Introduction to the Theory of Distributions

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1. Distributions in Hilbert Space
   - Hilbert Space
   - Sobolev Spaces
   - Some Properties of H^s Spaces
   - More on the Space $H^m(\Omega)$
   - Fourier Series and Periodic Distributions

Subsection 1

Hilbert Space

Hilbert Spaces

- A **Hilbert space** $\mathscr{H}$ is a Banach space whose norm is defined by an **inner product**.
- In a (complex) Hilbert space the **inner product** of any pair of vectors, $u, v \in \mathscr{H}$ is a complex number (w, v) with the following properties:
  (i) $(au + bv, w) = a(u, w) + b(v, w)$, for all $u, v, w \in \mathscr{H}$ and $a, b \in \mathbb{C}$;
  (ii) $(u, v) = \overline{(v, u)}$, for all $u, v \in \mathscr{H}$;
  (iii) $(u, u) > 0$, whenever $u \neq 0$.
- We clearly have $(u, av) = \overline{a}(u, v)$.
- The inner product of any vector with the zero vector is zero.
- The norm of any $u \in \mathscr{H}$, denoted by $\|u\|_{\mathscr{H}}$ or simply $\|u\|$, is defined by

$$\|u\| = \sqrt{(u, u)}.$$

- With this definition, the properties for the norm are satisfied.

Schwarz Inequality and the Parallelogram Law

- We have two additional properties of inner product spaces.
- The **Schwarz Inequality**: For all $u, v \in \mathscr{H}$,

$$|(u,v)| \leq \|u\| \|v\|;$$

- The **Parallelogram Law**: For all $u, v \in \mathscr{H}$,

$$\|w+v\|^2 + \|u-v\|^2 = 2(\|u\|^2 + \|v\|^2).$$

The Space $L^2(\Omega)$

- Let Ω be an open subset of $\mathbb{R}^n$.
- The space $L^2(\Omega)$ is an example of a Hilbert space.
- The inner product of any two functions f and g is defined by

$$(f,g) = \int_\Omega f(x)\overline{g}(x)dx.$$

Orthogonality

- Any two vectors $u, v \in \mathscr{H}$ are said to be **orthogonal** if

$$(u, v) = 0.$$

- The notion of orthogonality provides a geometric structure in the Hilbert space that generalizes that of the (finite dimensional) Euclidean space $\mathbb{R}^n$.

Continuity, Dual Space and Strong Convergence

- A linear functional T on the Hilbert space $\mathscr{H}$ is continuous if and only if
$$|T(\phi)| \leq M\|\phi\|_{\mathscr{H}}, \quad \phi \in \mathscr{H},$$
for some positive constant M.
- In the dual space $\mathscr{H}'$, we define the norm
$$\|T\|_{\mathscr{H}'} = \sup\{|T(\phi)| : \phi \in \mathscr{H}, \|\phi\|_{\mathscr{H}} = 1\}.$$
- This norm generates a topology on $\mathscr{H}'$.
- In $\mathscr{H}'$, equipped with this topology, convergence of the sequence (T_i) to 0 is equivalent to the uniform convergence of $T_i(\phi)$ to 0 on every bounded subset of $\mathscr{H}$.
- This was defined as **strong convergence** in $\mathscr{H}'$.

Riesz Representation Theorem

- For any vector ψ in $\mathscr{H}$, the map from $\mathscr{H}$ to $\mathbb{C}$ defined by $\phi \mapsto (\phi, \psi)$ is (by properties of the inner product) a linear functional on $\mathscr{H}$.
- The Schwarz inequality $|(\phi,\psi)| \leq \|\psi\|_{\mathscr{H}} \|\phi\|_{\mathscr{H}}$ shows its continuity.
- **Riesz Representation Theorem**:

  Every continuous linear functional on $\mathscr{H}$ is defined in this way.

  That is, to every continuous linear functional T on $\mathscr{H}$, there exists a unique vector $\psi \in \mathscr{H}$, such that

$$T(\phi) = (\phi, \psi), \quad \text{for all } \phi \in \mathscr{H},$$

  and

$$\|T\|_{\mathscr{H}'} = \|\psi\|_{\mathscr{H}}.$$

Consequences of the Riesz Representation Theorem

- The dual space $\mathscr{H}'$ of continuous linear functionals on $\mathscr{H}$ is also a Hilbert space.
- The correspondence $\psi \leftrightarrow T_\psi$ defines a norm-preserving bijection, or isometry, between $\mathscr{H}$ and $\mathscr{H}'$.
- Even though $\mathscr{H}$ and $\mathscr{H}'$ may be identified as sets, they cannot be identified as linear spaces.
- Indeed the linear combination $a_1\psi_1 + a_2\psi_2$ in $\mathscr{H}$ corresponds to the **conjugate linear combination** $\overline{a}_1 T_{\psi_1} + \overline{a}_2 T_{\psi_2}$, unless of course $\mathscr{H}$ is a real Hilbert space.

Reflexivity of Hilbert Spaces

- The **second dual** of $\mathscr{H}$, $\mathscr{H}'' = (\mathscr{H}')'$ composed of the continuous linear functionals on $\mathscr{H}'$ may be identified with $\mathscr{H}$.
- Linearity in this case is restored to the correspondence between the elements of $\mathscr{H}$ and the elements of $\mathscr{H}''$.
- This is the **reflexive property** of the Hilbert space.

Subsection 2

Sobolev Spaces

The Sobolev Space $H^m(\Omega)$

- For any integer $m \in \mathbb{N}_0$ and any open $\Omega \subseteq \mathbb{R}^n$, we define the **Sobolev space** $H^m(\Omega)$ to be the set of all functions $\phi \in L^2(\Omega)$ whose distributional derivatives $\partial^\alpha \phi$ are also in $L^2(\Omega)$, for every $\alpha \in \mathbb{N}_0^n$, with $|\alpha| \leq m$, i.e.,

$$H^m(\Omega) = \{\phi \in L^2(\Omega) : \partial^\alpha \phi \in L^2(\Omega), |\alpha| \leq m\}.$$

- Thus, $H^m(\Omega)$ is the subspace of distributions $\phi \in \mathscr{D}'(\Omega)$, such that $\partial^\alpha \phi \in L^2(\Omega)$, for all $|\alpha| \leq m$.
- We clearly have

$$\mathscr{D}'(\Omega) \supseteq L^2(\Omega) = H^0(\Omega) \supseteq H^1(\Omega) \supseteq H^2(\Omega) \supseteq \cdots.$$

Inner Product in Sobolev Space

- The **inner product** of two functions $\phi_1, \phi_2 \in H^m(\Omega)$ is defined by

$$(\phi_1, \phi_2)_m = \sum_{|\alpha| \le m} \int_\Omega \partial^\alpha \phi_1(x) \partial^\alpha \overline{\phi}_2(x) dx.$$

- The **norm** of $\phi \in H^m(\Omega)$ is given by

$$\|\phi\|_{m,2} = \sqrt{(\phi, \phi)_m} = \left[\sum_{|\alpha| \le m} \int_\Omega |\partial^\alpha \phi(x)|^2 dx \right]^{1/2}.$$

- The subscript 2 indicates the use of the L^2 norm.
- We have

$$\|\phi\|_{m,2}^2 = \sum_{|\alpha| \le m} \|\partial^\alpha \phi\|_2^2, \quad m \in \mathbb{N}_0.$$

- When $m = 0$, $\|\phi\|_{0,2} = \|\phi\|_2$.

The Banach Space $H^{m,p}(\Omega)$

- If L_2 is replaced by L^p in the definition, $1 \leq p < \infty$, then the resulting norm
$$\|\phi\|_{m,p} = \left[\sum_{|\alpha|\leq m} \|\partial^\alpha \phi\|_p^p\right]^{1/p}$$
generates the Banach spaces $H^{m,p}(\Omega)$.
- $H^{m,p}(\Omega)$ is a Hilbert space only when $p = 2$.
- We restrict to this case and write $H^m(\Omega)$ for $H^{m,2}(\Omega)$.
- Similarly $\|\cdot\|_{m,2}$ will be abbreviated to $\|\cdot\|_m = \|\cdot\|_{H^m}$, which should not be confused with the L^p norm $\|\cdot\|_p = \|\cdot\|_{L^p}$.
- Only L^2 will be relevant to the Hilbert space theory of distributions, and the L^2 norm $\|\cdot\|_2$ will henceforth be designated by $\|\cdot\|_0$.
- Thus, we can write
$$\|\phi\|_m = \left[\sum_{|\alpha|\leq m} \|\partial^\alpha \phi\|_0^2\right]^{1/2} = \left[\sum_{|\alpha|\leq m} \int_\Omega |\partial^\alpha \phi(x)|^2 dx\right]^{1/2}.$$

Example

- Let Ω be an open interval in $\mathbb{R}$ containing the closed interval $[a,b]$.

  Suppose f is the characteristic function of $[a,b]$.

  Then, for all $\phi \in \mathscr{D}(\Omega)$,

$$\langle f', \phi \rangle = \int_\Omega f(x)\phi'(x)dx = \int_a^b \phi'(x)dx = \phi(b) - \phi(a).$$

  So $f' = \delta_a - \delta_b$.

  Consequently, $f \notin H^1(\Omega)$.

  On the other hand, if:
  - Ω is bounded;
  - f is continuous on Ω;
  - f' is bounded except at a finite number of points in Ω,

  then $f \in H^1(\Omega)$.

$H^m(\Omega)$ is a Hilbert Space

Theorem

$H^m(\Omega)$ is a Hilbert space.

- $H^m(\Omega)$ is a normed linear space whose norm is derived from an inner product.

  So it suffices to show that $H^m(\Omega)$ is complete.

  Let (ϕ_k) be a Cauchy sequence in $H^m(\Omega)$.

  Thus, we have $\|\phi_k - \phi_j\|_m \to 0$.

  This implies, by the definition of $\|\cdot\|_m$, that

  $$\|\partial^\alpha \phi_k - \partial^\alpha \phi_j\|_0 \to 0, \quad |\alpha| \leq m.$$

  So the sequence $(\partial^\alpha \phi_k)$ is a Cauchy sequence in $L^2(\Omega)$, $\alpha \leq m$.

$H^m(\Omega)$ is a Hilbert Space (Cont'd)

- The sequence $(\partial^\alpha \phi_k)$ is a Cauchy sequence in $L^2(\Omega)$, $\alpha \leq m$.

  Since $L^2(\Omega)$ is complete, the sequence $(\partial^\alpha \phi_k)$ converges in $L^2(\Omega)$ to some function $\phi_\alpha \in L^2(\Omega)$.

  By the Schwarz inequality

$$\left| \int_\Omega [\partial^\alpha \phi_k(x) - \phi_\alpha(x)] \psi(x) dx \right| \leq \|\partial^\alpha \phi_k - \phi_\alpha\|_0 \|\psi\|_0, \ \psi \in \mathscr{D}(\Omega).$$

  We now see that, as $k \to \infty$,

$$\int_\Omega \partial^\alpha \phi_k(x) \psi(x) dx \to \int_\Omega \phi_\alpha(x) \psi(x) dx, \ \psi \in \mathscr{D}(\Omega).$$

$H^m(\Omega)$ is a Hilbert Space (Cont'd)

- But $f_k \to f$ in $L^2(\Omega)$ implies $f_k \to f$ in $\mathscr{D}'(\Omega)$.
  So we have, for all $\phi \in \mathscr{D}(\Omega)$,

$$\begin{array}{rcl} \int_\Omega \partial^\alpha \phi_k(x)\psi(x)dx & = & \langle \partial^\alpha \phi_k, \psi \rangle \\ & = & (-1)^{|\alpha|}\langle \phi_k, \partial^\alpha \psi \rangle \\ & \to & (-1)^{|\alpha|}\langle \phi, \partial^\alpha \psi \rangle. \\ & & (\phi = \lim \phi_k \text{ in } L^2(\Omega).) \end{array}$$

  But

$$(-1)^{|\alpha|}\langle \phi, \partial^\alpha \psi \rangle = \langle \partial^\alpha \phi, \psi \rangle = \int_\Omega \partial^\alpha \phi(x)\psi(x)dx.$$

  Hence, $\phi_\alpha = \partial^\alpha \phi$.
  So ϕ, which is clearly in $H^m(\Omega)$, is the limit of ϕ_k in the H^m norm.

Redefining Sobolev Space

- To apply Fourier transformation to $H^m(\Omega)$, we take $\Omega = \mathbb{R}^n$.
  A function f is in $H^m = H^m(\mathbb{R}^n)$ if and only if $\partial^\alpha f$ is in L^2, $|\alpha| \le m$.
  Hence $H^m \subseteq L^2 \subseteq \mathscr{S}'$. Using preceding results, we get

$$\begin{array}{rcl} \|f\|_m & = & [\sum_{|\alpha|\le m} \|\partial^\alpha f\|_0^2]^{1/2} \\ & = & (2\pi)^{-\frac{n}{2}}[\sum_{|\alpha|\le m} \|\widehat{\partial^\alpha f}\|_0^2]^{1/2} \\ & = & (2\pi)^{-\frac{n}{2}}[\sum_{|\alpha|\le m} \|\xi^\alpha \widehat{f}\|_0^2]^{1/2} \\ & \le & c_1 \|(1+|\xi|^2)^{\frac{1}{2}m} \widehat{f}\|_0, \end{array}$$

  where c_1 is a positive constant (which depends on m).
  Similarly, there is a positive constant c_2, such that

$$\|(1+|\xi|^2)^{\frac{1}{2}m} \widehat{f}\|_0 \le c_2 (2\pi)^{-n/2} \left[\sum_{|\alpha|\le m} \|\xi^\alpha \widehat{f}\|_0^2\right]^{1/2} = c_2 \|f\|_m.$$

  So a tempered distribution f is in H^m if and only if $(1+|\xi|^2)^{\frac{1}{2}m} \widehat{f} \in L^2$.

Redefining Sobolev Space (Cont'd)

- We redefine the **Sobolev space** H^m, $m \in \mathbb{N}_0$ as the space of tempered distributions $f \in \mathscr{S}'$, such that
$$(1+|\xi|^2)^{\frac{1}{2}m}\widehat{f} \in L^2.$$
The scalar product is
$$(f,g)_{\widehat{m}} = \int (1+|\xi^2)^m \widehat{f}(\xi)\overline{\widehat{g}}(\xi)d\xi.$$
The norm is
$$\|f\|_{\widehat{m}} = \left[\int (1+|\xi|^2)^m |\widehat{f}(\xi)|^2 d\xi\right]^{1/2}.$$
- Note that the norms $\|\cdot\|_m$ and $\|\cdot\|_{\widehat{m}}$, though equivalent, are not equal.
- In particular we note, by Plancherel's Theorem,
$$\|f\|_{\widehat{0}} = (2\pi)^{\frac{n}{2}}\|f\|_0.$$

The Space $H^s(\mathbb{R}^n)$

- The definition of H^m in the preceding slide is equivalent to the original when $m \geq 0$ is an integer and $\Omega = \mathbb{R}^n$, but allows for an extension to any real.

Definition

For any $s \in \mathbb{R}$, we define $H^s(\mathbb{R}^n)$ to be the tempered distributions whose Fourier transforms are square-integrable with respect to the measure $(1+|\xi|^2)^s d\xi$, i.e.,

$$H^s(\mathbb{R}^n) = \{f \in \mathscr{S}' : (1+|\xi|^2)^{\frac{1}{2}s}\widehat{f}(\xi) \in L^2(\mathbb{R}^n)\}.$$

H^s is a Hilbert Space

Claim: H^s, equipped with the inner product

$$(f,g)_{\hat{s}} = \int (1+|\xi|^2)^s \hat{f}(\xi)\overline{\hat{g}}(\xi)d\xi$$

and the norm

$$\|f\|_{\hat{s}} = \sqrt{(f,f)_{\hat{s}}} = \left[\int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi\right]^{1/2},$$

is a Hilbert space.

Suppose (f_k) is a Cauchy sequence in H^s.

Then $(1+|\xi|^2)^{\frac{1}{2}s}\hat{f}_k$ is a Cauchy sequence in L^2.

By the completeness of L^2, $(1+|\xi|^2)^{\frac{1}{2}s}\hat{f}_k$ converges to some $g \in L^2$.

Therefore, $f_k \to f = \mathscr{F}^{-1}[(1+|\xi|^2)^{-s/2}g]$ in H^s, for every $s \in \mathbb{R}$.

But $(1+|\xi|^2)^{\frac{1}{2}s}\hat{f} = g$ is in L^2. Thus, f is in H^s.

Inclusions Between Sobolev Spaces

- Suppose $s \geq 0$.

  Then
  $$\|f\|_0 = (2\pi)^{-\frac{n}{2}} \|f\|_{\widehat{0}} \leq (2\pi)^{-\frac{n}{2}} \|f\|_{\widehat{s}}.$$

  Thus, $H^s \subseteq L^2$.

- In general, we have the following inclusion relations.

Theorem

For all real numbers s and t with $s > t$, we have $\mathscr{S} \subseteq H^s \subseteq H^t \subseteq \mathscr{S}'$ and the identity mappings $\mathscr{S} \to H^s \to H^t \to \mathscr{S}'$ are continuous. Furthermore, $\mathscr{S}$ is dense in H^s, for all $s \in \mathbb{R}$.

- The inclusion relations between the spaces as sets are obvious.

  It is also clear that if $\phi_k \to 0$ in $\mathscr{S}$ then $\|\phi_k\|_{\widehat{s}} \to 0$, for any $s \in \mathbb{R}$.

  But $\|\phi_k\|_{\widehat{t}} \leq \|\phi_k\|_{\widehat{s}}$, whenever $t < s$. This implies that $\|\phi_k\|_{\widehat{t}} \to 0$.

Inclusions Between Sobolev Spaces (Cont'd)

- For any $\psi \in \mathscr{S}$, $\mathscr{F}^{-1}(\psi)$ is also in $\mathscr{S}$.

  We have

$$\begin{aligned} \langle \phi_k, \psi \rangle &= \langle \widehat{\phi}_k, \mathscr{F}^{-1}(\psi) \rangle \\ &= \langle (1+|x|^2)^{\frac{t}{2}} \widehat{\phi}_k, (1+|x|^2)^{-\frac{t}{2}} \mathscr{F}^{-1}(\psi) \rangle \\ &\leq \|\phi_k\|_{\widehat{t}} \|(1+|x|^2)^{-\frac{t}{2}} \mathscr{F}^{-1}(\psi)\|_0. \end{aligned}$$

  This means that $\phi_k \to 0$ in $\mathscr{S}'$ when $\phi_k \to 0$ in H^t.

Inclusions Between Sobolev Spaces (Cont'd)

- Finally, if $f \in H^s$, then $(1+|\xi|^2)^{\frac{s}{2}}\widehat{f}$ is in L^2.

  But $\mathscr{S}$ is dense in L^2.

  So there is a sequence (ϕ_k) in $\mathscr{S}$, such that $\phi_k \to (1+|\xi|^2)^{\frac{s}{2}}\widehat{f}$ in L^2.

  But $\widehat{\psi}_k = (1+|\xi|^2)^{-\frac{s}{2}}\phi_k$ is in $\mathscr{S}$, for every $s \in \mathbb{R}$.

  Hence,
$$\|(1+|\xi|^2)^{\frac{s}{2}}(\widehat{f}-\widehat{\psi}_k)\|_0 \to 0.$$
  So $\|f-\psi_k\|_{\widehat{s}} \to 0$, where (ψ_k) is clearly a sequence in $\mathscr{S}$.
- We know that C_0^∞ is dense in $\mathscr{S}$.

  By the theorem, it is also dense in H^s.

Corollary

H^s is the completion of C_0^∞ under the norm $\|\cdot\|_{\widehat{s}}$.

The Topological Dual of H^s

- As a Hilbert space H^s has a dual space with respect to the inner product $(f,g) \mapsto (f,g)_{\hat{s}}$ which may be identified with H^s.
- That space is not the same as its dual in the bilinear form $(f,g) \mapsto \langle f,g\rangle = (f,\overline{g})_0$ except when $s=0$ and the space is real.

Characterization of the Topological Dual of H^s

Theorem

H^{-s} represents the topological dual of H^s, for all $s \in \mathbb{R}$, and

$$|\langle f, \phi\rangle| \le \frac{1}{(2\pi)^n}\|f\|_{\widehat{-s}}\|\phi\|_{\widehat{s}}, \quad \text{for all } \phi \in H^s,\ f \in H^{-s}.$$

- A function $f \in \mathscr{S}$ defines a continuous linear functional on $\mathscr{S}$ by setting, for all $\phi \in \mathscr{S}$,

$$\begin{aligned} T_f(\phi) &= \langle f, \phi\rangle \\ &= \int f(x)\phi(x)dx \\ &= \tfrac{1}{(2\pi)^n}\int \widehat{f}(\xi)\widehat{\phi}(-\xi)d\xi, \end{aligned}$$

where the last equality follows from Parseval's relation.

The Topological Dual of H^s (Cont'd)

- We write

$$\widehat{f}(\xi)\widehat{\phi}(-\xi) = (1+|\xi|^2)^{-\frac{s}{2}}\widehat{f}(\xi)(1+|\xi|^2)^{\frac{s}{2}}\widehat{\phi}(-\xi).$$

Using Schwarz' inequality, we obtain

$$|\langle f,\phi\rangle| \leq \frac{1}{(2\pi)^n}\|f\|_{\widehat{-s}}\|\phi\|_{\widehat{s}}.$$

Since $\mathscr{S}$ is dense in H^s, for all s, the bilinear form $\langle f,\phi\rangle$ can be extended from $\mathscr{S}\times\mathscr{S}$ to $H^{-s}\times H^s$, with the inequality still valid.

Since the dual of H^s is a subset of $\mathscr{S}'$, H^{-s} is a subset of $(H^s)'$.

The Topological Dual of H^s (Cont'd)

- To show that $(H^s)' \subseteq H^{-s}$, let $T \in (H^s)'$ be arbitrary.

  Then, by the Riesz Representation Theorem for the Hilbert space H^s, there is a function $f \in H^s$, such that, for all $\phi \in H^s$,

$$\begin{aligned} T(\phi) &= (\phi, f)_{\hat{s}} \\ &= \int (1+|\xi|^2)^s \widehat{\phi}(\xi) \overline{\widehat{f}}(\xi) d\xi \\ &= \int \phi(x) \overline{h}(x) dx, \end{aligned}$$

  where $h(x) = (2\pi)^n \mathscr{F}^{-1}((1+|\xi|^2)^s \widehat{f}(\xi))$.

  Now the function

$$(1+|\xi|^2)^{-\frac{s}{2}} \widehat{h}(\xi) = (2\pi)^n (1+|\xi|^2)^{\frac{s}{2}} \widehat{f}(\xi)$$

  is in L^2. This means that $\overline{h}$ is in H^{-s}.

  Moreover, it represents T in the sense that $T(\phi) = \langle \overline{h}, \phi \rangle$, for all $\phi \in H$.

Characterization of Distributions in H^{-m}

- When s is a nonnegative integer we have the following characterization of H^{-s}.

Theorem

$f \in H^{-m}$, where $m \in \mathbb{N}_0$, if and only if f is a finite sum of derivatives of order less than or equal to m of L^2 functions.

- Let $f \in H^{-m}$.
  The function $(1+|\xi|^2)^{-\frac{m}{2}}\widehat{f}(\xi)$ is in L^2. Then

$$\begin{aligned}(1+|\xi|^2)^{\frac{m}{2}} &\leq (1+|\xi|)^m \\ &= [1+(\xi_1^2+\cdots+\xi_n^2)^{1/2}]^m \\ &\leq (1+|\xi_1|+\cdots+|\xi_n|)^m \\ &= 1+\sum_{1\leq|\alpha|\leq m} c_\alpha|\xi^\alpha|,\end{aligned}$$

  where c_α are nonnegative integers and α is a multi-index in $\mathbb{N}_0^n$.

Characterization of Distributions in H^{-m} (Cont'd)

- We got
$$(1+|\xi|^2)^{\frac{m}{2}} \le 1 + \sum_{1\le|\alpha|\le m} c_\alpha |\xi^\alpha|.$$
Let
$$\widehat{g}(\xi) = \left(1 + \sum_{1\le|\alpha|\le m} c_\alpha |\xi^\alpha|\right)^{-1} \widehat{f}(\xi).$$
The preceding inequality implies that $\widehat{g}(\xi)$ satisfies
$$|\widehat{g}(\xi)| \le (1+|\xi|^2)^{-\frac{m}{2}} |\widehat{f}(\xi)|.$$
Hence g is also in L^2.

Characterization of Distributions in H^{-m} (Cont'd)

- Now we can write

$$\widehat{f}(\xi) = \left(1 + \sum_{1 \le |\alpha| \le m} c_\alpha |\xi^\alpha|\right) \widehat{g}(\xi) = \sum_{|\alpha| \le m} \xi^\alpha \widehat{g}_\alpha(\xi),$$

where $\widehat{g}_\alpha(\xi) = \begin{cases} \widehat{g}(\xi), & \text{when } |\alpha| = 0 \\ c_\alpha |\xi^\alpha| \xi^{-\alpha} \widehat{g}(\xi), & \text{when } 1 \le |\alpha| \le m \end{cases}.$

Clearly $\widehat{g}_\alpha$ is in L^2 whenever $\widehat{g}$ is in L^2.

Taking the inverse Fourier transform of $\widehat{f}$, gives

$$f(x) = \sum_{|\alpha| \le m} D^\alpha g_\alpha(x),$$

with $g_\alpha \in L^2$, for all $|\alpha| \le m$.

Conversely, assume $f = \sum_{|\alpha| \le m} \partial^\alpha g_\alpha$, with $g_\alpha \in L^2$.

Then $\partial^\alpha g_\alpha \in H^{-m}$, for all $|\alpha| \le m$. Consequently $f \in H^{-m}$.

The Spaces H^m, $m \in \mathbb{Z}$

- Every $g \in L^2$ is a distribution of order 0.

  Apply the inequality

  $$|\langle f, \phi \rangle| \leq M|\phi|_0$$

  where $M = \|f\|_0[\text{volume}(\text{supp}\phi)]$.

Corollary

Every element of H^m, $m \in \mathbb{Z}$, is a distribution of finite order.

- This result, of course, also follows from the inclusion $H^s \subseteq \mathscr{S}'$, for all $s \in \mathbb{R}$.

Example

- We know that
  - $\widehat{\delta} = 1$;
  - $(1 + |\xi|^2)^{\frac{1}{2}s} \in L^2(\mathbb{R})$ provided $s < -\frac{1}{2}n$.

  It follows that $\delta \in H^s$, for all $s < -\frac{1}{2}n$.

  When $n = 1$, the Dirac measure δ lies in $H^{-1}(\mathbb{R})$.

  Consequently, it a sum of the form $f_1 + f_2'$ with $f_1, f_2 \in L^2(\mathbb{R})$.

  One possible choice for these functions is given by

$$f_1(x) = \frac{1}{2} e^{-|x|}, \quad f_2(x) = \frac{1}{2} e^{-|x|} \operatorname{sgn} x.$$

  One uses the facts that
  - $(\operatorname{sgn} x)^2 = 1$ almost everywhere;
  - $(\operatorname{sgn} x)' = 2\delta$.

The Sobolev Imbedding Theorem

- For $s \geq 0$, H^s is a subspace of L^2 and its functions would be expected to achieve higher degrees of smoothness with increasing values of s, as their derivatives of higher order have to lie in L^2.

The Sobolev Imbedding Theorem

If $s > \frac{n}{2}$, then $H^s \subseteq C^0$ with continuous injection.

- The function $(1+|\xi|^2)^{-s}$ is integrable if and only if $s > \frac{1}{2}n$. Therefore, when $s > \frac{1}{2}n$ and $f \in H^s$, we have

$$\begin{aligned} \int |\widehat{f}(\xi)| d\xi &= \int (1+|\xi|^2)^{-\frac{s}{2}} (1+|\xi|^2)^{\frac{s}{2}} |\widehat{f}(\xi)| d\xi \\ &\leq \|f\|_{\widehat{s}} \int (1+|\xi|^2)^{-s} d\xi. \end{aligned}$$

(by the Schwarz inequality)

This implies that $\widehat{f}$ is in $L^1 \subseteq \mathscr{S}'$.

So the inverse Fourier of f exists and satisfies $f(x) = \frac{1}{(2\pi)^n} \widehat{\widehat{f}}(-x)$.

The Sobolev Imbedding Theorem (Cont'd)

- We saw $\widehat{f}$ is in L^1.

  So its Fourier transform

  $$\widehat{\widehat{f}}(x) = \int e^{-i\langle x,\xi\rangle}\widehat{f}(\xi)d\xi$$

  is continuous on $\mathbb{R}^n$.

  It follows that f is also continuous on $\mathbb{R}^n$.

  We show that the topology of H^s, $s > \frac{1}{2}n$, is stronger than that of C^0.

  Let $\|f_k\|_{\widehat{s}} \to 0$. By the preceding inequality, $\|\widehat{f_k}\|_{L^1} \to 0$.

  But the Fourier transformation is continuous from L^1 to C^0.

  Hence $f_k(x) = \frac{1}{(2\pi)^n}\widehat{\widehat{f}}_k(-x) \to 0$ in C^0.

Remarks on the Imbedding Theorem

- As an element of H^s, f is really a class of functions which are equal almost everywhere.

  By writing equations $f(x) = \frac{1}{(2\pi)^n}\hat{\hat{f}}(-x)$, we are choosing the continuous representative of that class.

  This is actually the sense in which the inclusion $H^s \subseteq C^0$ should be understood in the theorem.
- We know $\hat{f} \in L^1$.

  The Riemann-Lebesgue Lemma yields $\hat{\hat{f}} \to 0$ as $|x| \to \infty$.

  Thus, when $s > \frac{1}{2}n$, H^s actually lies in the subspace C_∞^0 of C^0 which consists of all continuous functions on $\mathbb{R}^n$ that vanish at ∞.

More on the Sobolev Imbedding Theorem

Corollary

If $s > \frac{1}{2}n + k$, where k is a nonnegative integer, then $H^s \subseteq C^k$, with continuous injection.

- If $f \in H^s$, then $\partial^\alpha f \in H^{s-|\alpha|}$.

  Suppose $s > \frac{1}{2}n + k$ and $|\alpha| \leq k$.

  Then $s - |\alpha| \geq s - k > \frac{1}{2}n$.

  By the theorem, $\partial^\alpha f \in C^0$.

  Now the distributional derivative coincides with the ordinary derivative when it is continuous.

  We conclude that $f \in C^k$.

Example

- If $u(x) = e^{-|x|}$, $x \in \mathbb{R}$, then

$$\widehat{u}(\xi) = \frac{2}{1+\xi^2}.$$

So $u \in H^s$ if and only if $(1+\xi^2)^{\frac{1}{2}(s-2)} \in L^2$.

This yields that $u \in H^s$ if and only if $s < \frac{3}{2}$.

With $n = 1$, Sobolev's Imbedding Theorem guarantees the continuity of u but not its differentiability.

This is consistent with the fact that $e^{-|x|}$ is continuous but not differentiable on $\mathbb{R}$.

The Spaces H^∞ and $H^{-\infty}$

- Define
$$H^\infty(\mathbb{R}^n) = \bigcap_{s\in\mathbb{R}} H^s(\mathbb{R}^n), \quad H^{-\infty}(\mathbb{R}^n) = \bigcup_{s\in\mathbb{R}} H^s(\mathbb{R}^n).$$
- By the above corollary, H^∞ is a subspace of C^∞.
- A function ϕ in C^∞ lies in H^∞ if $\partial^\alpha \phi \in L^2$, for all $\alpha \in \mathbb{N}_0^n$.
- This means $\phi(x)$ and all its partial derivatives tend to 0 as $|x| \to \infty$.

The Topology on H^∞

- The topologies of H^∞ and $H^{-\infty}$ are defined so that the inclusion relations
$$H^\infty \subseteq H^s \subseteq H^{-\infty}$$
for any real number s become imbeddings.
- We define the topology of H^∞ to be the weakest locally convex topology such that the identity mapping from H^∞ to H^s is continuous for every s.
- This is the projective limit topology of $\{H^s : s \in \mathbb{R}\}$ introduced previously.

The Topology on $H^{-\infty}$

- We saw that for $s > t$, $H^s \subseteq H^t$, with continuous embeddings.
- As a result, $\bigcap_{|s|\le m} H^s = H^m$ and $\bigcup_{|s|\le m} H^s = H^{-m}$.
- We also saw that $(H^m)' = H^{-m}$.
- We conclude that
$$\Big(\bigcap_{|s|\le m} H^s\Big)' = \bigcup_{|s|\le m} H^s, \quad \text{for all } m \in \mathbb{N}.$$
- Therefore, $(H^\infty)' = H^{-\infty}$.
- This defines the topology of $H^{-\infty}$ as the inductive limit of the topologies on $\{H^s : s \in \mathbb{R}\}$.
- Recall that this is the strongest locally convex topology, such that the identity map from H^s to H^∞ is continuous, for every s.
- As we have seen in connection with $\mathscr{D}_F$ and $\mathscr{D}'_F$, these two methods of defining a topology on a linear space, that is the projective limit and the inductive limit, generally produce dual topological vector spaces.

Example

- Since $\mathscr{D} \subseteq \mathscr{S} \subseteq H^{\infty} \subseteq \mathscr{E}$ and $\mathscr{D}$ is dense in $\mathscr{E}$ we have the inclusions
$$\mathscr{E}' \subseteq H^{-\infty} \subseteq \mathscr{S}' \subseteq \mathscr{D}'.$$
- We relate, next, the order of a distribution in $H^s \cap \mathscr{E}'$ to s.
- From this, we can also obtain $\mathscr{E}' \subseteq H^{-\infty}$.

  Example: Let $T \in \mathscr{E}'$. Since every distribution with compact support is of finite order, suppose that the order of T is m. We prove that $T \in H^s$ if $s \leq -\frac{1}{2}n - m$.

  According to a previous theorem, $\widehat{T}$ is a C^{∞} function, given by $\widehat{T}(\xi) = \langle T_x, e^{-i\langle x,\xi\rangle}\rangle$.

  Since T is of order m, there exists a compact set $K \subseteq \mathbb{R}^n$ and a positive constant M, such that
$$|\widehat{T}(\xi)| \leq M \sum_{|\alpha| \leq m} \sup\{|\partial^{\alpha} e^{-i\langle x,\xi\rangle}| : x \in K\} \leq M \sum_{|\alpha| \leq m} |\xi^{\alpha}|.$$

Example (Cont'd)

- Therefore, for some positive constants M_1 and M_2

$$|\widehat{T}(\xi)|^2 \le M^2 \left(\sum_{|\alpha| \le m} |\xi^\alpha| \right)^2 \le M_1 \sum_{|\alpha| \le m} |\xi^\alpha|^2 \le M_2 (1 + |\xi|^2)^m.$$

The last inequality $M_1 \sum_{|\alpha| \le m} |\xi^\alpha|^2 \le M_2(1+|\xi|^2)^m$ follows from

$$\begin{array}{rcl} |\xi^\alpha|^2 & = & \xi_1^{2\alpha_1} \cdots \xi_n^{2\alpha_n} \\ & \le & (1+\xi_1^2+\cdots+\xi_n^2)^{\alpha_1} \cdots (1+\xi_1^2+\cdots+\xi_n^2)^{\alpha_n} \\ & = & (1+\xi_1^2+\cdots+\xi_n^2)^{\alpha_1+\cdots+\alpha_n} \\ & \stackrel{|a| \le m}{\le} & (1+|\xi|^2)^m. \end{array}$$

Hence, $(1+|\xi|^2)^s |\widehat{T}|^2 \le M_2(1+|\xi|^2)^{m+s}$.
So $T \in H^s$ when $(1+|\xi|^2)^{m+s} \in L^1$. I.e., when $m + s < -\frac{1}{2}n$.
Thus all distributions with compact support and zero order are contained in H^s for $s < -\frac{1}{2}n$. (δ is included in this set. So, in view of a previous example, this estimate cannot be made any sharper.)

Subsection 3

Some Properties of H^s Spaces

The Operator $P(D)$

- Sobolev spaces provide a way of measuring the differentiability properties of functions on $\mathbb{R}^n$.
- From the definition of the Sobolev space $H^s = H^s(\mathbb{R}^n)$, we have, for all $f \in H^s$,

$$\begin{array}{rcl} \|\partial_k f\|_{\widehat{s-1}} & = & \|\xi_k(1+|\xi|^2)^{\frac{1}{2}(s-1)}\widehat{f}\|_0 \\ & \leq & \|(1+|\xi|^2)^{\frac{s}{2}}\widehat{f}\|_0 \\ & = & \|f\|_{\widehat{s}}. \end{array}$$

- So the differential operator ∂_k, where $k \in \{1,\ldots,n\}$, is a continuous linear operator from H^s to H^{s-1}.
- So, for P a polynomial on $\mathbb{R}^n$ with constant coefficients and degree $\leq m$, $P(D)$ is a continuous linear operator from H^s to H^{s-m}.
- When the polynomial P has no zeros in $\mathbb{R}^n$, the mapping $P(D) : H^s \to H^{s-m}$ is also bijective, as the next example illustrates.

The Operator $k^2 - \Delta$

Claim: The operator $k^2 - \Delta$, $k \neq 0$, is a homeomorphism from H^{s+2} onto H^s.

(i) The continuity of $(k^2 - \Delta) : H^{s+2} \to H^s$ is obvious since this operator has constant coefficients.

(ii) To show that $(k^2 - \Delta)$ is bijective, let $(k^2 - \Delta)u = 0$, for some $u \in H^{s+2}$.
Then $(k^2 + |\xi|^2)\widehat{u} = 0$. Since $k \neq 0$, $\widehat{u} = 0$.
Hence, $u = 0$. So $(k^2 - \Delta)$ is injective.
Suppose $v \in H^s$. Then $u = \frac{\widehat{v}}{k^2 + |\xi|^2} \in \mathscr{S}'$. Also $(k^2 - \Delta)\mathscr{F}^{-1}(u) = v$.
Thus, if $\mathscr{F}^{-1}(u) \in H^{s+2}$, then $k^2 - \Delta$ is surjective.

The Operator $k^2 - \Delta$ (Cont'd)

(ii) For surjectivity, it suffices to show $\mathscr{F}^{-1}(u) \in H^{s+2}$.

This relies on the following.

$$\begin{array}{rcl} (1+|\xi|^2)^{\frac{1}{2}(s+2)}|\mathscr{F}(\mathscr{F}^{-1}(u))| & = & (1+|\xi|^2)^{\frac{1}{2}s+1}|u| \\ & = & (1+|\xi|^2)^{\frac{1}{2}s+1}\frac{|\widehat{v}|}{k^2+|\xi|^2} \\ & \le & c(1+|\xi|^2)^{\frac{1}{2}s+1}\frac{|\widehat{v}|}{1+|\xi|^2} \\ & = & c(1+|\xi|^2)^{\frac{s}{2}}|\widehat{v}|. \end{array}$$

Moreover, $(1+|\xi|^2)^{\frac{s}{2}}\widehat{v} \in L^2$, since $v \in H^s$.

(iii) We have seen that $k^2 - \Delta$ is a continuous bijection from H^{s+2} to H^s.

The spaces H^{s+2} and H^s are Banach spaces.

By the Open Mapping Theorem, $(k^2-\Delta)^{-1} : H^s \to H^{s+2}$ is continuous.

Therefore $k^2 - \Delta$ is a homeomorphism.

Multiplication of an H^s Distribution

- To allow the coefficients in the differential operator $P(D): H^s \to H^{s-m}$ to be functions, we investigate the feasibility of multiplying the elements of H^s by such functions.

Theorem

The mapping from $\mathscr{S} \times H^s$ into H^s, defined by $(\phi, u) \mapsto \phi u$, is bilinear and continuous on $\mathscr{S}$ and H^s separately.

- Note that $\langle \phi v, u\rangle = \langle v, \phi u\rangle$ for all $u \in H^s$, $v \in H^{-s}$ and $\phi \in \mathscr{S}$.
  So it suffices to consider the case when $s \geq 0$.
  Let ϕ and u be in $\mathscr{S}$.
  Then their Fourier transforms $\widehat{\phi}$ and $\widehat{u}$ are also in $\mathscr{S}$.
  With $\mathscr{F}(\phi u) = \frac{1}{(2\pi)^n}\widehat{\phi} * \widehat{u}$, we have

$$(1+|\xi|^2)^{\frac{s}{2}}|\mathscr{F}(\phi u)(\xi)| \leq \frac{1}{(2\pi)^n}\int (1+|\xi|^2)^{\frac{s}{2}}|\widehat{\phi}(\eta)\widehat{u}(\xi-\eta)|d\eta.$$

Multiplication of an H^s Distribution (Cont'd)

- Now we have

$$\begin{aligned} 1+|\xi|^2 &= 1+|\xi-\eta+\eta|^2 \\ &\le 1+|\xi-\eta|^2+2|\xi-\eta||\eta|+|\eta|^2 \\ &\le 1+|\xi-\eta|^2+2|\eta|(1+|\xi-\eta|^2)+|\eta|^2 \\ &\le (1+|\xi-\eta|^2)(1+|\eta|)^2. \end{aligned}$$

We can use this inequality and integrate with respect to ξ.

$$\begin{aligned} &\frac{1}{(2\pi)^n}\int(1+|\xi|^2)^{\frac{s}{2}}|\widehat{\phi}(\eta)\widehat{u}(\xi-\eta)|d\eta \\ &\le \frac{1}{(2\pi)^n}\int(1+|\xi-\eta|^2)^{\frac{s}{2}}|\widehat{u}(\xi-\eta)|(1+|\eta|)^s|\widehat{\phi}(\eta)|d\eta \\ &\le \frac{1}{(2\pi)^n}\|u\|_{\widehat{s}}\int(1+|\eta|)^s|\widehat{\phi}(\eta)|d\eta. \end{aligned}$$

Multiplication of an H^s Distribution (Cont'd)

- We obtained

$$\|\phi u\|_{\hat{s}} \leq \frac{1}{(2\pi)^n} \|u\|_{\hat{s}} \int (1+|\eta|)^s |\widehat{\phi}(\eta)| d\eta.$$

But $\mathscr{S}$ is dense in H^s.

So this inequality may be extended by continuity to all u in H^s.

So ϕu is in H^s and depends continuously on $\phi \in \mathscr{S}$ and $u \in H^s$.

Corollary

If P is a polynomial on $\mathbb{R}^n$, with coefficients in $\mathscr{S}$ and degree m, then $P(D)$ is a continuous linear differential operator from H^s to H^{s-m}.

Order of an Operator

- Given any real number t, a linear operator L defined on $H^{-\infty}$ is said to have **order** t if it maps H^s into H^{s-t}, for every $s \in \mathbb{R}$.
- The following list contains some examples.
  - The differential operator ∂^α has order $|\alpha|$.
  - Given a polynomial P of degree m, with coefficients in $\mathscr{S}$, the operator $P(D)$ has order m.
  - Let f be defined on $\mathbb{R}^n$ and bounded (almost everywhere). Then the mapping $u \mapsto v$, defined by $\widehat{v} = f\widehat{u}$, is an operator of order 0.
  - On the other hand, let $\widehat{v} = (1 + |\xi|^2)^{\frac{t}{2}}\widehat{u}$. Then the mapping $u \mapsto v$ is an operator of order t. The inverse operator has order $-t$.
  - By the preceding theorem, the mapping $u \mapsto fu$, with $f \in \mathscr{S}$, is an operator of order 0.

Convolutions

- Let $u \in H^s \subseteq \mathscr{S}'$ and $v \in \mathscr{E}'$.
- Then the convolution product $u * v$ is well-defined.
- Moreover, we have $\mathscr{F}(u * v) = \widehat{v}\widehat{u}$, with $\widehat{v}$ in C^∞.
- In general $\widehat{v}$ is not bounded on $\mathbb{R}^n$.
- So neither is $(1+|\xi|^2)^{\frac{t}{2}}\widehat{v}(\xi)$, for any $t \in \mathbb{R}$.
- Suppose we restrict $v \in \mathscr{E}'$ so that

$$\|v\|_{\widehat{t},\infty} = \sup_{\xi \in \mathbb{R}^n} (1+|\xi|^2)^{\frac{t}{2}} |\widehat{v}(\xi)| < \infty.$$

- Then the set $\{v \in \mathscr{E}' : \|v\|_{\widehat{t},\infty} < \infty\}$ is a linear subspace of $\mathscr{E}'$ on which $\|\cdot\|_{\widehat{t},\infty}$ defines a norm.
- The closure of this subspace in $\mathscr{S}'$ under the norm $\|\cdot\|_{\widehat{t},\infty}$ is a normed linear subspace of $\mathscr{S}'$, which we denote by $H^{t,\infty}$.

$$H^{t,\infty} = \{v \in \mathscr{S}' : \|v\|_{\widehat{t},\infty} < \infty\}.$$

$H^{s,\infty}$ vs. L^∞ and H^s vs. L^2

- The notation is suggested by that of the Banach space L^∞ of measurable functions on $\mathbb{R}^n$ which are bounded almost everywhere.
- The norm of $f \in L^\infty$ is defined as the essential supremum of $|f|$ on $\mathbb{R}^n$.
- It follows that
$$|f(x)| \leq \|f\|_{L^\infty} = \|f\|_\infty$$
holds almost everywhere in $\mathbb{R}^n$.
- The defining equation implies, for all $u \in H^{s,\infty}$,
$$\|u\|_{\widehat{s},\infty} = \|(1+|\xi|^2)^{\frac{s}{2}}\widehat{u}\|_\infty.$$
- This suggests that $H^{s,\infty}$ is related to L^∞ in the same way that H^s is related to L^2.

Spaces for Convolutions of Distributions

Theorem

The convolution $(u,v)\mapsto u*v$ is a bilinear mapping of $H^s\times H^{t,\infty}$ into H^{s+t} which is continuous on H^s and $H^{t,\infty}$ separately.

- Let $u\in H^s$ and $v\in H^{t,\infty}$. Then

$$(1+|\xi|^2)^{\frac{1}{2}(s+t)}\mathscr{F}(u*v)(\xi)=(1+|\xi|^2)^{\frac{s}{2}}\widehat{u}(\xi)(1+|\xi|^2)^{\frac{t}{2}}\widehat{v}(\xi).$$

$$\begin{aligned}\int(1+|\xi|^2)^{\frac{1}{2}(s+t)}|\mathscr{F}(u*v)(\xi)|d\xi &\\ &=\int(1+|\xi|^2)^{\frac{s}{2}}|\widehat{u}(\xi)|(1+|\xi|^2)^{\frac{t}{2}}|\widehat{v}(\xi)|d\xi\\ &\leq \sup_\xi(1+|\xi|^2)^{\frac{t}{2}}|\widehat{v}(\xi)|\int(1+|\xi|^2)^{\frac{s}{2}}|\widehat{u}(\xi)|d\xi.\end{aligned}$$

So we get $\|u*v\|_{\widehat{s+t}}\leq\|u\|_{\widehat{s}}\|v\|_{\widehat{t},\infty}$.

Corollary

When $u\in H^s$ and $v\in\mathscr{S}$, then $u*v\in H^\infty$.

Locally H^s Distributions

- Because the distributions in H^s for real values of s, are defined through their Fourier transforms, they are necessarily distributions in $\mathbb{R}^n$.
- We can also consider distributions "locally in" H^s.

Definition

Let Ω be an open subset of $\mathbb{R}^n$. A distribution $u \in \mathscr{D}'(\Omega)$ is said to be in $H^s_{\text{loc}}(\Omega)$ if, for every bounded open set ω in Ω, with $\overline{\omega} \subseteq \Omega$, there is a distribution $v \in H^s$, such that $u = v$ on ω.

- The distributions in $H^s_{\text{loc}}(\Omega)$ enjoy the smoothness properties of H^s on Ω without being subjected to its global integrability condition.
- Moreover, any distribution in $H^s_{\text{loc}}(\Omega)$ with compact support is necessarily in $H^s(\Omega)$.

Characterization of Locally H^s Distributions

Theorem

$u \in H^s_{\text{loc}}(\Omega)$ if and only if $\phi u \in H^s$, for every $\phi \in C_0^\infty(\Omega)$.

- Suppose $u \in H^s_{\text{loc}}(\Omega)$ and $\phi \in C_0^\infty(\Omega)$.
  Then there is a $v \in H^s$, such that $u = v$ on $\text{supp}\phi$.
  By the preceding theorem, ϕv lies in H^s. Thus, so does ϕu.
  Suppose, conversely, $\phi u \in H^s$, for all $\phi \in C_0^\infty(\Omega)$.
  Let ω be any bounded open set in Ω, whose closure lies in Ω.
  Then we can choose $\phi \in C_0^\infty(\Omega)$ with $\phi = 1$ on $\overline{\omega}$.
  Moreover, $u = \phi u \in H^s$ on ω.

Corollary

$H^s \subseteq H^s_{\text{loc}}(\Omega)$, for every $\Omega \subseteq \mathbb{R}^n$.

- When $u \in H^s$ and ϕ is any function in $C_0^\infty(\Omega)$, a previous theorem implies that $\phi u \in H^s$. From the preceding theorem, $u \in H^s_{\text{loc}}(\Omega)$.

$\bigcap_{s\in\mathbb{R}} H^s_{\text{loc}}(\Omega) = C^\infty(\Omega)$

- For any open set $\Omega \subseteq \mathbb{R}^n$, $\bigcap_{s\in\mathbb{R}} H^s_{\text{loc}}(\Omega) = C^\infty(\Omega)$.
  - (i) We show, first, that, if $s > \frac{1}{2}n + k$, then $H^s_{\text{loc}}(\Omega) \subseteq C^k(\Omega)$.
    Suppose $u \in H^s_{\text{loc}}(\Omega)$.
    For any $x \in \Omega$, let U be a bounded neighborhood of x.
    Let $\phi \in C_0^\infty(\Omega)$ be such that $\phi = 1$ on U.
    Then, by previous results, $\phi u \in H^s(\mathbb{R}^n) \subseteq C^k(\mathbb{R}^n)$.
    Since $\phi = 1$ on U, this implies that $u \in C^k(U)$, for every U.
    Therefore, $u \in C^k(\Omega)$. Thus, $\bigcap_{s\in\mathbb{R}} H^s_{\text{loc}}(\Omega) \subseteq C^\infty(\Omega)$.
  - (ii) We now show the inclusion in the other direction.
    Let $u \in C^\infty(\Omega)$.
    For any $\phi \in C_0^\infty(\Omega)$, the product ϕu is in $C_0^\infty \subseteq \mathscr{S} \subseteq H^s$, for all s.
    Thus, by the theorem, $u \in H^s_{\text{loc}}(\Omega)$, for every s.
    Therefore, $u \in \bigcap_{s\in\mathbb{R}} H^s_{\text{loc}}(\Omega)$.

$\bigcup H^s_{\text{loc}}(\Omega) = \mathscr{D}'_F(\Omega)$

- We show $\bigcup H^s_{\text{loc}}(\Omega) = \mathscr{D}'_F(\Omega)$, where $\mathscr{D}'_F(\Omega)$ is the space of distributions in $\mathscr{D}'(\Omega)$ of finite order.
  (i) Let $u \in H^s_{\text{loc}}(\Omega)$. Take $\phi \in \mathscr{D}(\Omega)$ with compact support K.
  There exists $v \in H^s$, such that $u = v$ in a neighborhood of K.
  Define u on $\mathscr{D}(\Omega)$ by $\langle u, \phi \rangle = \langle v, \phi \rangle$.
  It is straightforward to verify that u is a distribution in Ω.
  By a previous theorem, v can be expressed as a finite sum of derivatives of order $\leq |s| + 1$ of L^2 functions.
  So u has finite order.
  (ii) Let $u \in \mathscr{D}'_F(\Omega)$.
  By a previous theorem, u is a derivative of a continuous function in Ω.
  But any continuous function is locally square integrable.
  So u is locally a derivative of finite order, say m, of an L^2 function.
  For any $\phi \in C_0^\infty(\Omega)$, ϕu is also a finite sum of derivatives of order $\leq m$ of L^2 functions. Therefore, ϕu lies in H^{-m}.
  Thus, by the theorem, $u \in H^{-m}_{\text{loc}}$.

Elliptic Linear Differential Operators

- We saw that a linear differential operator L of order m, with coefficients in $\mathscr{S}$ maps H^s into H^{s-m}.
- We do not know whether $Lu \in H^s$ implies that $u \in H^{s+m}$, i.e., whether L^{-1} is an operator of order $-m$.
- This is not true in general.

  Consider, e.g., the equation $\partial_x \partial_y u = 0$ on $\mathbb{R}^2$.

  It is satisfied by the sum $u(x,y) = f(x) + g(y)$ of any pair of differentiable functions on $\mathbb{R}$.

  Thus, although $\partial_x \partial_y u \in H^\infty$, the function u is not necessarily in H^∞.
- When L is elliptic, we have a *local regularity theorem*.
- The linear differential operator

$$L = \sum_{|\alpha| \le m} c_\alpha \partial^\alpha$$

  is **elliptic** if $\sum_{|\alpha|=m} c_\alpha \xi^\alpha \neq 0$, whenever $\xi \neq 0$.

The Local Regularity Theorem

The Local Regularity Theorem

Let $L = \sum_{|\alpha| \le m} c_\alpha \partial^\alpha$ be a linear elliptic differential operator in Ω of order m with coefficients $c_\alpha \in C^\infty(\Omega)$. If $Lu = f \in H^s_{\text{loc}}(\Omega)$, for some $s \in \mathbb{R}$, then $u \in H^{s+m}_{\text{loc}}(\Omega)$.

- We only prove the simpler case in which the leading coefficients in L, i.e., those c_α in which $\alpha = m$, are constants.
  Let $\phi \in C_0^\infty(\Omega)$ be arbitrary and $t \le s + m - 1$.
  Claim: If $\psi u \in H^t$, for some $\psi \in C_0^\infty(\Omega)$ which is 1 on an open set containing $\text{supp}\phi$, then $\phi u \in H^{t+1}$.
  Consider the distribution $v = L(\phi u) - \phi L u = L(\phi u) - \phi f$.
  It has its support in $\text{supp}\phi$.
  So u may be replaced by ψu in this equation to give
  $$v = L(\phi\psi u) - \phi L(\psi u) = \sum_{|\alpha| \le m} c_\alpha [\partial^\alpha(\phi\psi u) - \phi \partial^\alpha(\psi u)].$$

The Local Regularity Theorem (Cont'd)

- We are working with

$$v = \sum_{|\alpha| \le m} c_\alpha [\partial^\alpha(\phi\psi u) - \phi\partial^\alpha(\psi u)].$$

Note that the derivatives of ψu of order m cancel out.

So this sum is a linear combination of derivatives of ψu of orders $\le m-1$ with coefficients in $C_0^\infty(\mathbb{R}^n)$.

Since $\psi u \in H^t$, we have $v \in H^{t-m+1}$.

Now $\phi f \in H^s$ and $t-m+1 \le s$. Thus, $\phi f \in H^{t-m+1}$.

We conclude that $L(\phi u) = v + \phi f \in H^{t-m+1}$.

From this we wish to conclude that $\phi u \in H^{t+1}$.

Write $L = P(\partial) + Q(\partial)$, where:

  - P is defined by $P(y) = \sum_{|\alpha|=m} c_\alpha y^\alpha$, $y \in \mathbb{R}^n$;
  - Q is a polynomial of degree $\le m-1$.

The Local Regularity Theorem (Cont'd)

- P, by assumption, has constant coefficients.

  So, for any $w \in H^s$,

$$\begin{array}{rcl}\mathscr{F}(P(\partial)w) & = & P(i\xi)\widehat{w} \\ & = & [|\xi|^{-m}(1+|\xi|^m)-|\xi|^{-m}]P(i\xi)\widehat{w} \\ & = & \mathscr{F}([P_2(\partial)-P_1(\partial)]w),\end{array}$$

  where $P_1(\partial)$, $P_2(\partial)$ are operators on $H^{-\infty}(\Omega)$ defined on $\mathbb{R}^n-\{0\}$ by

$$\mathscr{F}(P_1(\partial)w)=|\xi|^{-m}P(i\xi)\widehat{w}, \quad \mathscr{F}(P_2(\partial)w)=(1+|\xi|^m)|\xi|^{-m}P(i\xi)\widehat{w}.$$

  P is homogeneous of degree m whose only zero is $\xi=0$.

  So both $\frac{P(i\xi)}{|\xi|^m}$ and $\frac{|\xi|^m}{P(i\xi)}$ are bounded functions on $\mathbb{R}^n-\{0\}$.

  Hence, $P_1(\partial)$ and $P_1^{-1}(\partial)$ are operators of order 0.

The Local Regularity Theorem (Cont'd)

- On the other hand, we have

$$1+|\xi|^m = (1+|\xi|^2)^{\frac{m}{2}} \frac{1+|\xi|^m}{(1+|\xi|^2)^{\frac{m}{2}}},$$

where $(1+|\xi|^m)(1+|\xi|^2)^{-\frac{m}{2}}$ and its reciprocal are bounded on $\mathbb{R}^n$.

The mapping $w \mapsto z$ defined by $\widehat{z} = (1+|\xi|^2)^{\frac{m}{2}} \widehat{w}$ is of order m.

So the same is true of the mapping defined by

$$\widehat{z} = (1+|\xi|^2)^{\frac{m}{2}} g(\xi) \widehat{w},$$

where g and its inverse are bounded in $\mathbb{R}^n$.

So $P_2(\partial)$ is an operator of order m whose inverse has order $-m$.

The Local Regularity Theorem (Cont'd)

- We now have

$$\begin{array}{rcl}[P_2(\partial)-P_1(\partial)+Q(\partial)](\phi u) & = & [P(\partial)+Q(\partial)](\phi u) \\ & = & L(\phi u)\in H^{t-m+1}.\end{array}$$

We also have $\phi u=\phi\psi u$ and $\psi u\in H^t$.

So, by a previous theorem, $\phi u\in H^t$.

But $Q(\partial)$ has order $m-1$ and $P_1(\partial)$ has order 0.

So $[Q(\partial)-P_1(\partial)](\phi u)\in H^{t-m+1}$.

Therefore, $P_2(\partial)(\phi u)\in H^{t-m+1}$.

But $P_2^{-1}(\partial)$ has order $-m$.

Hence, $\phi u\in H^{t+1}$.

It remains to show that $\phi u\in H^{s+m}$.

The Local Regularity Theorem (Cont'd)

Claim: $\phi u \in H^{s+m}$.

Choose $\phi_0 \in C_0^\infty(\Omega)$, such that $\phi_0 = 1$ on a neighborhood of suppϕ.

Call that neighborhood U_0.

Now $\phi_0 u$ has compact support. So it lies in H^t, for some t.

We may take t to be $s+m-k$, for some positive integer k.

Choose open sets $U_1, \ldots, U_k$, such that:

- U_j properly contains $\overline{U}_{j+1}$, for $0 \le j \le k-1$;
- $\overline{U}_k = \text{supp}\phi$.

Finally, choose the C_0^∞ functions $\phi_1, \ldots, \phi_k$, such that:

- $\phi_j = 1$ on U_j and supp$\phi_j = \overline{U}_{j-1}$, for $1 \le j \le k-1$;
- $\phi_k = \phi$.

From the preceding argument, we conclude that

$$\phi_1 u \in H^{t+1}, \phi_2 u \in H^{t+2}, \ldots, \phi_k u = \phi u \in H^{t+k} = H^{s+m}.$$

Consequences of the Local Regularity Theorem

- Denote $\bigcap H^s_{\text{loc}}(\Omega)$ by $H^\infty_{\text{loc}}(\Omega)$ and $\bigcup H^s_{\text{loc}}(\Omega)$ by $H^{-\infty}_{\text{loc}}(\Omega)$.
- We know that $H^\infty_{\text{loc}}(\Omega) = C^\infty(\Omega)$ and $H^{-\infty}_{\text{loc}}(\Omega) = \mathscr{D}'_F(\Omega)$.
- Thus, the theorem yields the following.

Corollary

If $Lu \in C^\infty(\Omega)$, then $u \in C^\infty(\Omega)$. Hence any solution of the homogeneous equation $Lu = 0$ is in $C^\infty(\Omega)$. In particular, every harmonic distribution in $\mathscr{D}'(\Omega)$ is a C^∞ harmonic function in Ω.

Corollary

Any fundamental solution of L, i.e., a solution of $LE = \delta$ on $\mathbb{R}^n$, is infinitely differentiable on $\mathbb{R}^n - \{0\}$.

Subsection 4

More on the Space $H^m(\Omega)$

The Space $H^m(\Omega)$

- $H^m(\Omega)$ is the Hilbert space of functions on Ω, such that

$$\partial^\alpha u \in L^2(\Omega), \quad \text{for all } |\alpha| \le m.$$

- It is equipped with the norm

$$\|u\|_m = \left[\sum_{|\alpha|\le m} \|\partial^\alpha u\|_0^2\right]^{1/2}, \quad m \in \mathbb{N}_0.$$

- A preceding theorem implies that $H^1(\mathbb{R}) \subseteq C^0(\mathbb{R})$.
- It is shown, next, that the same inclusion holds when $\mathbb{R}$ is replaced by any open interval in $\mathbb{R}$.

$H^1(a,b) \subseteq C^0(a,b)$

- Let $f \in H^1(a,b)$.
  Define the function g on (a,b) by $g(x) = \int_a^x f'(t)dt$.
  Then g is continuous and $g' = f'$ in the sense of distributions.
  In fact, for all $\phi \in \mathscr{D}(a,b)$,

$$\begin{array}{rcl} \langle g', \phi \rangle & = & -\langle g, \phi' \rangle \\ & = & -\int_a^b [\int_a^x f'(t)dt]\phi'(x)dx \\ & = & -\int_a^b \int_a^b H(x-t)f'(t)\phi'(x)dtdx \\ & = & -\int_a^b \left[\int_t^b \phi'(x)dx\right] f'(t)dt \\ & = & \int_a^b \phi(t)f'(t)dt \\ & = & \langle f', \phi \rangle. \end{array}$$

  Therefore, $g = f + \text{constant}$ and f is continuous a.e. in (a,b).
  Thus, $H^1(a,b) \subseteq C^0(a,b)$.
- The analogous statement does not hold when $n \geq 2$.

The Space $H_0^m(\Omega)$

- We have seen that $C_0^\infty(\mathbb{R}^n)$ is dense in $H^m(\mathbb{R}^n)$.
- This is not true of Ω in general.
- It is not even true that $C^\infty(\overline{\Omega}) \cap H^m(\Omega)$ is dense in $H^m(\Omega)$, where $C^\infty(\overline{\Omega})$ denotes the restriction to $\overline{\Omega}$ of the functions in $C^\infty(\mathbb{R}^n)$, unless $\partial\Omega$ is smooth enough.
- The advantage of $\mathbb{R}^n$ in this respect is that it has no boundary.
- We now define $H_0^m(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$.
- So we have $H_0^m(\mathbb{R}^n) = H^m(\mathbb{R}^n)$.
- In general, however, $H_0^m(\Omega)$ is a proper closed subspace of $H^m(\Omega)$.
- So $H_0^m(\Omega)$ is a Hilbert space in the induced structure.

$H^1_0(\Omega)$ versus $H^1(\Omega)$

- We show $H^1_0(\Omega) \neq H^1(\Omega)$, if Ω is a bounded set in $\mathbb{R}^n$.

  Claim: Let $u \in H^1_0(\Omega)$. Define u_0 in $\mathbb{R}^n$ by

$$u_0(x) = \begin{cases} u(x), & \text{if } x \in \Omega \\ 0, & \text{if } x \in \mathbb{R}^n - \Omega \end{cases}.$$

  Then $u_0 \in H^1(\mathbb{R}^n)$.

  Let ϕ be in $C_0^\infty(\Omega)$. Then $\phi_0 \in C_0^\infty(\mathbb{R}^n)$ and $\|\phi_0\|_1 = \|\phi\|_1$.

  Consider the map $\lambda_1 : C_0^\infty(\Omega) \to H^1(\mathbb{R}^n)$, defined by $\lambda_1(\phi) = \phi_0$.

  It follows that λ_1 is a linear isometry which extends by continuity to a continuous linear map from $H^1_0(\Omega)$ to $H^1(\mathbb{R}^n)$.

  Now $u \in H^1_0(\Omega)$.

  So there is a sequence (u_k) in $C_0^\infty(\Omega)$ which converges to u in $H^1(\Omega)$.

$H_0^1(\Omega)$ versus $H^1(\Omega)$ (Cont'd)

- By the continuity of λ_1, $\lambda_1(u_k) \to \lambda_1(u)$ in $H^1(\mathbb{R}^n)$.

  Hence, $\lambda_1(u_k) \to \lambda_1(u)$ in $L^2(\mathbb{R}^n)$.

  Consequently, there is a subsequence $(u_{k'})$ of (u_k), such that $\lambda_1(u_{k'}) \to \lambda_1(u)$ a.e. in $\mathbb{R}^n$.

  Hence, $u_0 = \lambda_1(u)$ lies in $H^1(\mathbb{R}^n)$.

  To finish the proof, let:
  - Ω be a bounded open set in $\mathbb{R}^n$;
  - $u = 1$ on Ω.

  Then $u \in H^1(\Omega)$.

  By the first example of the set, $u_0 \notin H^1(\mathbb{R}^n)$.

  Therefore, by the claim, $u \notin H_0^1(\Omega)$.

  We conclude that $H_0^1(\Omega) \neq H^1(\Omega)$.

The Operator λ_m

- Let Ω be an open set in $\mathbb{R}^n$ and $u \in H_0^m(\Omega)$.

  Define $\lambda_m(u) = u_0$, where, as before,

  $$u_0(x) = \begin{cases} u(x), & \text{if } x \in \Omega \\ 0, & \text{if } x \in \mathbb{R}^n - \Omega \end{cases}.$$

  Then $\lambda_m(u) \in H^m(\mathbb{R}^n)$.

  Moreover, by a previous result, $\lambda_m(u) \in C^k(\mathbb{R}^n)$, if $m > \frac{1}{2}n + k$.

  But $\lambda_m(u) = u$ on Ω.

  So $H_0^m(\Omega) \subseteq C^k(\overline{\Omega})$ when $m > \frac{1}{2}n + k$.

  Consider the special case when $m = 0$.

  We know that $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$.

  So, in this case, $H_0^0(\Omega)$ coincides with $H^0(\Omega)$.

The "Negative Norm"

- Any function $v \in H_0^0(\Omega) = L^2(\Omega)$ defines a continuous linear functional T_v on $H_0^m(\Omega)$ by

$$T_v(u) = \langle v, u \rangle = (v, \overline{u})_0, \quad \text{for all } u \in H_0^m(\Omega).$$

By the Schwarz inequality, $|\langle v, u \rangle| \leq \|v\|_0 \|u\|_0 \leq \|v\|_0 \|u\|_m$.
So T_v is bounded by $\|v\|_0$.
We define the "negative norm" of $v \in L^2(\Omega)$ by

$$\|v\|_{-m} = \sup_{u \in H_0^m(\Omega)} \frac{|\langle v, u \rangle|}{\|u\|_m}.$$

By definition, $|\langle v, u \rangle| \leq \|v\|_{-m} \|u\|_m$.
Now $\|u\|_0 \leq \|u\|_m$. So we have

$$\|v\|_{-m} \leq \sup_{u \in H_0^m(\Omega)} \frac{|\langle v, u \rangle|}{\|u\|_0} = \|v\|_0.$$

We can verify that $\|\cdot\|_{-m}$ satisfies the properties of a norm.

The Space $H^{-m}(\Omega)$

- Define $H^{-m}(\Omega)$ to be the completion of $L^2(\Omega)$ in the norm $\|\cdot\|_{-m}$.

Theorem

The dual space $(H_0^m)'(\Omega)$ of the space $H_0^m(\Omega)$ may be identified with $H^{-m}(\Omega)$, for all $m \geq 0$.

- Let F be the set of continuous linear functionals T_v on $H_0^m(\Omega)$ defined by

$$T_v(u) = \langle v, u \rangle = (v, \overline{u})_0, \quad u \in H_0^m(\Omega).$$

This is clearly a subspace of the Hilbert space $(H_0^m)'(\Omega)$.
We now show that it is a dense subspace.
Suppose F is not dense in $(H_0^m)'(\Omega)$. Then there is a nonzero $S \in (H_0^m)''(\Omega)$, such that $S(T_v) = 0$, for all $T_v \in F$.
By reflexivity applied to $H_0^m(\Omega)$, there is $w \in H_0^m(\Omega)$, such that

$$S(T) = T(w), \quad \text{for all } T \in (H_0^m)'(\Omega).$$

The Space $H^{-m}(\Omega)$ (Cont'd)

- Now we get, for all $v \in L^2(\Omega)$,

$$\langle v, w \rangle = T_v(w) = S(T_v) = 0.$$

But $H_0^m(\Omega) \subseteq L^2(\Omega)$.

So we can choose $v = w$.

We conclude that $w = 0$.

This however, contradicts $S \neq 0$.

Now we have $\overline{F} = (H_0^m)'(\Omega)$.

So $(H_0^m)'(\Omega)$ can be identified with $H^{-m}(\Omega)$ by the correspondence

$$T_v \leftrightarrow v \quad \text{and} \quad \|T_v\| = \|v\|_{-m}.$$

Characterization of $H^{-m}(\Omega)$

- We characterize membership in $H^{-m}(\Omega)$ by showing

$$v \in H^{-m}(\Omega) \quad \text{if and only if} \quad v = \sum_{|\alpha| \le m} \partial^\alpha v_\alpha, \text{ where } v_\alpha \in L^2(\Omega).$$

- We use the preceding theorem.
- We prove that:

A distribution T belongs to $(H_0^m)'(\Omega)$
if and only if
T is of the form T_v, where $v = \sum_{|\alpha| \le m} \partial^\alpha v_\alpha$, $v_\alpha \in L^2(\Omega)$.

Characterization of $H^{-m}(\Omega)$ (Part (i))

(i) Let T be a distribution of the form T_v with $v = \sum_{|\alpha|\le m} \partial^\alpha v_\alpha$.
Let $u \in C_0^\infty(\Omega)$. Then

$$T_v(u) = \left\langle \sum_{|\alpha|\le m} \partial^\alpha v_\alpha, u \right\rangle = \sum_{|\alpha|\le m} (-1)^{|\alpha|} \langle v_\alpha, \partial^\alpha u\rangle.$$

We know $C_0^\infty(\Omega)$ is dense in $H_0^\infty(\Omega)$.
So the equality holds even when $u \in H_0^\infty(\Omega)$.
Hence,

$$|T_v(u)| = \left| \sum_{|\alpha|\le m} (-1)^{|\alpha|} (v_\alpha, \partial^\alpha \overline{u})_0 \right| \le \sum_{|\alpha|\le m} \|v_\alpha\|_0 \|u\|_m.$$

This clearly shows that T_v lies in $(H_0^m)'(\Omega)$.

Characterization of $H^{-m}(\Omega)$ (Part (ii))

(ii) Suppose $T \in (H_0^m)'(\Omega)$.

Then there exists $g \in H_0^m(\Omega)$, such that, for all $f \in H_0^m(\Omega)$,

$$T(f) \stackrel{\text{Riesz}}{=} (f,g)_m = \sum_{|\alpha| \le m} (\partial^\alpha f, \partial^\alpha g)_0.$$

In particular, if $\phi \in C_0^\infty(\Omega) \subseteq H_0^m(\Omega)$,

$$T(\phi) = \sum_{|\alpha| \le m} (\partial^\alpha \phi, \partial^\alpha g)_0 = \sum_{|\alpha| \le m} (-1)^{|\alpha|} \langle \phi, \partial^{2\alpha} \overline{g} \rangle = \sum_{|\alpha| \le m} \langle \phi, \partial^\alpha g_\alpha \rangle,$$

where $g_\alpha = (-1)^\alpha \partial^\alpha \overline{g} \in L^2(\Omega)$.

But $C_0^\infty(\Omega)$ is dense in $H_0^m(\Omega)$.

So T has the form T_v, with $v = \sum_{|\alpha| \le m} \partial^\alpha g_\alpha$, $g_\alpha \in L^2(\Omega)$.

Example

- Consider now the differential operator

$$(1-\Delta)^m : H^m(\Omega) \to \mathscr{D}'(\Omega),$$

where Δ is the Laplacian operator in $\mathbb{R}^n$ and $m \in \mathbb{N}_0$.

For $u \in H^m(\Omega)$ and $\phi \in C_0^\infty(\Omega)$,

$$\begin{aligned} \langle (1-\Delta)^m u, \phi \rangle &= \sum_{|\alpha| \le m} c_\alpha (-1)^{|\alpha|} \langle \partial^{2\alpha} u, \phi \rangle \\ &= \sum_{|\alpha| \le m} c_\alpha \langle \partial^\alpha u, \partial^\alpha \phi \rangle \\ &= \sum_{|\alpha| \le m} c_\alpha (\partial^\alpha u, \partial^\alpha \overline{\phi})_0, \end{aligned}$$

where c_α are the binomial coefficients of $(1 - \sum_{k=1}^n \partial_k^2)^m$.

Example (Cont'd)

- Now $1 \leq c_\alpha \leq c_m$, for some integer c_m.

  So we have

$$|\langle(1-\Delta)^m u, \phi\rangle| \leq c_m \sum_{|\alpha|\leq m} \|\partial^\alpha u\|_0 \|\partial^\alpha \phi\|_0 \leq c_m \|u\|_m \|\phi\|_m.$$

  This means that the mapping

$$\phi \mapsto \langle(1-\Delta)^m u, \phi\rangle = ((1-\Delta)^m u, \overline{\phi})_0$$

  is a continuous linear functional on $C_0^\infty(\Omega)$, bounded in the H^m norm.

  It may therefore be extended by continuity to $H_0^m(\Omega)$.

  Thus, $(1-\Delta)^m u \in (H_0^m)'(\Omega) = H^{-m}(\Omega)$.

  We conclude that the linear differential operator $(1-\Delta)^m$ maps $H^m(\Omega)$ continuously into $H^{-m}(\Omega)$.

Subsection 5

Fourier Series and Periodic Distributions

Inner Product of Linear Combinations of Exponentials

- Let
$$u(x) = \sum_{|\alpha|\leq k} a_\alpha e^{i\langle\alpha,x\rangle}$$
be a finite sum of exponential functions with:
  - $x \in \mathbb{R}^n$;
  - $\alpha = (\alpha_1,\ldots,\alpha_n) \in \mathbb{Z}^n$;
  - $|\alpha| = \sum_{j=1}^n |\alpha_j|$;
  - $\langle\alpha,x\rangle = \alpha_1 x_1 + \cdots + \alpha_n x_n$.
- The coefficients a_α are complex numbers which satisfy $a_{-\alpha} = \overline{a}_\alpha$ when u is a real function.
- For any integer m, we define the **inner product** of u with $v(x) = \sum_{|\alpha|\leq k} b_\alpha e^{i\langle\alpha,x\rangle}$, by
$$(u,v)_m = (2\pi)^n \sum_{|\alpha|\leq k} (1+|\alpha|^2)^m a_\alpha \overline{b}_\alpha$$

Some Observations

- We have
$$(u,v)_0 = (2\pi)^n \sum_{|\alpha|\le k} a_\alpha \overline{b}_\alpha = \int u(x)\overline{v}(x)dx,$$
where the integral is taken over the cube $[-\pi,\pi]^n$.
For this, it suffices o notice that, for $a \in \mathbb{Z}$,
$$\int e^{iax}dx = \begin{cases} (2\pi)^n, & \text{if } a = 0 \\ 0, & \text{if } a \neq 0 \end{cases}.$$
- We also have
$$a_\alpha = \frac{1}{(2\pi)^n}(u, e^{i\langle\alpha,x\rangle})_0.$$

The Norm Generated by the Inner Product

- The norm generated by this inner product is

$$\|u\|_m = \sqrt{(u,u)_m} = (2\pi)^{n/2}\left[\sum_{|\alpha|\leq k}(1+|\alpha|^2)^m |a_\alpha|^2\right]^{1/2}.$$

- The Schwarz inequality gives

$$|(u,v)_m| \leq \|u\|_m \|v\|_m.$$

- It may be generalized to

$$|(u,v)_m| \leq \|u\|_{m+\ell}\|v\|_{m-\ell},$$

  for any integer ℓ.

The Space $\widetilde{H}^m$

- Let $\widetilde{H}^m$ be the completion of the linear space of trigonometric polynomials of the form

$$u(x) = \sum_{|\alpha| \le k} a_\alpha e^{i\langle \alpha, x \rangle}$$

under the norm $\|\cdot\|_m$.

- $\widetilde{H}^m$ is a Hilbert space.
- The elements of $\widetilde{H}^m$ are represented by infinite sums of the form $\sum a_\alpha e^{i\langle \alpha, x \rangle}$, such that the norm

$$(2\pi)^{n/2} \left[\sum (1 + |\alpha|^2)^m |a_\alpha|^2\right]^{1/2}$$

is finite.

Some Remarks

- When m is a nonnegative integer, this implies the convergence of the series in the $L^2([-\pi,\pi]^n)$ norm.
- In his case
$$\sum a_\alpha e^{i\langle a,x\rangle}$$
is a Fourier series expansion of a periodic function in $\mathbb{R}^n$ whose Fourier coefficients, in the classical sense, are a_α.
- The choice of $(2\pi)^n$ as the period is arbitrary.
- It can be changed by an appropriate change of scale of x.

The Operator ∂^α

- For every multi-index $\alpha \in \mathbb{N}_0^n$, we have

$$\partial^\alpha \sum a_\beta e^{i\langle\beta,x\rangle} = \sum (i\beta)^\alpha a_\beta e^{i\langle\beta,x\rangle},$$

  where $(i\beta)^\alpha = (i\beta_1)^{\alpha_1}(i\beta_2)^{\alpha_2}\cdots(i\beta_n)^{\alpha_n}$.
- So we obtain

$$\begin{array}{rcl} \|\partial^\alpha \sum a_\beta e^{i\langle\beta,x\rangle}\|_0^2 & = & (2\pi)^n \sum |\beta^\alpha a_\beta|^2 \\ & \leq & (2\pi)^n \sum (1+|\beta|^2)^{|\alpha|} |a_\beta|^2 \\ & = & \|\sum a_\beta e^{i\langle\beta,x\rangle}\|_{|\alpha|}^2. \end{array}$$

- In other words, for all $u \in \widetilde{H}^m$, $|\alpha| \leq m$,

$$\|\partial^\alpha u\|_0 \leq \|u\|_{|\alpha|}.$$

The Operator ∂^α (Cont'd)

- More generally, for all $u \in \widetilde{H}^m$, ℓ and α, such that $\ell + |\alpha| \le m$,

$$\|\partial^\alpha u\|_\ell \le \|u\|_{\ell+|\alpha|}.$$

- This implies, in particular, that ∂^α is a bounded linear operator from $\widetilde{H}^m$ to $\widetilde{H}^{m-|\alpha|}$, $|\alpha| \le m$.
- If $u \in \widetilde{H}^m$ and $\ell < m$, then $\|u\|_\ell \le \|u\|_m$.
- So, if $\ell < m$, $\widetilde{H}^m \subseteq \widetilde{H}^\ell$.
- $\widetilde{H}^0$ is therefore the space of periodic functions which are square integrable over $[-\pi,\pi]^n$.
- It obviously includes $\widetilde{C}^0$, the continuous periodic functions in $\mathbb{R}^n$.

Example

- When $m > 0$, $\widetilde{H}^m$ is the space of periodic functions whose (distributional) derivatives up to order m are square integrable.
- To characterize $\widetilde{H}^m$ when $m < 0$, we first consider some examples.

  Example: Let $\sum a_\alpha e^{i\langle\alpha,x\rangle} \in \widetilde{H}^m$, $m \geq 0$. Then, there exists a $u \in \widetilde{H}^m$, such that, for any $\varepsilon > 0$, when k is large enough,

$$\| u(x) - \sum_{|\alpha|\leq k} a_\alpha e^{i\langle\alpha,x\rangle} \|_m < \varepsilon.$$

  Consequently

$$\begin{array}{rcl} \|\partial^\beta u - \sum_{|\alpha|\leq k}(i\alpha)^\beta a_\alpha e^{i\langle\alpha,x\rangle}\|_0 & = & \|\partial^\beta [u - \sum_{|\alpha|\leq k} a_\alpha e^{i\langle\alpha,x\rangle}]\|_0 \\ & \leq & \|u - \sum_{|\alpha|\leq k} a_\alpha e^{i\langle\alpha,x\rangle}\|_{|\beta|} \\ & \leq & \|u - \sum_{|\alpha|\leq k} a_\alpha e^{i\langle\alpha,x\rangle}\|_m \\ & < & \varepsilon. \end{array}$$

  Therefore, $\sum_{|\alpha|\leq k}(i\alpha)^\beta a_\alpha e^{i\langle\alpha,x\rangle} \to \partial^\beta u$ in $\widetilde{H}^0 = \widetilde{L}^2$, for all $|\beta| \leq m$.

Example

- We have
$$(1-\Delta)\sum a_\alpha e^{i\langle\alpha,x\rangle}=\sum(1+|\alpha|^2)a_\alpha e^{i\langle\alpha,x\rangle}.$$
Thus,
$$(1-\Delta)^{-1}\sum a_\alpha e^{i\langle\alpha,x\rangle}=\sum(1+|\alpha|^2)^{-1}a_\alpha e^{i\langle\alpha,x\rangle}.$$
Therefore, for all $\ell\in\mathbb{Z}$,
$$(1-\Delta)^{\ell}\sum a_\alpha e^{i\langle\alpha,x\rangle}=\sum(1+|\alpha|^2)^{\ell}a_\alpha e^{i\langle\alpha,x\rangle}.$$
Thus, for any pair of trigonometric polynomials
$$u_k=\sum_{|\alpha|\le k}a_\alpha e^{i\langle\alpha,x\rangle}\quad\text{and}\quad v_k=\sum_{|\alpha|\le k}b_\alpha e^{i\langle\alpha,x\rangle},$$
we have
$$((1-\Delta)^{\ell}u_k,v_k)_m=(u_k,(1-\Delta)^{\ell}v_k)_m=(u_k,v_k)_{m+\ell}.$$

Example (Cont'd)

- We got

$$((1-\Delta)^{\ell}u_k, v_k)_m = (u_k, (1-\Delta)^{\ell}v_k)_m = (u_k, v_k)_{m+\ell}.$$

Taking $v_k = (1-\Delta)^{\ell}u_k$,

$$\|(1-\Delta)^{\ell}u_k\|_m^2 = (u_k, (1-\Delta)^{2\ell}u_k)_m = (u_k, u_k)_{m+2\ell}.$$

Equivalently,

$$\|(1-\Delta)^{\ell}u_k\|_m = \|u_k\|_{m+2\ell}.$$

But the trigonometric polynomials are dense in $\widetilde{H}^m$.
So this equation can be extended by continuity to $\widetilde{H}^m$.
Hence, for any $u \in \widetilde{H}^m$,

$$\|(1-\Delta)^{\ell}u\|_m = \|u\|_{m+2\ell}, \quad \ell, m \in \mathbb{Z}.$$

Example (Cont'd)

- For any $u \in \widetilde{H}^m$,

$$\|(1-\Delta)^{\ell} u\|_m = \|u\|_{m+2\ell}, \quad \ell, m \in \mathbb{Z}.$$

In particular, if m is replaced by $-m$ and $\ell = m$, then

$$\|(1-\Delta)^m u\|_{-m} = \|u\|_m, \quad m \in \mathbb{Z}.$$

This equation implies that the linear mapping $(1-\Delta)^m : \widetilde{H}^m \to \widetilde{H}^{-m}$, $m \in \mathbb{Z}$ is bijective and norm preserving.

Since $\widetilde{H}^m$ is a Hilbert space, the Riesz representation theorem provides another norm-preserving isomorphism between $\widetilde{H}^m$ and $(\widetilde{H}^m)'$.

So there is a norm-preserving isomorphism between $(\widetilde{H}^m)'$ and $\widetilde{H}^{-m}$.

This allows identification of these two spaces for all integers m.

The Dual of the Real Space $\widetilde{H}^m$

Theorem

For all $m \geq 0$, $\widetilde{H}^{-m}$ is the dual space of the real Hilbert space $\widetilde{H}^m$ with respect to the inner product $(\cdot,\cdot)_0$ in the sense that T is a continuous linear functional on $\widetilde{H}^m$ if and only if there is a unique $v \in \widetilde{H}^{-m}$, such that $T(u) = \langle v, u \rangle = (v, u)_0$, $u \in \widetilde{H}^m$. Furthermore, $\|T\| = \|v\|_{-m}$.

- Suppose $\widetilde{H}^m$ is real. Then the coefficients satisfy $a_{-\alpha} = \overline{a}_\alpha$.
  Let T be a continuous linear functional on $\widetilde{H}^m$ in the (weak) topology defined by $(\cdot,\cdot)_0$. Then, for any $u \in \widetilde{H}^m$ and $m \geq 0$,

  $$|T(u)| \leq M\|u\|_0 \leq M\|u\|_m.$$

  Hence T is also continuous in the (strong) topology of $\widetilde{H}^m$ defined by $(\cdot,\cdot)_m$. By the Riesz Representation Theorem, there is a unique $v \in (\widetilde{H}^m)' = \widetilde{H}^{-m}$, such that:
  - $T(u) = (v, u)_0$;
  - $\|T\| = \sup\{(v, u)_0 : \|u\|_m = 1\} = \|v\|_{-m}$.

Example

- Consider the trigonometric polynomial

$$f_p(x) = \sum_{k=-p}^{p} e^{ikx}, \quad x \in \mathbb{R}.$$

It converges in $\widetilde{H}^m$, whenever $m \leq -1$.

According to a previous theorem, the limit to which f_p converges in $\widetilde{H}^{-1}$ may be determined by considering the (weak) limit

$$\lim_{p\to\infty} \sum_{-p}^{p} (e^{ikx}, \phi(x))_0 = \lim_{p\to\infty} \sum_{-p}^{p} (e^{-ikx}, \phi(x))_0,$$

where ϕ is an arbitrary function in $\widetilde{H}^1$.

But $(e^{ikx}, \phi(x))_0$ is the expansion coefficient a_k of ϕ.

So this limit is simply $\phi(0)$.

Thus $\lim f_p = \widetilde{\delta}$, where $\widetilde{\delta}$ is a periodic version of the Dirac distribution.

Its m-th derivative is $\widetilde{\delta}^{(m)} = \sum (ik)^m e^{ikx}$.

Sobolev Imbedding for $\widetilde{H}^m$

Theorem

If $m > \frac{1}{2}n$, then $\widetilde{H}^m$ is a subspace of $\widetilde{C}^0$.

- Let $m > \frac{1}{2}n$ and $u = \sum a_\alpha e^{i\langle\alpha,x\rangle} \in \widetilde{H}^m$. Then

$$\begin{aligned}(\sum|a_\alpha|)^2 &= [\sum(1+|\alpha|^2)^{-\frac{1}{2}m}(1+|\alpha|^2)^{\frac{1}{2}m}|a_\alpha|]^2 \\ &\leq [\sum(1+|\alpha|^2)^{-m}][\sum(1+|\alpha|^2)^m|a_\alpha|^2] \\ &= \sum(1+|\alpha|^2)^{-m}\frac{1}{(2\pi)^n}\|u\|_m^2 < \infty.\end{aligned}$$

  Thus, the Fourier series of u converges uniformly.
  Since $e^{i\langle\alpha,x\rangle}$ is continuous, the sum $\sum a_\alpha e^{i\langle\alpha,x\rangle}$ is continuous.
- This result is generalized to

Theorem

If $m > \frac{1}{2}n + k$, where $k \geq 0$ is an integer, then $\widetilde{H}^m$ is a subspace of $\widetilde{C}^k$.

The Spaces $\widetilde{H}^\infty$ and $\widetilde{H}^{-\infty}$

- Setting $\widetilde{H}^\infty = \bigcap \widetilde{H}^m$, we conclude that $\widetilde{H}^\infty = \widetilde{C}^\infty$.
- $\widetilde{H}^\infty$ is a locally convex topological vector space in the projective limit topology defined by $\{\widetilde{H}^m : m \in \mathbb{N}_0\}$.
- Similarly, we define $\widetilde{H}^{-\infty} = \bigcup \widetilde{H}^m$.
- $\widetilde{H}^{-\infty}$ represents the dual of $\widetilde{H}^\infty$ in the inner product $(\cdot,\cdot)_0$.

  This may be seen by using the argument that was used for H^s.

  We have $\widetilde{H}^m \subseteq H^\ell$, when $\ell < m$.

  Thus, for any positive integer m,

$$\bigcup_{|k| \le m} \widetilde{H}^k = \widetilde{H}^{-m}, \qquad \bigcap_{|k| \le m} \widetilde{H}^k = \widetilde{H}^m.$$

  A previous theorem implies that $\bigcup_{|k| \le m} \widetilde{H}^k = (\bigcap_{|k| \le m} \widetilde{H}^k)'$.

  Since m was arbitrary, $\widetilde{H}^{-\infty} = (\widetilde{H}^\infty)'$.

Distributions in $\widetilde{H}^{-\infty}$

- $\widetilde{H}^{-\infty}$ is the space of periodic distributions, whose elements are continuous linear functionals on $\widetilde{C}^{\infty}$ in the (weak) topology defined by $\langle\cdot,\cdot\rangle = (\cdot,\cdot)_0$.
- That means $u \in \widetilde{H}^{-\infty}$ if, for every sequence ϕ_k in $\widetilde{C}^{\infty}$, such that $\|\phi_k\|_m \to 0$, for all $m \geq 0$,

$$|(u,\phi_k)_0| \leq \|u\|_{-m}\|\phi_k\|_m \to 0, \quad \text{for all } m.$$

- Equivalently, $u \in \widetilde{H}^{-\infty}$ if, for every sequence ϕ_k in $\widetilde{C}^{\infty}$, such that $\partial^{\alpha}\phi_k \to 0$ uniformly for all $\alpha \in \mathbb{N}_0^n$,

$$|(u,\phi_k)_0| \leq \|u\|_{-m}\|\phi_k\|_m \to 0, \quad \text{for all } m.$$

$\widetilde{H}^{-\infty}$ and Duality

- When m is a positive integer, we have already shown that $\widetilde{H}^{-m}$ represents the continuous linear functionals on $\widetilde{H}^m$.
- But $\widetilde{H}^\infty = \widetilde{C}^\infty$ is dense in $\widetilde{H}^m$.
- So $\widetilde{H}^{-m}$ may also be identified with the subspace of $\widetilde{H}^{-\infty}$ consisting of the distributions u for which $(u, \phi_k)_0 \to 0$ whenever ϕ_k is a sequence in $\widetilde{C}^\infty$ which converges to 0 in $\widetilde{H}^m$.

Periodicity and Representation in $\widetilde{H}^{-\infty}$

- The translation of $u \in \widetilde{H}^{-\infty}$ by $(2\pi,\ldots,2\pi)$ is denoted by $\tau_{2\pi}$.
- It satisfies, for all $\phi \in \widetilde{C}^{\infty}$,

$$\langle \tau_{2\pi} u, \phi \rangle = \langle u, \tau_{-2\pi} \phi \rangle = \langle u, \phi \rangle.$$

- Recall that:
  - Trigonometric polynomials are complete in $\widetilde{H}^m$, for every m;
  - $\widetilde{H}^{-\infty} = \bigcup \widetilde{H}^m$.
- So trigonometric polynomials are complete in $\widetilde{H}^{-\infty}$, in the sense that every $u \in \widetilde{H}^{-\infty}$ is the limit as $k \to \infty$ of a sum

$$\sum_{|\alpha| \le k} b_\alpha e^{i\langle \alpha, x \rangle}.$$

Representation in $\widetilde{H}^{-\infty}$ (Cont'd)

- Note that any $\phi \in \widetilde{C}^\infty$ is represented by a series

$$\sum a_\alpha e^{i\langle\alpha,x\rangle},$$

where $|\alpha|^m a_\alpha \xrightarrow{|\alpha|\to\infty} 0$, for all $m \geq 0$. Hence,

$$\langle u,\phi\rangle = \sum (u, a_\alpha e^{i\langle\alpha,x\rangle})_0 = \sum \overline{a}_\alpha (u, e^{i\langle\alpha,x\rangle})_0,$$

where $a_\alpha = \frac{1}{(2\pi)^n}(\phi(x), e^{i\langle\alpha,x\rangle})_0$ are the Fourier coefficients of ϕ.

Denote $(u, e^{i\langle\alpha,x\rangle})_0$ by $(2\pi)^n b_\alpha$. We then have

$$\begin{aligned} \langle u,\phi\rangle &= (2\pi)^n \sum b_\alpha \overline{a}_\alpha \\ &= (\sum b_\alpha e^{i\langle\alpha,x\rangle}, \sum a_\alpha e^{i\langle\alpha,x\rangle})_0 \\ &= (\sum b_\alpha e^{i\langle\alpha,x\rangle}, \phi)_0. \end{aligned}$$

Thus, u is represented by the series $u(x) = \sum b_\alpha e^{i\langle\alpha,x\rangle}$.

Example (Estimating Approximation Error)

- Let $n = 1$. Assume $f(x) = \sum a_k e^{ikx} \in \widetilde{H}^m$, $m \geq 1$. Then

$$\begin{aligned} \left| f(x) - \textstyle\sum_{|k| \leq \ell} a_k e^{ikx} \right| &\leq \textstyle\sum_{|k| > \ell} |a_k| \\ &\leq \left[\textstyle\sum_{|k|>\ell} (1+k^2)^m |a_k|^2\right]^{1/2} \left[\textstyle\sum_{|k|>\ell} (1+k^2)^{-m}\right]^{1/2} \\ &\leq \tfrac{1}{\sqrt{2\pi}} \|f\|_m \left[\textstyle\sum_{|k|>\ell} (1+k^2)^{-m}\right]^{1/2}. \end{aligned}$$

Now

$$\sum_{|k|>\ell} (1+k^2)^{-m} \leq 2 \int_\ell^\infty \frac{dt}{(1+t^2)^m} \leq 2 \int_\ell^\infty \frac{1}{t^{2m}} dt = \frac{2}{2m-1} \frac{1}{\ell^{2m-1}}.$$

This yields the estimate $\sup_{x \in \mathbb{R}} |f(x) - \sum_{|k| \leq \ell} a_k e^{ikx}| \leq c \ell^{-m+\frac{1}{2}}$, where c is a positive constant which depends on f and m.

Thus $\ell^{-m+\frac{1}{2}}$ indicates the rate of convergence of the Fourier series for f as ℓ increases. The greater the (positive) integer m, the smoother the function f and the faster the convergence of the series.

Inclusions Involving $\widetilde{H}^m$ and $\widetilde{C}^m$

Claim: The inclusion relations $\widetilde{C}^m \subseteq \widetilde{H}^m \subseteq \widetilde{C}^{m-1}$ can also be shown to hold when $n = 1$ and $m \geq 1$.

Let $f(x) = \sum a_k e^{ikx} \in \widetilde{C}^m$. Then

$$f^{(m)}(x) = \sum (ik)^m a_k e^{ikx} \in \widetilde{C}^0 \subseteq \widetilde{H}^0 = \widetilde{L}^2.$$

Hence, $\sum k^{2m} |a_k|^2 < \infty$.

Consequently, $\sum (1 + k^2)^m |a_k|^2 < \infty$.

This means that $f \in \widetilde{H}^m$.

The inclusion $\widetilde{H}^m \subseteq \widetilde{C}^{m-1}$ follows from a previous theorem.

Example

- Let $\widetilde{L}^1$ be the space of periodic functions which are locally integrable. We show that $\widetilde{L}^1 \subseteq \widetilde{H}^{-1}$ when $n = 1$.
  Let $u = \sum a_k e^{ik} \in \widetilde{L}^1$. Let $\phi(x) = \sum b_k e^{ikx}$ is any function in $\widetilde{H}^1$.
  Then $\langle u, \phi \rangle = (u, \overline{\phi})_0 = 2\pi \sum a_k b_k$.
  By the Riemann-Lebesgue Lemma, $a_k \to 0$ as $|k| \to \infty$.
  So there is a positive integer ℓ, such that $|a_k| \le 1$, for all $|k| \ge \ell$.
  Note that:
  - The series $\sum (1+k^2)^{-1}$ converges;
  - The series $\sum (1+k^2)|b_k|^2$ converges since $\phi \in \widetilde{H}^1$.

  Therefore, given $\varepsilon > 0$, for large enough ℓ, we have

$$\begin{array}{rcl} \sum_{|k| \ge \ell} |a_k b_k| & \le & \sum_{|k| \ge \ell} |b_k| \\ & \le & [\sum_{|k| \ge \ell} (1+k^2)^{-1}]^{1/2} [\sum_{|k| \ge \ell} (1+k^2)|b_k|^2]^{1/2} \\ & < & \varepsilon. \end{array}$$

  Hence, $|\langle u, \phi \rangle| \le 2\pi \sum |a_k b_k| \le c\|\phi\|_1$, for some constant c.
  So u defines a continuous linear functional on $\widetilde{H}^1$, i.e., $u \in \widetilde{H}^{-1}$.

Product of $\widetilde{C}^\infty$ by $\widetilde{H}^m$

- Since $L^2_{\text{loc}} \subseteq L^1_{\text{loc}}$ we also have $\widetilde{L}^2 \subseteq \widetilde{L}^1$.
  Combining this with the preceding inclusions, we ge

$$\widetilde{H}^1 \subseteq \widetilde{C}^0 \subseteq \widetilde{L}^2 \subseteq \widetilde{L}^1 \subseteq \widetilde{H}^{-1}.$$

Theorem

If $\phi \in \widetilde{C}^\infty$ and $u \in \widetilde{H}^m$, then $\phi u \in \widetilde{H}^m$ and $\|\phi u\|_m \leq c\|u\|_m$, where c is a constant which depends on ϕ and m.

- If $m \geq 0$, then, by Leibniz's Formula, $\|\phi u\|_m$ is bounded by a constant multiple of $\|u\|_m$.
  If $m < 0$, then

$$\|\phi u\|_m = \sup\left\{\frac{(\phi u, v)_0}{\|v\|_{|m|}} : v \in \widetilde{H}^{|m|}\right\} \leq \sup_{v \neq 0} \frac{\|u\|_m \|\phi v\|_{|m|}}{\|v\|_{|m|}}.$$

  But $\|\phi v\|_{|m|} \leq c\|v\|_{|m|}$. Hence, $\|\phi u\|_m \leq c\|u\|_m$.

Differential Operators on $\widetilde{H}^m$

- Since ∂^α maps $\widetilde{H}^m$ continuously into $\widetilde{H}^{m-|\alpha|}$, this theorem implies

Corollary

The linear differential operator of order ℓ, $L = \sum_{|\alpha| \leq \ell} a_\alpha \partial^\alpha$, with coefficients in $\widetilde{C}^\infty$ maps $\widetilde{H}^m$ continuously into $\widetilde{H}^{m-\ell}$, with $\|Lu\|_{m-\ell} \leq c\|u\|_m$, $u \in \widetilde{H}^m$.

- The converse, i.e., that $Lu \in \widetilde{H}^m$ implies that $u \in \widetilde{H}^{m+\ell}$ is true, provided L is elliptic.

  In that case, it may be deduced from the Local Regularity Theorem.
- The similarity of these results with those obtained earlier in this chapter are due to a striking analogy between Fourier series and Fourier transforms.

  In the series, the weight function $(1+|\xi|^2)^s$ in the integral which defines the inner product in H^s corresponds to $(1+|\alpha|^2)^m$ in the sum

$$(u,v)_m = (2\pi)^n \sum_{|\alpha| \leq k} (1+|\alpha|^2)^m a_\alpha \overline{b}_\alpha.$$