Introduction to the Theory of Distributions

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1. Fourier Transforms and Tempered Distributions
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Subsection 1

The Classical Fourier Transformation in L^2

The Fourier Transformation in $L^1(\mathbb{R}^n)$

- Fix Ω to be $\mathbb{R}^n$ and write L^p, $\mathscr{D}$, $\mathscr{D}'$, etc. for $L^p(\mathbb{R}^n)$, $\mathscr{D}(\mathbb{R}^n)$, $\mathscr{D}'(\mathbb{R}^n)$, etc.
- For $x = \langle x_1, \ldots, x_n \rangle, \xi = \langle \xi_1, \ldots, \xi_n \rangle \in \mathbb{R}^n$, let

$$\langle x, \xi \rangle = \sum_{j=1}^{n} x_i \xi_i.$$

- The **Fourier transform** of a function $f \in L^1$ is a function $\mathscr{F}(f) = \widehat{f}$ on $\mathbb{R}^n$ defined by

$$\widehat{f}(\xi) = \int e^{-i\langle x, \xi \rangle} f(x) dx, \quad \xi \in \mathbb{R}^n.$$

- The **Fourier transformation** is the mapping

$$\mathscr{F} : f \mapsto \widehat{f}$$

defined, so far, on L^1.

Properties of the Fourier Transform

Lemma

If $f \in L^1$, then, for all $\xi \in \mathbb{R}^n$, $|\widehat{f}(\xi)| \leq \|f\|_1$.

- By definition,

$$\widehat{f}(\xi) = \int e^{-i\langle x,\xi\rangle} f(x)dx.$$

So we have

$$\begin{aligned} |\widehat{f}(\xi)| &= |\textstyle\int e^{-i\langle x,\xi\rangle} f(x)dx| \\ &\leq \textstyle\int |e^{-i\langle x,\xi\rangle}||f(x)|dx \\ &= \textstyle\int |f(x)|dx \\ &= \|f\|_1. \end{aligned}$$

The Riemann-Lebesgue Lemma

Lemma (Riemann-Lebesgue Lemma)

If $f \in L^1$ is an integrable function, then $|\widehat{f}(\xi)| \to 0$ as $|\xi| \to \infty$.

- We prove the lemma for $n = 1$.
  Assume, first, that $f \in C_0^0(\mathbb{R})$.
  Starting from the definition and substituting $y = x - \frac{\pi}{\xi}$, we get

$$\begin{aligned} \widehat{f}(\xi) &= \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx = \int_{-\infty}^{\infty} f(y + \tfrac{\pi}{\xi}) e^{-iy\xi} e^{-i\pi} dy \\ &= -\int_{-\infty}^{\infty} f(y + \tfrac{\pi}{\xi}) e^{-iy\xi} dy. \end{aligned}$$

  So $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-ix\xi} dx = -\int_{-\infty}^{\infty} f(x + \frac{\pi}{\xi}) e^{-ix\xi} dx$.
  Taking means, we get $|\widehat{f}(\xi)| \le \frac{1}{2} \int_{-\infty}^{\infty} |f(x) - f(x + \frac{\pi}{\xi})| dx$.
  By continuity, $|f(x) - f(x + \frac{\pi}{\xi})| \xrightarrow{|\xi| \to \infty} 0$.
  By the Lebesgue Dominated Convergence Theorem, $|\widehat{f}(\xi)| \xrightarrow{|\xi| \to \infty} 0$.

The Riemann-Lebesgue Lemma (Cont'd)

- Now suppose that $f \in L^1$.

  The key result is that C_0^0 is dense in L^1.

  So, given $\varepsilon > 0$, there exists $g \in C_0^0$, such that $\|f - g\|_1 < \varepsilon$.

  Thus, using the preceding slide, we get

$$\begin{aligned} |\widehat{f}(\xi)| &= |\int_{-\infty}^{\infty} f(x)e^{-ix\xi}dx| \\ &= |\int_{-\infty}^{\infty}(f(x)-g(x)+g(x))e^{-ix\xi}dx| \\ &\leq |\int_{-\infty}^{\infty}(f(x)-g(x))e^{-ix\xi}dx| + |\int_{-\infty}^{\infty}g(x)e^{-ix\xi}dx| \\ &\leq \varepsilon + |\int_{-\infty}^{\infty}g(x)e^{-ix\xi}dx| \\ &\xrightarrow{|\xi|\to\infty} \varepsilon + 0. \end{aligned}$$

  Since $\varepsilon > 0$ was arbitrary, $|\widehat{f}(\xi)| \to 0$ as $|\xi| \to \infty$.

Properties of the Fourier Transform

Proposition

Let $f \in L^1$ and $\xi_k \to \xi$ in $\mathbb{R}^n$.

(a) $\widehat{f}(\xi_k) \to \widehat{f}(\xi)$;
(b) $\widehat{f}$ is bounded an continuous on $\mathbb{R}^n$;
(c) $\widehat{f}(\xi) \to 0$ as $|\xi| \to \infty$.

(a) Suppose ξ_k is a sequence in $\mathbb{R}^n$ which converges to ξ.

$$|\widehat{f}(\xi_k) - \widehat{f}(\xi)| \leq \int |f(x)||e^{-i\langle x,\xi_k\rangle} - e^{-i\langle x,\xi\rangle}|dx.$$

Moreover, $|e^{-i\langle x,\xi_k\rangle} - e^{-i\langle x,\xi\rangle}| \xrightarrow{\xi_k \to \xi} 0$.

By Lebesgue's Dominated Convergence Theorem, $\widehat{f}(\xi_k) \to \widehat{f}(\xi)$.

(b) By a previous lemma and Part (a).

(c) By the preceding lemma.

Remark on Integrability

- In general $\widehat{f}$ may not be integrable.

  Example: Consider the function

$$f(x) = \begin{cases} 1, & \text{if } x \in (-1,1) \\ 0, & \text{otherwise} \end{cases}.$$

We have over $\mathbb{R}$

$$\begin{aligned} \widehat{f}(\xi) &= \int_{-\infty}^{\infty} e^{-ix\xi} f(x) dx \\ &= \int_{-1}^{1} e^{-ix\xi} dx \\ &= -\frac{1}{i\xi} e^{-ix\xi} \Big|_{-1}^{1} \\ &= -\frac{1}{i\xi}(e^{-i\xi} - e^{i\xi}) \\ &= \frac{2\sin\xi}{\xi}. \end{aligned}$$

This function is not in L^1.

The Inverse Fourier Transform

- When $\widehat{f} \in L^1$,

$$f(x) = \frac{1}{(2\pi)^n} \int e^{i\langle \xi, x \rangle} \widehat{f}(\xi) d\xi$$

  almost everywhere.
- The right-hand side is continuous.
- If we assume that f, besides being integrable, is also continuous, then the equality holds everywhere.

The Fourier Transform as a Map From L^1 To C_∞^0

- Suppose $f, g \in L^1$.
- We have the linearity property

$$\mathscr{F}(af + bg) = a\widehat{f} + b\widehat{g}, \quad a, b \in \mathbb{C}.$$

- Let C_∞^0 be the Banach space of continuous functions on $\mathbb{R}^n$ which tend to 0 at ∞, equipped with the norm

$$\|f\| = |f|_0 = \sup\{|f(x)| : x \in \mathbb{R}^n\}.$$

- The Fourier transformation $\mathscr{F}$ satisfies the inequality

$$\|\mathscr{F}(f)\| = |\widehat{f}|_0 \le \|f\|_1.$$

- It is therefore an injective, continuous linear map from L^1 to C_∞^0.

Preparing for an Extension to Distributions

- Suppose $f, g \in L^1$.
- Then $\widehat{g}$ is bounded.
- So $f\widehat{g} \in L^1$.
- By Fubini's Theorem,

$$\begin{aligned} \int f(x)\widehat{g}(x)dx &= \int f(x)\int g(\xi)e^{-i\langle x,\xi\rangle}d\xi dx \\ &= \int g(\xi)\int f(x)e^{-i\langle x,\xi\rangle}dx d\xi. \end{aligned}$$

Therefore,

$$\int f(x)\widehat{g}(x)dx = \int g(\xi)\widehat{f}(\xi)d\xi.$$

Idea of the Extension of $\mathscr{F}$

- We would like to extend the definition of the Fourier transformation from L^1 to $\mathscr{D}'$.
- Viewing f as a distribution and g as a test function, we may consider applying the formula
$$\langle \widehat{f}, g\rangle = \langle f, \widehat{g}\rangle.$$
- Here we run into some problems.
  - Suppose g in $\mathscr{D}$. Then $\widehat{g}$ is analytic. So it cannot have compact support unless it is identically zero.
    This indicates that $\mathscr{D}$ is too small as a space of test functions.
    Equivalently, $\mathscr{D}'$ is too large for the purpose of extension.
  - Suppose g is taken in $\mathscr{E}$. Then it may not be integrable. As a consequence, its Fourier transform may not exist.
    So it would seem that $\mathscr{E}$ is too big as a space of test functions.

Thus, a new space of test functions larger than $\mathscr{D}$ and smaller than $\mathscr{E}$ seems to be suitable for an extension of the Fourier transformation.

Constraints on a Space of Test Functions

- An appropriate test function space, call it X, should meet certain conditions in order to serve our purpose.
  (i) X should be a subspace of C^∞ in order that the distributions in X' have derivatives of all orders;
  (ii) The Fourier transformation should be "well behaved" on X, in the sense that it maps X onto itself;
  (iii) Since $\partial_k \mathscr{F}(\phi) = -i\mathscr{F}(x_k\phi)$, X should be closed under multiplication by polynomials.
- With these conditions, we should also choose X as small as possible, in order that X' be as large as possible.

Subsection 2

Tempered Distributions

Rapidly Decreasing Functions

- A function $\phi \in C^\infty$ is said to be **rapidly decreasing** if

$$\sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \phi(x)| < \infty,$$

  for all pairs of multi-indices α and β.
- This is equivalent to the condition that

$$\lim_{|x| \to \infty} |x^\alpha \partial^\beta \phi(x)| = 0.$$

- It is also equivalent to the condition that

$$\sup_{|\beta| \le m} \sup_{x \in \mathbb{R}^n} (1 + |x|^2)^m |\partial^\beta \phi(x)| < \infty, \text{ for all } m \in \mathbb{N}_0.$$

The Space of Rapidly Decreasing Functions

- We use $\mathscr{S}$ to denote the set of all rapidly decreasing functions.
- $\mathscr{S}$ is a linear space under the usual operations of addition and multiplication by scalars.
- A function in $\mathscr{S}$ approaches 0 as $|x| \to \infty$ faster than any power of $\frac{1}{|x|}$.

  Example: An example of a function in $\mathscr{S}$ is $e^{-|x|}$.

The Topology on $\mathscr{S}$

- For any $\phi \in \mathscr{S}$, we define the seminorms

$$p_{\alpha\beta}(\phi) = \sup_{x \in \mathbb{R}^n} |x^\alpha \partial^\beta \phi(x)|,$$

  with $\alpha, \beta \in \mathbb{N}_0^n$.
- The countable family $\{p_{\alpha\beta}\}$ defines a Hausdorff, locally convex, topology on $\mathscr{S}$ which is metrizable and complete.
- With this topology, $\mathscr{S}$ is, therefore, a Fréchet space.
- A sequence (ϕ_k) converges to 0 in $\mathscr{S}$ if and only if $x^\alpha \partial^\beta \phi_k(x) \to 0$ uniformly on $\mathbb{R}^n$ as $k \to \infty$.
- If ϕ is in $\mathscr{S}$, then $x^\alpha \partial^\beta \phi$ is in $\mathscr{S}$, for any pair $\alpha, \beta \in \mathbb{N}_0^n$.

Inclusion Relations Between $\mathscr{D},\mathscr{S},\mathscr{E}$

Theorem

The topological vector spaces $\mathscr{D}$, $\mathscr{S}$ and $\mathscr{E}$ are related by $\mathscr{D}\subseteq\mathscr{S}\subseteq\mathscr{E}$, with continuous injection. Moreover, $\mathscr{D}$ is a dense subspace of $\mathscr{S}$ and $\mathscr{S}$ is a dense subspace of $\mathscr{E}$.

- The inclusion relations clearly hold between $\mathscr{D},\mathscr{S}$ and $\mathscr{E}$ as sets.

  Let (ϕ_k) be a sequence in $\mathscr{D}$ which converges to 0. Then, there is a compact set $K\subseteq\mathbb{R}^n$, such that (ϕ_k) lies in $\mathscr{D}_K$ and converges to 0 in $\mathscr{D}_K$. Hence, $\phi_k\to 0$ in $\mathscr{S}$.

  Let (ϕ_k) be a sequence in $\mathscr{S}$ which converges to 0. Then, for any $\alpha\in\mathbb{N}_0^n$, $\partial^\alpha\phi_k\to 0$ uniformly on every compact subset of $\mathbb{R}^n$. This means that (ϕ_k) converges to 0 in $\mathscr{E}$.

  The first part of the theorem is now proved.

  The second part follows from the simple observation that $\mathscr{D}$ is dense in $\mathscr{E}$ as has already been shown.

Density of $\mathscr{S}$ in L^p

Theorem

$\mathscr{S}$ is a dense subspace of L^p, $1 \leq p < \infty$, with the identity map from $\mathscr{S}$ into L^P continuous.

- Let $\phi \in \mathscr{S}$. Then $(1+|x|^2)^m \phi$ is in $\mathscr{S}$, for any $m > 0$. So $\phi \in L^p$.
  Let $\phi_k \to 0$ in $\mathscr{S}$. Then

$$\sup_{x \in \mathbb{R}^n} (1+|x|^2)^m |\phi_k(x)|^p \to 0,$$

  for every m as $k \to \infty$. When $m > \frac{1}{2}n$, $(1+|x|^2)^{-m}$ is integrable.
  We then have

$$\begin{array}{rcl} \|\phi_k\|_p^p & = & \int (1+|x|^2)^m |\phi_k(x)|^p (1+|x|^2)^{-m} dx \\ & \leq & M \sup_{x \in \mathbb{R}^n} (1+|x|^2)^m |\phi_k(x)|^p. \end{array}$$

  Therefore, $\phi_k \to 0$ in L^p.
  Since $\mathscr{D}$ is dense in L^p, by a previous result, so is $\mathscr{S}$.

Convolution of Functions in $\mathscr{S}$

- The convolution $\phi * \psi$ of any pair of functions ϕ, ψ in $\mathscr{S}$ is well defined in $\mathbb{R}^n$ and is in fact an $\mathscr{S}$ function.

  To see this, note that the integral

$$(\phi * \psi)(x) = \int \phi(x-y)\psi(y)dy$$

  is uniformly convergent in $\mathbb{R}^n$. Therefore, we can write

$$\begin{aligned} \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta (\phi * \psi)(x)| &\leq \int \sup_{x\in\mathbb{R}^n} |x^\alpha \partial^\beta \phi(x-y)||\psi(y)|dy \\ &\leq M \int |\psi(y)|dy < \infty. \end{aligned}$$

Tempered Distributions

- A previous theorem implies that the relation $\mathscr{E}' \subseteq \mathscr{S}' \subseteq \mathscr{D}'$ must hold between the topological dual spaces with the identity maps from $\mathscr{E}'$ to $\mathscr{S}'$ and from $\mathscr{S}'$ to $\mathscr{D}'$ continuous.
- Further, every locally integrable function f on $\mathbb{R}^n$ defines a distribution in $\mathscr{D}'$ by $\phi \mapsto \int f\phi$, $\phi \in \mathscr{D}$.
- For f to define a distribution in $\mathscr{S}'$ by $\phi \mapsto \int f\phi$, $\phi \in \mathscr{S}$, it must, additionally, satisfy a growth condition at ∞.
- f cannot grow faster than some power of x as $|x| \to \infty$, since, otherwise, the integral $\int f\phi$ will not be defined.

  Example: The exponential function $e^{|x|}$ does not define a distribution in $\mathscr{S}'$.
- Loosely speaking, we can say that the elements of $\mathscr{S}'$ are the distributions of polynomial growth as $|x| \to \infty$.
- Hence they are called **tempered distributions**.

Tempered Distributions Defined by Polynomials

(i) Any polynomial function f on $\mathbb{R}^n$ defines a tempered distribution by the formula

$$\langle f, \phi \rangle = \int f(x)\phi(x)x, \quad \phi \in \mathscr{S}.$$

Indeed, let:

- k be the degree of the polynomial f;
- $m > \frac{1}{2}(n+k)$.

Then we have

$$\begin{aligned} |\langle f, \phi \rangle| &\leq \int |f(x)\phi(x)| dx \\ &\leq M \sup_{x \in \mathbb{R}^n} (1+|x|^2)^m |\phi(x)|, \end{aligned}$$

with $M = \|f(x)(1+|x|^2)^{-m}\|_1$.

Multiplication of Distributions

(ii) The same definitions and properties of **convergence**, **differentiation**, **translation** and **reflection in the origin** which were given in $\mathscr{D}'$ apply to the elements of $\mathscr{S}'$.

- Since $\mathscr{S}$ is closed under multiplication by polynomials, we can define the **product of a polynomial** P on $\mathbb{R}^n$ **with a tempered distribution** by
$$PT(\phi) = T(P\phi), \quad \phi \in \mathscr{S}.$$
- This definition clearly extends to any C^∞ function f with polynomial growth at ∞, i.e., an $f \in C^\infty$ for which there is a positive integer m, such that $|x|^{-m}|\partial^\alpha f(x)|$ remains bounded as $|x| \to \infty$, for all $\alpha \in \mathbb{N}_0^n$.
- Thus, the linear space of multipliers of $\mathscr{D}'$, which is C^∞, is also "tempered" by a growth condition before it can serve as a linear space of multipliers of $\mathscr{S}'$.

$\mathscr{S}$ is a Subspace of L^p, $1 \leq p \leq \infty$

(iii) Suppose $1 \leq p < \infty$ and $\phi \in \mathscr{S}$.

Then for any positive integer m,

$$|\phi(x)| = (1+|x|^2)^{-m}(1+|x|^2)^m|\phi(x)| \leq M(1+|x|^2)^{-m},$$

where $M = \sup\{(1+|x|^2)^m|\phi(x)| : x \in \mathbb{R}^n\}$.

Now $|\phi|^p$ is integrable if $m > \frac{1}{2}\frac{n}{p}$.

Hence, $\mathscr{S} \subseteq L^p$.

Moreover, any $\phi \in \mathscr{S}$ is bounded on $\mathbb{R}^n$.

So we also have $\mathscr{S} \subseteq L^\infty$.

Thus, $\mathscr{S}$ is a subspace of L^p, for $1 \leq p \leq \infty$.

Extension from L^p to $\mathscr{S}'$

(iv) We prove that $L^p \subseteq \mathscr{S}'$, for $1 \le p \le \infty$.

Suppose $f \in L^p$ and ϕ is any C^∞ function with compact support K.

$$\begin{array}{rcl} |\langle f, \phi \rangle| & = & |\int_K f(x)\phi(x)dx| \\ & = & |\int \phi(x) I_K(x) f(x) dx| \\ & \stackrel{\text{Hölder}}{\le} & M|\phi|_0 \|f\|_p. \end{array}$$

Thus, f defines a continuous linear functional on C_0^∞ in the topology induced by $\mathscr{S}$.

But C_0^∞ is dense in $\mathscr{S}$.

So f can be extended to a continuous linear functional of $\mathscr{S}$.

- More generally, any locally integrable function f, such that $|x|^{-m}|f(x)|$ is bounded (almost everywhere) as $|x| \to \infty$, for some positive integer m, defines a distribution in $\mathscr{S}'$.

Non-Necessity of the Boundedness Condition

- Consider the function $f(x) = e^x \sin(e^x)$, $x \in \mathbb{R}$.

  Note that for no positive integer m, does $x^{-m}|f(x)| = x^{-m}e^x|\sin(e^x)|$ remain bounded as $x \to \infty$.

  Hence, $f(x)$ cannot be dominated at ∞ by a polynomial.

  However, if $\phi \in \mathscr{S}(\mathbb{R})$, then

$$\begin{array}{rcl} |\int f(x)\phi(x)dx| & = & |\int e^x \sin(e^x)\phi(x)dx| \\ & = & |\int \phi(x)d(-\cos(e^x))| \\ & = & |\int \cos(e^x)\phi'(x)dx| \\ & \leq & \int |\phi'(x)|dx \\ & = & \int (1+x^2)|\phi'(x)|\frac{1}{1+x^2}dx \\ & \leq & M\sup(1+x^2)|\phi'(x)|. \end{array}$$

  Thus f defines a distribution in $\mathscr{S}'(\mathbb{R})$.

Tempered Distributions as Derivatives

(v) The inclusion $\mathscr{S}' \subseteq \mathscr{D}'_F$.

- Clearly, $\mathscr{D}_F \subseteq \mathscr{S}$;
- Moreover, convergence in $\mathscr{D}_F$ implies convergence in $\mathscr{S}$.

Thus, every tempered distribution is of finite order.

By a previous theorem, we conclude that every tempered distribution is a derivative of some continuous function of polynomial growth.

Examples:

- Consider again the tempered distribution $e^x \sin(e^x)$.
  It is the first derivative of the bounded function $-\cos(e^x)$.
- The powers x_+^λ, x_-^λ and $|x|^\lambda$ are examples of tempered distributions.
  Each is dominated at $\pm\infty$ by $|x|^m$, if $m \geq \mathrm{Re}\lambda$.

Subsection 3

Fourier Transform in $\mathscr{S}$

Differentiation of Fourier Transforms

- Since $\mathscr{S} \subseteq L^1$, the Fourier transform $\widehat{\phi}$ of any $\phi \in \mathscr{S}$ exists.

  Moreover,

$$\begin{aligned} \partial_k \widehat{\phi}(\xi) &= \partial_k \int e^{-i\langle x,\xi\rangle} \phi(x) dx \\ &= \int \frac{\partial}{\partial \xi_k} e^{-i\langle x,\xi\rangle} \phi(x) dx \\ &= -i \int e^{-i\langle x,\xi\rangle} x_k \phi(x) dx \\ &= -i \mathscr{F}(x_k \phi). \end{aligned}$$

  The second equality, where differentiation is carried inside the integral, is justified by the uniform convergence of the integral as a function of ξ.

Fourier Transforms of Derivatives

- We also have

$$\begin{aligned} \mathscr{F}(\partial_k\phi)(\xi) &= \int e^{-i\langle x,\xi\rangle}\partial_k\phi(x)dx \\ &= i\xi_k\int e^{-i\langle x,\xi\rangle}\phi(x)dx \\ &\quad \text{(integration by-parts)} \\ &= i\xi_k\widehat{\phi}(\xi). \end{aligned}$$

- Using the notation $D_k = -i\partial_k$, we have the relations

$$\mathscr{F}(D_k\phi) = \xi_k\mathscr{F}(\phi), \quad \mathscr{F}(x_k\phi) = -D_k\mathscr{F}(\phi).$$

- This process may be repeated any number of times, and with respect to any index, giving, for all $\alpha = (\alpha_1,\ldots,\alpha_n)$ and $D^\alpha = (-i)^{|\alpha|}\partial^\alpha$,

$$\begin{aligned} \mathscr{F}(D^\alpha\phi) &= \xi^\alpha\mathscr{F}(\phi), \\ \mathscr{F}(x^\alpha\phi) &= (-1)^{|\alpha|}D^\alpha\mathscr{F}(\phi). \end{aligned}$$

The Fourier Transformation on $\mathscr{S}$

Theorem

The Fourier transformation is a continuous linear map from $\mathscr{S}$ into $\mathscr{S}$.

- For any $\phi \in \mathscr{S}$ and $\alpha, \beta \in \mathbb{N}_0^n$, we have the relations

$$\mathscr{F}(D^\alpha \phi) = \xi^\alpha \mathscr{F}(\phi), \quad \mathscr{F}(x^\alpha \phi) = (-1)^{|\alpha|} D^\alpha \mathscr{F}(\phi), \quad D^\alpha = (-i)^{|\alpha|} \partial^\alpha.$$

These imply

$$\begin{aligned}
\xi^\alpha D^\beta \widehat{\phi}(\xi) &= \xi^\alpha (-1)^{|\beta|} \mathscr{F}(x^\beta \phi) = \mathscr{F}(D^\alpha (-x)^\beta \phi) \\
&= \int e^{-i\langle x,\xi\rangle} D^\alpha [(-x)^\beta \phi(x)] dx; \\
|\xi^\alpha D^\beta \widehat{\phi}(\xi)| &\leq \int |D^\alpha [x^\beta \phi(x)]| dx \\
&= \int (1+|x|^2)^{-m} (1+|x|^2)^m |D^\alpha [x^\beta \phi(x)]| dx.
\end{aligned}$$

We can choose m, so that $\int (1+|x|^2)^{-m} dx = M < \infty$.
Then $|\xi^\alpha D^\beta \widehat{\phi}(\xi)| \leq \sup_{x \in \mathbb{R}^n} (1+|x|^2)^m |D^\alpha [x^\beta \phi(x)]| M$.
But ϕ is in $\mathscr{S}$. So the right side is finite. Hence, $\widehat{\phi}$ is in $\mathscr{S}$.
Now $\mathscr{F}$ is linear and $\widehat{\phi} \to 0$ as $\phi \to 0$ in $\mathscr{S}$. So $\mathscr{F}$ is continuous on $\mathscr{S}$.

Example: A Special Fourier Transform

Proposition

We have

$$\mathscr{F}\left(e^{-\frac{1}{2}|x|^2}\right) = (2\pi)^{n/2} e^{-\frac{1}{2}|\xi|^2}.$$

- Let $\gamma(x) = e^{-\frac{1}{2}|x|^2}$, $x \in \mathbb{R}^n$.
  For $n = 1$, γ satisfies the differential equation $\gamma'(x) + x\gamma(x) = 0$, $x \in \mathbb{R}$.
  Taking the Fourier transform on the left, using

  $$\mathscr{F}(D^\alpha \phi) = \xi^\alpha \mathscr{F}(\phi), \quad \mathscr{F}(x^\alpha \phi) = (-1)^{|\alpha|} D^\alpha \mathscr{F}(\phi), \quad D^\alpha = (-i)^{|\alpha|}\partial^\alpha,$$

  we obtain

  $$\begin{array}{rcl} \mathscr{F}(\gamma'(x) + x\gamma(x)) & = & \mathscr{F}(\gamma'(x)) + \mathscr{F}(x\gamma(x)) \\ & = & \xi \mathscr{F}(\gamma(x)) + (-D\mathscr{F}(\gamma(x))) \\ & = & \xi \widehat{\gamma}(\xi) + \widehat{\gamma}'(\xi). \end{array}$$

  Thus, $\xi\widehat{\gamma}(\xi) + (\widehat{\gamma})'(\xi) = 0$, $\xi \in \mathbb{R}$.

Example (Cont'd)

- We found $\xi\widehat{\gamma}(\xi)+(\widehat{\gamma})'(\xi)=0$, $\xi\in\mathbb{R}$.

  Its solution is given by $\widehat{\gamma}(\xi)=ce^{-\frac{1}{2}\xi^2}$.

  The initial condition gives $c=\widehat{\gamma}(0)=\int_{-\infty}^{\infty}e^{-\frac{1}{2}x^2}dx=(2\pi)^{1/2}$.

  Therefore, $\widehat{\gamma}(\xi)=(2\pi)^{1/2}e^{-\frac{1}{2}\xi^2}$.
- Suppose, next, that $n\geq 1$.

  Then we can write

$$\begin{array}{rcl}\widehat{\gamma}(\xi) & = & \int_{\mathbb{R}^n}\prod_{k=1}^n e^{-ix_k\xi_k}e^{-\frac{1}{2}x_k^2}dx\\ & = & \prod_{k=1}^n\int_{-\infty}^{\infty}e^{-ix_k\xi_k-\frac{1}{2}x_k^2}dx_k\\ & = & \prod_{k=1}^n\widehat{\gamma}(\xi_k)\\ & = & (2\pi)^{\frac{1}{2}n}e^{-\frac{1}{2}|\xi|^2}.\end{array}$$

The Fourier Inversion Formula in $\mathscr{S}$

Theorem

If $\phi \in \mathscr{S}$, then

$$\phi(x) = \mathscr{F}^{-1}(\widehat{\phi})(x) = (2\pi)^{-n} \int e^{i\langle x,\xi\rangle} \widehat{\phi}(\xi) d\xi.$$

- For any $\phi, \psi \in \mathscr{S}$, we have, using Fubini's Theorem,

$$\begin{aligned} \int \widehat{\phi}(x)\psi(x)e^{i\langle \xi,x\rangle} dx &= \int [\int e^{-i\langle y,x\rangle}\phi(y)dy]\psi(x)e^{i\langle \xi,x\rangle} dx \\ &= \int \phi(y)[\int e^{-i\langle y-\xi,x\rangle}\psi(x)dx]dy \\ &= \int \phi(y)\widehat{\psi}(y-\xi)dy \\ &= \int \phi(\xi+y)\widehat{\psi}(y)dy. \end{aligned}$$

Furthermore, when $\psi \in \mathscr{S}$ and $\varepsilon > 0$,

$$\mathscr{F}(\psi(\varepsilon x))(y) = \int e^{-i\langle y,x\rangle}\psi(\varepsilon x)dx = \int e^{-i\left\langle y,\frac{\xi}{\varepsilon}\right\rangle}\psi(\xi)\frac{1}{\varepsilon^n}d\xi = \frac{1}{\varepsilon^n}\widehat{\psi}\left(\frac{y}{\varepsilon}\right).$$

The Fourier Inversion Formula in $\mathscr{S}$ (Cont'd)

- Using this, we get

$$\begin{aligned}\int \widehat{\phi}(x)\psi(\varepsilon x)e^{i\langle \xi,x\rangle}dx &= \int \phi(\xi+y)\mathscr{F}(\psi(\varepsilon x))(y)dy\\ &= \int \phi(\xi+y)\widehat{\psi}(\tfrac{y}{\varepsilon})\tfrac{1}{\varepsilon^n}dy\\ &= \int \phi(\xi+y)\widehat{\psi}(\tfrac{y}{\varepsilon})d(\tfrac{y}{\varepsilon})\\ &= \int \phi(\xi+\varepsilon y)\widehat{\psi}(y)dy.\end{aligned}$$

  Since these integrals are uniformly convergent, we can take the limit as $\varepsilon \to 0$ inside the integral sign.

  The result is $\psi(0)\int \widehat{\phi}(x)e^{i\langle \xi,x\rangle}dx = \phi(\xi)\int \widehat{\psi}(y)dy$.

  If we choose $\psi(x) = e^{-\frac{1}{2}|x|^2}$, then:

  - $\psi(0) = 1$.
  - $\int \widehat{\psi}(y)dy = (2\pi)^{\frac{1}{2}n}\int e^{-\frac{1}{2}|y|^2}dy = (2\pi)^n$.

  So we get $\int \widehat{\phi}(x)e^{i\langle \xi,x\rangle}dx = \phi(\xi)(2\pi)^n$.

A Topological Isomorphism

- We showed that $\mathscr{F}$ is a continuous linear map from $\mathscr{S}$ into $\mathscr{S}$.
- We also showed that an inversion formula exists.
- Thus, the Fourier transformation defines a **topological isomorphism** from $\mathscr{S}$ onto $\mathscr{S}$.
- This means that it is a bijection from $\mathscr{S}$ to $\mathscr{S}$ which, in addition, preserves:
  - The algebraic properties of the linear space $\mathscr{S}$ (linearity);
  - The topological properties of $\mathscr{S}$ (homeomorphism).

Properties of Fourier Transforms

- Recall that $\phi\psi$ and $\phi * \psi$ are both in $\mathscr{S}$ when $\phi, \psi \in \mathscr{S}$.

Theorem

If $\phi, \psi \in \mathscr{S}$, then:

(a) $\int \widehat{\phi}\psi = \int \phi\widehat{\psi}$;

(b) $\int \phi\overline{\psi} = (2\pi)^{-n} \int \widehat{\phi}\overline{\widehat{\psi}}$; **(Parseval's Relation)**

(c) $\mathscr{F}(\phi * \psi) = \widehat{\phi}\widehat{\psi}$;

(d) $\mathscr{F}(\phi\psi) = (2\pi)^{-n}\widehat{\phi} * \widehat{\psi}$.

(a) We get the conclusion from the following upon setting $\xi = 0$.

$$
\begin{aligned}
\int \widehat{\phi}(x)\psi(x)e^{i\langle \xi, x\rangle} dx &= \int \int \phi(y)e^{-i\langle y, x\rangle} dy \psi(x) e^{i\langle \xi, x\rangle} dx \\
&= \int \phi(y) \int \psi(x) e^{-i\langle y-\xi, x\rangle} dx dy \\
&= \int \phi(y)\widehat{\psi}(y-\xi) dy \\
&= \int \phi(y+\xi)\widehat{\psi}(y) dy.
\end{aligned}
$$

Properties of Fourier Transforms (b)

(b) We have

$$\begin{array}{rcl} \widehat{\overline{\widehat{\psi}}}(\xi) & = & \int e^{-i\langle \xi,x\rangle}\overline{\widehat{\psi}}(x)dx \\ & = & \overline{\int e^{i\langle \xi,x\rangle}\widehat{\psi}(x)dx} \\ & = & \overline{(2\pi)^n\psi(x)} \\ & = & (2\pi)^n\overline{\psi}(x). \end{array}$$

Now in Part (a), replace ψ by $(2\pi)^{-n}\overline{\widehat{\psi}}$ to get

$$\begin{array}{rcl} (2\pi)^{-n}\int\widehat{\phi}\overline{\widehat{\psi}} & = & (2\pi)^{-n}\int\phi\widehat{\overline{\widehat{\psi}}} \\ & = & (2\pi)^{-n}\int\phi(2\pi)^n\overline{\psi} \\ & = & \int\phi\overline{\psi}. \end{array}$$

Properties of Fourier Transforms (c)

(c) Using Fubini's Theorem, we get

$$\begin{aligned}
\mathscr{F}(\phi * \psi)(\xi) &= \int e^{-i\langle \xi, x\rangle}(\phi * \psi)(x)dx \\
&= \int e^{-i\langle \xi, x\rangle}[\int \phi(y)\psi(x-y)dy]dx \\
&= \int \phi(y)[\int e^{-i\langle \xi, x\rangle}\psi(x-y)dx]dy \\
&= \int \phi(y)[\int e^{-i\langle \xi, y+\eta\rangle}\psi(\eta)d\eta]dy \\
&= \int e^{-i\langle \xi, y\rangle}\phi(y)dy \int e^{-i\langle \xi, \eta\rangle}\psi(\eta)d\eta \\
&= \widehat{\phi}(\xi)\widehat{\psi}(\xi).
\end{aligned}$$

Properties of Fourier Transforms (d)

(d) The inversion formula gives

$$\begin{array}{rcl} \phi(x) & = & (2\pi)^{-n}\int e^{i\langle x,\xi\rangle}\widehat{\phi}(\xi)d\xi = (2\pi)^{-n}\widehat{\widehat{\phi}}(-x); \\ \widehat{\widehat{\phi}}(x) & = & (2\pi)^n\phi(-x). \end{array}$$

Using Part (c), we now get

$$\begin{array}{rcl} (\widehat{\phi}*\widehat{\psi})(\xi) & = & \mathscr{F}^{-1}(\widehat{\widehat{\phi}}\widehat{\widehat{\psi}})(\xi) \\ & = & (2\pi)^{-n}\int e^{i\langle\xi,x\rangle}\widehat{\widehat{\phi}}(x)\widehat{\widehat{\psi}}(x)dx \\ & = & (2\pi)^{n}\int e^{i\langle\xi,x\rangle}\phi(-x)\psi(-x)dx \\ & = & (2\pi)^{n}\int e^{-i\langle\xi,x\rangle}\phi(x)\psi(x)dx \\ & = & (2\pi)^n\mathscr{F}(\phi\psi)(\xi). \end{array}$$

Example

- The equation $\mathscr{F}(\phi * \psi) = \widehat{\phi}\widehat{\psi}$ can be used to construct two nonzero functions $\phi, \psi \in \mathscr{S}$, such that $\phi * \psi = 0$.

  Let $\phi_0, \psi_0 \neq 0$ be in $\mathscr{D}$, such that $\text{supp}\phi_0 \cap \text{supp}\psi_0 = \emptyset$.

  Define $\phi = \mathscr{F}^{-1}(\phi_0)$ and $\psi = \mathscr{F}^{-1}(\psi_0)$.

  Since $\phi_0, \psi_0 \in \mathscr{S}$ and $\mathscr{F}$ is bijective, ϕ and ψ are in $\mathscr{S}$.

  We now have

  $$\mathscr{F}(\phi * \psi) = \mathscr{F}(\phi)\mathscr{F}(\psi) = \phi_0\psi_0 = 0.$$

  This implies that $\phi * \psi = 0$.

- On the other hand, suppose $\phi \in \mathscr{S}$ and $\phi * \phi = 0$.

  Then $0 = \mathscr{F}(\phi * \phi) = [\mathscr{F}(\phi)]^2$.

  So $\mathscr{F}(\phi) = 0$.

  Therefore, $\phi = 0$.

Subsection 4

Fourier Transform in $\mathscr{S}'$

Fourier Transform of a Distribution

Definition

For any $T \in \mathscr{S}'$, the Fourier transform $\mathscr{F}(T) = \widehat{T}$ is defined by

$$\widehat{T}(\phi) = T(\widehat{\phi}), \quad \phi \in \mathscr{S}.$$

- Note that:
  - $\widehat{\phi} \in \mathscr{S}$, for every $\phi \in \mathscr{S}$;
  - The Fourier transformation is continuous on $\mathscr{S}$.

  It now follows that $\widehat{T} \in \mathscr{S}'$, for every $T \in \mathscr{S}'$.

Fourier Transform of a Distribution

- $\mathscr{S}$ can be considered a subspace of $\mathscr{S}'$.
  The function $\psi \in \mathscr{S}$ corresponds to $T_\psi \in \mathscr{S}'$.
  In this case
  $$\widehat{T}_\psi(\phi) = T_\psi(\widehat{\phi}) \stackrel{\int \widehat{\phi}\psi = \int \phi\widehat{\psi}}{=} T_{\widehat{\psi}}(\phi).$$
  Hence, $\widehat{T}_\psi = T_{\widehat{\psi}}$.
- $\mathscr{F} : \mathscr{S}' \to \mathscr{S}'$ is continuous in the (weak) topology of $\mathscr{S}'$.
  This follows from the continuity of $\mathscr{F} : \mathscr{S} \to \mathscr{S}$.
  Suppose $T_k \to T$ in $\mathscr{S}'$ and $\phi \in \mathscr{S}$.
  Then
  $$\widehat{T}_k(\phi) = T_k(\widehat{\phi}) \to T(\widehat{\phi}) = \widehat{T}(\phi).$$
  Thus, if $T_k \to T$ in $\mathscr{S}'$, then $\widehat{T}_k \to \widehat{T}$ in $\mathscr{S}'$.
  This means that $\mathscr{F} : \mathscr{S}' \to \mathscr{S}'$ is continuous in the topology of $\mathscr{S}'$.

The Case of an L^1 Function

- Suppose f is an L^1 function. Then $\widehat{f}$ is a C_∞^0 function.
  Therefore, $T_{\widehat{f}} \in \mathscr{S}'$.
  Hence, for any $\phi \in \mathscr{S}$,

$$\begin{aligned} T_{\widehat{f}}(\phi) &= \int \widehat{f}(\xi)\phi(\xi)d\xi \\ &= \int \left[\int e^{-i\langle \xi,x\rangle} f(x)dx\right]\phi(\xi)d\xi \\ &= \int f(x)\left[\int e^{-i\langle x,\xi\rangle}\phi(\xi)d\xi\right]dx \\ &= \int f(x)\widehat{\phi}(x)dx \\ &= T_f(\widehat{\phi}). \end{aligned}$$

  So, for all $\phi \in \mathscr{S}$, $\widehat{T}_f(\phi) = T_f(\widehat{\phi}) = T_{\widehat{f}}(\phi)$ (i.e., the Fourier transform of T, as a distribution, coincides with its transform as an L^1 function).

The Fourier Transformation in $\mathscr{S}'$

Theorem

The Fourier transformation $\mathscr{F}$ from $\mathscr{S}'$ to $\mathscr{S}'$ with the inversion formula

$$\hat{\hat{T}} = (2\pi)^n \check{T}, \quad T \in \mathscr{S}',$$

is a topological isomorphism.

- We define the inverse Fourier transform of $T \in \mathscr{S}'$ by

$$\mathscr{F}^{-1}(T)(\phi) = T(\mathscr{F}^{-1}(\phi)), \quad \phi \in \mathscr{S}.$$

Then $\mathscr{F}^{-1}$ is also a continuous map from $\mathscr{S}'$ into $\mathscr{S}'$.
Moreover, $\mathscr{F}^{-1}(\hat{T})(\hat{\phi}) = \hat{T}(\mathscr{F}^{-1}(\hat{\phi})) = \hat{T}(\phi) = T(\hat{\phi})$.
Using equation $\hat{\hat{\phi}}(x) = (2\pi)^n \phi(-x)$, we get, for all $\phi \in \mathscr{S}$,

$$\hat{\hat{T}}(\phi) = T(\hat{\hat{\phi}}) = (2\pi)^n T(\check{\phi}) = (2\pi)^n \check{T}(\phi).$$

Hence, $\hat{\hat{T}} = (2\pi)^n \check{T}$, $T \in \mathscr{S}'$.

Properties of the Fourier Transform in $\mathscr{S}'$

- The definition of the Fourier transform of a tempered distribution by duality carries the properties of the Fourier transformation in $\mathscr{S}$ into $\mathscr{S}'$.
- Recall the equations

$$\mathscr{F}(D^\alpha \phi) = \xi^\alpha \mathscr{F}(\phi), \quad \mathscr{F}(x^\alpha \phi) = (-1)^{|\alpha|} D^\alpha \mathscr{F}(\phi), \quad D^\alpha = (-i)^{|\alpha|} \partial^\alpha.$$

- Recall, also, that, for every $T \in \mathscr{S}'$, multiplication of T by any polynomial P has been defined by

$$PT(\phi) = T(P\phi), \quad \phi \in \mathscr{S}.$$

- Hence, we have, for every $T \in \mathscr{S}'$,

$$\begin{aligned} \mathscr{F}(D^\alpha T) &= \xi^\alpha \mathscr{F}(T); \\ \mathscr{F}(x^\alpha T) &= (-1)^{|\alpha|} D^\alpha \mathscr{F}(T). \end{aligned}$$

Example

- For any $\phi \in \mathscr{S}$, we have
$$\langle \widehat{\delta}, \phi \rangle = \langle \delta, \widehat{\phi} \rangle = \widehat{\phi}(0) = \langle 1, \phi \rangle.$$
  Hence, $\widehat{\delta} = 1$.
  We know that $\widehat{\widehat{\delta}} = (2\pi)^n \check{\delta} = (2\pi)^n \delta$.
  So $\widehat{1} = \widehat{\widehat{\delta}} = (2\pi)^n \delta$.
- Now let $\alpha \in \mathbb{N}^n$.
  We know $\mathscr{F}(D^\alpha T) = \xi^\alpha \mathscr{F}(T)$, $\mathscr{F}(x^\alpha T) = (-1)^{|\alpha|} D^\alpha \mathscr{F}(T)$.
  Hence the results above may be generalized to
$$\begin{aligned} \mathscr{F}(D^\alpha \delta) &= \xi^\alpha, \\ \mathscr{F}(x^\alpha) &= (-1)^{|\alpha|} (2\pi)^n D^\alpha \delta. \end{aligned}$$

Even and Odd Distributions

- A distribution $T \in \mathscr{D}'$ is said to be:
  - **even** if $\check{T} = T$, in the sense that $T(\check{\phi}) = T(\phi)$, for every $\phi \in \mathscr{D}$;
  - **odd** if $\check{T} = -T$, in the sense that $T(\check{\phi}) = -T(\phi)$, for every $\phi \in \mathscr{D}$.
- When T is an even distribution in $\mathscr{S}'$, for any $\phi \in \mathscr{S}$,

$$\widehat{T}(\check{\phi}) = \widehat{T(\check{\phi})} = \overline{T(\widehat{\phi})} \stackrel{T \text{ even}}{=} T(\widehat{\phi}) = \widehat{T}(\phi).$$

  Therefore $\widehat{T}$ is even.

  Conversely, if $\widehat{T}$ is even, we can also show that T is even.
- Similarly, T is odd if and only if $\widehat{T}$ is odd.
- Taking into account $\widehat{\widehat{T}} = (2\pi)^n \check{T}, \quad T \in \mathscr{S}'$, we also get

$$\mathscr{F}(T) = \begin{cases} (2\pi)^n \mathscr{F}^{-1}(T), & \text{if } T \text{ is even} \\ -(2\pi)^n \mathscr{F}^{-1}(T), & \text{if } T \text{ is odd} \end{cases}.$$

Example

- Let $T = \text{pv}(\frac{1}{x})$, $x \in \mathbb{R}$.
  Claim: T is odd.
  If $\phi \in \mathscr{D}(\mathbb{R})$, then

$$\begin{aligned} \langle T, \check{\phi} \rangle &= \lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{1}{x} \phi(-x) dx \\ &= -\lim_{\varepsilon \to 0} \int_{|x| \geq \varepsilon} \frac{1}{x} \phi(x) dx \\ &= -\langle T, \phi \rangle. \end{aligned}$$

Example (Cont'd)

Claim: For $T = \text{pv}(\frac{1}{x})$, $\widehat{T} = -2\pi i H + \pi i$.

We have

$$\begin{aligned}\langle xT, \phi\rangle &= \langle T, x\phi\rangle = \lim_{\varepsilon\to 0}\int_{|x|\geq\varepsilon}\phi(x)dx \\ &= \int\phi(x)dx = \langle 1, \phi\rangle.\end{aligned}$$

We conclude that $xT = 1$. Therefore, $\mathscr{F}(xT) = \widehat{1} = 2\pi\delta$.

But $\mathscr{F}(xT) = -D\widehat{T} = i\frac{d\widehat{T}}{d\xi}$. Hence, $\frac{d\widehat{T}}{d\xi} = -2\pi i\delta$.

This implies that $\widehat{T} = -2\pi i H + c$, for some constant c.

But $\widehat{T}$ is odd. So this constant satisfies $-2\pi i + c = -c$. Thus,

$$\widehat{T} = -2\pi i H + \pi i.$$

Example (Cont'd)

- We saw that for $T = \text{pv}(\frac{1}{x})$, $\widehat{T} = -2\pi i H + \pi i$.
  Claim: We have $\mathscr{F}^{-1}(H) = \frac{1}{2}\delta - \frac{1}{2\pi i}\text{pv}\frac{1}{x}$.
  The expressions for $\widehat{H}$ and $\mathscr{F}^{-1}(H)$ can now be derived.

$$\begin{aligned} -2\pi i\widehat{H} + 2\pi^2 i\delta &= -2\pi i\widehat{H} + \pi i\widehat{1} = \widehat{\widehat{T}} \\ &= 2\pi\check{T} \quad (\text{since } \widehat{\widehat{T}} = (2\pi)^n\check{T}) \\ &= -2\pi T. \quad (\text{since } T \text{ is odd}) \end{aligned}$$

Hence, $\widehat{H} = \pi\delta - i\text{pv}\frac{1}{x}$. On the other hand,

$$\begin{aligned} T &= \mathscr{F}^{-1}(\widehat{T}) \\ &= -2\pi i\mathscr{F}^{-1}(H) + \pi i\mathscr{F}^{-1}(1) \\ &= -2\pi i\mathscr{F}^{-1}(H) + \pi i\delta. \end{aligned}$$

Therefore,

$$\mathscr{F}^{-1}(H) = \frac{1}{2}\delta - \frac{1}{2\pi i}\text{pv}\frac{1}{x}.$$

Subsection 5

Fourier Transform in L^2

L^2 Norm and Inner Product

- Let Ω be an open subset of $\mathbb{R}^n$.
- $L^2(\Omega)$ is the Banach space of (**Lebesgue**) **square integrable complex functions** on Ω under the norm
$$\|f\|_2 = \left[\int_\Omega |f(x)|^2 dx\right]^{1/2}.$$
- The Schwarz inequality gives, for all $f, g \in L^2(\Omega)$,
$$\left|\int_\Omega f(x)\overline{g(x)}dx\right| \le \|f\|_2 \|g\|_2.$$
- Consequently, the complex number
$$(f, g) = \int_\Omega f(x)\overline{g}(x)dx$$
is always finite.
- It is called the **inner product** of f, g in L^2.

Some Properties and Remarks

- We have
$$(f,f)=\int_\Omega |f(x)|^2dx=\|f\|_2^2.$$
- We use L^2 to denote $L^2(\mathbb{R}^n)$.
- L_2 is not a subspace of L_1.
  So the definition $\widehat{f}(\xi)=\int e^{-i\langle x,\xi\rangle}f(x)dx$ cannot be applied to all L^2 functions.
- Suppose, on the other hand, that $f\in L^1\cap L^2$.
  Then $\widehat{f}$ is also in L^2.
  So Parseval's relation gives
$$\|f\|_2=(2\pi)^{-n/2}\|\widehat{f}\|_2.$$

Plancherel's Theorem

- Parseval's relation $\int \phi\overline{\psi} = (2\pi)^{-n}\int \widehat{\phi}\overline{\widehat{\psi}}$, which was proved in $\mathscr{S}$, will now be shown to hold in L^2 as a subspace of $\mathscr{S}$.

Theorem (Plancherel)

If $f \in L^2$, then $\widehat{f} \in L^2$ and

$$\|\widehat{f}\|_2 = (2\pi)^{n/2}\|f\|_2.$$

- When we set $\psi = \phi$ in Parseval's relation, we obtain

$$\|\phi\|_2 = (2\pi)^{-n/2}\|\widehat{\phi}\|_2, \quad \phi \in \mathscr{S}.$$

C_0^∞ is dense in L^2. Also, $C_0^\infty \subseteq \mathscr{S} \subseteq L^2$. Thus, $\mathscr{S}$ is also dense in L^2. Moreover, convergence in $\mathscr{S}$ implies convergence in L^2. So the preceding equation may be extended to L^2.

Parseval's Relation in L^2

- Recall that, for all $f, g \in L^2$,
  - (**Parallelogram Law**) $\|f+g\|_2^2 = (f+g, f+g) = \|f\|_2^2 + 2\mathrm{Re}(f,g) + \|g\|_2^2$;
  - (**Plancheret's Theorem**) $\|\widehat{f}\|_2 = (2\pi)^{n/2}\|f\|_2$.

Corollary (Parseval's Relation)

For all $f, g \in L^2$,

$$(\widehat{f}, \widehat{g}) = (2\pi)^n (f, g).$$

- We have (for real f, g)

$$\begin{array}{rcl} 2(\widehat{f}, \widehat{g}) & = & \|\widehat{f}+\widehat{g}\|_2^2 - \|\widehat{f}\|_2^2 - \|\widehat{g}\|_2^2 \\ & = & (2\pi)^n\|f+g\|_2^2 - (2\pi)^n\|f\|_2^2 - (2\pi)^n\|g\|_2^2 \\ & = & (2\pi)^n(\|f+g\|_2^2 - \|f\|_2^2 - \|g\|_2^2) \\ & = & (2\pi)^n 2(f,g). \end{array}$$

Then we may reason by real and imaginary parts.

Example

- Suppose $f \in \mathscr{S}'$ satisfies the following differential equation in $\mathbb{R}^n$, where $c > 0$,
$$(-\Delta + c)f = g.$$
If $g \in L^2$, then we can show that $f \in L^2$.
More generally, $D_k^m f \in L^2$, for all $0 \le m \le 2$, $1 \le k \le n$.
We have
$$\mathscr{F}[(-\Delta + c)f] = \mathscr{F}[(D_1^2 + \cdots + D_n^2 + c)f] = (\xi_1^2 + \cdots + \xi_n^2 + c)\widehat{f}.$$
By hypothesis, $(-\Delta + c)f \in L^2$. So $(|\xi|^2 + c)\widehat{f} \in L^2$. Hence
$$(|\xi|^2 + 1)\widehat{f} = \frac{|\xi|^2 + 1}{|\xi|^2 + c}(|\xi|^2 + c)\widehat{f} \in L^2.$$
With $\xi = (\xi_1, \ldots, \xi_n) \in \mathbb{R}^n$, $|\xi_k|^m \le |\xi|^2 + 1$, $0 \le m \le 2$, $1 \le k \le n$.
This implies that $\mathscr{F}(D_k^m f) = \xi_k^m \widehat{f} \in L^2$. Hence, $D_k^m f \in L^2$.

Subsection 6

Fourier Transform in $\mathscr{E}'$

Analytic and Entire Functions

- Let f be defined on an open connected set Ω in $\mathbb{C}^n$.
- f is **analytic** in Ω if, for all $k \in \{1,\ldots,n\}$, with $z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_n$ all fixed, the function

$$f_k(z_k) = f(z_1,\ldots,z_{k-1},z_k,z_{k+1},\ldots,z_n)$$

of the single variable z_k is analytic on

$$\{z_k \in \mathbb{C} : z = (z_1,\ldots,z_{k-1},z_k,z_{k+1},\ldots,z_n) \in \Omega\}.$$

- When f is analytic in $\mathbb{C}^n$, it is called **entire**.

Analytic Functions and Power Series

- As in the single variable theory, if f is analytic in Ω, it has a power series expansion about every point $c \in \Omega$,

$$f(z) = \sum_{\alpha} a_\alpha (z-c)^\alpha,$$

valid for every point z in the open ball

$$B(c,r) = \left\{ z \in \Omega : |z-c| = \left[\sum_{k=1}^{n} |z_k - c_k|^2 \right]^{1/2} < r \right\},$$

for some positive number r.
- The summation index α runs through $\mathbb{N}_0^n$.
- The a_α are the Taylor coefficients

$$a_\alpha = \frac{1}{\alpha!} \partial_z^\alpha f(c).$$

The Cauchy-Riemann Equations

- Let f be defined on an open connected set Ω in $\mathbb{C}^n$.
- When $z_k = x_k + iy_k$, we shall use the notation

$$\begin{aligned} \partial_{z_k} &= \tfrac{1}{2}(\partial_{x_k} - i\partial_{y_k}); \\ \overline{\partial}_{z_k} &= \partial_{\overline{z}_k} = \tfrac{1}{2}(\partial_{x_k} + i\partial_{y_k}), \quad k = 1, \ldots, n, \end{aligned}$$

- The Cauchy-Riemann equations take the form

$$\overline{\partial}_{z_k} f = \frac{1}{2}\left[\frac{\partial f}{\partial x_k} + i\frac{\partial f}{\partial y_k}\right] = 0, \quad k = 1, \ldots, n.$$

- When Ω is an open subset of $\mathbb{R}^n$, we shall say that f is (real) analytic in Ω if it has a power series expansion about every point $c \in \Omega$, with z replaced by $x \in B(c, r) \subseteq \mathbb{R}^n$.

  This is so if and only if the function f can be extended to an open neighborhood of Ω in $\mathbb{C}^n$, where f is (complex) analytic.

Analyticity of the Fourier Transform in $\mathscr{E}'$

Theorem

The Fourier transform of $T \in \mathscr{E}'$ is an analytic function in $\mathbb{R}^n$ given by

$$\widehat{T}(\xi) = T_x(e^{-i\langle x,\xi\rangle}).$$

Furthermore, the right-hand side may be extended as an analytic function to $\mathbb{C}^n$, known as the **Fourier-Laplace transform** of T.

- As a function of ξ, $T_x(e^{-i\langle x,\xi\rangle})$ is in C^∞.
  Thus, it remains to show that the claimed equation holds in $\mathscr{S}'$.
  By definition, for any $\phi \in \mathscr{D}$, we have $\widehat{T}(\phi) = T(\widehat{\phi})$.
  If we consider ϕ as an element in $\mathscr{E}'$, then, by applying a previous theorem to distributions with compact support:

$$\begin{aligned}\langle \widehat{T}(\xi), \phi\rangle &= \langle T_x, \widehat{\phi}\rangle = T_x(\textstyle\int e^{-i\langle \xi,x\rangle}\phi(\xi)d\xi) \\ &= \textstyle\int T_x(e^{-i\langle \xi,x\rangle})\phi(\xi)d\xi = \langle T_x(e^{-i\langle \xi,x\rangle}), \phi\rangle.\end{aligned}$$

Analyticity of the Fourier Transform in $\mathscr{E}'$ (Cont'd)

- We got, by working with $\phi \in \mathscr{D}$,

$$\widehat{T}(\xi) = T_x(e^{-i\langle x,\xi\rangle}).$$

But $\mathscr{D}$ is dense in $\mathscr{S}$. So the equation holds in $\mathscr{S}'$.

By replacing ξ by $\zeta = \xi + i\eta$, $\widehat{T}$ may be extended into $\mathbb{C}^n$.

There, it is also a C^∞ function of ζ.

$\partial_{\zeta_k}\widehat{T}$ and $\overline{\partial}_{\zeta_k}\widehat{T}$ may be computed by differentiating $e^{-i\langle x,\zeta\rangle}$.

The exponential function is entire.

Therefore, the same holds for $\widehat{T}(\zeta)$.

Hence, $\widehat{T}$ is analytic in $\mathbb{R}^n$.

Example

- Let T be a distribution in $\mathbb{R}$, such that $T^{(m)} = \delta$, for some $m > 0$.

  Applying the Fourier transformation and taking into account $\mathscr{F}(D^\alpha T) = \xi^\alpha \mathscr{F}(T)$, we get

  $$(i\xi)^m \widehat{T} = 1.$$

  This gives

  $$\widehat{T} = \frac{1}{(i\xi)^m}.$$

  Now $\widehat{T}$ is singular at $\xi = 0$.

  By the preceding theorem, T cannot have compact support.

  In other words, any fundamental solution of the operator $\frac{d^m}{dx^m}$ in $\mathbb{R}$ cannot have compact support.

Example

- Suppose T is a distribution with compact support such that $\langle T_x, x^\alpha\rangle = 0$, for every $\alpha \in \mathbb{N}_0^n$.
  We prove that $T = 0$, and thereby conclude that the set of all polynomials in $\mathbb{R}^n$ with constant coefficients is dense in C^∞.

(i) By hypothesis, $T \in \mathscr{E}'$.
By the theorem, $\widehat{T} \in \mathscr{E}'$ can be extended as an analytic function $f(\zeta)$ in $\mathbb{C}^n$, such that $f(\zeta) = T_x(e^{-i\langle x,\zeta\rangle})$. For any $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha f(\zeta) = T_x(\partial_\zeta^\alpha e^{-i\langle x,\zeta\rangle}) = (-i)^{|\alpha|} T_x(x^\alpha e^{-i\langle x,\zeta\rangle}).$$

At $\zeta = 0$, for all $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha f(0) = (-i)^{|\alpha|} T_x(x^\alpha) = 0.$$

But f is an entire function in $\mathbb{C}^n$. So it is represented by the power series $f(\zeta) = \sum_\alpha \frac{1}{\alpha!}\partial^\alpha f(0)\zeta^\alpha = 0$, for any $\zeta \in \mathbb{C}^n$.
Thus f, and therefore $\widehat{T}$ vanishes identically.
Since the Fourier transformation is injective in $\mathscr{S}'$, $T = 0$.

Example (Cont'd)

(ii) Let $\mathscr{P}$ be the set of all polynomials in $\mathbb{R}^n$ with constant coefficients.

Assume that $\overline{\mathscr{P}}$ is a proper subset of C^∞.

By the Hahn-Banach theorem, there exists a nonzero continuous linear functional T on C^∞, such that

$$\langle T, P \rangle = 0, \quad \text{for every } P \in \overline{\mathscr{P}}.$$

This implies, in particular, that T is a nonzero distribution with compact support which satisfies $\langle T, x^\alpha \rangle = 0$, for every $\alpha \in \mathbb{N}^n$.

However, this contradicts Part (i).

Convolution of $\mathscr{S}'$ by $\mathscr{E}'$

Theorem

If $T_1 \in \mathscr{S}'$ and $T_2 \in \mathscr{E}'$, then $T_1 * T_2 \in \mathscr{S}'$ and

$$\mathscr{F}(T_1 * T_2) = \mathscr{F}(T_2)\mathscr{F}(T_1),$$

the right-hand side being a well-defined distribution because $\mathscr{F}(T_2)$ is C^∞.

- Let $\phi \in \mathscr{D}$. By properties of convolution and preceding results:
  - $(T_1 * T_2)(\phi) = (T_1 * T_2 * \check{\phi})(0)$;
  - $(T_2 * \check{\phi})(x) = T_2(\tau_x \phi)$ is a C_0^∞ function.

  Moreover, we have

$$\begin{aligned}
(T_1 * T_2 * \check{\phi})(0) &= \langle T_{1_y}, (T_2 * \check{\phi})(-y)\rangle \\
&= \langle T_{1_y}, T_2(\tau_{-y}\phi)\rangle \\
&= \langle T_{1_y}, \check{T}_2(\tau_y \check{\phi})\rangle \\
&= \langle T_{1_y}, (\check{T}_2 * \phi)(y)\rangle.
\end{aligned}$$

  Therefore $(T_1 * T_2)(\phi) = T_1(\check{T}_2 * \phi)$.

Convolution of $\mathscr{S}'$ by $\mathscr{E}'$ (Cont'd)

- We found $(T_1 * T_2)(\phi) = T_1(\check{T}_2 * \phi)$.

  Let ϕ be in $\mathscr{S}$.

  Then
  $$(\check{T}_2 * \phi)(x) = \check{T}_2(\tau_x \check{\phi}) = T_2(\tau_{-x}\phi).$$

  So $\check{T}_2 * \phi$ is also in $\mathscr{S}$.

  This holds, since, if T_2 is of order m, then
  $$\sup_{\substack{x \in \mathbb{R}^n \\ |\alpha+\beta| \leq k}} |x^\alpha \partial^\beta (\check{T}_2 * \phi)(x)| \leq M_k \sup_{\substack{x \in \mathbb{R}^n \\ |\alpha+\beta| \leq k+m}} |x^\alpha \partial^\beta \phi(x)|.$$

  Thus, $T_1 * T_2$ is a continuous linear functional on $\mathscr{S}$.

Convolution of $\mathscr{S}'$ by $\mathscr{E}'$ (Cont'd)

- We now compute its Fourier transform.

  Let $\phi \in \mathscr{S}$ so that $\widehat{\phi}$ is also in $\mathscr{S}$.

  By a previous equation

$$\begin{array}{rcl} (T_1 * T_2)(\widehat{\phi}) & = & T_1(\check{T}_2 * \widehat{\phi}); \\ (\check{T}_2 * \widehat{\phi})(x) & = & T_2(\tau_{-x}\widehat{\phi}) = T_{2_y}(\tau_{-x}\widehat{\phi}(y)) = T_{2_y}(\widehat{\phi}(x+y)). \end{array}$$

  If $\phi \in \mathscr{D}$, then we can write

$$\begin{array}{rcl} T_{2_y}(\widehat{\phi}(x+y)) & = & T_{2_y}(\int e^{-i\langle x+y,\xi\rangle}\phi(\xi)d\xi) \\ & = & \int T_{2_y}(e^{-i\langle y,\xi\rangle})\phi(\xi)e^{-i\langle x,\xi\rangle}d\xi \\ & = & \int \widehat{T}_2(\xi)\phi(\xi)e^{-i\langle x,\xi\rangle}d\xi. \\ & & (\widehat{T}(\xi) = T_x(e^{-i\langle x,\xi\rangle})) \end{array}$$

Convolution of $\mathscr{S}'$ by $\mathscr{E}'$ (Conclusion)

- Similarly, for $\phi \in \mathscr{D}$,

$$\begin{array}{rcl}(T_1 * T_2)(\widehat{\phi}) & = & T_{1_x}(T_{2_y}(\widehat{\phi}(x+y))) \\ & = & T_{1_x}(\int \widehat{T}_2(\xi)\phi(\xi)e^{-i\langle x,\xi\rangle}d\xi) \\ & = & \int \widehat{T}_1(\xi)\widehat{T}_2(\xi)\phi(\xi)d\xi \\ & = & \widehat{T}_1\widehat{T}_2(\phi).\end{array}$$

Since $\mathscr{D}$ is dense in $\mathscr{S}$, this equation holds for all $\phi \in \mathscr{S}$.

But, for all $\phi \in \mathscr{S}$,

$$(T_1 * T_2)(\widehat{\phi}) = \widehat{T_1 * T_2}(\phi).$$

So $\widehat{T_1 * T_2} = \widehat{T}_1\widehat{T}_2$.

Example (Part (i))

(i) Let $T_a = \frac{1}{2}(\delta_a + \delta_{-a})$, for some real number a.

To find the Fourier transform of T_a, we shall first compute $\widehat{\delta}_a$.

We have, for all $\phi \in \mathscr{D}(\mathbb{R})$,

$$\begin{aligned} \langle \widehat{\delta}_a, \phi \rangle &= \langle \delta_a, \widehat{\phi} \rangle \\ &= \widehat{\phi}(a) \\ &= \int e^{-ixa} \phi(x) dx \\ &= \langle e^{-iax}, \phi \rangle. \end{aligned}$$

Hence, $\widehat{\delta}_a(\xi) = e^{-ia\xi}$.

It follows that

$$\widehat{T}_a(\xi) = \frac{1}{2}(e^{-ia\xi} + e^{ia\xi}) = \cos a\xi.$$

Example (Part (ii))

(ii) We verify that
$$\mathscr{F}(T_a * T_b) = \mathscr{F}(T_a)\mathscr{F}(T_b).$$
We use
$$\begin{aligned}(\delta_a * \delta_b)(x) &= \int \delta_a(y)\delta_b(x-y)dy \\ &= \delta_b(x-a) \\ &= \tau_a\delta_b(x) \\ &= \delta_{a+b}(x).\end{aligned}$$
Now we get
$$\begin{aligned}T_a * T_b &= (\tfrac{1}{2}(\delta_a+\delta_{-a})) * (\tfrac{1}{2}(\delta_a+\delta_{-a})) \\ &= \tfrac{1}{4}(\delta_{a+b}+\delta_{-(a+b)}+\delta_{a-b}+\delta_{-(a-b)}).\end{aligned}$$
So
$$\begin{aligned}\mathscr{F}(T_a * T_b) &= \tfrac{1}{4}[2\cos(a+b)\xi + 2\cos(a-b)\xi] \\ &= \cos a\xi \cos b\xi \\ &= \mathscr{F}(T_a)\mathscr{F}(T_b).\end{aligned}$$

Example (Part (iii))

(iii) Now compute the Fourier transforms of $\sin x$ and $\cos x$.

$$\begin{aligned}
\mathscr{F}(\cos x) &= \mathscr{F}(\cos(1x)) \\
&= \widehat{\widehat{T_1}} \\
&= 2\pi \check{T}_1 \\
&= \pi(\check{\delta}_1 + \check{\delta}_{-1}) \\
&= \pi(\delta_{-1} + \delta_1); \\
\mathscr{F}(\sin x) &= \mathscr{F}(-iD\cos x) \\
&= -i\xi\mathscr{F}(\cos x) \\
&= -i\pi\xi(\delta_1 + \delta_{-1}) \\
&= i\pi(\delta_{-1} - \delta_1).
\end{aligned}$$

The Paley-Wiener-Schwartz Theorem

The Paley-Wiener-Schwartz Theorem

(i) If $T \in \mathscr{E}'$ and $\operatorname{supp} T \subseteq \{x \in \mathbb{R}^n : |x| \leq r\} = \overline{B}(0,r)$, then there is a constant M and a nonnegative integer N, such that

$$|\widehat{T}(\zeta)| \leq M(1+|\zeta|)^N e^{r|\mathrm{Im}\zeta|}, \quad \zeta \in \mathbb{C}^n.$$

(ii) Conversely, every entire function in $\mathbb{C}^n$ satisfying the preceding inequality is the Fourier-Laplace transform of a distribution with support contained in $\overline{B}(0,r)$.

(iii) If $T \in C_0^\infty$ and $\operatorname{supp} T \subseteq \overline{B}(0,r)$, then, for every integer $m \geq 0$, there is a constant M_m, such that

$$|\widehat{T}(\zeta)| \leq M_m(1+|\zeta|)^{-m} e^{r|\mathrm{Im}\zeta|}, \quad \zeta \in \mathbb{C}^n.$$

(iv) Conversely, every entire function in $\mathbb{C}^n$ satisfying the equation above, for every $m \in \mathbb{N}_0$ is the Fourier-Laplace transform of a C_0^∞ function with support contained in $\overline{B}(0,r)$.

Proof of Paley-Wiener-Schwartz Theorem Part (i)

(i) Let $K = \operatorname{supp} T \subseteq \overline{B}(0,r)$.
Let ψ be a C_0^∞ function which equals 1 on a neighborhood of K.
Then we have $T(\phi) = T(\psi\phi)$, for all $\phi \in \mathscr{E}$.
Now $\psi\phi$ is in $\mathscr{D}$. By a previous theorem, T is of finite order on $\mathscr{D}$.
So there is an integer $N \geq 0$ and a constant M_1, such that

$$|T(\phi)| = |T(\psi\phi)| \leq M_1 |\psi\phi|_N.$$

Suppose $\operatorname{supp}\psi = K_0 \supseteq K^\circ \supseteq K$.
By Leibniz's formula, there exists $M_2 > 0$, such that

$$|\psi\phi|_N \leq M_2 \sup\{|\partial^\alpha \phi(x)| : x \in K_0, |\alpha| \leq N\}.$$

Since the inequality is true, for every K_0, such that $K_0^\circ \supseteq K$, it is holds for K.

Proof of Paley-Wiener-Schwartz Theorem (Part (i) Cont'd)

- Setting $\phi(x) = e^{-i\langle x,\zeta\rangle}$ and $\zeta = \xi + i\eta$, we obtain

$$\begin{array}{rcl} \sup\{|\partial^\alpha \phi(x)| : x \in K, |\alpha| \le N\} & = & \sup\{|\partial^\alpha e^{-i\langle x,\xi+i\eta\rangle}| : x \in K, |\alpha| \le N\} \\ & \le & \sup\{|\zeta|^{|\alpha|} e^{\langle x,\eta\rangle} : |x| \le r, |\alpha| \le N\} \\ & \le & (1+|\zeta|)^N e^{r|\eta|}. \end{array}$$

Applying the preceding three inequalities, we get

$$\begin{array}{rcl} |\widehat{T}(\zeta)| & = & |T_x(e^{-i\langle x,\zeta\rangle})| \\ & \le & M_1|\psi e^{-i\langle x,\zeta\rangle}|_N \\ & \le & M_2 M_1 \sup\{|\partial^\alpha e^{-i\langle x,\zeta\rangle}| : x \in K, |\alpha| \le N\} \\ & \le & M_2 M_1 (1+|\zeta|)^N e^{r|\mathrm{Im}\zeta|}. \end{array}$$

Proof of Paley-Wiener-Schwartz Theorem Part (ii)

(ii) Suppose T is a C_0^∞ function.

Then we can use $\mathscr{F}(D^\alpha \phi) = \xi^\alpha \mathscr{F}(\phi)$, to write, for any $\alpha \in \mathbb{N}_0^n$,

$$\zeta^\alpha \widehat{T}(\zeta) = \int e^{-i\langle x,\zeta\rangle} D^\alpha T(x) dx.$$

Assume, moreover, that $\operatorname{supp} T$ in $\overline{B}(0,r)$.

Then the expression above yields

$$|\zeta^\alpha \widehat{T}(\zeta)| \leq M e^{r|\eta|},$$

for some constant M.

From this, Part (ii) follows.

Proof of Paley-Wiener-Schwartz Theorem Part (iii)

(iii) Suppose that, for all m, there exists M_m, such that

$$|\widehat{T}(\zeta)| \le M_m(1+|\zeta|)^{-m} e^{r|\mathrm{Im}\zeta|}, \quad \zeta \in \mathbb{C}^n.$$

Then the integral

$$(2\pi)^{-n} \int \widehat{T}(\xi) e^{i\langle x,\xi\rangle} d\xi$$

is absolutely convergent on $\mathbb{R}^n$.

It clearly defines the inverse Fourier transform $T(x)$ of $\widehat{T}(\xi)$.

Now, for $\alpha \in \mathbb{N}_0^n$,

$$\partial^\alpha T(x) = (-i)^{|\alpha|}(2\pi)^{-n} \int \widehat{T}(\xi) \xi^\alpha e^{i\langle x,\xi\rangle} d\xi$$

is also absolutely convergent.

We conclude that T is in C^∞.

Proof of Paley-Wiener-Schwartz Theorem (Part (iii) Cont'd)

- We show, next, that T has compact support.
  The preceding integrand extends to an entire function on $\mathbb{C}^n$.
  So we can use Cauchy's Theorem with each variable $\zeta_1, \ldots, \zeta_n$ to shift the integration from $\mathbb{R}^n$ into $\mathbb{C}^n$.
  For any fixed $\eta \in \mathbb{R}^n$, we get

$$T(x) = (2\pi)^{-n} \int \widehat{T}(\xi + i\eta) e^{i\langle x, \xi + i\eta\rangle} d\xi.$$

  Using the hypothesis, with $m = n + 1$,

$$\begin{array}{rcl} |T(x)| & \leq & (2\pi)^{-n} M_{n+1} e^{-\langle x,\eta\rangle + r|\eta|} \int (1 + |\xi|)^{-n-1} d\xi \\ & \leq & M e^{r|\eta| - \langle x,\eta\rangle}. \end{array}$$

  Taking $\eta = tx$ we get

$$|T(x)| \leq M e^{-t|x|(r-|x|)}.$$

  Letting $t \to \infty$, we get $T(x) = 0$, for all $x \in \mathbb{R}^n$, with $|x| > r$.
  Therefore, the support of T must lie in $\overline{B}(0, r)$.

Proof of Paley-Wiener-Schwartz Theorem Part (iv)

(iv) Let $\widehat{T}(\zeta)$ be an entire function which satisfies

$$|\widehat{T}(\zeta)| \le M(1+|\zeta|)^N e^{r|\mathrm{Im}\zeta|}.$$

Then $\widehat{T}(\xi)$ has polynomial growth at ∞. So it lies in $\mathscr{S}'$.

Its inverse Fourier transform T must also be in $\mathscr{S}'$.

We show, next, that $\mathrm{supp}\, T$ is compact.

We regularize T using the C^∞ functions β_λ, $\lambda > 0$, satisfying $\mathrm{supp}\beta_\lambda \subseteq \overline{B}(0,\lambda)$.

Now $T_\lambda = T * \beta_\lambda$ is in C^∞.

Its Fourier transform, according to a previous theorem, is $\widehat{T}_\lambda = \widehat{\beta}_\lambda \widehat{T}$.

For each $\lambda > 0$, $\widehat{T}_\lambda(\xi)$ extends to an analytic function on $\mathbb{C}^n$.

Proof of Paley-Wiener-Schwartz Theorem (Part (iv) Cont'd)

- $\widehat{T}$ satisfies, for some M and $N \geq 0$,

$$|\widehat{T}(\zeta)| \leq M(1+|\zeta|)^N e^{r|\text{Im}\zeta|}, \quad \zeta \in \mathbb{C}^n.$$

β_λ satisfies, for all $m \geq 0$ and some M_m,

$$|\widehat{\beta}_\lambda(\zeta)| \leq M_m(1+|\zeta|)^{-m} e^{\lambda|\text{Im}\zeta|}, \quad \zeta \in \mathbb{C}^n.$$

So $\widehat{T}_\lambda$ must satisfy, for $m = 0, 1, 2, \ldots$ and $\zeta \in \mathbb{C}^n$,

$$|\widehat{T}_\lambda(\zeta)| \leq M M_m (1+|\zeta|)^{N-m} e^{(r+\lambda)|\text{Im}\zeta|}.$$

Choosing m greater than N, we see that $\widehat{T}_k$ satisfies the hypothesis of Part (iii) with r replaced by $r+\lambda$.
So, by Part (iii), $\text{supp}\, T_\lambda \subseteq \overline{B}(0, r+\lambda)$.
Since $T_\lambda \to T$ as $\lambda \to 0$,

$$\text{supp}\, T \subseteq \bigcap \{\overline{B}(0, r+\lambda) : \lambda > 0\} = \overline{B}(0, r).$$

Subsection 7

The Cauchy-Riemann Operator

Fourier Transformation with Respect to Some Variables

- Suppose $T \in \mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, with $n_1 + n_2 = n$.
- The **Fourier transform** $\mathscr{F}_1(T)$ **of** T **with respect to** $x \in \mathbb{R}^{n_1}$ is defined, for all $\phi \in \mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, by

$$\langle \mathscr{F}_1(T), \phi \rangle = \langle T, \mathscr{F}_1(\phi) \rangle.$$

- $\mathscr{F}_1(\phi)$ is well defined by the integral formula

$$\mathscr{F}_1(\phi(\cdot, y))(\xi) = \int_{\mathbb{R}^{n_1}} e^{-i\langle x, \xi \rangle} \phi(x, y) dx, \quad \xi \in \mathbb{R}^{n_1},\ y \in \mathbb{R}^{n_2}.$$

- $\mathscr{F}_1(\phi(\cdot, y))(\xi)$ is also denoted by $\widehat{\phi}(\xi, y)$.
- It lies in $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

Partial Differentiation

- Given $T \in \mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$, $\mathscr{F}_1(T) \in \mathscr{S}'(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$.

  Claim: If ∂_y^α is a partial differential operator in $y \in \mathbb{R}^{n_2}$, then

  $$\mathscr{F}_1(\partial_y^\alpha T) = \partial_y^\alpha \mathscr{F}_1(T).$$

  We have, for all $\phi \in \mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$,

  $$\begin{aligned} \langle \mathscr{F}_1(\partial_y^\alpha T), \phi \rangle &= \langle \partial_y^\alpha T, \mathscr{F}_1(\phi) \rangle \\ &= (-1)^{|\alpha|} \langle T, \partial_y^\alpha \mathscr{F}_1(\phi) \rangle \\ &= (-1)^{|\alpha|} \langle T, \mathscr{F}_1(\partial_y^\alpha \phi) \rangle \\ &= (-1)^{|\alpha|} \langle \mathscr{F}_1(T), \partial_y^\alpha \phi \rangle \\ &= \langle \partial_y^\alpha \mathscr{F}_1(T), \phi \rangle. \end{aligned}$$

  We note that the commutation of $\mathscr{F}_1$ with ∂_y^α on $\mathscr{S}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ is based on the linearity and continuity of $\mathscr{F}_1$.

Example

- Consider the differential operator in $\mathbb{R}$ of order m with constant coefficients

$$L = \sum_{k=0}^{m} c_k D^k.$$

If $u \in \mathscr{E}'(\mathbb{R})$ satisfies $Lu = 0$, then, upon transformation,

$$0 = \mathscr{F}(Lu) = \sum_{k=0}^{m} c_k \xi^k \widehat{u}.$$

Hence, $\widehat{u}(\xi) = 0$ except possibly at the zeros of the polynomial

$$c_0 + c_1\xi + \cdots + c_m\xi^m.$$

But u has compact support.

So $\widehat{u}$ is continuous. Thus, $\widehat{u}$ must vanish in all $\mathbb{R}$.

It follows that the ordinary differential equation $Lu = 0$ has only the trivial solution in $\mathscr{E}'$.

Example

- Consider the differential operator in $\mathbb{R}^n$ of order m with constant coefficients

$$L = \sum_{|\alpha| \leq m} c_\alpha D^\alpha.$$

Let $u \in \mathscr{S}'$ be a solution of $Lu = 0$.

The application of the Fourier transformation gives

$$0 = \mathscr{F}\left(\sum c_\alpha D^\alpha u\right) = \left(\sum c_\alpha \xi^\alpha\right) \widehat{u} = P(\xi)\widehat{u},$$

where $P(\xi)$ is the polynomial $\sum_{|\alpha| \leq m} c_\alpha \xi^\alpha$.

Suppose $P(\xi) = 0$ only when $\xi = 0$. Then $\operatorname{supp} \widehat{u} \subseteq \{0\}$.

By a previous theorem, $\widehat{u} = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha \delta$, for some k.

By taking the inverse Fourier transform, $u = \sum_{|\alpha| \leq k} b_\alpha x^\alpha$.

Thus, the only solution of $Lu = 0$ in $\mathscr{S}'$ for this type of operator is a polynomial. In other words, the fundamental solution of L in $\mathscr{S}'$ is unique up to an additive polynomial.

The Cauchy-Riemann Operator

- Consider the **Cauchy-Riemann operator** in $\mathbb{R}^2$,

$$\overline{\partial} = \frac{1}{2}(\partial_1 + i\partial_2).$$

The polynomial

$$P(i\xi) = \frac{1}{2} i(\xi_1 + i\xi_2)$$

vanishes only at $\xi = 0$.

So this operator is an example of the preceding slide.

Its fundamental solution in $\mathscr{S}'(\mathbb{R}^2)$ is unique up to an additive polynomial.

But every entire function f satisfies $\overline{\partial} f = 0$ in $\mathbb{R}^2$.

Hence, the fundamental solution of $\overline{\partial}$ in $\mathscr{D}'(\mathbb{R}^2)$ is unique up to an additive entire function.

Example

- We show that $\frac{1}{\pi z} = \frac{1}{\pi(x+iy)}$ is a fundamental solution of the Cauchy-Riemann operator in the plane.
  Since $\frac{1}{|z|} = \frac{1}{r} \in L^1_{\text{loc}}(\mathbb{R}^2)$, $\frac{1}{z}$ defines a distribution in $\mathbb{R}^2$.
  For any $\phi \in \mathscr{D}(\mathbb{R}^2)$,
  $$\left\langle \overline{\partial}\frac{1}{z}, \phi \right\rangle = -\left\langle \frac{1}{z}, \overline{\partial}\phi \right\rangle = -\frac{1}{2}\int_{\mathbb{R}^2} \frac{1}{x+iy}\left(\frac{\partial\phi}{\partial x} + i\frac{\partial\phi}{\partial y}\right) dxdy.$$
  We change to polar coordinates. Let $\widetilde{\phi}(r,\theta) = \phi(x,y)$.
  Recall that $\frac{\partial}{\partial x} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}$, $\frac{\partial}{\partial y} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial\theta}$.
  Therefore, we obtain
  $$\left\langle \overline{\partial}\frac{1}{z}, \phi \right\rangle = -\frac{1}{2}\int_0^{2\pi}\int_0^{\infty} \frac{1}{re^{i\theta}}\left[e^{i\theta}\frac{\partial\widetilde{\phi}}{\partial r} + \frac{i}{r}e^{i\theta}\frac{\partial\widetilde{\phi}}{\partial\theta}\right] rdrd\theta.$$

Example (Cont'd)

- With $\widetilde{\phi}(r,\theta)=\phi(x,y)$,

$$\left\langle \overline{\partial}\frac{1}{z},\phi\right\rangle = -\frac{1}{2}\int_0^{2\pi}\int_0^{\infty}\frac{1}{re^{i\theta}}\left[e^{i\theta}\frac{\partial\widetilde{\phi}}{\partial r}+\frac{i}{r}e^{i\theta}\frac{\partial\widetilde{\phi}}{\partial\theta}\right]rdrd\theta.$$

By Fubini's Theorem,

$$\begin{aligned}\left\langle \overline{\partial}\tfrac{1}{z},\phi\right\rangle &= -\tfrac{1}{2}\int_0^{2\pi}\int_0^{\infty}\tfrac{\partial\widetilde{\phi}}{\partial r}drd\theta-\tfrac{1}{2}i\int_0^{\infty}\tfrac{1}{r}\int_0^{2\pi}\tfrac{\partial\widetilde{\phi}}{d\theta}d\theta dr\\ &= -\tfrac{1}{2}[-2\pi\widetilde{\phi}(0)]-0, \quad \text{since } \widetilde{\phi}(r,2\pi)=\widetilde{\phi}(r,0)\\ &= \pi\phi(0).\end{aligned}$$

Therefore, $\overline{\partial}(\frac{1}{\pi z})=\delta$.

It follows that any fundamental solution E of $\overline{\partial}$ in $\mathscr{D}'(\mathbb{R}^2)$ is of the form $E(z)=\frac{1}{\pi z}+h(z)$, where h is an entire function in $\mathbb{C}$.

Subsection 8

Fourier Transforms and Homogeneous Distributions

Dualizing a Linear Mapping

- Let Λ be a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^n$.
- Let $F(\mathbb{R}^n)$ be the linear space of complex functions on $\mathbb{R}^n$.
- We define the map $\Lambda^* : F(\mathbb{R}^n) \to F(\mathbb{R}^n)$ by

$$\Lambda^* f(x) = f(\Lambda x), \quad f \in F(\mathbb{R}^n).$$

- Λ^* is also linear. For all $f, g \in F(\mathbb{R}^n)$ and $a, b \in \mathbb{C}$,

$$\begin{aligned} \Lambda^*(af + bg)(x) &= (af + bg)(\Lambda x) \\ &= a\Lambda^* f(x) + b\Lambda^* g(x) \\ &= (a\Lambda^* f + b\Lambda^* g)(x). \end{aligned}$$

- Λ may be represented by a real $n \times n$ matrix, determined by the basis that we choose for $\mathbb{R}^n$.
- It is nonsingular if the null space of Λ is $\{0\} \subseteq \mathbb{R}^n$. In this case:
  - The determinant $\det\Lambda$ is nonzero.
  - The inverse map Λ^{-1} exists and is a linear map from $\mathbb{R}^n$ to $\mathbb{R}^n$.

Continuity of Λ^*

Claim: If Λ is nonsingular, then Λ^* maps $\mathscr{S}$ continuously onto $\mathscr{S}$.
Let ϕ, ψ be functions in $\mathscr{S}$. Then

$$\begin{aligned} \langle \Lambda^* \psi, \phi \rangle &= \int \psi(\Lambda x) \phi(x) dx \\ &= \int \psi(y) \phi(\Lambda^{-1} y) \tfrac{1}{|\det \Lambda|} dy \\ &= \int \psi(y) \tfrac{1}{|\det \Lambda|} \Lambda^{-1*} \phi(y) dy. \end{aligned}$$

This shows that

$$\langle \Lambda^* \psi, \phi \rangle = \left\langle \psi, \frac{1}{|\det \Lambda|} \Lambda^{-1*} \phi \right\rangle.$$

Now note that $\frac{1}{|\det \Lambda|} \Lambda^{-1*} \phi$ is in $\mathscr{S}$, if ϕ is in $\mathscr{S}$.
So the function ψ in the preceding equation may be extended by continuity from $\mathscr{S}$ to $\mathscr{S}'$.

Inverse of Λ^*

- We have, for every $f \in F(\mathbb{R}^n)$,

$$\begin{aligned} f(x) &= f(\Lambda^{-1}\Lambda x) \\ &= \Lambda^* f(\Lambda^{-1}x) \\ &= \Lambda^*\Lambda^{-1*}(x). \end{aligned}$$

  Therefore,

$$\Lambda^{-1*} = \Lambda^{*-1}.$$

The Fourier Transform of the Dual

- For any $\phi \in \mathscr{S}$, we have (denoting by Λ^T the transpose of Λ)

$$\begin{aligned}\mathscr{F}(\Lambda^*\phi)(\xi) &= \int e^{-i\langle \xi, x\rangle}\phi(\Lambda x)dx \\ &= \int e^{-i\langle \xi, \Lambda^{-1}y\rangle}\phi(y)\tfrac{1}{|\det\Lambda|}dy \\ &= \int e^{-i\langle \Lambda^{-1T}\xi, y\rangle}\phi(y)\tfrac{1}{|\det\Lambda|}dy \\ &= \tfrac{1}{|\det\Lambda|}\widehat{\phi}(\Lambda^{-1T}\xi).\end{aligned}$$

Thus,

$$\widehat{\Lambda^*\phi} = \frac{1}{|\det\Lambda|}(\Lambda^{-1T})^*\widehat{\phi}, \quad \phi \in \mathscr{S}.$$

Now $\mathscr{F}\Lambda^*$ and $\frac{1}{|\det\Lambda|}(\Lambda^{-1T})^*\mathscr{F}$ are equal and continuous on $\mathscr{S}$. So they may be extended by continuity to $\mathscr{S}'$ to obtain

$$\widehat{\Lambda^* T} = \frac{1}{|\det\Lambda|}(\Lambda^{-1T})^*\widehat{T}, \quad T \in \mathscr{S}'.$$

Reflection Operator

- Consider the **reflection operator**

$$\Lambda x = -x, \quad x \in \mathbb{R}^n.$$

It is linear and continuous, for any $t \in \mathbb{R}$.

If $T \in \mathscr{D}'$, then $\Lambda^* T$ is the distribution defined by

$$\langle \Lambda^* T, \phi \rangle = \left\langle T, \frac{1}{|\det \Lambda|} \Lambda^{-1*} \phi \right\rangle, \quad \phi \in \mathscr{D}.$$

In this case we have:

- $\det \Lambda = (-1)^n$;
- $\Lambda^{-1} = \Lambda$.

So we get

$$\langle \Lambda^* T, \phi \rangle = \left\langle T, \frac{1}{|(-1)^n|} \Lambda^* \phi \right\rangle = \langle T, \check{\phi} \rangle = \langle \check{T}, \phi \rangle.$$

Scaling Operators

- A more general example is the transformation

$$\Lambda_t x = tx, \quad x \in \mathbb{R}^n.$$

It is linear and continuous, for any $t \in \mathbb{R}$, but singular when $t = 0$.

If $T \in \mathscr{D}'$ and $t \neq 0$, then $\Lambda_t^* T$ is the distribution defined by

$$\langle \Lambda_t^* T, \phi \rangle = \left\langle T, \frac{1}{|\det \Lambda_t|} \Lambda_t^{-1*} \phi \right\rangle, \quad \phi \in \mathscr{D}.$$

In this case we have:
  - $\det \Lambda_t = t^n$;
  - $\Lambda^{-1} = \Lambda_{1/t}$.

So we get

$$\langle \Lambda_t^* T, \phi \rangle = \left\langle T, \frac{1}{t^n} \Lambda_{1/t}^* \phi \right\rangle.$$

Homogeneous Functions and Distributions

- Let d be a complex number.
- A function f on $\mathbb{R}^n$ is **homogeneous of degree** d if
$$f(tx) = t^d f(x).$$
- A distribution T is **homogeneous of degree** d if
$$\Lambda_t^* T = t^d T, \quad \text{for any } t > 0.$$

Homogeneous Functions vs. Homogeneous Distributions

Claim: The two definitions coincide when the function is locally integrable in $\mathbb{R}^n$, in the sense that $\Lambda_t^* f = t^d f$ if and only if $f(tx) = t^d f(x)$ a.e.

We have, for all $\phi \in \mathscr{D}$,

$$\begin{aligned} \langle \Lambda_t^* f, \phi \rangle &= \langle f, \tfrac{1}{t^n} \Lambda_{1/t}^* \phi \rangle \\ &= \int f(x) \tfrac{1}{t^n} \phi(\tfrac{x}{t}) dx \\ &= \int f(ty) \phi(y) dy. \end{aligned}$$

Suppose, first, $f(tx) = t^d f(x)$ a.e..

Then $\langle \Lambda_t^* f, \phi \rangle = \int t^d f(y) \phi(y) dy = \langle t^d f, \phi \rangle$.

So $\Lambda_t^* f = t^d f$.

Conversely, assume $\Lambda_t^* f = t^d f$.

Then, for all $\phi \in \mathscr{D}$, $\int f(ty) \phi(y) dy = \int t^d f(y) \phi(y) dy$.

Hence, by a previous result, $f(ty) = t^d f(y)$ a.e..

Example

(i) Let $\{T_1, \ldots, T_m\}$ be a set of nonzero distributions in $\mathbb{R}^n$, such that T_k, $1 \leq k \leq m$, is homogeneous of real degree d_k and $d_k \neq d_j$, if $k \neq j$.

Claim: The set $\{T_1, \ldots, T_m\}$ is linearly independent over $\mathbb{C}$.

Let $a_1 T_1 + \cdots + a_m T_m = 0$. Without loss of generality, assume that $d_1 > d_2 > \cdots > d_m$. For any $\phi \in \mathscr{D}$, we have

$$0 = \left\langle \Lambda_t^* \sum_{k=1}^{m} a_k T_k, \phi \right\rangle = \sum_{k=1}^{m} a_k \langle \Lambda_t^* T_k, \phi \rangle = \sum_{k=1}^{m} a_k t^{d_k} \langle T_k, \phi \rangle.$$

If the coefficients a_k do not all vanish, let $i \geq 1$ be the smallest integer for which $a_i \neq 0$.

- If $i = m$, then $\langle T_m, \phi \rangle = 0$. So $T_m = 0$, a contradiction.
- If $1 \leq i < m$, then $a_i \langle T_i, \phi \rangle + \sum_{k=i}^{m} a_k t^{d_k - d_i} \langle T_k, \phi \rangle = 0$, for all $t > 0$ and $\phi \in \mathscr{D}$. Letting $t \to \infty$, we obtain $a_i \langle T_i, \phi \rangle = 0$. But $a_i \neq 0$. Hence, $T_i = 0$, again a contradiction.

Example

(ii) We show that $\partial^\alpha \delta$ is homogeneous of degree $-n-|\alpha|$.

We have, for all $\phi \in \mathscr{D}$,

$$\begin{array}{rcl} \langle \Lambda_t^* \partial^\alpha \delta, \phi \rangle & = & \langle \partial^\alpha \delta, \frac{1}{t^n} \Lambda_{1/t}^* \phi \rangle \\ & = & \frac{1}{t^n} \langle \partial^\alpha \delta, \phi(\frac{x}{t}) \rangle \\ & = & (-1)^{|\alpha|} \frac{1}{t^n} \langle \delta, \partial^\alpha \phi(\frac{x}{t}) \rangle \\ & = & (-1)^{|\alpha|} \frac{1}{t^n} \frac{1}{t^{|\alpha|}} (\partial^\alpha \phi)(0) \\ & = & \frac{1}{t^{n+|\alpha|}} \langle \partial^\alpha \delta, \phi \rangle. \end{array}$$

Therefore,

$$\Lambda_t^* \partial^\alpha \delta = \frac{1}{t^{n+|\alpha|}} \partial^\alpha \delta.$$

In view of Part (i), we conclude that the distributions $\delta, \delta', \ldots, \delta^{(m)}$ on $\mathbb{R}$ are linearly independent.

Example

- For $\lambda \geq 0$ we show that

$$x_+^\lambda = x^\lambda H, \quad x \in \mathbb{R},$$

is homogeneous of degree λ.

We have

$$\begin{array}{rcl} \langle \Lambda_t^* x_+^\lambda, \phi \rangle & = & \langle x_+^\lambda, \frac{1}{t} \Lambda_{1/t}^* \phi \rangle \\ & = & \frac{1}{t} \int_0^\infty x^\lambda \phi(\frac{x}{t}) dx \\ & = & \frac{1}{t} \int_0^\infty t^\lambda y^\lambda \phi(y) t dy \\ & = & \langle t^\lambda x_+^\lambda, \phi \rangle. \end{array}$$

Hence

$$\Lambda_t^* x_+^\lambda = t^\lambda x_+^\lambda.$$

Derivatives and Transforms of Homogeneous Distributions

Theorem

If $T \in \mathscr{S}'$ is homogeneous of degree d, then $\partial_k T$ is homogeneous of degree $d-1$ and $\widehat{T}$ is homogeneous of degree $-n-d$.

- Let $\phi \in \mathscr{S}$ be homogeneous of degree d and t be a positive number. Then, by the chain rule, $\partial_k[\phi(tx)] = t(\partial_k\phi)(tx)$. Hence,

$$\Lambda_t^*(\partial_k\phi)(x) = (\partial_k\phi)(tx) = \frac{1}{t}\partial_k[\phi(tx)] = t^{d-1}(\partial_k\phi)(x).$$

  This means that $\partial_k\phi$ is homogeneous of degree $d-1$.

- To obtain the result for $T \in \mathscr{S}'$, suppose the degree of T is d. We first note that, for all $\phi \in \mathscr{S}$,

$$\partial_k(\Lambda_t^*\phi)(x) = \partial_k[\phi(tx)] = t(\partial_k\phi)(tx) = t\Lambda_t^*(\partial_k\phi)(x).$$

Derivatives and Transforms of Homogeneous Distributions

- Keeping in mind $\Lambda_t^{-1} = \Lambda_{1/t}$, we get

$$\begin{aligned}
\Lambda_t^* \partial_k T(\phi) &= \partial_k T\left(\frac{1}{|\det \Lambda_t|} \Lambda_t^{-1*} \phi\right) \\
&= -T(|\det \Lambda_{1/t}| \partial_k \Lambda_{1/t}^* \phi) \\
&= -\frac{1}{t} T(|\det \Lambda_{1/t}| \Lambda_{1/t}^* \partial_k \phi) \\
&= -\frac{1}{t} \Lambda_t^* T(\partial_k \phi) \\
&= \frac{1}{t} \partial_k \Lambda_t^* T(\phi) \\
&= t^{d-1} \partial_k T(\phi).
\end{aligned}$$

Thus $\partial_k T$ has degree $d-1$.

Using the relations $\det \Lambda_t = t^n$ and $\Lambda_t^T = \Lambda_t$,

$$\widehat{\Lambda_t^* T} = \frac{1}{|\det \Lambda|} (\Lambda^{-1T})^* \widehat{T} = \frac{1}{t^n} \Lambda_{1/t}^* \widehat{T}, \quad T \in \mathscr{S}'.$$

If T is homogeneous of degree d, $t^d \widehat{T} = \frac{1}{t^n} \Lambda_{1/t}^* \widehat{T}$. So $\Lambda_t^* \widehat{T} = \frac{1}{t^{n+d}} \widehat{T}$.

Example

- Consider the function

$$f(z) = \frac{1}{z} = \frac{1}{x+iy}.$$

It is locally integrable in the plane.

Clearly, $|f(z)| < 1$ when $|z| > 1$.

Hence, f defines a tempered distribution in $\mathbb{R}^2$.

We compute its Fourier transform.

$$\mathscr{F}(zf) = \mathscr{F}(1) = \hat{\hat{\delta}} = (2\pi)^2 \check{\delta} = (2\pi)^2 \delta.$$

Recalling the operator $\overline{\partial} = \frac{1}{2}(\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial \eta})$, we have,

$$\begin{aligned} \mathscr{F}(zf) &= \mathscr{F}(xf) + i\mathscr{F}(yf) = i\frac{\partial}{\partial \xi}\hat{f} - \frac{\partial}{\partial \eta}\hat{f} \\ &= i(\frac{\partial}{\partial \xi} + i\frac{\partial}{\partial \eta})\hat{f} = 2i\overline{\partial}\hat{f}. \end{aligned}$$

Therefore, $\frac{i\hat{f}}{2\pi^2}$ is a fundamental solution of the operator $\overline{\partial}$.

Example (Cont'd)

- By a previous example,

$$\frac{i}{2\pi}\widehat{f}(\zeta) = \frac{1}{\zeta} + h(\zeta),$$

where h is an entire function.
But f is homogeneous of degree -1 in $\mathbb{R}^2$.
By the theorem, $\widehat{f}$ is homogeneous of degree $-2+1=-1$.
If h is not identically 0, it must also have degree -1.
Hence,

$$h(t\zeta) = \frac{h(\zeta)}{t}, \quad t>0.$$

This becomes unbounded as $t \to 0$.
Thus, $h=0$.
So

$$\mathscr{F}\left(\frac{1}{z}\right) = \widehat{f}(\zeta) = -\frac{2\pi i}{\zeta}.$$

Orthogonal Transformations

- A linear transformation $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ is said to be **orthogonal** if

$$\Lambda^T = \Lambda^{-1}.$$

- If Λ is orthogonal, then so is Λ^{-1} and $\det \Lambda = \pm 1$.

  Claim: The transformation Λ is orthogonal if and only if it is norm-preserving.

  An orthogonal transformation Λ satisfies, for all $x \in \mathbb{R}^n$,

$$|\Lambda x|^2 = \langle \Lambda x, \Lambda x \rangle = \langle x, \Lambda^T \Lambda x \rangle = \langle x, x \rangle = |x|^2.$$

  Thus, $|\Lambda x| = |x|$.

  Conversely, suppose $|\Lambda x| = |x|$, for all $x \in \mathbb{R}^n$.

  Then $\Lambda^T \Lambda =$ identity. This implies that Λ is orthogonal.

Invariance

- A distribution $T \in \mathscr{D}'$ is **invariant** under the transformation $\Lambda : \mathbb{R}^n \to \mathbb{R}^n$ if
$$\Lambda^* T = T.$$
- A function $f : \mathbb{R}^n \to \mathbb{C}$ is called **rotation-invariant**, or **spherically symmetric**, if there exists a function $g : [0, \infty) \to \mathbb{C}$, such that
$$f(x) = g(|x|), \quad \text{for all } x \in \mathbb{R}^n.$$

  Claim: A function is rotation invariant if and only if it is invariant under orthogonal transformations.

  Suppose f is rotation-invariant. Then
$$\Lambda^* f(x) = f(\Lambda x) = g(|\Lambda x|) = g(|x|) = f(x).$$

  So f is invariant under any orthogonal transformation Λ.

  Conversely, a rotation in $\mathbb{R}^n$ is an orthogonal transformation.

Invariance of the Fourier Transform

Theorem

If $T \in \mathscr{S}'$ is invariant under orthogonal transformations, then $\widehat{T}$ is also invariant under orthogonal transformations.

- Suppose Λ is an orthogonal transformation.
  If T is any distribution in $\mathscr{S}'$, then
  $$\widehat{\Lambda^* T} = \frac{1}{|\det \Lambda|} (\Lambda^{-1\,T})^* \widehat{T} = \Lambda^* \widehat{T}.$$
  Consequently,
  $$\Lambda^* T = T \quad \text{if and only if} \quad \widehat{\Lambda^* T} = \widehat{T} \quad \text{if and only if} \quad \Lambda^* \widehat{T} = \widehat{T}.$$
- When a distribution is represented by a rotation-invariant function, the distribution is also said to be **rotation-invariant**.
- The theorem implies that if $T \in \mathscr{S}'$ is rotation invariant and $\widehat{T}$ is a function, then $\widehat{T}$ is also rotation invariant.