Introduction to Dynamical Systems

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1. Basic Notions and Examples
   - The Notion of a Dynamical System
   - Examples With Discrete Time
   - Examples With Continuous Time
   - Invariant Sets

Subsection 1

The Notion of a Dynamical System

Dynamical Systems With Discrete Time

Definition

Any map $f : X \to X$ is called a **dynamical system with discrete time** or simply a **dynamical system**.

- We define recursively
$$f^{n+1} = f \circ f^n, \text{ for each } n$$
- We also write $f^0 = \mathrm{Id}$, where Id is the identity map.
- Clearly,
$$f^{m+n} = f^m \circ f^n, \text{ for every } m, n \in \mathbb{N}_0,$$
where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Additional Notation for Invertible Maps

- Let $f : X \to X$ be a dynamical system.
- If f is invertible, we define

$$f^{-n} = (f^{-1})^n, \quad \text{for each } n \in \mathbb{N}.$$

- In the case of invertible f,

$$f^{m+n} = f^m \circ f^n, \quad \text{for every } m, n \in \mathbb{Z}.$$

Pair of Dynamical Systems

- Consider dynamical systems

$$f : X \to X \quad \text{and} \quad g : Y \to Y.$$

- Define a new dynamical system

$$h : X \times Y \to X \times Y$$

by

$$h(x, y) = (f(x), g(y)).$$

- Note that if f and g are invertible, then the map h is also invertible.
- Its inverse is given by

$$h^{-1}(x, y) = (f^{-1}(x), g^{-1}(y)).$$

Semiflows and Flows

Definition

A **semiflow** is a family of maps $\varphi_t : X \to X$ for $t \geq 0$, such that:

- $\varphi_0 = \mathrm{Id}$;
- $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for every $t, s \geq 0$.

A **flow** is a family of maps $\varphi_t : X \to X$ for $t \in \mathbb{R}$, such that:

- $\varphi_0 = \mathrm{Id}$;
- $\varphi_{t+s} = \varphi_t \circ \varphi_s$, for every $t, s \in \mathbb{R}$.

Dynamical Systems With Continuous Time

Definition

A **dynamical system with continuous time** or simply a **dynamical system** is a family of maps φ_t that is a flow or a semiflow.

- We note that if φ_t is a flow, then
$$\varphi_t \circ \varphi_{-t} = \varphi_{-t} \circ \varphi_t = \varphi_0 = \mathrm{Id}.$$
- Thus, in the case of a flow, each map φ_t is invertible and its inverse is given by
$$\varphi_t^{-1} = \varphi_{-t}.$$

Example

- Given $y \in \mathbb{R}^n$, consider the maps $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$ defined by

$$\varphi_t(x) = x + ty, \quad t \in \mathbb{R},\ x \in \mathbb{R}^n.$$

Clearly, $\varphi_0 = \text{Id}$.

Moreover,

$$\begin{array}{rcl}\varphi_{t+s}(x) & = & x + (t+s)y \\ & = & (x + sy) + ty \\ & = & (\varphi_t \circ \varphi_s)(x).\end{array}$$

In other words, the family of maps φ_t is a flow.

Pair of Flows

- Consider two flows

$$\varphi_t : X \to X \quad \text{and} \quad \psi_t : Y \to Y, \quad t \in \mathbb{R}.$$

- The family of maps

$$\alpha_t : X \times Y \to X \times Y$$

defined, for each $t \in \mathbb{R}$, by

$$\alpha_t(x, y) = (\varphi_t(x), \psi_t(y))$$

is also a flow.
- Moreover,

$$\alpha_t^{-1}(x, y) = (\varphi_{-t}(x), \psi_{-t}(y)).$$

- The expression dynamical system is used to refer both to dynamical systems with discrete time and to ones with continuous time.

Subsection 2

Examples With Discrete Time

The Circle

- The **circle** S^1 is defined to be $\mathbb{R}/\mathbb{Z}$.
- This is the real line with two points $x, y \in \mathbb{R}$ identified if $x - y \in \mathbb{Z}$.
- In other words, $S^1 = \mathbb{R}/\mathbb{Z} = \mathbb{R}/\sim$, where $\sim$ is the equivalence relation on $\mathbb{R}$ defined by

$$x \sim y \quad \text{iff} \quad x - y \in \mathbb{Z}.$$

The Circle (Cont'd)

- The corresponding equivalence classes, which are the elements of S^1, can be written in the form

$$[x] = \{x + m : m \in \mathbb{Z}\}.$$

- In particular, one can introduce the operations

$$[x] + [y] = [x + y] \quad \text{and} \quad [x] - [y] = [x - y].$$

- One can also identify S^1 with

$$[0, 1]/\{0, 1\},$$

where the endpoints of the interval $[0, 1]$ are identified.

Rotations of the Circle

Definition

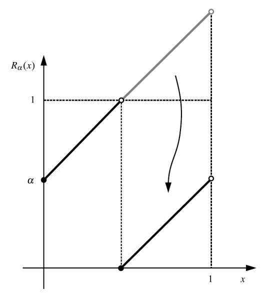
Given $\alpha \in \mathbb{R}$, we define the **rotation** $R_\alpha : S^1 \to S^1$ by

$$R_\alpha([x]) = [x + \alpha].$$

- Sometimes, we also write

$$R_\alpha(x) = x + \alpha \mod 1,$$

  thus identifying $[x]$ with its representative in the interval $[0, 1)$.
- The map R_α could also be called a **translation of the interval**.



- $R_\alpha : S^1 \to S^1$ is invertible, with inverse $R_\alpha^{-1} = R_{-\alpha}$.

Periodic Points

Definition

Let $f : X \to X$ be a map and $q \in \mathbb{N}$.

- A point $x \in X$ is said to be a q-**periodic point** of f if $f^q(x) = x$;
- A point $x \in X$ is a **periodic point** of f if it is q-periodic for some $q \in \mathbb{N}$.

- Note that fixed points, i.e., points $x \in X$, such that $f(x) = x$, are q-periodic, for any $q \in \mathbb{N}$.
- Moreover, a q-periodic point is kq-periodic for any $k \in \mathbb{N}$.

Definition

A periodic point is said to have **period** q if:

- It is q-periodic;
- It is not ℓ-periodic for any $\ell < q$.

Periodic Points of the Rotations of the Circle

Proposition

Given $\alpha \in \mathbb{R}$:

1. if $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, then R_α has no periodic points;
2. if $\alpha = \frac{p}{q} \in \mathbb{Q}$ with p and q coprime, then all points of S^1 are periodic for R_α and have period q.

- Note that $[x] \in S^1$ is q-periodic if and only if $[x + q\alpha] = [x]$.

  That is, if and only if $q\alpha \in \mathbb{Z}$.

  Both properties follow easily from this observation.

Expanding Maps of the Circle

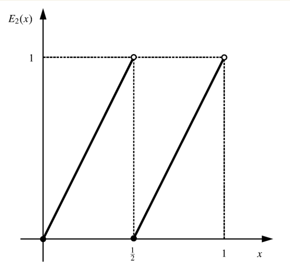
Definition

Given an integer $m > 1$, the **expanding map** $E_m : S^1 \to S^1$ is defined by

$$E_m(x) = mx \mod 1.$$

Example:
For $m = 2$, we have

$$E_2(x) = \begin{cases} 2x, & \text{if } x \in [0, \frac{1}{2}), \\ 2x - 1, & \text{if } x \in [\frac{1}{2}, 1). \end{cases}$$



Periodic Points of the Expanding Maps

- We determine the periodic points of the expanding map E_m.
- Note that, for $x \in S^1$,

$$E_m^q(x) = m^q x \mod 1.$$

- So a point x is q-periodic if and only if

$$m^q x - x = (m^q - 1)x \in \mathbb{Z}.$$

- Hence, the q-periodic points of the expanding map E_m are

$$x = \frac{p}{m^q - 1}, \text{ for } p = 1, 2, \ldots, m^q - 1.$$

Periodic Points of the Expanding Maps (Cont'd)

- Let $n_m(q)$ be the number of periodic points of E_m with period q.
- This number can be computed easily for each given q.
- For example, if q is prime, then

$$n_m(q) = m^q - m.$$

The n-Torus

- Given $n \in \mathbb{N}$, the n-**torus** or simply the **torus** is defined to be

$$\mathbb{T}^n = \mathbb{R}^n/\mathbb{Z}^n = \mathbb{R}^n/\sim,$$

where $\sim$ is the equivalence relation on $\mathbb{R}^n$ defined by

$$x \sim y \quad \text{iff} \quad x - y \in \mathbb{Z}^n.$$

- The elements of $\mathbb{T}^n$ are thus the equivalence classes

$$[x] = \{x + y : y \in \mathbb{Z}^n\},$$

with $x \in \mathbb{R}^n$.

The Endomorphism of the Torus T_A

- Let A be an $n \times n$ matrix with entries in $\mathbb{Z}$.

Definition

The **endomorphism of the torus** $T_A : \mathbb{T}^n \to \mathbb{T}^n$ is defined by

$$T_A([x]) = [Ax], \text{ for } [x] \in \mathbb{T}^n.$$

We say that T_A is the **endomorphism of the torus induced by** A.

- Since A is a linear transformation,

$$x - y \in \mathbb{Z}^n \quad \text{implies} \quad Ax - Ay \in \mathbb{Z}^n.$$

- So

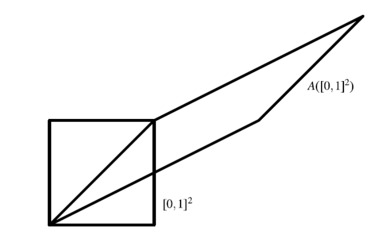
$$y \in [x] \quad \text{implies} \quad Ay \in [Ax].$$

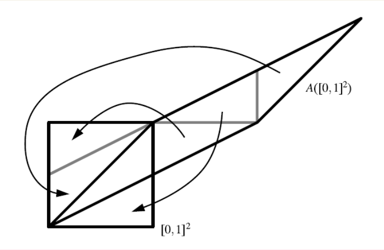
- Hence, T_A is well defined.

Invertibility of the Endomorphism

- In general, T_A may not be invertible, even if A is invertible.
- When T_A is invertible, we also say that it is the **automorphism of the torus induced by** A.

  Example: We represent in the figure the automorphism of the torus $\mathbb{T}^2$ induced by the matrix $A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$.





Periodic Points of the Automorphisms of the Torus

Proposition

Let $T_A : \mathbb{T}^n \to \mathbb{T}^n$ be an automorphism of the torus induced by a matrix A without eigenvalues with modulus 1. Then the periodic points of T_A are the points with rational coordinates, i.e., the elements of $\mathbb{Q}^n/\mathbb{Z}^n$.

- Let $[x] = [(x_1, \ldots, x_n)] \in \mathbb{T}^n$ be a periodic point.
  Then, there exist $q \in \mathbb{N}$ and $y = (y_1, \ldots, y_n) \in \mathbb{Z}^n$, such that
  $$A^q x = x + y.$$
  Equivalently, $(A^q - \mathrm{Id})x = y$.
  By hypothesis, A has no eigenvalues with modulus 1.
  Hence, the matrix $A^q - \mathrm{Id}$ is invertible.
  So we can write $x = (A^q - \mathrm{Id})^{-1} y$.
  Also, $A^q - \mathrm{Id}$ has only integer entries.
  Hence, each entry of $(A^q - \mathrm{Id})^{-1}$ is a rational number.
  Thus, $x \in \mathbb{Q}^n$.

Periodic Points of the Automorphisms (Cont'd)

- Now we assume that $[x]=[(x_1,\ldots,x_n)]\in\mathbb{Q}^n/\mathbb{Z}^n$.
  Let $(x_1,\ldots,x_n)=(\frac{p_1}{r},\ldots,\frac{p_n}{r})$, with $p_1,\ldots,p_n\in\{0,1,\ldots,r-1\}$.
  Since A has only integer entries, for each $q\in\mathbb{N}$, we have

$$A^q(x_1,\ldots,x_n)=\left(\frac{p_1'}{r},\ldots,\frac{p_n'}{r}\right)+(y_1,\ldots,y_n)$$

  for some $p_1',\ldots,p_n'\in\{0,1,\ldots,r-1\}$ and $(y_1,\ldots,y_n)\in\mathbb{Z}^n$.
  Now the number of points of the form of x is r^n.
  So, there exist $q_1,q_2\in\mathbb{N}$, with $q_1\neq q_2$, such that

$$A^{q_1}(x_1,\ldots,x_n)-A^{q_2}(x_1,\ldots,x_n)\in\mathbb{Z}^n.$$

  Assuming, without loss of generality, that $q_1>q_2$, we obtain

$$A^{q_1-q_2}(x_1,\ldots,x_n)-(x_1,\ldots,x_n)\in\mathbb{Z}^n.$$

  Thus, $T_A^{q_1-q_2}([x])=[x]$.

Limit of the Proposition

- The preceding proposition cannot be extended to arbitrary endomorphisms of the torus.

  Example: Consider the endomorphism of the torus $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ induced by the matrix

$$A = \left(\begin{array}{cc} 3 & 1 \\ 1 & 1 \end{array} \right).$$

  We have $\det A = 2$.

  So T_A is not an automorphism.

  Observe that

$$T_A\left(0, \frac{1}{2}\right) = \left(\frac{1}{2}, \frac{1}{2}\right), \quad T_A\left(\frac{1}{2}, \frac{1}{2}\right) = (0, 0), \quad T_A(0, 0) = (0, 0).$$

  The rational coordinate points $(0, \frac{1}{2})$ and $(\frac{1}{2}, \frac{1}{2})$ are not periodic.

  On the other hand, A has eigenvalues $2 + \sqrt{2}$ and $2 - \sqrt{2}$.

  None of these eigenvalues has modulus 1.

Subsection 3

Examples With Continuous Time

Autonomous Differential Equations

- An **autonomous** (ordinary) **differential equation** is a differential equation not depending explicitly on time.
- Such equations give rise naturally to the concept of a flow.

Proposition

Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function such that, given $x_0 \in \mathbb{R}^n$, the initial value problem

$$\begin{cases} x' = f(x), \\ x(0) = x_0 \end{cases}$$

has a unique solution $x(t, x_0)$ defined for $t \in \mathbb{R}$. Then the family of maps $\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$ defined, for each $t \in \mathbb{R}$, by

$$\varphi_t(x_0) = x(t, x_0)$$

is a flow.

Autonomous Differential Equations (Cont'd)

- Given $s \in \mathbb{R}$, consider the function $y : \mathbb{R} \to \mathbb{R}^n$ defined by

$$y(t) = x(t + s, x_0).$$

We have:

  - $y(0) = x(s, x_0)$;
  - For $t \in \mathbb{R}$,

$$y'(t) = x'(t + s, x_0) = f(x(t + s, x_0)) = f(y(t)).$$

So, the function y is also a solution of the equation $x' = f(x)$.

By hypothesis, the initial value problem has a unique solution.

It follows that

$$y(t) = x(t, y(0)) = x(t, x(s, x_0)).$$

Autonomous Differential Equations (Cont'd)

- We obtained

$$y(t) = x(t, y(0)) = x(t, x(s, x_0)).$$

Equivalently,

$$x(t + s, x_0) = x(t, x(s, x_0)),$$

for $t, s \in \mathbb{R}$ and $x_0 \in \mathbb{R}^n$.

It follows that

$$\varphi_{t+s} = \varphi_t \circ \varphi_s.$$

Moreover,

$$\varphi_0(x_0) = x(0, x_0) = x_0.$$

That is, $\varphi_0 = \mathrm{Id}$.

This shows that the family of maps φ_t is a flow.

Example

- Consider the differential equation

$$\begin{cases} x' = -y, \\ y' = x. \end{cases}$$

Suppose $(x, y) = (x(t), y(t))$ is a solution.

Then

$$(x^2 + y^2)' = 2xx' + 2yy' = -2xy + 2yx = 0.$$

Thus, there exists a constant $r \geq 0$, such that

$$x(t)^2 + y(t)^2 = r^2.$$

Example (Cont'd)

- Write

$$x(t) = r\cos\theta(t), \quad y(t) = r\sin\theta(t),$$

where θ is some differentiable function.

Now $x' = -y$ yields

$$-r\theta'(t)\sin\theta(t) = -r\sin\theta(t).$$

Hence, $\theta'(t) = 1$.

So there exists a constant $c \in \mathbb{R}$, such that

$$\theta(t) = t + c.$$

Now write $(x_0, y_0) = (r\cos c, r\sin c) \in \mathbb{R}^2$.

Example (Cont'd)

- We obtain

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} r\cos(t+c) \\ r\sin(t+c) \end{pmatrix} = \begin{pmatrix} \cos t \cdot r\cos c - \sin t \cdot r\sin c \\ \sin t \cdot r\cos c + \cos t \cdot r\sin c \end{pmatrix} = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

  Notice that

$$R(t) = \begin{pmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{pmatrix}$$

  is a rotation matrix for each $t \in \mathbb{R}$.

  Moreover, $R(0) = \mathrm{Id}$.

Example (Cont'd)

- It follows from the proposition that the family of maps $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\varphi_t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = R(t) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

  is a flow.

  Incidentally, the identity $\varphi_{t+s} = \varphi_t \circ \varphi_s$ is equivalent to the identity between rotation matrices

$$R(t+s) = R(t)R(s).$$

Example

- Consider the differential equation

$$\begin{cases} x' = y, \\ y' = x. \end{cases}$$

Suppose $(x, y) = (x(t), y(t))$ is a solution.

Then

$$(x^2 - y^2)' = 2xx' - 2yy' = 2xy - 2yx = 0.$$

Thus, there exists a constant $r \geq 0$, such that

$$x(t)^2 - y(t)^2 = r^2 \quad \text{or} \quad x(t)^2 - y(t)^2 = -r^2.$$

Example: Case I

- We consider the first case

$$x(t)^2 - y(t)^2 = r^2.$$

- We can write

$$x(t) = r\cosh\theta(t) \quad \text{and} \quad y(t) = r\sinh\theta(t),$$

where θ is some differentiable function.

The equation $x' = y$ yields

$$r\theta'(t)\sinh\theta(t) = r\sinh\theta(t).$$

Hence, $\theta(t) = t + c$, for some constant $c \in \mathbb{R}$.

Write, also $(x_0, y_0) = (r\cosh c, r\sinh c) \in \mathbb{R}^2$.

Example: Case I (Cont'd)

- Then

$$\begin{aligned}
\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} &= \begin{pmatrix} r\cosh(t+c) \\ r\sinh(t+c) \end{pmatrix} \\
&= \begin{pmatrix} \cosh t \cdot r\cosh c + \sinh t \cdot r\sinh c \\ \sinh t \cdot r\cosh c + \cosh t \cdot \sinh c \end{pmatrix} \\
&= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix} \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} \\
&= S(t) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix},
\end{aligned}$$

where $S(t) = \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}$.

Example: Case II

- We now consider the second case

  $$x(t)^2 - y(t)^2 = -r^2.$$

  We can write

  $$x(t) = r \sinh \theta(t) \quad \text{and} \quad y(t) = r \cosh \theta(t).$$

  As in the first case, we find that

  $$\theta(t) = t + c,$$

  for some constant $c \in \mathbb{R}$.

  Write $(x_0, y_0) = (r \sinh c, r \cosh c) \in \mathbb{R}^2$.

Example: Case II (Cont'd)

- Then

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} r\sinh(t+c) \\ r\cosh(t+c) \end{pmatrix} = \begin{pmatrix} \sinh t \cdot r\cosh c + \cosh t \cdot r\sinh c \\ \cosh t \cdot r\cosh c + \sinh t \cdot r\sinh c \end{pmatrix} = S(t)\begin{pmatrix} x_0 \\ y_0 \end{pmatrix}.$$

Notice, also, that $S(0) = \mathrm{Id}$.

Example (Cont'd)

- It follows that the family of maps $\psi_t : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$\psi_t \begin{pmatrix} x_0 \\ y_0 \end{pmatrix} = S(t) \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$$

is a flow.

In particular, it follows from the identity $\psi_{t+s} = \psi_t \circ \psi_s$ that

$$S(t+s) = S(t)S(s), \quad \text{for } t, s \in \mathbb{R}.$$

From Continuous to Discrete Time

- Let $\varphi_t : X \to X$ be a flow.
- For each $T \in \mathbb{R}$, consider the map

$$f = \varphi_T : X \to X.$$

- f is a dynamical system with discrete time.
- We note that f is invertible.
- Its inverse is given by $f^{-1} = \varphi_{-T}$.
- Similarly, let $\varphi_t : X \to X$ be a semiflow.
- Consider, for each $T \geq 0$, the map

$$f = \varphi_T : X \to X.$$

- It is a dynamical system with discrete time.

Suspension Semi-Flows: The Set Y

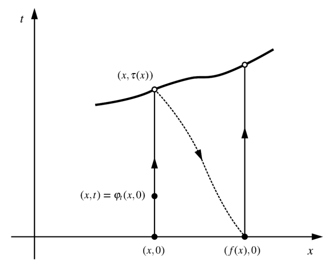
- Let $f : X \to X$ be a dynamical system with discrete time.
- Let $\tau : X \to \mathbb{R}^+$ be a function.
- Define
$$Z = \{(x,t) \in X \times \mathbb{R} : 0 \leq t \leq \tau(x)\}.$$
- Consider the set Y obtained from Z by identifying the points $(x, \tau(x))$ and $(f(x), 0)$, for each $x \in \mathbb{R}$.
- More precisely, we define
$$Y = Z/\sim,$$
where $\sim$ is the equivalence relation on Z defined by
$$(x,t) \sim (y,s) \quad \text{iff} \quad y = f(x),\ t = \tau(x) \text{ and } s = 0.$$

Suspension Semi-Flows

Definition

The **suspension semiflow** $\varphi_t : Y \to Y$ **over** f **with height** τ is defined for each $t \geq 0$ by

$$\varphi_t(x,s) = (x, s+t),\ \ s+t \in [0, \tau(x)].$$



- Each suspension semiflow is indeed a semiflow.
- If f is invertible, the family of maps φ_t, for $t \in \mathbb{R}$, is a flow.
- It is then called the **suspension flow over** f **with height** τ.

Poincaré Sections

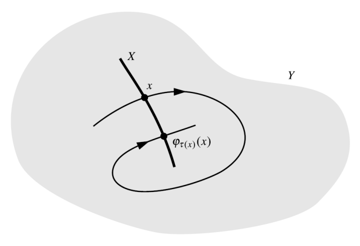
- Given a semiflow $\varphi_t : Y \to Y$, sometimes one can construct a dynamical system with discrete time $f : X \to X$, such that the semiflow can be seen as a suspension semiflow over f.

Definition

A set $X \subseteq Y$ is said to be a **Poincaré section** for a semiflow $\varphi_t : Y \to Y$ if

$$\tau(x) := \inf \{t > 0 : \varphi_t(x) \in X\} \in \mathbb{R}^+,$$

for each $x \in X$, with the convention that $\inf \emptyset = +\infty$. The number $\tau(x)$ is called the **first return time** of x to the set X.



- Thus, the first return time to X is a function $\tau : X \to \mathbb{R}^+$.
- The definition assumes that each point of X returns to X.

Poincaré Maps

- Given a Poincaré section, one can introduce a corresponding Poincaré map.

Definition

Given a Poincaré section X for a semiflow φ_t, we define its **Poincaré map** $f : X \to X$ by

$$f(x) = \varphi_{\tau(x)}(x).$$

Differential Equations on the Torus $\mathbb{T}^2$

- We also consider a class of differential equations on $\mathbb{T}^2$.
- Recall that two vectors $x, y \in \mathbb{R}^2$ represent the same point of the torus $\mathbb{T}^2$ if and only if $x - y \in \mathbb{Z}^2$.

  Example: Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be C^1 functions such that, for all $x, y \in \mathbb{R}$, $k, \ell \in \mathbb{Z}$:
  - $f(x + k, y + \ell) = f(x, y)$;
  - $g(x + k, y + \ell) = g(x, y)$.

  Then the differential equation in the plane $\mathbb{R}^2$ given by

$$\begin{cases} x' = f(x, y), \\ y' = g(x, y) \end{cases}$$

  can be seen as a differential equation on $\mathbb{T}^2$.

  It has unique solutions (that are global, that is, they are defined for $t \in \mathbb{R}$ since the torus is compact).

Differential Equations on the Torus $\mathbb{T}^2$ (Cont'd)

- Let $\varphi_t : \mathbb{T}^2 \to \mathbb{T}^2$ be the corresponding flow.

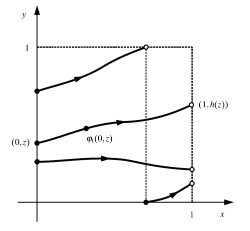
  Assume that f takes only positive values.

  Then each solution $\varphi_t(0, z) = (x(t), y(t))$ of the equation crosses infinitely often the line segment $x = 0$.

  Thus, $x = 0$ is a Poincaré section for φ_t.

  The first intersection (for $t > 0$) occurs at the time

$$T_z = \inf\{t > 0 : x(t) = 1\}.$$



One can use the C^1 dependence of the solutions of a differential equation on the initial conditions to show that h is a diffeomorphism.

This is, a bijective differentiable map with differentiable inverse.

Subsection 4

Invariant Sets

Invariant Sets

Definition

Given a map $f : X \to X$, a set $A \subseteq X$ is said to be:

1. f-**Invariant** if $f^{-1}A = A$, where
$$f^{-1}A = \{x \in X : f(x) \in A\};$$
2. **Forward** f-**invariant** if $f(A) \subseteq A$;
3. **Backward** f-**invariant** if $f^{-1}A \subseteq A$.

Example

- Consider the rotation $R_\alpha : S^1 \to S^1$.

  Consider the set
  $$\gamma(x) = \{R_\alpha^n(x) : n \in \mathbb{Z}\}.$$

  For $\alpha \in \mathbb{Q}$, it is finite and R_α-invariant.

  More generally, if $\alpha \in \mathbb{Q}$, then a nonempty set $A \subseteq X$ is R_α-invariant if and only if it is a union of sets of the form $\gamma(x)$.

  For example, the set $\mathbb{Q}/\mathbb{Z}$ is R_α-invariant.

  For $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, each set $\gamma(x)$ is also R_α-invariant, but now it is infinite.

  Again, a nonempty set $A \subseteq X$ is R_α-invariant if and only if it is a union of sets of the form $\gamma(x)$.

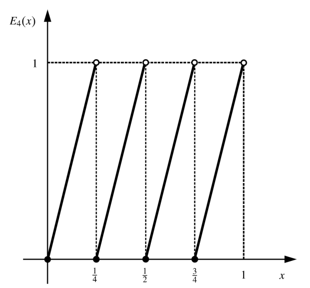
  One can show that each set $\gamma(x)$ is dense in S^1.

  Thus, the closed R_α-invariant sets are $\emptyset$ and S^1.

Example

- Now we consider the expanding map $E_4 : S^1 \to S^1$, given by

$$E_4(x) = \begin{cases} 4x, & \text{if } x \in [0, 1/4), \\ 4x - 1, & \text{if } x \in [1/4, 2/4), \\ 4x - 2, & \text{if } x \in [2/4, 3/4), \\ 4x - 3, & \text{if } x \in [3/4, 1). \end{cases}$$



For example, the set

$$A = \bigcap_{n \geq 0} E_4^{-n}([0, 1/4] \cup [2/4, 3/4])$$

is forward E_4-invariant.

We note that A is a Cantor set, that is, A is a closed set without isolated points and containing no intervals.

Orbits and Semiorbits

Definition

For a map $f : X \to X$, given a point $x \in X$, the set

$$\gamma^+(x) = \gamma_f^+(x) = \{f^n(x) : n \in \mathbb{N}_0\}$$

is called the **positive semiorbit** of x.
Moreover, when f is invertible,

$$\gamma^-(x) = \gamma_f^-(x) = \{f^{-n}(x) : n \in \mathbb{N}_0\}$$

is called the **negative semiorbit** of x.
The set

$$\gamma(x) = \gamma_f(x) = \{f^n(x) : n \in \mathbb{Z}\}$$

is called the **orbit** of x.

Orbits and Invariance

Claim: When f is invertible, a nonempty set $A \subseteq X$ is f-invariant if and only if it is a union of orbits.

By definition, $A \subseteq X$ is f-invariant if and only if

$$x \in A \quad \text{iff} \quad x \in f^{-1}(A) \quad \text{iff} \quad f(x) \in A.$$

By induction, and f's invertibility, this is equivalent to

$$x \in A \quad \text{iff} \quad \{f^n(x) : n \in \mathbb{Z}\} \subseteq A \quad \text{iff} \quad \gamma(x) \in A.$$

Thus, a nonempty set $A \subseteq X$ is f-invariant if and only if

$$A = \bigcup_{x \in A} \gamma(x).$$

Invariance for Flows and Semiflows

Definition

Given a flow $\Phi = (\varphi_t)_{t\in\mathbb{R}}$ of X, a set $A \subseteq X$ is said to be **Φ-invariant** if

$$\varphi_t^{-1}A = A, \text{ for } t \in \mathbb{R}.$$

Given a semiflow $\Phi = (\varphi_t)_{t\geq 0}$ of X, a set $A \subseteq X$ is said to be **Φ-invariant** if

$$\varphi_t^{-1}A = A, \text{ for } t \geq 0.$$

- In the case of flows, since $\varphi_t^{-1} = \varphi_{-t}$ for $t \in \mathbb{R}$, a set $A \subseteq X$ is **Φ-invariant** if and only if

$$\varphi_t(A) = A, \text{ for } t \in \mathbb{R}.$$

Example

- Consider the differential equation

$$\begin{cases} x' = 2y^3, \\ y' = -3x. \end{cases}$$

Each solution $(x, y) = (x(t), y(t))$ satisfies

$$\begin{aligned} (3x^2 + y^4)' &= 6xx' + 4y^3y' \\ &= 12xy^3 - 12y^3x = 0. \end{aligned}$$

Consider, for each set $I \subseteq \mathbb{R}^+$, the union

$$A = \bigcup_{a \in I} \{(x, y) \in \mathbb{R}^2 : 3x^2 + y^4 = a\}.$$

It is clearly invariant with respect to the flow determined by the differential equations.

Orbits and Semiorbits for a Semiflow

Definition

For a semiflow $\Phi = (\varphi_t)_{t \geq 0}$ of X, given a point $x \in X$, the set

$$\gamma^+(x) = \gamma^+_\Phi(x) = \{\varphi_t(x) : t \geq 0\}$$

is called the **positive semiorbit** of x.
Moreover, for a flow $\Phi = (\varphi_t)_{t \in \mathbb{R}}$ of X,

$$\gamma^-(x) = \gamma^-_\Phi(x) = \{\varphi_{-t}(x) : t \geq 0\}$$

is called the **negative semiorbit** of x. Further, the set

$$\gamma(x) = \gamma_\Phi(x) = \{\varphi_t(x) : t \in \mathbb{R}\}$$

is called the **orbit** of x.