end{array}$$

The Tensor Product Distribution (Cont'd)

- Thus, using the displayed inequality, we obtain

$$|T_1(T_2(\phi))| \leq M_1 M_2 \sup_{\substack{|\gamma|\leq m \\ x\in K}} |\partial^\gamma \phi| = M_1 M_2 |\phi|_m,$$

  where $\gamma = \alpha + \beta$ and $m = m_1 + m_2$.

  This inequality holds for all $\phi \in \mathscr{D}_K$ and all $K = K_1 \times K_2 \subseteq \Omega_1 \times \Omega_2$.

  By a previous theorem, the linear functional defined on $\mathscr{D}(\Omega_1 \times \Omega_2)$ by $\phi \mapsto T_1(T_2(\phi))$ is a distribution in $\Omega_1 \times \Omega_2$.

  Similarly, the linear functional defined on $\mathscr{D}(\Omega_1 \times \Omega_2)$ by $\phi \mapsto T_2(T_1(\phi))$ also lies in $\mathscr{D}'(\Omega_1 \times \Omega_2)$.

  Now $T_1(T_2(\phi_1 \otimes \phi_2)) = T_1(\phi_1) T_2(\phi_2) = T_2(T_1(\phi_1 \otimes \phi_2))$, $\phi_i \in \mathscr{D}(\Omega_i)$.

  Hence, by uniqueness, $T_1(T_2(\phi)) = T_2(T_1(\phi))$, for all $\phi \in \mathscr{D}(\Omega_1 \times \Omega_2)$.

The Direct or Tensor Product of Distributions

- With $T_i \in \mathscr{D}'(\Omega_i)$, the distribution $T_1 \otimes T_2 = T_2 \otimes T_1$ is called the **direct**, or **tensor**, **product** of T_1 and T_2.
- Strictly speaking, $T_1 \otimes T_2$ and $T_2 \otimes T_1$ act on two different spaces and their equality should be understood as the equality of their images.
- We finally show that $\operatorname{supp}(T_1 \otimes T_2) = (\operatorname{supp} T_1) \times (\operatorname{supp} T_2)$.

  Let $\operatorname{supp} T_i = K_i$, $i = 1, 2$. Suppose $\phi \notin K_1 \times K_2$.

  Then $\phi \notin \mathscr{D}(\Omega_1) \times K_2$ or $\phi \notin K_1 \times \mathscr{D}(\Omega_2)$.

  Consequently, $(T_1 \otimes T_2)(\phi) = 0$. I.e., $\phi \notin \operatorname{supp}(T_1 \otimes T_2)$.

  Hence, $\operatorname{supp}(T_1 \otimes T_2) \subseteq K_1 \times K_2$.

  Now $(T_1 \otimes T_2)(\phi_1 \otimes \phi_2) = T_1(\phi_1) T_2(\phi_2)$, for all $\phi_i \in \mathscr{D}(\Omega_i)$.

  Hence, $K_1 \times K_2 \subseteq \operatorname{supp}(T_1 \otimes T_2)$.

Example

- Given $\xi \in \Omega_1$ and $\eta \in \Omega_2$, we have

$$\begin{array}{rcl} \text{supp}\delta_\xi & = & \{\xi\}; \\ \text{supp}(\delta_\xi \otimes \delta_\eta) & = & \{(\xi,\eta)\}. \end{array}$$

  This implies that $\delta_\xi \otimes \delta_\eta = \delta_{(\xi,\eta)}$.

Subsection 3

Convolution

Problem with the Domain of a Convolution

- We wish to extend the definition of the convolution of a C_0^∞ function with a locally integrable function.
- We define (tentatively) the **convolution** of two distributions T_1 and T_2 on $\mathbb{R}^n$ by setting, for all $\phi \in \mathscr{D}(\mathbb{R}^n)$.

$$(T_1 * T_2)(\phi) = (T_1 \otimes T_2)(\phi(x+y)) = T_1(T_2(\phi(x+y))).$$

- If ϕ is in $\mathscr{D}(\mathbb{R}^n)$, $\psi(x,y) = \phi(x+y)$ is a C^∞ function in $\mathbb{R}^{2n}$.
- The boundedness of $\text{supp}\phi$ does not guarantee the boundedness of $\{(x,y) \in \mathbb{R}^{2n} : x+y \in \text{supp}\phi\}$.
- So $\phi(x+y)$ as a function of (x,y) does not have compact support in $\mathbb{R}^{2n}$.
- Therefore, the right-hand side is not necessarily bounded unless $\text{supp}(T_1 \otimes T_2) = (\text{supp}\, T_1) \times (\text{supp}\, T_2)$ intersects $\text{supp}(\phi(x+y))$ in a bounded set.

Defining a Convolution

- If K is the support of ϕ, then

$$\operatorname{supp}(\phi(x+y)) = \{(x,y) \in \mathbb{R}^{2n} : x+y \in K\}.$$

- Suppose either T_1 or T_2 has compact support.
- Then the intersection of $(\operatorname{supp} T_1) \times (\operatorname{supp} T_2)$ with $\operatorname{supp}(\phi(x+y))$ is compact.

  In fact, if either x or y is bounded and $x+y$ is bounded, then both x and y are bounded.
- In that case the right-hand side in

$$(T_1 * T_2)(\phi) = (T_1 \otimes T_2)(\phi(x+y)) = T_1(T_2(\phi(x+y)))$$

  is well defined.
- Moreover, since $T_1 \otimes T_2 = T_2 \otimes T_1$, we have $T_1 * T_2 = T_2 * T_1$.
- Thus, the equation defines the convolution of two distributions $T_1, T_2 \in \mathscr{D}(\mathbb{R}^n)$ provided at least one of them has compact support.

Definition of a Convolution

- Let T_i be defined by $f_i \in L^1_{\text{loc}}(\mathbb{R}^n)$, $i = 1,2$, and either f_1 or f_2 has compact support.
- Then

$$\begin{aligned}(T_1 * T_2)(\phi) &= T_1(T_2(\phi(x+y))) \\ &= \int f_1(x) \int f_2(y)\phi(x+y)dy\,dx \\ &= \int\int f_1(x-y)f_2(y)\phi(x)dy\,dx \\ &= \langle f_1 * f_2, \phi\rangle.\end{aligned}$$

- Here

$$(f_1 * f_2)(x) = \int f_1(x-y)f_2(y)dy = \int f_1(y)f_2(x-y)dy$$

is a locally integrable function which represents the distribution $T_1 * T_2$ and extends the definition of $f_1 * f_2$.

On the Necessity of Compact Support

- Although the convolution of two distributions is always well defined when one of them has compact support, this condition is not always necessary.

  Example: Let g be bounded (measurable), with $M = \sup|g|$.

  Let $f \in L^1(\mathbb{R}^n)$.

  Then
  $$\int f(y)g(x-y)dy \le M\|f\|_1.$$

  Thus, g may be convoluted with f
- Naturally, this result holds if g is merely bounded almost everywhere in $\mathbb{R}^n$.

Convolution by $L^\infty(\Omega)$

- The linear space of complex measurable functions on Ω which are bounded almost everywhere is denoted by $L^\infty(\Omega)$.
- $L^\infty(\Omega)$ becomes a normed linear space if we define the norm of $g \in L^\infty(\Omega)$, called the **essential supremum** of g, by

$$\|g\|_\infty = \inf\{M : |g(x)| \le M \text{ a.e. in } \Omega\}.$$

- Thus, we can state that, if $f \in L^1(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$, then

$$|f * g| \le \|f\|_1 \|g\|_\infty.$$

- So, if $f \in L^1(\mathbb{R}^n)$ and $g \in L^\infty(\mathbb{R}^n)$, then $f * g \in L^\infty(\mathbb{R}^n)$ and

$$\|f * g\|_\infty \le \|f\|_1 \|g\|_\infty.$$

Convolution of $L^1(\mathbb{R}^n)$ Functions

- When f and g are both in $L^1(\mathbb{R}^n)$ it is not obvious that their convolution $(f*g)(x)=\int f(x-y)g(y)dy$ exists.
  E.g., at $x=0$, if we take $f(-y)=g(y)$, this integral may diverge, since not every integrable function is square integrable.
- We show that the function $F(x)=\int f(x-y)g(y)dy$ exists for almost all x in $\mathbb{R}^n$ by showing that $F=f*g\in L^1(\mathbb{R}^n)$.
  Let $F_k(x)=\displaystyle\int_{|y|\le k} f(x-y)g(y)dy$. So $|F_k(x)|\le\displaystyle\int_{|y|\le k}|f(x-y)g(y)|dy$.
  Now we get

$$\begin{array}{rcl}\int|F_k(x)|dx & \le & \int[\int_{|y|\le k}|f(x-y)g(y)|dy]dx \\ & = & \int_{|y|\le k}[\int|f(x-y)|dx]|g(y)|dy \\ & = & \|f\|_1\int_{|y|\le k}|g(y)|dy \\ & \le & \|f\|_1\|g\|_1.\end{array}$$

  In the limit as $k\to\infty$, we obtain $\|F\|_1=\|f*g\|_1\le\|f\|_1\|g\|_1$.

Convolution of $L^1(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$

- Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, $1 \le p \le \infty$.
- We saw that, if $p = 1, \infty$, $f * g \in L^p(\mathbb{R}^n)$ and

$$\|f * g\|_p \le \|f\|_1 \|g\|_p.$$

- Now consider $1 < p < \infty$.

  Since $g \in L^p(\mathbb{R}^n)$, we get $|g|^p \in L^1(\mathbb{R}^n)$.

  Hence, since $f \in L^1(\mathbb{R}^n)$, we have a.e.

$$\int |f(x-y)||g(y)|^p dy < \infty.$$

  Therefore, as a function of y, the product $|f(x-y)|^{1/p}|g(y)|$ lies in $L^p(\mathbb{R}^n)$, for almost all x.

  Let q be such that $\frac{1}{p} + \frac{1}{q} = 1$.

  Since $|f| \in L^1(\mathbb{R}^n)$, we have (for almost all x) $|f(x-y)|^{1/q} \in L^q(\mathbb{R}^n)$.

Convolution of $L^1(\mathbb{R}^n)$ and $L^p(\mathbb{R}^n)$, $1 \le p \le \infty$

- Now, by Hölder's inequality, the function

$$|f(x-y)g(y)| = [|f(x-y)|^{1/p}|g(y)|][|f(x-y)|^{1/q}]$$

lies in $L^1(\mathbb{R}^n)$, for almost all x.
For such values of x, let $h(x) = \int f(x-y)g(y)dy$.
Hölder's inequality then gives

$$\begin{array}{rcl} |h(x)| & \le & \int |f(x-y)||g(y)|dy \\ & \le & [\int |f(x-y)||g(y)|^p dy]^{1/p}[\int |f(x-y)|dy]^{1/q}; \\ |h(x)|^p & \le & [\int |f(x-y)||g(y)|^p dy]\|f\|_1^{p/q}; \\ \int |h(x)|^p dx & \le & \|f\|_1^{p/q} \int \int [|f(x-y)||g(y)|^p dy]dx \\ & = & \|f\|_1^{p/q} \int [\int |f(x-y)|dx]|g(y)|^p dy \\ & = & \|f\|_1^{p/q} \|f\|_1 \|g\|_p^p \\ & = & \|f\|_1^p \|g\|_p^p. \end{array}$$

Thus, $h = f * g \in L^p(\mathbb{R}^n)$ and $\|f * g\|_p \le \|f\|_1 \|g\|_p$.

Example

- Let $T \in \mathscr{D}'(\mathbb{R}^n)$. Then, for any $\phi \in \mathscr{D}(\mathbb{R}^n)$,

$$\begin{aligned} (\delta * T)(\phi) &= (\delta \otimes T)(\phi(x+y)) \\ &= T_y(\delta(\phi(x+y))) \\ &= T_y(\phi(y)) \\ &= T(\phi). \end{aligned}$$

Thus, δ is the unit element of the product operation $*$.

Furthermore,

$$\begin{aligned} (\partial^\alpha \delta) * T(\phi) &= T_y((\partial^\alpha \delta)_x \phi(x+y)) \\ &= T_y((-1)^{|\alpha|} \partial^\alpha \phi(y)) \\ &= \partial^\alpha T(\phi). \end{aligned}$$

Therefore, $(\partial^\alpha \delta) * T = \partial^\alpha T = \delta * \partial^\alpha T$.

Basic Property 1

1. $\operatorname{supp}(T_1 * T_2) \subseteq \operatorname{supp} T_1 + \operatorname{supp} T_2$.

   Let $\operatorname{supp} T_i = E_i$, $i = 1,2$. Suppose, without loss of generality, that E_i is compact and E_2 is closed.

   First, we show that the set $E_1 + E_2 = \{x + y : x \in E_1, y \in E_2\}$ is closed.

   Let $(x_k + y_k)$, $x_k \in E_1$, $y_k \in E_2$, be a sequence converging to a point a.

   Since E_1 is compact, (x_k) has a subsequence (x_k'), with $x_k \to x \in E_1$.

   Now both (x_k') and the corresponding subsequence $(x_k' + y_k')$ converge.

   Since E_2 is closed, their difference (y_k') also converges to some $y \in E_2$.

   Thus, $a = x + y$ is in $E_1 + E_2$. So $E_1 + E_2$ is closed.

   Thus, $\Omega = \mathbb{R}^n - (E_1 + E_2)$ is open.

   Now for any $(x,y) \in \operatorname{supp}(T_1 \otimes T_2) = E_1 \times E_2$, we have $x + y \in E_1 + E_2$.

   So $\operatorname{supp}(T_1 \otimes T_2)$ does not intersect $\operatorname{supp}(\phi(x+y))$, for any $\phi \in \mathscr{D}(\Omega)$.

   Hence $T_1 * T_2$ vanishes on $\mathscr{D}(\Omega)$ and its support must be in $E_1 + E_2$.

   In particular, if T_1 and T_2 have compact support, so does $T_1 * T_2$.

Basic Property 2

2. $T_1 * (T_2 * T_3) = (T_1 * T_2) * T_3 = T_1 * T_2 * T_3$, for $T_1, T_2, T_3 \in \mathscr{D}(\mathbb{R}^n)$ and at least two of the three distributions have compact support.

   Both $T_1 * T_2$ and $T_2 * T_3$ are in $\mathscr{D}'(\mathbb{R}^n)$.

   The convolutions $T_1 * (T_2 * T_3)$ and $(T_1 * T_2) * T_3$ are well defined.

   To show that they are equal, note that, for any $\phi \in \mathscr{D}(\mathbb{R}^n)$,

$$\begin{array}{rcl}[T_1 * (T_2 * T_3)](\phi) & = & [T_1 \otimes (T_2 * T_3)](\phi(x + y')) \\ & = & [T_1 \otimes (T_2 \otimes T_3)](\phi(x + y + z)) \\ & = & [(T_1 \otimes T_2) \otimes T_3](\phi(x + y + z)) \\ & = & [(T_1 * T_2) * T_3](\phi).\end{array}$$

- This associative property of $*$, implies that the linear space $\mathscr{E}'(\mathbb{R}^n)$ is a commutative and associative algebra under the convolution product, with δ as its unit element.

Basic Property 3

3. For T_1 and T_2, with at least one having compact support,

$$\begin{aligned} \partial^\alpha(T_1 * T_2) &= (\partial^\alpha\delta) * (T_1 * T_2) \\ &= ((\partial^\alpha\delta) * T_1) * T_2 \\ &= (\partial^\alpha T_1) * T_2 \\ &= T_1 * (\partial^\alpha T_2). \end{aligned}$$

This follows directly from equation

$$(\partial^\alpha\delta) * T = \partial^\alpha T = \delta * \partial^\alpha T$$

and the commutative and associative properties of $*$.

Basic Property 4: Translations in $\mathbb{R}^n$

4. Let f is a function on $\mathbb{R}^n$ and h is any point in $\mathbb{R}^n$.
   The **translation** τ_h of f by h is the function $\tau_h f$ defined on $\mathbb{R}^n$ by

$$\tau_h f(x) = f(x-h).$$

- We clearly have $\tau_h \phi \in C_0^\infty(\mathbb{R}^n)$, whenever $\phi \in C_0^\infty(\mathbb{R}^n)$.
  We define the **translation** of the distribution $T \in \mathscr{D}'(\mathbb{R}^n)$ by

$$(\tau_h T)(\phi) = T(\tau_{-h}\phi), \qquad \phi \in \mathscr{D}(\mathbb{R}^n).$$

  which is again a distribution in $\mathbb{R}^n$.
- When the distribution T is defined by a locally integrable function $f(x)$, its translation $\tau_h T$ is clearly defined by $f(x-h)$.
- In the case of the Dirac measure, we have

$$\tau_h \delta(\phi) = \delta(\tau_{-h}\phi) = \phi(h) = \delta_h(\phi).$$

  This implies that $\tau_h \delta = \delta_h$.

Translations in $\mathbb{R}^n$ (Cont'd)

- More generally, for any $T \in \mathscr{D}'(\mathbb{R}^n)$

$$\begin{array}{rcl} \tau_h T(\phi) & = & T(\tau_{-h}\phi) \\ & = & T_x(\phi(x+h)) \\ & = & T_x(\delta_h(\phi(x+y))) \\ & = & (\delta_h * T)(\phi). \end{array}$$

  Therefore, $\tau_h T = \delta_h * T$, $T \in \mathscr{D}'(\mathbb{R}^n)$.

- If either T_1 or T_2 has compact support, this gives

$$\begin{array}{rcll} \tau_h(T_1 * T_2) & = & \delta_h * (T_1 * T_2) & \text{(preceding property)} \\ & = & (\delta_h * T_1) * T_2 & \text{(associativiy)} \\ & = & (\tau_h T_1) * T_2 & \text{(preceding property)} \\ & = & T_1 * (\tau_h T_2). & \text{(commutativity)} \end{array}$$

Convolutions of Multiple Distributions

- Even though we can sometimes define the convolution product of several distributions where more than one is without compact support, such products may not satisfy all the properties listed above.

  Example: Let 1 denote the distribution represented by the constant function 1 on $\mathbb{R}^n$. Then $(H*\delta')*1$ and $H*(\delta'*1)$ are both well defined distributions but they are not equal.

$$\begin{aligned}(H*\delta')*1 &= (H'*\delta)*1 \\ &= (\delta*\delta)*1 \\ &= \delta*1 \\ &= 1;\end{aligned}$$

$$\begin{aligned}H*(\delta'*1) &= H*(\delta*1') \\ &= H*0 \\ &= 0.\end{aligned}$$

Cancelations

- The equality $\delta' * 1 = 0$ also shows that, if T_1 and T_2 are two nonzero distributions, it may happen that $T_1 * T_2 = 0$.
- In other words, the equality $S_1 * T = S_2 * T$, for some $T \neq 0$, does not necessarily imply that $S_1 = S_2$.
- On the positive side, suppose:
  - $T \in \mathscr{D}'(\mathbb{R}^n)$;
  - $S_1, S_2 \in \mathscr{E}'(\mathbb{R}^n)$, such that

$$S_1 * T = S_2 * T = \delta.$$

  Then we have

$$\begin{aligned} S_1 &= \delta * S_1 \\ &= (S_2 * T) * S_1 \\ &= S_2 * (T * S_1) \\ &= S_2 * \delta \\ &= S_2. \end{aligned}$$

Convolutions in $\mathscr{D}'_+(\mathbb{R})$

- Let $\mathscr{D}'_+(\mathbb{R}) = \{T \in \mathscr{D}'(\mathbb{R}) : \operatorname{supp} T \subseteq [0,\infty)\}$.
- If $T, S \in \mathscr{D}'_+(\mathbb{R})$, we can still define the convolution of T and S by

$$\langle S * T, \phi\rangle = \langle S_x, \langle T_y, \phi(x+y)\rangle\rangle, \quad \phi \in \mathscr{D}(\mathbb{R}).$$

For fixed x and $\phi \in \mathscr{D}(\mathbb{R})$, $\phi(x+y)$ has compact support in y.

So $\psi(x) = \langle T_y, \phi(x+y)\rangle$ is a well-defined function in $C^\infty(\mathbb{R})$.

Moreover, $\operatorname{supp}\psi$ is bounded from above.

Suppose $y \in \operatorname{supp} T$ and $x+y \in \operatorname{supp}\phi \subseteq [-M, M]$.
Then $y \geq 0$ and $|x+y| \leq M$. Hence, $x \leq x+y \leq M$.

Thus, $\operatorname{supp} S \subseteq [0,\infty)$ intersects $\operatorname{supp}\psi \subseteq (-\infty, M]$ in a bounded set.

So we can define $\langle S * T, \phi\rangle = \langle S, \psi\rangle$ as $\lim \langle S, \phi_n \psi\rangle$, where ϕ_n is a C_0^∞ function which equals 1 on $[-n, n]$.

Note that $\operatorname{supp}(S * T) \subseteq \operatorname{supp} S + \operatorname{supp} T \subseteq [0,\infty)$.

Hence, $S * T \in \mathscr{D}'_+(\mathbb{R})$, i.e., $\mathscr{D}'_+(\mathbb{R})$ is closed under the operation $*$.

Inverse of a Distribution

- Let $T \in \mathscr{D}'(\mathbb{R}^n)$.
- A distribution $S \in \mathscr{D}'(\mathbb{R}^n)$ is called an **inverse** of T in $\mathscr{D}'(\mathbb{R}^n)$ with respect to the binary operation $*$, and denoted by T^{-1}, if

$$S * T = \delta.$$

- We saw that in $\mathscr{E}'$, if such an inverse exists, it is unique.
- It is also unique in any subspace of $\mathscr{D}'$, where the convolution product is a commutative and associative algebra, such as $\mathscr{D}'_+$.

Examples of Inverse Distributions

- We look at the possibility of inverting some simple distributions in $\mathbb{R}$.
  (i) Let $S * H = \delta$. Then

$$\delta' = (S * H)' = S * H' = S * \delta = S.$$

  Hence, $H^{-1} = \delta'$.
  Similarly, $(\delta')^{-1} = S^{-1} = H$.

  (ii) Let $S * (\delta' - \lambda\delta) = \delta$. Then

$$\begin{gathered} S * \delta' - \lambda S * \delta = \delta \\ (S * \delta)' - \lambda S * \delta = \delta \\ S' - \lambda S = \delta. \end{gathered}$$

  Set $S = e^{\lambda x} T$. Then

$$\begin{gathered} S' = \lambda e^{\lambda x} T + e^{\lambda x} T' \\ e^{\lambda x} T' = S' - \lambda S = \delta \\ T' = \delta \\ T = H. \end{gathered}$$

  Therefore, $(\delta' - \lambda\delta)^{-1} = S = e^{\lambda x} H$.

Subsection 4

Regularization of Distributions

The Reflection of a Function in 0

- For any function f on $\mathbb{R}^n$, we define its **reflection** in 0 as the function $\check{f}$ defined on $\mathbb{R}^n$ by
$$\check{f}(x) = f(-x).$$
- We extend this definition to $\mathscr{D}'(\mathbb{R}^n)$ by duality,
$$\check{T}(\phi) = T(\check{\phi}), \qquad \phi \in \mathscr{D}(\mathbb{R}^n).$$

Convolution of $\mathscr{D}'(\mathbb{R}^n)$ by $C_0^\infty(\mathbb{R}^n)$

Theorem

For all $T \in \mathscr{D}'(\mathbb{R}^n)$ and all $\psi \in C_0^\infty(\mathbb{R}^n)$, the convolution $(T * \psi) = T(\tau_x \check{\psi})$ is in $C^\infty(\mathbb{R}^n)$.

- For any $\phi \in \mathscr{D}(\mathbb{R})$, we have

$$\begin{array}{rcl} (T*\psi)(\phi) & = & T_x(\langle \psi(y), \phi(x+y)\rangle); \\ \langle \psi(y), \phi(x+y)\rangle & = & \int \psi(y)\phi(x+y)dy \\ & = & \int \psi(\xi - x)\phi(\xi)d\xi \\ & = & \langle \psi(\xi - x), \phi(\xi)\rangle \\ & = & \langle \check{\psi}(x-\xi), \phi(\xi)\rangle \\ & = & \langle \tau_\xi \check{\psi}(x), \phi(\xi)\rangle. \end{array}$$

  Hence, $(T*\psi)(\phi) = T_x(\langle \tau_\xi \check{\psi}(x), \phi(\xi)\rangle) = \langle T(\tau_\xi \check{\psi}), \phi(\xi)\rangle$.
  Furthermore, $(T*\psi)(x) = T(\tau_x \check{\psi}) = T_y(\psi(x-y))$ is a $C^\infty(\mathbb{R}^n)$ function, by a preceding corollary.

Consequences

Corollary

$T(\phi) = (T * \check{\phi})(0)$, for every $\phi \in \mathscr{D}(\mathbb{R}^n)$ and $T \in \mathscr{D}'(\mathbb{R}^n)$.

- As a consequence, if $T * \phi = 0$, for every $\phi \in \mathscr{D}(\mathbb{R}^n)$, then $T = 0$.

Corollary

Let $T \in \mathscr{E}'(\mathbb{R}^n)$.

(a) If $\psi \in C^\infty(\mathbb{R}^n)$, then $(T * \psi) = T(\tau_x \check{\psi})$ is in $C^\infty(\mathbb{R}^n)$.

(b) If $\psi \in C_0^\infty(\mathbb{R}^n)$, then $(T * \psi)(x)$ is in $C_0^\infty(\mathbb{R}^n)$.

- For Part (a) multiply ψ by a $C_0^\infty(\mathbb{R}^n)$ function equal to 1 on $\operatorname{supp} T$. Part (b) follows from the fact $\operatorname{supp}(T * \psi) \subseteq \operatorname{supp} T + \operatorname{supp} \psi$.

Convolution of $\mathscr{D}^{m\prime}(\mathbb{R}^n)$ by $C_0^m(\mathbb{R}^n)$

- If $T \in \mathscr{D}^{m\prime}(\mathbb{R}^n)$ and $\psi \in C_0^m(\mathbb{R}^n)$, $(T * \psi) = T(\tau_x \check{\psi})$ still holds and the convolution $T * \psi$ is then continuous in $\mathbb{R}^n$.

  Suppose (x_i) is a sequence in $\mathbb{R}^n$ which converges to x.

$$\begin{aligned} \lim (T * \psi)(x_i) &= \lim \langle T_y, \psi(x_i - y)\rangle \\ &= \langle T_y, \lim \psi(x_i - y)\rangle \\ &\quad (T \text{ continuous on } \mathscr{D}^m(\mathbb{R}^n)) \\ &= \langle T_y, \psi(x - y)\rangle \\ &= (T * \psi)(x). \end{aligned}$$

- If $T \in \mathscr{D}^{m\prime}(\mathbb{R}^n)$ has compact support, then we can take ψ in $C^m(\mathbb{R}^n)$ and reach the same conclusion.

Corollary

If $T \in \mathscr{D}^{m\prime}(\mathbb{R}^n)$ and $\psi \in C^m(\mathbb{R}^n)$, then $(T * \psi)(x) = \langle T_y, \psi(x - y)\rangle$ is a continuous function in $\mathbb{R}^n$, provided T or ψ has compact support.

The Function β_λ

- Recall the definition of the C^∞ function $\alpha(x) = \begin{cases} e^{-\frac{1}{1-|x|^2}}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$ that has support in $\overline{B}(0,1)$, with finite positive integral over $\mathbb{R}^n$.
- Recall, also, the definition $\beta(x) = \frac{\alpha(x)}{\int \alpha(x)dx}$, another C^∞ function with support $\overline{B}(0,1)$, satisfying $\int \beta(x)dx = 1$.
- Finally, recall the function

$$\beta_\lambda(x) = \frac{1}{\lambda^n}\beta\left(\frac{x}{\lambda}\right), \quad \text{for } \lambda > 0.$$

- We have $\beta_\lambda \in \mathscr{D}(\mathbb{R}^n)$, $\text{supp}(\beta_\lambda) = \overline{B}(0,\lambda)$ and

$$\int \beta_\lambda(x)dx = \int \beta(x)dx = 1.$$

The Convolution $T * \beta_\lambda$

Theorem

For any $T \in \mathscr{D}'(\mathbb{R}^n)$ the $C^\infty(\mathbb{R}^n)$ function $T * \beta_\lambda$ converges strongly to T as $\lambda \to 0$, i.e., $(T * \beta_\lambda)(\phi)$ converges to $T(\phi)$ uniformly on every bounded subset of $\mathscr{D}(\mathbb{R}^n)$.

- $T * \beta_\lambda$ is in $C^\infty(\mathbb{R}^n)$ by the preceding theorem.

  Let E be any bounded subset of $\mathscr{D}(\mathbb{R}^n)$. By previous theorems:
  - There is a compact K in $\mathbb{R}^n$, such that E is bounded in $\mathscr{D}_K(\mathbb{R}^n)$;
  - For every $\phi \in E$, the support of $\beta_\lambda * \phi$ lies in a λ-neighborhood of K.

  If $\lambda \in (0,1)$, then there is a compact K_0, such that $K \subseteq K_0 \subseteq \mathbb{R}^n$ and $\text{supp}(\beta_\lambda * \phi) \subseteq K_0$, for all $\phi \in E$.

  Let m be any nonnegative integer. $\partial^\alpha \phi(x) \in \mathscr{D}_K(\mathbb{R}^n)$, $|\alpha| \leq m$. So there is $\varepsilon = \varepsilon(m) > 0$, such that $\partial^\alpha \phi(x-y) \in \mathscr{D}_{K_0}(\mathbb{R}^n)$, $y \in B(0,\varepsilon)$.

  The function $\partial^\alpha \phi(x-y) \xrightarrow{y \to 0} \partial^\alpha \phi(x)$ uniformly on K_0, $|\alpha| \leq m$.

The Convolution $T * \beta_\lambda$ (Cont'd)

- We also have

$$\begin{array}{rcl} |(\beta_\lambda * \partial^\alpha \phi - \partial^\alpha \phi)(x)| & = & \left| \int \beta_\lambda(y)[\partial^\alpha \phi(x-y) - \partial^\alpha \phi(x)] dy \right| \\ & \leq & \int \beta_\lambda(y) |\partial^\alpha \phi(x-y) - \partial^\alpha \phi(x)| dy. \end{array}$$

For all values of λ in $(0, \varepsilon)$, $\operatorname{supp} \beta_\lambda \subseteq \overline{B}(0, \varepsilon)$.

So the integration may be performed over $B(0, \varepsilon)$.

Thus, the left-hand side tends to 0 uniformly as $\lambda \to 0$, for all x in K and all $|\alpha| \leq m$.

Using a preceding corollary,

$$\begin{array}{rcl} (T * \beta_\lambda - T)(\phi) & = & (T * \beta_\lambda) * \check{\phi}(0) - (T * \check{\phi})(0) \\ & = & T * (\beta_\lambda * \check{\phi} - \check{\phi})(0) \\ & = & T(\beta_\lambda * \phi - \phi). \end{array}$$

For the last equality, $\beta_\lambda * \check{\phi} = \check{\beta_\lambda} * \check{\phi} = \widecheck{\beta_\lambda * \phi}$, since β_λ is even.

The Convolution $T * \beta_\lambda$ (Conclusion)

- As $\lambda \to 0$, $\beta_\lambda * \phi \to \phi$ uniformly for all $\phi \in E$.
  Therefore, $T * \beta_\lambda - T$ converges to 0 uniformly on E.

Corollary

If $T \in \mathscr{E}'(\mathbb{R}^n)$, then $T * \beta_\lambda$ converges uniformly to T on every bounded subset of $\mathscr{E}(\mathbb{R}^n)$.

Regularization

Example: By setting $T=\delta$ in the theorem, we see that β_λ converges strongly to δ in both $\mathscr{D}'(\mathbb{R}^n)$ and $\mathscr{E}'(\mathbb{R}^n)$.

- Previously, the convolution of a locally integrable function f with β_λ was called a **regularization** of f.
- Extending the notion to distributions, we call

$$T*\beta_{1/k}=T*\gamma_k$$

a **regularizing sequence** of functions for the distribution $T\in\mathscr{D}'(\mathbb{R}^n)$.

Example: γ_k is a regularizing sequence for δ.

- In consequence, if $T*\phi=0$, for every $\phi\in\mathscr{D}(\mathbb{R}^n)$, then

$$T=T*\delta=\lim T*\gamma_k=0.$$

Vanishing Derivative on $\mathbb{R}$

- We reestablish that T' vanishes in $\mathbb{R}$ only if T is a constant (a.e.).

  Let $T \in \mathscr{D}'(\mathbb{R})$ satisfy $T' = 0$.

  Let γ_k be a regularizing sequence for δ.

  The C^∞ function $T * \gamma_k$ satisfies $(T * \gamma_k)' = T' * \gamma_k = 0$ in $\mathbb{R}$, for every k.

  So $T * \gamma_k = c_k$, for some constant c_k.

  Now $c_k = T * \gamma_k \to T$ in $\mathscr{D}'$.

  We show that the sequence of constants c_k also converges in $\mathbb{C}$.

  Let $\phi \in \mathscr{D}(\mathbb{R})$, such that $\int \phi(x) dx = 1$.

  The sequence $c_k = \langle c_k, \phi \rangle$ converges in $\mathbb{C}$ because c_k converges in $\mathscr{D}'$.

  Hence, its limit, the constant $\lim c_k$, coincides with T.

Vanishing Derivative on $\mathbb{R}$ (Remark)

- In general, the convergence of a sequence of functions f_k to f in $\mathscr{D}'$ does not imply that its pointwise limit is f, or that it is even a function (recall the sequence $\sin kx$ which converges to 0 in $\mathscr{D}'$).
- However, when f_k is constant, we have just shown that both assertions can be made.

Linearity of a Distribution

- The result leads to the conclusion that, if $T^{(k)} = 0$, then T is (almost everywhere) a polynomial of degree less than k.
- When $k = 2$, we can use a regularization process, which can be generalized from $\mathbb{R}$ to $\mathbb{R}^n$.

  Example: If $T \in \mathscr{D}'(\mathbb{R})$ satisfies $T'' = 0$, we shall show that T is a linear function a.e.

  For any $\phi \in \mathscr{D}(\mathbb{R})$, we know that $T * \phi$ is a C^∞ function and that

  $$(T * \phi)'' = T'' * \phi = 0.$$

  Therefore, $T * \phi$ is a linear function of the form $(T * \phi)(x) = ax + b$.

Linearity of a Distribution (Cont'd)

- We saw that, for all $\phi \in \mathscr{D}(\mathbb{R})$, $(T * \phi)(x)$ is of he form $ax + b$.
  Let $h(x) = ax + b$, $x \in \mathbb{R}$.
  Now β is a C^∞ function supported in $[-1, 1]$, with $\int \beta(x)dx = 1$.
  So we obtain

$$\begin{array}{rcl}(h * \beta)(x) & = & \int_{-\infty}^{\infty} h(x-y)\beta(y)dy \\ & = & \int_{-\infty}^{\infty} [a(x-y)+b]\beta(y)dy \\ & = & ax+b,\end{array}$$

  since $\int_{-\infty}^{\infty} y\beta(y)dy = 0$, the integrand being an odd function.
  Thus, $h * \beta = h$.
  Let $\beta_{1/k} \in \mathscr{D}(\mathbb{R})$ be the regularizing sequence defined previously.
  Then, taking into account what was shown above,

$$(T * \beta) * \beta_{1/k} = (T * \beta_{1/k}) * \beta = T * \beta_{1/k}.$$

  In the limit as $k \to \infty$, we obtain $T = T * \beta$ a.e.
  Since $T * \beta$ is a linear function, so is the distribution T (a.e.).

Characterization of Convolutions by $\mathscr{D}(\mathbb{R}^n)$

Theorem

For any $T \in \mathscr{D}'(\mathbb{R}^n)$, the linear map L from $\mathscr{D}(\mathbb{R}^n)$ to $\mathscr{E}(\mathbb{R}^n)$ defined by $L(\phi) = T * \phi$ is continuous and commutes with the translation τ_h, $h \in \mathbb{R}^n$. Conversely, if L is a continuous linear map from $\mathscr{D}'(\mathbb{R}^n)$ to $\mathscr{E}(\mathbb{R}^n)$ which commutes with τ_h, then there is a unique $T \in \mathscr{D}'(\mathbb{R}^n)$, such that $L(\phi) = T * \phi$, $\phi \in \mathscr{D}(\mathbb{R}^n)$.

(i) For any sequence $\phi_k \to \phi$ in $\mathscr{D}_K$, we have

$$\lim (T * \phi_k)(x) = \lim T(\tau_x \check{\phi}_k) = T(\tau_x \check{\phi}) = (T * \phi)(x).$$

The second equality because both T and τ_x are continuous.
If $T \in \mathscr{D}'(\mathbb{R}^n)$, then, by a previous theorem, for all $\phi \in \mathscr{D}(\mathbb{R}^n)$,

$$\begin{aligned}(T * \tau_h \phi)(x) &= T(\tau_x (\widecheck{\tau_h \phi})) = T(\tau_x \tau_{-h} \check{\phi}) \\ &= T(\tau_{x-h} \check{\phi}) = (T * \phi)(x-h) = \tau_h (T * \phi)(x).\end{aligned}$$

Thus, $L\tau_h = \tau_h L$.

Characterization of Convolutions by $\mathscr{D}(\mathbb{R}^n)$ (Cont'd)

(ii) Suppose L is a continuous linear map from $\mathscr{D}(\mathbb{R}^n)$ to $\mathscr{E}(\mathbb{R}^n)$ which commutes with τ_h. Then the map

$$\phi \mapsto L(\check{\phi})(0)$$

is a continuous linear function on $\mathscr{D}(\mathbb{R}^n)$. So there is $T \in \mathscr{D}'(\mathbb{R}^n)$, such that

$$L(\check{\phi})(0) = T(\phi), \qquad \phi \in \mathscr{D}(\mathbb{R}^n).$$

Now we have

$$\begin{aligned} L(\phi)(x) &= \tau_{-x} L(\phi)(0) = L(\tau_{-x}\phi)(0) \\ &= T(\widecheck{\tau_{-x}\phi}) = T(\tau_x \check{\phi}) = (T * \phi)(x). \end{aligned}$$

The uniqueness of T follows from the observation that $T * \phi = 0$, for all $\phi \in \mathscr{D}(\mathbb{R}^n)$, implies that $T = 0$.

Subsection 5

Local Structure of Distributions

Distributions as Derivatives of Continuous Functions

- We saw that the Dirac distribution on $\mathbb{R}$ is the second derivative of the continuous function $x_+ = xH(x)$.
- From a previous theorem, we conclude that every distribution on $\mathbb{R}$ with support $\{0\}$ is a finite linear combination of derivatives of x_+.
- More generally, we can show that every distribution is, locally, a derivative of some continuous function.
- In this sense distributions are the natural generalization of continuous functions, achieved by supplementing these functions with their (distributional) derivatives of all orders.

Notation

- For $x \in \mathbb{R}^n$ and $k \in \mathbb{N}$, we define

$$\begin{array}{rcl} (x_i)_+^k & = & x_i^k H(x_i), \quad i = 1, \ldots, n; \\ x^k & = & x_1^k x_2^k \cdots x_n^k; \\ x_+^k & = & (x_1)_+^k (x_2)_+^k \cdots (x_n)_+^k; \\ \partial^k & = & \partial_1^k \partial_2^k \cdots \partial_n^k. \end{array}$$

- For all $i = 1, \ldots, n$, $\frac{1}{(k-1)!} \partial_i^k (x_i)_+^{k-1} = \delta$ is the Dirac measure on $\mathbb{R}$.
- So in $\mathbb{R}^n$,

$$\partial^k E_k = \delta,$$

where
  - $E_k = \frac{1}{[(k-1)!]^n} x_+^{k-1}$ is in $C^{k-2}(\mathbb{R}^n)$;
  - δ is the Dirac measure on $\mathbb{R}^n$, which is the tensor product of $\delta \in \mathscr{D}'(\mathbb{R})$ with itself n times.

Local Representation of a Distribution

Theorem

If $T \in \mathscr{D}'(\mathbb{R}^n)$ and K is a compact subset of $\mathbb{R}^n$, then there is a continuous function f on $\mathbb{R}^n$ and a multi-index $\alpha \in \mathbb{N}_0^n$, such that

$$T(\phi) = \langle \partial^\alpha f, \phi \rangle = (-1)^{|\alpha|} \langle f, \partial^\alpha \phi \rangle, \quad \phi \in \mathscr{D}_K.$$

- Let $\psi \in C_0^\infty(\mathbb{R}^n)$ and $\psi = 1$ on a neighborhood of K. The distribution ψT equals T on K and has compact support. Therefore it is of finite order, say m. We can now write

$$\psi T = \delta * \psi T = (\partial^{m+2} E_{m+2}) * \psi T = \partial^{m+2}(E_{m+2} * \psi T).$$

Now $E_{m+2} \in C^m(\mathbb{R}^n)$. The distribution ψT being of order m, may be extended to a continuous linear functional on $C_0^m(\mathbb{R}^n)$ in the topology of $\mathscr{D}^m(\mathbb{R}^n)$. Since ψT has compact support, by a previous theorem, the convolution $E_{m+2} * \psi$ is a continuous function on $\mathbb{R}^n$. $E_{m+2} * \psi$ represents the desired function f.

The Case of Compact Support

- When T has compact support this result takes a global form.

Corollary

If $T \in \mathscr{E}'(\mathbb{R}^n)$, then there is a continuous function f on $\mathbb{R}^n$ and a multi-index α, such that $T = \partial^\alpha f$.

- If $\operatorname{supp} T = K$ is compact, then T is of finite order, say m.

  By the theorem, $T = \partial^{m+2} f$, where $f = E_{m+2} * T \in C^0(\mathbb{R}^n)$.

An Example on Compact Support

- Let $T \in \mathscr{E}'(\mathbb{R})$ be a distribution of compact support.

  As a consequence, T has finite order, say m.

  We have
  - $E_{m+2} = \frac{1}{(m+1)!} x_+^{m+1} \in C^m(\mathbb{R})$;
  - $E_{m+1}^{(m+2)} = \delta$.

  Suppose $f = T * E_{m+2}$. Then

  $$T = T * \delta = T * E_{m+2}^{(m+2)} = f^{(m+2)}.$$

  But $T \in \mathscr{E}'(\mathbb{R})$ is of order m.

  So it can be extended to a bounded linear functional on $C^m(\mathbb{R})$.

  Hence, using equation $(T * \psi)(x) = T(\tau_x \check{\psi})$, we can write

  $$f(x) = \langle T_y, E_{m+2}(x - y) \rangle,$$

  which is clearly continuous.

An Example on Compact Support (Remarks)

- In the example, even though T has compact support, the continuous function f which satisfies $T = f^{(m+2)}$ may not have compact support.

  In fact, when $T = \delta$, then $f = E_{m+2}$, which has support $[0,\infty)$.
- Note, also, the relation between the order of differentiation of f which is needed to represent T, namely $m+2$, and the order of T.

Remarks on the Order of the Derivative and the Domain

- The representation $T = \partial^\alpha f$ in the statement of both the theorem and its corollary is not unique.

  The choice $\alpha = (\alpha_1, \ldots, \alpha_n) = (m+2, \ldots, m+2)$ always works when f is chosen to be $E_{m+2} * T$, but obviously there are other possibilities.
- The second point worth noting is that this representation remains valid whether K is taken in $\mathbb{R}^n$ or in any of its open subsets.

  Hence the theorem and its corollary still hold if $\mathbb{R}^n$ is replaced by Ω.
- The corollary remains valid if the distribution T is merely of finite order, as we will next show.

Locally Finite Partitions of Unity

- Let Ω be any open set in $\mathbb{R}^n$.
- An open covering $\{\Omega_i : i \in \mathbb{N}\}$ of Ω is called **locally finite** if every compact subset of Ω intersects at most a finite number of Ω_i.
- Following a procedure outlined previously, we can construct a sequence of functions ψ_i, in $C_0^\infty(\Omega)$, such that, for each $i \in \mathbb{N}$, $\text{supp}\psi_i \subseteq \Omega_i$, $0 \le \psi_i \le 1$, and
$$\sum_{i=1}^{\infty} \psi_i(x) = 1, \quad \text{for every } x \in \Omega.$$
- Since any $x \in \Omega$ lies in at most a finite number of the sets Ω_i, this sum has only a finite number of nonzero terms.
- The collection $\{\psi_i\}$ is called a **locally finite partition of unity in Ω subordinate to the cover** $\{\Omega_i\}$.

Finite Order and Global Derivative Representation

Theorem

If $T \in \mathscr{D}'(\Omega)$ is of finite order, then there exists a continuous function f in Ω and a multi-index α, such that $T = \partial^\alpha f$ in Ω.

- Suppose $T \in \mathscr{D}'(\Omega)$ is of order m.

  Let $\{\psi_i\}$ be a locally finite partition of Ω subordinate to the cover $\{\Omega_i\}$.

  Then $T = \sum \psi_i T = \sum T_i$, where: $T_i := \psi_i T$ is a distribution:
  - with compact support in Ω_i;
  - of order $m_i \leq m$, since its order cannot exceed the order of T.

  By the corollary to the theorem, it is represented in Ω by

  $$T_i = \partial^{m_i+2}(E_{m_i+2} * T_i) = \partial^{m+2}(E_{m+2} * T_i),$$

  where the convolution of E_{m+2} and T is well defined because $T_i = \psi_i T$ can be extended as 0 into $\mathbb{R}^n - \Omega_i$.

Finite Order and Global Derivative Representation (Cont'd)

- Now $E_{m+2} * T_i = f_i$ is a continuous function in Ω.

  Moreover, T is represented by the sum

  $$T = \sum_i T_i = \sum_i \partial^{m+2} f_i = \partial^{m+2} \sum_i f_i.$$

  Since any compact set in Ω intersects the supports of at most a finite number of the functions f_i, this sum over f is finite.

  Therefore, the function $g = \sum_i f_i$ is continuous in Ω.

Another Global Version

- If the distribution T is not of finite order, the representation

$$T = \sum \partial^{m+2} f_i$$

  is still valid.
- So we obtain a global version of the theorem.

Corollary

For every $T \in \mathscr{D}'(\Omega)$, there exist continuous functions f_i in Ω and multi-indices $\alpha_i \in \mathbb{N}_0^n$, such that $T = \sum \partial^{\alpha_i} f_i$, in the sense that

$$\langle T, \phi \rangle = \sum_{i=1}^{N} (-1)^{|\alpha_i|} \langle f_i, \partial^{\alpha_i} \phi \rangle, \quad \text{for all } \phi \in \mathscr{D}(\Omega),$$

where the (finite) integer N depends on suppϕ.

Subsection 6

Applications to Differential Equations

Existence of Primitive Distributions in $\mathbb{R}$

- Recall, for a given $T \in \mathscr{D}'(\Omega)$, the distribution S which satisfies $\partial_k S(\phi) = T(\phi)$, for every $\phi \in \mathscr{D}(\Omega)$, is called a **primitive** of T.

Theorem

Any distribution in $\mathscr{D}'(\mathbb{R})$ has a primitive distribution which is unique up to an additive constant.

- Let $T \in \mathscr{D}'(\mathbb{R})$. We wish to determine a distribution S, such that

$$S'(\phi) = -S(\phi') = T(\phi), \quad \phi \in \mathscr{D}(\mathbb{R}).$$

This determines S on the space

$$\mathscr{D}_0(\mathbb{R}) = \{\psi \in \mathscr{D}(\mathbb{R}) : \psi = \phi', \text{ for some } \phi \in \mathscr{D}(\mathbb{R})\}.$$

We have already seen that $\psi \in \mathscr{D}_0(\mathbb{R})$ if and only if $\int_{-\infty}^{\infty} \psi(x) dx = 0$. Let ϕ_0 be a fixed function in $\mathscr{D}(\mathbb{R})$, such that $\langle 1, \phi_0 \rangle = 1$.

Existence of Primitive Distributions in $\mathbb{R}$ (Cont'd)

For any $\phi \in \mathscr{D}(\mathbb{R})$, we can write

$$\phi(x) = \phi(x) - \langle 1, \phi \rangle \phi_0(x) + \langle 1, \phi \rangle \phi_0(x) = \underbrace{\psi(x)}_{\in \mathscr{D}_0(\mathbb{R})} + \underbrace{\langle 1, \phi \rangle \phi_0(x)}_{\in \mathscr{D}(\mathbb{R}) - \mathscr{D}_0(\mathbb{R})}.$$

We first define S on $\mathscr{D}_0(\mathbb{R})$ by

$$S(\psi) = -T(\chi), \quad \text{where } \chi(x) = \int_{-\infty}^{x} \psi(t) dt \in \mathscr{D}(\mathbb{R}).$$

Then we extend the definition to $\mathscr{D}(\mathbb{R})$ by

$$S(\phi) = -T(\chi) + \langle c, \phi \rangle,$$

where c is an arbitrary complex constant.

If S is a distribution, then, for all $\phi \in \mathscr{D}(\mathbb{R})$,

$$S'(\phi) = -S(\phi') = -\left[-T\left(\int_{-\infty}^{x} \phi'(t) dt\right) + \langle c, \phi' \rangle\right] = T(\phi) + 0.$$

This means that S is a primitive of T.

Existence of Primitive Distributions in $\mathbb{R}$ (Cont'd)

Claim: S is in $\mathscr{D}'(\mathbb{R})$.

Let (ϕ_k) be any sequence in $\mathscr{D}(\mathbb{R})$ which converges to 0.

This implies that:

- $\text{supp}\phi_k$ is in some fixed compact set $K \subseteq \mathbb{R}$, for all k;
- $\partial^\alpha \phi_k \to 0$ uniformly on K.

So $\langle 1, \phi_k \rangle \to 0$. Therefore, in $\mathscr{D}(\mathbb{R})$,

$$\begin{aligned} \psi_k(x) &= \phi_k(x) - \langle 1, \phi_k \rangle \phi_0(x) \to 0; \\ \chi_k(x) &= \int_{-\infty}^x \psi_k(t) dt \to 0. \end{aligned}$$

Hence,

$$S(\phi_k) = -T(\chi_k) + \langle c, \phi_k \rangle \to 0.$$

This proves that S is a distribution in $\mathbb{R}$.

Existence of Primitive Distributions in $\mathbb{R}$ (Conclusion)

Claim Uniqueness.

Suppose S_1 and S_2 are two primitives of T.

Then for any $\phi \in \mathscr{D}(\mathbb{R})$,

$$(S_1 - S_2)'(\phi) = S_1'(\phi) - S_2'(\phi) = T(\phi) - T(\phi) = 0.$$

By a preceding result, we conclude that $S_1 - S_2$ must be a constant.

Linear Partial Differential Equations of Order m

- Let L be a **linear partial differential operator of order** $m \geq 1$ of the form
$$L = \sum_{|\alpha| \leq m} c_\alpha(x) \partial^\alpha,$$
where $\alpha \in \mathbb{N}_0^n$ and c_α are C^∞ functions on $\mathbb{R}^n$.
- L clearly maps $\mathscr{D}'(\Omega)$ into $\mathscr{D}'(\Omega)$.
- The corresponding equation
$$Lu = f$$
where f is generally given as a distribution in $\Omega \subseteq \mathbb{R}^n$, is called a **linear partial differential equation of order** m.
- The restriction to *linear* differential equations is necessary because we cannot define multiplication in $\mathscr{D}'$ as a natural extension of multiplication of functions.

Strong and Weak Solutions of a Differential Equation

- In the classical theory, by a "solution" to the differential equation $Lu = f$ in Ω we mean a function which is differentiable up to order m in Ω and satisfies the equation in the sense of equality of functions.
- We demand here a little more smoothness.
- We call u a **strong solution** of $Lu = f$ in Ω if $u \in C^m(\Omega)$ and the (continuous) function Lu equals f in Ω.
- A **weak solution** of $Lu = f$ is a distribution $u \in \mathscr{D}'(\Omega)$ which satisfies $Lu = f$ in the sense of distributions, i.e., in the sense that

$$\langle Lu, \phi \rangle = \langle f, \phi \rangle, \text{ for all } \phi \in \mathscr{D}(\Omega).$$

- Every strong solution of $Lu = f$ is also a weak solution.
  - Any continuous function defines a distribution in $\mathscr{D}'$;
  - All its continuous derivatives coincide with its corresponding distributional derivatives.
- We ask whether there are weak solutions of the equation $Lu = f$ which are not strong solutions.

Example

- Consider the ordinary differential equation $xu' = 0$ on $\mathbb{R}$.

  It has the strong solution $u = c_1$.

  The function $u(x) = c_2 H(x)$ satisfies the equation as a distribution.

  $$u' = c_2 H' = c_2 \delta.$$

  So, for all $\phi \in \mathscr{D}(\mathbb{R})$,

  $$\langle xu', \phi \rangle = c_2 \langle x\delta, \phi \rangle = c_2 \langle \delta, x\phi \rangle = 0.$$

  Hence, $u = c_1 + c_2 H$ is a weak solution of $xu' = 0$.
- This solution violates the (classical) rule that an ordinary differential equation of order 1 has a general solution with one arbitrary constant.
- It would seem that this "rule" no longer holds when distributions are admitted to the class of solutions.

Characterization of Strong Solutions

Theorem

Let L be a linear differential operator of order m and u be a weak solution of $Lu = f$ in Ω. If $u \in C^m(\Omega)$ and $f \in C^0(\Omega)$, then u is also a strong solution of the equation.

- A weak solution of $Lu = f$, u satisfies $\langle Lu, \phi \rangle = \langle f, \phi \rangle$, for $\phi \in \mathscr{D}(\Omega)$.
  Equivalently, $\int_\Omega (Lu - f)\phi = 0$, for all $\phi \in \mathscr{D}(\Omega)$.
  We must show $Lu - f = 0$ on Ω. If not, there exists $x \in \Omega$, where $Lu(x) - f(x) \neq 0$. But $Lu - f$ is continuous.
  So there is a neighborhood U of x, where $Lu - f$ does not vanish.
  Now we can choose $\phi \in \mathscr{D}(\Omega)$ to be a positive function supported in U.
  For such a choice, we would have
  $$\int_\Omega (Lu - f)\phi = \int_U (Lu - f)\phi \neq 0$$
  in contradiction to the equality above.

Example

- Suppose f and g are continuous functions on $I = (a, b)$.
  Suppose T is a distribution satisfying the differential equation
  $$T' + fT = g.$$
  We show that T is a C^1 function which, consequently, is a strong solution of the equation.
  Choose a function $\phi \in C^1(I)$, such that $\phi' = f$.
  The function $u(x) = ce^{-\phi(x)}$, c constant, satisfies $u' + fu = 0$.
  Using the method of variation of parameters to construct a solution of $u' + fu = g$, we now assume that c is a function of x.
  Then the equation $u' + fu = g$ is satisfied if
  $$c'e^{-\phi} - ce^{-\phi}\phi' + ce^{-\phi}f = g$$
  $$c'e^{-\phi} = g$$
  $$c(x) = \int_{x_0}^{x} e^{\phi(t)} g(t) dt, \ x_0 \text{ fixed in } (a, b).$$

Example (Cont'd)

- Since $c \in C^1(I)$, $u(x) = c(x)e^{-\phi(x)}$ is in $C^1(I)$ and $u' + fu = g$.
  Let T be a distribution on (a,b), such that $T' + fT = g$.
  With $\phi' = f$, the distribution $S = e^{\phi}(T - u)$ satisfies

$$\begin{array}{rcl} S' & = & [e^{\phi}(T-u)]' \\ & = & e^{\phi}\phi'(T-u) + e^{\phi}(T'-u') \\ & = & e^{\phi}[(T'+fT)-(u'+fu)] \\ & = & 0. \end{array}$$

  Therefore, S is a constant, say λ.
  Hence, $T = u + \lambda e^{-\phi} \in C^1(I)$ is a strong solution of $T' + fT = g$.

Example

- Consider the differential operator in $\mathbb{R}$ with constant coefficients
$$L = \frac{d^m}{dx^m} + c_1 \frac{d^{m-1}}{dx^{m-1}} + \cdots + c_{m-1} \frac{d}{dx} + c_m.$$
Let $\lambda_1, \ldots, \lambda_m$ be the roots of $P(x) = x^m + c_1 x^{m-1} + \cdots + c_{m-1} x + c_m$.
We show that
$$u = He^{\lambda_1 x} * He^{\lambda_2 x} * \cdots * He^{\lambda_m x}$$
is a solution of the ordinary differential equation $Lu = \delta$.
We have
$$P(x) = (x - \lambda_1)(x - \lambda_2) \cdots (x - \lambda_m).$$
So
$$L = \left(\frac{d}{dx} - \lambda_1 \right) \left(\frac{d}{dx} - \lambda_2 \right) \cdots \left(\frac{d}{dx} - \lambda_m \right).$$

Example (Cont'd)

- It now follows that

$$\begin{aligned} L\delta &= (\tfrac{d}{dx}-\lambda_1)(\tfrac{d}{dx}-\lambda_2)\cdots(\tfrac{d}{dx}-\lambda_m)\delta \\ &= (\tfrac{d}{dx}-\lambda_1)(\tfrac{d}{dx}-\lambda_2)\cdots(\tfrac{d}{dx}-\lambda_m)\delta*\delta*\cdots*\delta \\ &= (\tfrac{d}{dx}-\lambda_1)\delta*(\tfrac{d}{dx}-\lambda_2)\delta*\cdots*\cdots(\tfrac{d}{dx}-\lambda_m)\delta \\ &= (\delta'-\lambda_1\delta)(\delta'-\lambda_2\delta)\cdots(\delta'-\lambda_m\delta). \end{aligned}$$

  Write

$$Lu = Lu*\delta = u*L\delta.$$

  Then u satisfies $Lu=\delta$ if $u*L\delta=\delta$.

  So $Lu=\delta$ if $u=(L\delta)^{-1}$.

Example (Cont'd)

- We saw $Lu = \delta$ if $u = (L\delta)^{-1}$.

  Now we rely on the following two facts, the first of which was established in a previous example.
  - $(\delta' - \lambda\delta)^{-1} = e^{\lambda x} H$;
  - $(v * w)^{-1} = v^{-1} * w^{-1}$.

  Then we can compute

$$\begin{aligned} u &= [(\delta' - \lambda_1\delta) * (\delta' - \lambda_2\delta) * \cdots * (\delta' - \lambda_m\delta)]^{-1} \\ &= (\delta' - \lambda_1\delta)^{-1} * (\delta' - \lambda_2\delta)^{-1} * \cdots * (\delta' - \lambda_m\delta)^{-1} \\ &= He^{\lambda_1 x} * He^{\lambda_2 x} * \cdots * He^{\lambda_m x}. \end{aligned}$$

A Special Case

- When $c_1 = c_2 = \cdots = c_m = 0$, then $\lambda_1 = \lambda_2 = \cdots = \lambda_m = 0$.
  So we retrieve the solution of $\frac{d^m}{dx^m}u = \delta$ as given by
  $$H * H * \cdots * H = \frac{1}{(m-1)!}x_+^{m-1} = \frac{1}{(m-1)!}Hx^{m-1}.$$
  Suppose v is another solution of $Lu = \delta$.
  Then $\frac{d^m}{dx^m}(v-u) = 0$.
  By a previous example, $v-u$ is a polynomial of degree $\leq m-1$.
  Hence,
  $$u = \frac{1}{(m-1)!}x^{m-1} + b_1x^{m-1} + b_2x^{m-2} + \cdots + b_m$$
  is the general solution of $\frac{d^m u}{dx^m} = \delta$, where $b_1, \ldots, b_m$ are arbitrary constants.
  These constants may be evaluated by imposing conditions on u and its derivatives at one or more points in $\mathbb{R}$.

Example of a Second-Order Differential Equation

- Let $E \in \mathscr{D}'(\mathbb{R})$ satisfy the differential equation

$$\frac{d^2}{dx^2} E = \delta,$$

where δ is the Dirac distribution on $\mathbb{R}$.

We found that one solution to the equation is given by $x_+ = xH(x)$.

Any other solution E will satisfy the homogeneous equation

$$\frac{d^2}{dx^2}[E - xH(x)] = 0$$

It must, therefore, have the form

$$E(x) = xH(x) + ax + b,$$

which is a continuous function on $\mathbb{R}$.

The arbitrary constants a and b may be determined by imposing boundary conditions on E.

Example of a Second-Order Differential Equation

- Consider the differential equation $u'' = f$, for given $f \in L^1(0,1)$.
  Its solution in $(0,1)$ may be constructed by using the result of the previous example.
  If f is extended into $\mathbb{R}$ by setting $f = 0$ outside $(0,1)$, then $f \in L^1(\mathbb{R})$.
  We have
  $$(f * E)'' = f * E'' = f * \delta = f.$$
  Thus, one solution of the equation is given by
  $$\begin{array}{rcl} u(x) & = & (f * E)(x) \\ & = & \int_0^1 (x-\xi)H(x-\xi)f(\xi)d\xi \\ & = & \int_0^x (x-\xi)f(\xi)d\xi, \quad 0 \le x \le 1. \end{array}$$
  The general solution is therefore
  $$u(x) = \int_0^x (x-\xi)f(\xi)d\xi + ax + b.$$

Example (Fundamental Solutions)

- The requirement that f be integrable on $(0,1)$ in $u'' = f$ is, of course, not necessary.
- It was only made in order to allow us to express the convolution $f * g$ as an integral.
- We could have assumed that f is a distribution on $(0,1)$ which can be extended to a distribution in $\mathbb{R}$ with compact support in $[0,1]$.
- In fact, equation $\frac{d^2}{dx^2}E = \delta$ is really a special case of equation $u'' = f$, where we chose f to be δ.
- The resulting solution E is called a **fundamental solution** of the differential operator $\frac{d^2}{dx^2}$.
- The function $He^{\lambda_1 x} * \cdots * He^{\lambda_m x}$ shown to satisfy the m-th order equation $(\frac{d^m}{dx^m} + c_1 \frac{d^{m-1}}{dx^{m-1}} + \cdots + c_{m-1}\frac{d}{dx} + c_m)u = \delta$ is a fundamental solution of the operator $\frac{d^m}{dx^m} + c_1 \frac{d^{m-1}}{dx^{m-1}} + \cdots + c_{m-1}\frac{d}{dx} + c_m$.

Fundamental Solutions of a Differential Operator

- Recall the linear partial differential operator of order $m \geq 1$

$$L = \sum_{|\alpha| \leq m} c_\alpha(x) \partial^\alpha,$$

  where $\alpha \in \mathbb{N}_0^n$ and c_α are C^∞ functions on $\mathbb{R}^n$.
- E is a **fundamental solution** of the operator L, if $E \in \mathscr{D}'(\mathbb{R}^n)$ and

$$LE = \delta.$$

- The importance of the fundamental solution lies in the fact that it allows solving the more general equation $Lu = f$.
- If E is a fundamental solution of L and f is a distribution with compact support in $\Omega \subseteq \mathbb{R}^n$, then

$$L(f * E) = f * LE = f * \delta = f.$$

- So $f * E$ is a solution of the differential equation $Lu = f$.

Example

- Consider the operator
$$L = \frac{d^2}{dx^2} + \omega^2,$$
where ω is a (nonzero) constant.
We determine its fundamental solution.
The solution of $Lu = 0$ is a linear combination of $\cos\omega x$ and $\sin\omega x$.
Let $f_1 = a\cos\omega x$ and $f_2 = b\sin\omega x$.
Based on the work of a previous example, we assume that one solution of $LE = \delta$ is given by
$$E(x) = \begin{cases} a\cos\omega x, & x \le 0 \\ b\sin\omega x, & x > 0 \end{cases}.$$
Continuity of E at $x = 0$ gives $f_1(0) = f_2(0)$, i.e., $a = 0$.
For E' to have a unit jump discontinuity at $x = 0$, we must have $f_2'(0) - f_1'(0) = 1$, or $b\omega = 1$. Therefore, $E(x) = \frac{1}{\omega}H(x)\sin\omega x$.

Example (Cont'd)

- The solution of the differential equation $Lu = f$, where $f \in \mathscr{E}'(\mathbb{R})$, is now given by $u = f * E$.

  When $\text{supp} f \subseteq [0,1]$ and f is integrable, we can write

$$\begin{aligned} u(x) &= \frac{1}{\omega}\int_0^1 f(\xi)H(x-\xi)\sin\omega(x-\xi)d\xi \\ &= \frac{1}{\omega}\int_0^x f(\xi)\sin\omega(x-\xi)d\xi, \quad 0 \le x \le 1. \end{aligned}$$

  The general solution of $Lu = f$ is therefore

$$u(x) = \frac{1}{\omega}\int_0^x f(\xi)\sin(x-\xi)d\xi + c_1\cos\omega x + c_2\sin\omega x,$$

  where c_1 and c_2 are arbitrary constants which may be determined by imposing appropriate boundary conditions on u.

Example

- The general linear ordinary differential operator of order 2 with constant coefficients is given by

$$L = c_1 \frac{d^2}{dx^2} + c_2 \frac{d}{dx} + c_3, \quad c_1 \neq 0.$$

From the classical theory are C^∞ functions, we can find two linearly independent solutions of $Lu = 0$, say w_1 and w_2.

Assume that a fundamental solution of L has the form

$$E(x) = \begin{cases} aw_1(x), & x \leq 0 \\ bw_2(x), & x > 0 \end{cases}.$$

We see that in order to satisfy $LE = \delta$, we must have

$$bw_2(0) - aw_1(0) = 0, \quad bw_2'(0) - aw_1'(0) = \frac{1}{c_1}.$$

Example (Cont'd)

- We must have

$$\left\{ \begin{array}{lcrcl} bw_2(0) & - & aw_1(0) & = & 0 \\ bw_2'(0) & - & aw_1'(0) & = & \frac{1}{c_1} \end{array} \right\}$$

Let

$$W(x) = w_1(x)w_2'(x) - w_1'(x)w_2(x).$$

be the Wronskian of the solutions w_1 and w_2 of $Lu = 0$.

Since w_1 and w_2 are independent, $W(x) \neq 0$.

So we have

$$a = \frac{w_2(0)}{c_1 W(0)}, \quad b = \frac{w_1(0)}{c_1 W(0)}.$$

Formal Adjoints

- Let L be the general linear differential operator

$$L = \sum_{|\alpha| \leq m} c_\alpha(x) \partial^\alpha,$$

where $\alpha \in \mathbb{N}_0^n$ and c_α are C^∞ functions on $\mathbb{R}^n$.

- Then we have, for all $T \in \mathscr{D}'(\Omega)$,

$$\begin{array}{rcl} \langle LT, \phi \rangle & = & \langle \sum_{|\alpha| \leq m} c_\alpha(x) \partial^\alpha T, \phi \rangle, \quad \phi \in \mathscr{D}(\Omega) \\ & = & \langle T, \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha (c_\alpha(x)\phi) \rangle \\ & = & \langle T, L^* \phi \rangle. \end{array}$$

- The operator L^*, defined by

$$L^* \phi = \sum_{|\alpha| \leq m} (-1)^{|\alpha|} \partial^\alpha (c_\alpha(x)\phi)$$

is known as the **formal adjoint** of L.

- We always have $(L^*)^* = L$.
- When $L^* = L$, we say that L is **formally self-adjoint**.

Formal Self-Adjointness: Ordinary of Order 2

- Consider the general linear ordinary differential operator of order 2

$$L = c_1(x)\frac{d^2}{dx^2} + c_2(x)\frac{d}{dx} + c_3.$$

- For self-adjointness, we must have, for every test function ϕ,

$$(c_1\phi)'' - (c_2\phi)' + c_3\phi = c_1\phi'' + c_2\phi' + c_3\phi.$$

- This equality is satisfied if and only if $c_2 = c_1'$.
- Thus, we arrive at

$$\begin{array}{rcl} L & = & c_1\frac{d^2}{dx^2} + c_1'\frac{d}{dx} + c_3 \\ & = & \frac{d}{dx}(c_1\frac{d}{dx}) + c_3. \end{array}$$

- Therefore, the general formally self-adjoint linear differential operator of order 2 on $\mathbb{R}$ is given by

$$L = \frac{d}{dx}\left(p\frac{d}{dx}\right) + q, \quad p \neq 0,\ q \text{ are } C^\infty \text{ on } \mathbb{R}.$$

Formal Self-Adjointness: Ordinary of Order 2 (Cont'd)

- Let w_1 and w_2 be two linearly independent solutions of the homogeneous differential equation

$$Lu = (pu')' + qu = 0.$$

Then a fundamental solution of L can still be represented by

$$\begin{aligned} E(x) &= \begin{cases} aw_1(x), & x \leq 0 \\ bw_2(x), & x > 0 \end{cases} \\ &= aw_1(x) + [bw_2(x) - aw_1(x)]H(x). \end{aligned}$$

Formal Self-Adjointness: The Case of Order 2 (Cont'd)

- If we assume in

$$E(x)=\begin{cases} aw_1(x), & x\leq 0\\ bw_2(x), & x>0\end{cases}=aw_1(x)+[bw_2(x)-aw_1(x)]H(x).$$

that $aw_1(0)=bw_2(0)$, we get

$$\begin{array}{rcl} pE' & = & p[aw_1'+(bw_2-aw_1)'H+(bw_2-aw_1)\delta]\\ & = & p[aw_1'+(bw_2'-aw_1')H]. \end{array}$$

With $Lw_1=Lw_2=0$, we have

$$\begin{array}{rcl} LE & = & (pE')'+qE\\ & = & a(pw_1')'+aqw_1+[p(bw_2'-aw_1')H]'+q(bw_2-aw_1)H\\ & = & a(pw_1')'+aqw_1+p'(bw_2'-aw_1')H\\ & & +p(bw_2''-aw_1'')H+p(bw_2'-aw_1')\delta+q(bw_2-aw_1)H\\ & = & a[(pw_1')'+qw_1]-a[(pw_1')'+qw_1]H\\ & & +b[(pw_2')'+qw_2]+p(bw_2'-aw_1')\delta\\ & = & p(0)[bw_2'(0)-aw_1'(0)]\delta. \end{array}$$

Formal Self-Adjointness: The Case of Order 2 (Cont'd)

- We found
$$LE = p(0)[bw_2'(0) - aw_1'(0)]\delta.$$
Thus, if E is to be a fundamental solution, a and b must also satisfy
$$p(0)[bw_2'(0) - aw_1'(0)] = 1.$$
I.e., a and b must be the solutions of
$$\left\{ \begin{array}{rcrcl} aw_1(0) & - & bw_2(0) & = & 0 \\ -aw_1'(0) & + & bw_2'(0) & = & \frac{1}{p(0)} \end{array} \right\}.$$
Consequently,
$$a = \frac{w_2(0)}{p(0)W(0)}, \quad b = \frac{w_1(0)}{p(0)W(0)}.$$

Formal Self-Adjointness: The Case of Order 2 (Conclusion)

- Since $a = \frac{w_2(0)}{p(0)W(0)}, b = \frac{w_1(0)}{p(0)W(0)}$,

$$E(x) = \frac{1}{p(0)W(0)}\left\{w_2(0)w_1(x) + [w_1(0)w_2(x) - w_2(0)w_1(x)]H(x)\right\}.$$

The general solution in $(0,1)$ of the differential equation $(pu')' + qu = f$ with f integrable on $(0,1)$, is therefore

$$\begin{aligned} u(x) &= f * E(x) + c_1 w_1(x) + c_2 w_2(x) \\ &= \tfrac{1}{p(0)W(0)}[w_1(0)\textstyle\int_0^x f(\xi)w_2(x-\xi)d\xi \\ &\qquad + w_2(0)\textstyle\int_x^1 f(\xi)w_1(x-\xi)d\xi] \\ &\quad + c_1 w_1(x) + c_2 w_2(x), \quad 0 \le x \le 1. \end{aligned}$$

c_1 and c_2 may be determined from the boundary conditions on u.

The Case of PDEs

- The method that we have used for constructing a solution to the differential equation $Lu = f$ by taking the convolution of f with a fundamental solution of L works equally well when L is a partial differential operator.
- Recall the following previously obtained results

$$\begin{aligned} \Delta \log|x| &= (\partial_1^2 + \partial_2^2) \log|x| = 2\pi\delta, \\ \Delta \frac{1}{|x|} &= (\partial_1^2 + \partial_2^2 + \partial_3^2) \left(\frac{1}{\sqrt{x_1^2 + x_2^2 + x_3^2}} \right) = -4\pi\delta. \end{aligned}$$

- We may conclude, concerning the operator Δ, that:
  - A fundamental solution of Δ in $\mathbb{R}^2$ is $\frac{1}{2\pi} \log|x|$;
  - A fundamental solution of Δ in $\mathbb{R}^3$ is $-\frac{1}{4\pi|x|}$.

Example: The Poisson Equation

- Consider in $\mathbb{R}^3$ the partial differential equation $\Delta u = f$, known as the **nonhomogenous Laplace**, or **Poisson**, **equation**.
- It has a solution which is given by

$$u = f * \left(-\frac{1}{4\pi|x|}\right),$$

  when this convolution is well defined.
- The solution may be interpreted as the potential generated by f.
- When f is an integrable function with compact support, u is represented by the function

$$u(x) = -\frac{1}{4\pi}\int_{\mathbb{R}^3}\frac{f(\xi)}{|x-\xi|}d\xi.$$

- Clearly $u(x) \to 0$ as $|x| \to \infty$, i.e., away from the mass distribution.
- If u is to satisfy other boundary conditions, the solution has to be supplemented by a solution of $\Delta u = 0$.

Example: The Heat Equation

- The temperature distribution u on a slender, infinite conducting bar as a function of time t and position x may be described by

$$\begin{array}{rcl} \partial_t u & = & \partial_x^2 u, \quad (x,t) \in (-\infty,\infty) \times (0,\infty), \\ u(x,0) & = & g(x), \quad x \in (-\infty,\infty). \end{array}$$

  - The equation governs the heat flow along the bar for all $t > 0$;
  - g describes the initial temperature distribution at $t = 0$.
- The "fundamental solution" that we need to construct $u = f * E$, would have to satisfy

$$\begin{array}{rcl} (\partial_t - \partial_x^2)E(x,t) & = & 0, \quad (x,t) \in (-\infty,\infty) \times (0,\infty); \\ E(x,0) & = & \delta_{(x,0)}, \quad t = 0, \ -\infty < x < \infty. \end{array}$$

- Such an E is given by $E(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}}$.
  - Note that the first equation above is satisfied.
  - To satisfy the second, it suffices to show that $E(x,t)$ is a delta-convergent sequence as $t \to 0^+$.

Example: The Heat Equation (Cont'd)

- Recall hat we have shown $f_\lambda = \frac{1}{\sqrt{\pi\lambda}} e^{-x^2/\lambda} \xrightarrow{\lambda \to 0} \delta$.

  Setting $\lambda = 4t$, we get $E(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{x^2}{4t}} \xrightarrow{t \to 0^+} \delta$.
- In its dependence on x, $E(x,t)$ is a C^∞ function which decays exponentially as $|x| \to \infty$, for every $t > 0$.
- So the convolution

$$u(x,t) = (g * E)(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} g(\xi) e^{-\frac{(x-\xi)^2}{4t}} d\xi$$

  is well defined for a wide class of functions, including all locally integrable functions $g(x)$ whose growth as $|x| \to \infty$ is no faster than some power of x.
- It represents the temperature distribution u in $(-\infty,\infty) \times (0,\infty)$.
- It is clear that $u \to 0$ as $t \to \infty$ or as $|x| \to \infty$.

Example: The Wave Equation

- The motion of an infinite vibrating string is described by the **wave equation**
$$\partial_t^2 u = \partial_x^2 u, \quad -\infty < x < \infty,\ 0 < t < \infty.$$
- If the string is released with initial shape u_0 and initial velocity u_1 then we have the initial conditions
$$\begin{array}{rcl} u & = & u_0, \quad t=0,\ -\infty < x < \infty; \\ \partial_t u & = & u_1, \quad t=0,\ -\infty < x < \infty. \end{array}$$
- Let
$$\begin{array}{rcl} E_0 & = & \frac{1}{2}[H(x+t)-H(x-t)], \\ E_1 & = & \partial_t E_0 = \frac{1}{2}[\delta(x+t)+\delta(x-t)]. \end{array}$$
- Then the following differential equations hold in the upper half-plane $t>0,\ -\infty < x < \infty$,
$$\begin{array}{rcl} (\partial_t^2 - \partial_x^2)E_0 & = & 0; \\ (\partial_t^2 - \partial_x^2)E_1 & = & 0. \end{array}$$

Example: The Wave Equation

- When $t = 0$, we have

$$\begin{aligned} E_0 &= 0; \\ E_1 &= \delta; \\ \partial_t E_1 &= 0. \end{aligned}$$

Consequently, a solution of the boundary value problem is given by

$$u = u_0 * E_1 + u_1 * E_0,$$

where the convolution is taken with respect to x.

Example: The Wave Equation (Cont'd)

- We found $u = u_0 * E_1 + u_1 * E_0$, where

$$\begin{array}{rcl} E_0 & = & \frac{1}{2}[H(x+t) - H(x-t)], \\ E_1 & = & \partial_t E_0 = \frac{1}{2}[\delta(x+t) + \delta(x-t)]. \end{array}$$

- When $w_0 \in C^2(\mathbb{R})$ and $u_1 \in C^1(\mathbb{R}) \cap L^1(\mathbb{R})$, we can write this in the form

$$\begin{array}{rcl} u(x,t) & = & \frac{1}{2}[u_0(x-t) + u_0(x+t)] \\ & & + \frac{1}{2}\int_{-\infty}^{\infty} u_1(\xi)[H(x+t-\xi) - H(x-t-\xi)]d\xi \\ & = & \frac{1}{2}[u_0(x-t) + u_0(x+t)] + \frac{1}{2}\int_{x-t}^{x+t} u_1(\xi)d\xi. \end{array}$$

- It is straightforward to verify that this expression satisfies the wave equation and that $u(x,t) \to u_0(x)$ and $\partial_t u(x,t) \to u_t(x)$ as $t \to 0^+$.
- If the string is released from rest then $u_1 = 0$.

  In that case, the solution is the average of the two traveling waves $u_0(x-t)$ and $u_0(x+t)$, both having the same shape u_0 but traveling in opposite directions with velocities ± 1.

Classification of the Applications

- Second order partial differential equations with constant coefficients are classified as
  - **elliptic**;
  - **parabolic**;
  - **hyperbolic**.
- Typical examples of these are:
  - The Poisson equation for elliptic ones;
  - The heat equation for parabolic ones;
  - The wave equation for hyperbolic ones.
- In fact, in its homogeneous form, any second order partial differential equation with constant coefficients may be transformed, by an appropriate change of coordinates, to one of the following forms (Δ the Laplacian in $\mathbb{R}^n$):
  - $\Delta u = 0$; (**Laplace's Equation**)
  - $(\partial_t - \Delta)u = 0$; (**Heat Equation**)
  - $(\partial_t^2 - \Delta)u = 0$. (**Wave Equation**)

Classification: The Names

- Consider the three equations:
  - $\Delta u = 0$; (**Laplace's Equation**)
  - $(\partial_t - \Delta)u = 0$; (**Heat Equation**)
  - $(\partial_t^2 - \Delta)u = 0$. (**Wave Equation**)
- Replace ∂_t by τ and ∂_k by ξ_k in the above operators.
- The Laplacian Δ becomes a polynomial in $\xi_1, \ldots, \xi_n$ whose level surfaces are spherical or, up to a change of scale, elliptical.
- From this it follows that:
  - The Laplace operator corresponds to an elliptic surface $|\xi|^2 = 0$;
  - The heat operator corresponds to the parabolic surface $\tau - |\xi|^2 = 0$;
  - The wave operator to the hyperbolic surface $\tau^2 - |\xi|^2 = 0$.