# Introduction to Dynamical Systems

**George Voutsadakis**[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 500

1. Topological Dynamics
   - Topological Dynamical Systems
   - Limit Sets and Basic Properties
   - Topological Recurrence
   - Topological Entropy

## Subsection 1

# Topological Dynamical Systems

# Topological Dynamical Systems

## Definition

A continuous map

$$f : X \to X$$

is said to be a **topological dynamical system with discrete time** or, simply, a **topological dynamical system**.

When $f$ is a homeomorphism (that is, a bijective continuous map with continuous inverse), we also say that

$$f : X \to X$$

is an **invertible topological dynamical system**.

# Example

- Consider the circle $S^1 = \mathbb{R}/\mathbb{Z}$.

  The topology is the one induced from that of $\mathbb{R}$.

  Each rotation

  $$R_\alpha : S^1 \to S^1$$

  is a homeomorphism of the circle.

  More precisely, the topology of $S^1$ is generated by the sets of the form $(a, b)$ and $[0, a) \cup (b, 1]$, with $0 < a < b < 1$.

  The distance $d$ on $S^1$ is given by

  $$\begin{aligned} d(x, y) &= \min \{|(x + k) - (y + \ell)| : k, \ell \in \mathbb{Z}\} \\ &= \min \{|x - y - m| : m \in \mathbb{Z}\}. \end{aligned}$$

# Topological Flows

## Definition

Any flow (respectively, any semiflow)

$$\varphi_t : X \to X,$$

such that the map $(t, x) \mapsto \varphi_t(x)$ is continuous in $\mathbb{R} \times X$ (respectively, in $\mathbb{R}_0^+ \times X$) is said to be a **topological flow** (respectively, a **topological semiflow**).

Any topological flow or semiflow is also said to be a **topological dynamical system with continuous time** or, simply, a **topological dynamical system**.

- The continuity assumptions imply that each map $\varphi_t : X \to X$ is continuous.
- In the case of flows it is even a homeomorphism.

# Lipschitz Functions and Gronwall's Lemma

- Recall $f : \mathbb{R}^n \to \mathbb{R}^n$ is said to be a **Lipschitz function** if there exists an $L > 0$, such that

$$\|f(x) - f(y)\| \leq L\|x - y\|, \quad \text{for } x, y \in \mathbb{R}^n.$$

## Theorem (Gronwall's Lemma)

Let $u, v : [a, b] \to \mathbb{R}$ are continuous functions, with $v \geq 0$, such that

$$u(t) \leq c + \int_a^t u(s)v(s)ds, \quad t \in [a, b].$$

Then

$$u(t) \leq c \exp \int_a^t v(s)ds, \quad t \in [a, b].$$

# Example

- Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be a Lipschitz function with $f(0) = 0$.

  Consider the initial value problem

  $$\begin{cases} x' &= f(x), \\ x(0) &= x_0. \end{cases}$$

  It has a unique solution

  $$x(t, x_0).$$

  for each $x_0 \in \mathbb{R}^n$.

  Moreover, we have

  $$x(t, x_0) = x_0 + \int_0^t f(x(s, x_0))ds.$$

# Example (Cont'd)

- Therefore,

$$\begin{array}{rcl} \|x(t,x_0)\| & \leq & \|x_0\| + |\int_0^t \|f(x(s,x_0))\| ds| \\ & \leq & \|x_0\| + L|\int_0^t \|x(s,x_0)\| ds|. \end{array}$$

By Gronwall's Lemma,

$$\|x(t,x_0)\| \leq \|x_0\| e^{L|t|},$$

for $t$ in the domain of the solution.

This implies that the solution $\varphi_t(x_0) = x(t,x_0)$ is defined for $t \in \mathbb{R}$.

It follows from the continuous dependence of the solutions of a differential equation on the initial conditions that

$$\varphi_t : \mathbb{R}^n \to \mathbb{R}^n$$

is a topological dynamical system.

Subsection 2

# Limit Sets and Basic Properties

# Limit Sets in Discrete Time

- We begin with the case of discrete time.
- Let $f : X \to X$ be a map (not necessarily continuous).

## Definition

Given a point $x \in X$, the $\omega$-**limit set** of $x$ is defined by

$$\omega(x) = \omega_f(x) = \bigcap_{n\in\mathbb{N}} \overline{\{f^m(x) : m \geq n\}}.$$

Moreover, when $f$ is invertible, the $\alpha$-**limit set** of $x$ is defined by

$$\alpha(x) = \alpha_f(x) = \bigcap_{n\in\mathbb{N}} \overline{\{f^{-m}(x) : m \geq n\}}.$$

# Example

- Let $R_\alpha : S^1 \to S^1$ be a rotation of the circle.
  For $\alpha \in \mathbb{Q}$, we have $\omega(x) = \alpha(x) = \gamma(x)$, for $x \in S^1$.
  For $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, we have $\omega(x) = \alpha(x) = S^1$, for $x \in S^1$.
  To establish this, we must show that the sets

  $$\{R_\alpha^m(x) : m \geq n\} \quad \text{and} \quad \{R_\alpha^{-m}(x) : m \geq n\}$$

  are dense in $S^1$, for every $x \in S^1$ and $n \in \mathbb{N}$.
  Assume, first, that there exist integers $m_1 > m_2 \geq n$, such that

  $$R_\alpha^{m_1}(x) = R_\alpha^{m_2}(x).$$

  This is the same as $x + m_1\alpha = x + m_2\alpha \mod 1$.
  Equivalently, $m_1\alpha - m_2\alpha = m$, for some $m \in \mathbb{Z}$.
  Thus, $\alpha = \frac{m}{m_1 - m_2}$, contradicting the irrationality of $\alpha$.
  So, for each $n \in \mathbb{N}$, the points $R_\alpha^m(x)$ are pairwise distinct for $m \geq n$.

# Example (Cont'd)

- Take $\epsilon > 0$ and $N \in \mathbb{N}$, such that $\frac{1}{N} < \epsilon$.
  The points $R_\alpha^n(x), R_\alpha^{n+1}(x), \ldots, R_\alpha^{n+N}(x)$ are distinct.
  So there exist integers $i_1$ and $i_2$, such that $0 \leq i_1 < i_2 \leq N$ and
  $$d(R_\alpha^{n+i_1}(x), R_\alpha^{n+i_2}(x)) \leq \frac{1}{N} < \epsilon,$$
  where $d$ is the distance $d(x, y) = \min\{|x - y - m| : m \in \mathbb{Z}\}$.
  Hence,
  $$\begin{array}{rcl} d(R_\alpha^{i_2-i_1}(x), x) & = & d(R_\alpha^{i_2-i_1}(R_\alpha^{n+i_1}(x)), R_\alpha^{n+i_1}(x)) \\ & = & d(R_\alpha^{n+i_2}(x), R_\alpha^{n+i_1}(x)) \\ & < & \epsilon. \end{array}$$

# Example (Cont'd)

- So the sequence $x_m = R_\alpha^{m(i_2-i_1)}(x)$, with $m \in \mathbb{N}$, is $\epsilon$-dense in $S^1$.

  I.e., for each $y \in S^1$, there exists $m \in \mathbb{N}$, such that $d(y, x_m) < \epsilon$.

  Since $\epsilon$ is arbitrary, $\{R_\alpha^m(x) : m \geq n\}$ is dense in $S^1$.

  It remains to prove that

  $$\{R_\alpha^{-m}(x) : m \geq n\}$$

  is also dense in $S^1$.

  For this it is sufficient to repeat the above argument to show that there exist no integers $m_1 > m_2 \geq n$ with $R_\alpha^{-m_1}(x) = R_\alpha^{-m_2}(x)$.

  Alternatively, we may observe that this identity is equivalent to $R_\alpha^{m_1}(x) = R_\alpha^{m_2}(x)$.

# Example

Claim: Given $\alpha \in \mathbb{R} \backslash \mathbb{Q}$ and $\delta > 0$, there exist integers $p \in \mathbb{Z}$ and $q \in (0, \frac{1}{\delta}]$, such that $|\alpha - \frac{p}{q}| \leq \frac{\delta}{q}$.

Take an integer $N > 1$, such that $\frac{1}{N} \leq \delta$.

As in the preceding example, we find integers $m$ and $n$, such that $0 \leq n < m \leq N$ and

$$d(R_\alpha^m(0), R_\alpha^n(0)) < \frac{1}{N}.$$

Taking $q = m - n$, we obtain

$$\begin{array}{rcl} d(R_\alpha^q(0), 0) & = & d(R_\alpha^q(R_\alpha^n(0)), R_\alpha^n(0)) \\ & = & d(R_\alpha^m(0), R_\alpha^n(0)) \\ & < & \frac{1}{N} \leq \delta. \end{array}$$

# Example (Cont'd)

- We obtained
$$d(R_\alpha^q(0),0) < \frac{1}{N} \le \delta.$$
Finally, by the definition of $d$, there exists a $p \in \mathbb{Z}$, such that
$$|R_\alpha^q(0) - p| < \frac{1}{N} \le \delta.$$
But $\frac{1}{N} < 1$ and $R_\alpha^q(0) = q\alpha \mod 1$.
Therefore,
$$|q\alpha - p| < \frac{1}{N} \le \delta.$$
This, finally, gives
$$\left|\alpha - \frac{p}{q}\right| \le \frac{\delta}{q}.$$

# Example

- Now we consider the expanding map $E_2 : S^1 \to S^1$ and the point

$$x = 0.\boxed{0}\boxed{1}\boxed{00}\boxed{01}\boxed{10}\boxed{11}\boxed{000}\boxed{001}\boxed{010}\cdots$$

whose base-2 expansion comprises:
  - The sequence of all length 1 binary strings $(0, 1)$;
  - Followed by all length 2 binary strings $(00, 01, 10, 11)$;
  - Then all length 3 binary strings $(000, 001, 010, \ldots)$;

  $\vdots$

We have

$$E_2^m(0.x_1x_2\ldots) = 0.x_{m+1}x_{m+2}\ldots.$$

So each set $\{E_2^m(x) : m \geq n\}$ is dense in $S^1$. Thus, $\omega(x) = S^1$.

Note: The same happens when $x$ is replaced by any point in $S^1$ whose base-2 representation contains all finite binary strings, in any order.

# Example

- Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be the map given by

$$f(r\cos\theta, r\sin\theta) = \left( \frac{r}{r + \frac{1-r}{2}} \cos\left(\theta + \frac{\pi}{4}\right), \frac{r}{r + \frac{1-r}{2}} \sin\left(\theta + \frac{\pi}{4}\right) \right).$$

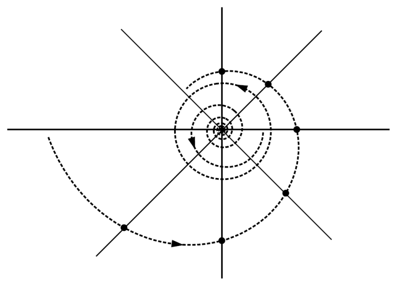
  One can easily verify that $f$ is invertible.

  Moreover, for all $n \in \mathbb{Z}$,

$$f^n(r\cos\theta, r\sin\theta) = \left( \frac{r}{r + \frac{1-r}{2^n}} \cos\left(\theta + \frac{n\pi}{4}\right), \frac{r}{r + \frac{1-r}{2^n}} \sin\left(\theta + \frac{n\pi}{4}\right) \right).$$

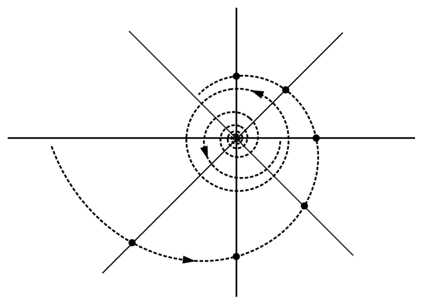
  Clearly, the origin $(r = 0)$ and the circle $r = 1$ are $f$-invariant sets.

# Example (Cont'd)



- For $r > 0$, we have
$$\lim_{n\to\infty} \frac{r}{r + \frac{1-r}{2^n}} = 1.$$
Thus, the $\omega$-limit set of a point $p = (r\cos\theta, r\sin\theta)$ outside the origin is
$$\omega(p) = \left\{\left(\cos\left(\theta + \frac{n\pi}{4}\right), \sin\left(\theta + \frac{n\pi}{4}\right)\right) : n = 0, 1, 2, \ldots, 7\right\}.$$

# Example (Cont'd)



- For $r \in (0, 1)$, we have

$$\lim_{n \to -\infty} \frac{r}{r + \frac{1-r}{2^n}} = 0.$$

Thus, the $\alpha$-limit set of any point in the region $0 < r < 1$ is the origin.

# Characterization of $\omega(x)$

- Recall that $X$ is a metric space, say with distance $d$.

## Proposition

Given a map $f : X \to X$, for each $x \in X$ the following properties hold:

1. $y \in \omega(x)$ if and only if there exists a sequence $n_k \nearrow \infty$ in $\mathbb{N}$ such that $f^{n_k}(x) \to y$ when $k \to \infty$;
2. If $f$ is continuous, then $\omega(x)$ is forward $f$-invariant.

- We have $\omega(x) = \bigcap_{m \geq 1} \overline{A_m}$, where $A_m = \{f^n(x) : n \geq m\}$.

  Let $y \in \omega(x)$. We consider two cases:

  1. Suppose $y \notin \bigcap_{m \geq 1} A_m$. Then there exists $p \geq 1$, such that $y \notin A_p$. Hence, $y \in \overline{A_p} \backslash A_p$. So there exists a sequence $n_k \nearrow \infty$ in $\mathbb{N}$ such that $f^{n_k}(x) \to y$ when $k \to \infty$.

# Characterization of $\omega(x)$ (Cont'd)

2. Suppose $y \in \bigcap_{m \geq 1} A_m$. Then, there exists $p \geq 1$, such that $y = f^p(x)$. Since $y \in A_m$, for $m > p$, there exists $q > p$, such that $y = f^q(x)$. Thus,

$$f^{(q-p)k}(f^p(x)) = y, \text{for } k \in \mathbb{N}.$$

Now the increasing sequence $n_k = (q-p)k + p$ satisfies $f^{n_k}(x) = y$.

Conversely, suppose there exists a sequence $n_k \nearrow \infty$ in $\mathbb{N}$, such that $f^{n_k}(x) \to y$ when $k \to \infty$. Then $y \in \overline{A_m}$, for every $m \in \mathbb{N}$.

Hence, $y \in \omega(x)$.

Now let us take $y \in \omega(x)$ and $n \in \mathbb{N}$. By Property 1, there exists a sequence $n_k \nearrow \infty$ in $\mathbb{N}$, such that $f^{n_k}(x) \to y$ when $k \to \infty$.

By the continuity of $f$, $f^{n_k+n}(x) \to f^n(y)$, when $k \to \infty$.

Hence $f^n(y) \in \omega(x)$. So $\omega(x)$ is forward $f$-invariant.

# Properties of $\omega(x)$

## Proposition

Let $f : X \to X$ be a continuous map. Suppose the positive semiorbit $\gamma^+(x)$ of a point $x \in X$ has compact closure. Then:

1. $\omega(x)$ is compact and nonempty;
2. $\inf\{d(f^n(x), y) : y \in \omega(x)\} \to 0$ when $n \to \infty$.

- Note that, by definition, the set $\omega(x)$ is closed.

  Now $\omega(x) \subseteq \overline{\gamma^+(x)}$ and, by hypothesis, $\overline{\gamma^+(x)}$ is compact.

  Thus, the set $\omega(x)$ is also compact.

  Next, consider the sequence $f^n(x)$.

  It is contained in the compact subset $\overline{\gamma^+(x)}$ of the metric space $X$.

  So there exists a convergent subsequence $f^{n_k}(x)$, with $n_k \nearrow \infty$.

  By Property 1 of the preceding proposition, the limit of $f^{n_k}(x)$ is in $\omega(x)$. So $\omega(x)$ is nonempty.

# Properties of $\omega(x)$ (Cont'd)

- Finally, suppose Property 2 does not hold.

  Then there would exist $\delta > 0$ and a sequence $n_k \nearrow \infty$, such that

  $$\inf \{d(f^{n_k}(x), y) : y \in \omega(x)\} \geq \delta, \quad k \in \mathbb{N}.$$

  But the set $\overline{\gamma^+(x)}$ is compact. So there would exist a convergent subsequence $f^{m_k}(x)$ of $f^{n_k}(x)$ whose limit, by Property 1 of the preceding proposition, is a point $p \in \omega(x)$.

  However, by the displayed inequality,

  $$d(f^{m_k}(x), y) \geq \delta, \quad k \in \mathbb{N}, \; y \in \omega(x).$$

  Thus, $d(p, y) \geq \delta$, for $y \in \omega(x)$. This is impossible, since $p \in \omega(x)$.

  This contradiction yields Property 2 of the proposition.

# Properties of $\alpha(x)$

## Proposition

Given an invertible map $f : X \to X$, for each $x \in X$ the following properties hold:

1. $y \in \alpha(x)$ if and only if there exists a sequence $n_k \nearrow \infty$ in $\mathbb{N}$ such that $f^{-n_k}(x) \to y$ when $k \to \infty$;
2. If $f$ has a continuous inverse, then $\alpha(x)$ is backward $f$-invariant.

## Proposition

Given an invertible map $f : X \to X$ with continuous inverse, if the negative semiorbit $\gamma^-(x)$ of a point $x \in X$ has compact closure, then:

1. $\alpha(x)$ is compact and nonempty;
2. $\inf \{d(f^n(x), y) : y \in \alpha(x)\} \to 0$ when $n \to -\infty$.

- The proofs involve applying the preceding to the map $g = f^{-1}$.

# Limits Sets for Continuous Time

## Definition

Given a semiflow $\Phi = (\varphi_t)_{t \geq 0}$ of $X$, the $\omega$-**limit set** of a point $x \in X$ is defined by

$$\omega(x) = \omega_\Phi(x) = \bigcap_{t>0} \overline{\{\varphi_s(x) : s > t\}}.$$

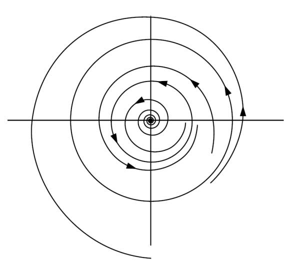
Moreover, given a flow $\Phi = (\varphi_t)_{t \in \mathbb{R}}$ of $X$, the $\alpha$-**limit set** of a point $x \in X$ is defined by

$$\alpha(x) = \alpha_\Phi(x) = \bigcap_{t<0} \overline{\{\varphi_s(x) : s < t\}}.$$

# Example

- Consider the differential equation in polar coordinates

$$\begin{cases} r' = r(r-1)(r-2), \\ \theta' = 1. \end{cases}$$



Note the following:

- $r' > 0$, for $r \in (0,1) \cup (2,+\infty)$;
- $r' < 0$ for $r \in (1,2)$.

# Example (Cont'd)

- Consider the sets

$$C_r = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = r^2\}, \quad r > 0.$$

Let $p \in C_r$.

We have:

$$\begin{array}{lll}
\alpha(p) = \{(0,0)\}, & \omega(p) = \{(0,0)\}, & \text{if } r = 0, \\
\alpha(p) = \{(0,0)\}, & \omega(p) = C_1, & \text{for } r \in (0,1), \\
\alpha(p) = C_1, & \omega(p) = C_1, & \text{for } r = 1, \\
\alpha(p) = C_2, & \omega(p) = C_1, & \text{for } r \in (1,2), \\
\alpha(p) = C_2, & \omega(p) = C_2, & \text{for } r = 2, \\
\alpha(p) = C_2, & \omega(p) = \emptyset, & \text{for } r > 2.
\end{array}$$

# Characterization of $\omega(x)$ in Continuous Time

## Proposition

Given a semiflow $\Phi = (\varphi_t)_{t\geq 0}$ of $X$, for each $x \in X$ the following properties hold:

1. $y \in \omega(x)$ if and only if there exists a sequence $t_k \nearrow +\infty$ in $\mathbb{R}^+$ such that $\varphi_{t_k}(x) \to y$ when $k \to \infty$;
2. If $\Phi$ is a topological semiflow, then $\omega(x)$ is forward $\Phi$-invariant.

- Both properties can be obtained repeating arguments in the proof of the corresponding proposition for the discrete case.

# Properties of $\omega(x)$ in Continuous Time

## Proposition

Let $\Phi = (\varphi_t)_{t\geq 0}$ be a topological semiflow of $X$. Suppose the positive semiorbit $\gamma^+(x)$ of a point $x \in X$ has compact closure. Then:

1. $\omega(x)$ is compact, connected and nonempty;
2. $\inf\{d(\varphi_t(x), y) : y \in \omega(x)\} \to 0$ when $t \to +\infty$.

- With the exception of the connectedness of the $\omega$-limit set, the remaining properties can be obtained repeating arguments in the proof of the discrete case.

# Properties of $\omega(x)$ in Continuous Time (Cont'd)

- We must show that $\omega(x)$ is connected.
  Suppose, to the contrary, that $\omega(x)$ is not connected.
  Then it can be written in the form

$$\omega(x) = A \cup B,$$

  for nonempty $A$ and $B$ such that

$$\overline{A} \cap B = A \cap \overline{B} = \emptyset.$$

  Since $\omega(x)$ is closed, we have

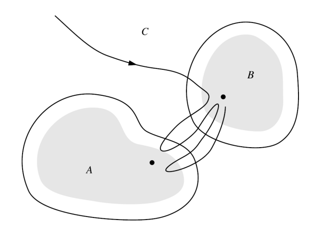
$$\begin{array}{rcl} \overline{A} & = & \overline{A} \cap \omega(x) \\ & = & \overline{A} \cap (A \cup B) \\ & = & (\overline{A} \cap A) \cup (\overline{A} \cap B) \\ & = & A. \end{array}$$

# Properties of $\omega(x)$ in Continuous Time (Cont'd)

- We have $\overline{A} = A$. Analogously $\overline{B} = B$.
  This shows that the sets $A$ and $B$ are also closed.
  This implies that they are at a positive distance, that is,
  $$\delta := \inf \{d(a,b) : a \in A, b \in B\} > 0.$$
  Now we consider the closed set
  $$C = \left\{ z \in X : d(z,y) \geq \frac{\delta}{4} \text{ for } y \in \omega(x) \right\}.$$
  Claim: $C \cap \{\varphi_s(x) : s > t\} \neq \emptyset$, for $t > 0$.
  Otherwise, the set $\{\varphi_s(x) : s > t\}$ would be completely contained in the $\frac{\delta}{4}$-neighborhood of $A$ or in the $\frac{\delta}{4}$-neighborhood of $B$.
  By the first property in the preceding proposition, we would have $\omega(x) \cap B = \emptyset$ or $\omega(x) \cap A = \emptyset$.
  This is impossible, since $\omega(x) = A \cup B$, with $A$ and $B$ nonempty.

# Properties of $\omega(x)$ in Continuous Time (Conclusion)

- It follows from the claim that there exists a sequence $t_k \nearrow +\infty$ such that $\varphi_{t_k}(x) \in C$ for $k \in \mathbb{N}$.
  Hence, it follows from the compactness of $C \cap \overline{\gamma^+(x)}$ and again from the first property in the preceding proposition that $C \cap \omega(x) \neq \emptyset$.




  On the other hand, it follows from the definition of $C$ that

$$C \cap \omega(x) = \emptyset.$$

  This contradiction shows that the set $\omega(x)$ is connected.

# Properties of $\alpha$-Limit Sets in Continuous Time

## Proposition

Given a flow $\Phi = (\varphi_t)_{t\in\mathbb{R}}$ of $X$, for each $x \in X$ the following properties hold:

1. $y \in \alpha(x)$ if and only if there exists a sequence $t_k \searrow -\infty$ in $\mathbb{R}$ such that $\varphi_{t_k}(x) \to y$ when $k \to \infty$;
2. If $\Phi$ is a topological flow, then $\alpha(x)$ is backward $\Phi$-invariant.

## Proposition

Let $\Phi = (\varphi_t)_{t\in\mathbb{R}}$ be a topological flow of $X$. Suppose the negative semiorbit $\gamma^-(x)$ of a point $x \in X$ has compact closure. Then:

1. $\alpha(x)$ is compact, connected and nonempty;
2. $\inf\{d(\varphi_t(x), y) : y \in \alpha(x)\} \to 0$ when $t \to -\infty$.

Subsection 3

## Topological Recurrence

# Recurrence

- Let $f : X \to X$ be a continuous map.

**Definition**

A point $x \in X$ is said to be (**positively**) **recurrent** (with respect to $f$) if $x \in \omega(x)$.

- By a previous proposition, a point $x$ is recurrent if and only if there exists a sequence $n_k \nearrow \infty$ in $\mathbb{N}$, such that $f^{n_k}(x) \to x$ when $k \to \infty$.
- Moreover, the set of recurrent points (with respect to $f$) is forward invariant.

  Indeed, suppose $f^{n_k}(x) \to x$ with $n_k \to \infty$ when $k \to \infty$.

  Then also $f^{n_k+n}(x) \to f^n(x)$ when $k \to \infty$, for $n \in \mathbb{N}$.

  Example: Any periodic point $x$ is recurrent, since $x \in \gamma^+(x) = \omega(x)$.

# Example

- Consider the rotation $R_\alpha : S^1 \to S^1$.
  When $\alpha$ is rational, all points are periodic.
  Thus, when $\alpha$ is rational all points are recurrent.
  When $\alpha$ is irrational, for each $x \in S^1$, we have $\omega(x) = S^1$.
  Again all points are recurrent.
- More generally, each point $x \in X$ with $\omega(x) = X$ is recurrent.
  Moreover, its positive semiorbit $\gamma^+(x)$ is dense in $X$.

# Topological Transitivity

## Definition

A map $f : X \to X$ is called **topologically transitive** if, given nonempty open sets

$$U, V \subseteq X,$$

there exists an $n \in \mathbb{N}$, such that

$$f^{-n}U \cap V \neq \emptyset.$$

# Properties of Topological Transitivity

## Theorem

Let $f : X \to X$ be a continuous map of a locally compact metric space with a countable basis. Then the following properties hold:

1. If $f$ is topologically transitive, then there exists an $x \in X$ whose positive semiorbit $\gamma^+(x)$ is dense in $X$;
2. If $X$ has no isolated points and there exists an $x \in X$ whose positive semiorbit $\gamma^+(x)$ is dense in $X$, then $f$ is topologically transitive.

- We first assume that $f$ is topologically transitive.

  Let $U \subseteq X$ be a nonempty open set.

  The union $\bigcup_{n \in \mathbb{N}} f^{-n}U$ intersects all open sets.

  So $\bigcup_{n \in \mathbb{N}} f^{-n}U$ is dense in $X$.

  Let $\{U_i\}_{i \in \mathbb{N}}$ be a countable basis of $X$.

# Topological Transitivity (Cont'd)

- Any locally compact metric space is a Baire space (i.e., it satisfies that any countable intersection of dense open sets is dense).

  So the set $Y = \bigcap_{i \in \mathbb{N}} \bigcup_{n \in \mathbb{N}} f^{-n} U_i$ is nonempty.

  Given $x \in Y$, we have $x \in \bigcup_{n \in \mathbb{N}} f^{-n} U_i$ for $i \in \mathbb{N}$.

  Thus, $\gamma^+(x) \cap U_i \neq \emptyset$, for $i \in \mathbb{N}$.

  This shows that the positive semiorbit of $x$ is dense in $X$.

  Now we assume that $X$ has no isolated points and that there exists an $x \in X$ with dense positive semiorbit.

  Let $U, V \subseteq X$ be nonempty open sets.

  By hypothesis, $X$ has no isolated points.

  So the semiorbit $\gamma^+(x)$ visits infinitely often $U$ and $V$.

  Hence, there exist $m, n \in \mathbb{N}$, $m > n$, with $f^m(x) \in U$ and $f^n(x) \in V$.

  Therefore, $x \in f^{-m} U \cap f^{-n} V = f^{-n}(f^{-(m-n)} U \cap V)$.

  So the set $f^{-(m-n)} U \cap V$ is nonempty.

# Example

- Clearly $S^1$ has no isolated points.

  Consider the rotation $R_\alpha : S^1 \to S^1$.

  By a previous example, if $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, then, for every $x \in S^1$, $\gamma^+(x)$ is dense in $S^1$.

  Therefore, by the theorem, for each $\alpha \in \mathbb{R} \backslash \mathbb{Q}$, the rotation $R_\alpha : S^1 \to S^1$ is topologically transitive.

# Dense Orbits and Dense Positive Semiorbits

**Theorem**

Let $X$ be a locally compact metric space with a countable basis and without isolated points. Let $f : X \to X$ be a homeomorphism. If there exists an $x \in X$ whose orbit $\gamma(x)$ is dense in $X$, then there exists a $y \in X$ whose positive semiorbit $\gamma^+(y)$ is dense in $X$.

- By hypothesis, $x$ is not isolated. So a dense orbit $\gamma(x)$ visits infinitely often each open neighborhood of $x$.
  Thus, there exists a sequence $n_k$, with $|n_k| \nearrow \infty$, such that
  $$f^{n_k}(x) \to x \quad \text{when } k \to \infty.$$
  By hypothesis, $f$ is a homeomorphism.
  So, we also have, for each $m \in \mathbb{Z}$,
  $$f^{n_k+m}(x) \to f^m(x) \quad \text{when } k \to \infty.$$

# Dense Orbits and Dense Positive Semiorbits (Cont'd)

- The sequence $n_k$ takes infinitely many positive values or infinitely many negative values (or both).
  - In the first case, the positive semiorbit $\gamma^+(x)$ is dense in $X$.
  - In the second case, the negative semiorbit $\gamma^-(x)$ is dense in $X$.

Let $U, V \subseteq X$ be nonempty open sets.

Now $\gamma^-(x)$ is dense and $X$ has no isolated points.

So there exist negative $m > n$, with $f^m(x) \in U$, $f^n(x) \in V$.

Hence,

$$x \in f^{-m}U \cap f^{-n}V = f^{-n}(f^{-(m-n)}U \cap V).$$

So the set $f^{-(m-n)}U \cap V$ is nonempty.

This shows that the map $f$ is topologically transitive.

By a previous theorem, there exists a dense positive semiorbit.

# Topological Mixing Maps

## Definition

A map $f : X \to X$ is called **topologically mixing** if, given nonempty open sets

$$U, V \subseteq X,$$

there exists an $n \in \mathbb{N}$, such that

$$f^{-m}U \cap V \neq \emptyset, \quad \text{for } m \geq n.$$

- Clearly, any topologically mixing map is also topologically transitive.

# Topological Transitivity Versus Mixing

- Let $R_\alpha : S^1 \to S^1$ be a rotation of the circle with $\alpha \in \mathbb{R} \setminus \mathbb{Q}$.

  Let $\varepsilon < \frac{1}{4}$ and consider the open interval $U = (x - \varepsilon, x + \varepsilon) \subseteq S^1$.

  We have that:
  - Each preimage $R_\alpha^{-n} U$ is an open interval of length $2\varepsilon < \frac{1}{2}$;
  - The orbit of $x$ is dense.

  Hence, there exists a sequence $n_k \nearrow \infty$ in $\mathbb{N}$, such that

  $$R_\alpha^{-n_k}(x) \to x + \frac{1}{2} \quad \text{when } k \to \infty.$$

  Thus, $R_\alpha^{-n_k} U \cap U = \emptyset$, for any sufficiently large $k$.

  This shows that the rotation $R_\alpha$ is not topologically mixing.

# Example

- Consider the expanding map $E_2 : S^1 \to S^1$.
  By a previous example, there exists a point $x \in S^1$ whose positive semiorbit $\gamma^+(x)$ is dense in $S^1$.
  By a previous theorem, the map $E_2$ is topologically transitive.
  Claim: $E_2$ is also topologically mixing.
  Let $U, V \subseteq S^1$ be nonempty open sets.
  Consider an open interval $I \subseteq V$ of the form
  $$I = (0.x_1x_2 \cdots x_n, 0.x_1x_2 \cdots x_n 11 \ldots),$$
  with the endpoints written in base 2.
  Let $y = 0.y_1y_2 \ldots \in U$. Take $x = 0.x_1x_2 \ldots x_n y_1 y_2 \ldots \in I$.
  We have $E_2^n(x) = y$. Hence, $x$ is in $E_2^{-n}U$.
  Therefore,
  $$E_2^{-n}U \cap V \supseteq E_2^{-n}U \cap I \neq \emptyset.$$
  This shows that the map $E_2$ is topologically mixing.

# Example

- Let $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ be an automorphism of the torus $\mathbb{T}^2$.
  Suppose that $|\mathrm{tr}A| > 2$.
  By invertibility, $A$ must be an invertible matrix with entries in $\mathbb{Z}$.
  So we have $\mathrm{det}A = \pm 1$.
  Note that
  $$\det(A - \lambda \mathrm{Id}) = \lambda^2 - \mathrm{tr}A\lambda + \mathrm{det}A.$$
  Thus, the eigenvalues of $A$ are given by
  $$\lambda_{1,2} = \frac{\mathrm{tr}A \pm \sqrt{(\mathrm{tr}A)^2 - 4\mathrm{det}A}}{2}.$$
  Since $|\mathrm{tr}A| > 2$, the eigenvalues are real numbers.
- Since $\lambda_1\lambda_2 = \pm 1$, there exists $\lambda > 1$, such that
  $$\{|\lambda_1|, |\lambda_2|\} = \{\lambda, \lambda^{-1}\}.$$

# Example (Cont'd)

Claim: $\lambda_1$ and $\lambda_2$ are irrational.

Clearly, $\lambda_1$ and $\lambda_2$ are rational if and only if

$$m^2 \pm 4 = \ell^2,$$

for some integer $\ell \in \mathbb{N}$, where $m = \mathrm{tr}A$.

Hence, $(m - \ell)(m + \ell) = \pm 4$.

Thus, since $m + \ell > m - \ell$,

$$m + \ell = 4 \text{ and } m - \ell = 1 \quad \text{or} \quad m + \ell = -1 \text{ and } m - \ell = -4.$$

It is easy to verify that these systems have no integer solutions.

This implies that $\lambda_1$ and $\lambda_2$ are irrational.

In particular, the eigendirections of $A$ have irrational slopes.

# Example (Cont'd)

- Now let $U, V \subseteq \mathbb{T}^2$ be nonempty open sets.

  Let $I \subseteq U$ be a line segment parallel to the eigendirection of $A$ corresponding to the eigenvalue with modulus $\lambda^{-1} < 1$.

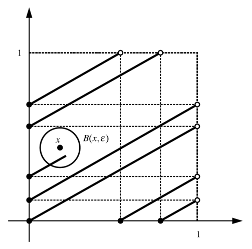
  Then $A^{-m}I \subseteq \mathbb{R}^2$ is a line segment of length $\lambda^m|I|$, where $|I|$ is the length of $I$.

  The eigendirection of $A$ corresponding to $\lambda^{-1}$ has irrational slope.

  Based on this, one can show that for any straight line $J \subseteq \mathbb{R}^2$ with this direction, the set $J/\mathbb{Z}^2$ is dense in $\mathbb{T}^2$.

  This implies that, given $\varepsilon > 0$, there exists an $L > 0$, such that for any line segment $J' \subseteq \mathbb{R}^2$ of length $L$ with that direction, the set $J'/\mathbb{Z}^2$ is $\varepsilon$-dense in $\mathbb{T}^2$.

  In other words, the $\varepsilon$-neighborhood of $J'/\mathbb{Z}^2$ coincides with $\mathbb{T}^2$.



# Example (Conclusion)

- Now take $\varepsilon > 0$ such that $V$ contains an open ball $B$ of radius $\varepsilon$. Recalling that $\lambda > 1$, take $n = n(\varepsilon) \in \mathbb{N}$, such that
$$\lambda^n |I| > L.$$
Since $\lambda^m |I| > L$, for $m \geq n$, by the $\varepsilon$-density of ${T_A}^{-m} I$ in $\mathbb{T}^2$, we obtain
$${T_A}^{-m} U \cap V \supseteq {T_A}^{-m} I \cap B \neq \emptyset, \quad m \geq n.$$
This shows that the automorphism of the torus $T_A$ is topologically mixing.

Subsection 4

# Topological Entropy

# Introduction

- We introduce the notion of the *topological entropy* of a dynamical system (with discrete time).
- Topological entropy measures how the orbits of a dynamical system move apart as time increases.
- So it can be seen as a measure of the complexity of the dynamics.
- We establish some *basic properties* of topological entropy.
- We illustrate its *computation* with several examples.
- We describe several alternative *characterizations* of topological entropy that are particularly useful for its explicit computation.
- We show that *topological entropy is a topological invariant*, i.e., it takes the same value for topologically conjugate dynamical systems.

# Topological Entropy

- Let $X$ be a compact metric space $X$, say with distance $d$.
- Let $f : X \to X$ be a continuous map.
- For each $n \in \mathbb{N}$, we introduce a new distance on $X$ by

$$d_n(x, y) = \max\{d(f^k(x), f^k(y)) : 0 \leq k \leq n-1\}.$$

## Definition

The **topological entropy** of $f$ is defined by

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon),$$

where $N(n, \varepsilon)$ is the largest number of points $p_1, \ldots, p_m \in X$, such that $d_n(p_i, p_j) \geq \varepsilon$, for $i \neq j$.

# Remarks on the Definition of Topological Entropy

- We defined
$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon).$$
Note: $N(n, \varepsilon)$ is always finite.
Let
$$B_1, B_2, \ldots$$
be a cover of $X$ by open balls of radius $\frac{\varepsilon}{2}$ in the distance $d_n$.
Since $X$ is compact, there exists a finite subcover, say $B_1', \ldots, B_m'$.
Thus, $N(n, \varepsilon) \leq m$.
Note: The function $\varepsilon \mapsto \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon)$ is nonincreasing.
Thus, the limit $\lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon)$ always exists.

# Example

- Let $R_\alpha : S^1 \to S^1$ be a rotation of the circle.

  Consider the distance

  $$d = \min \{|x - y - m| : m \in \mathbb{Z}\}.$$

  We have

  $$d(R_\alpha(x), R_\alpha(y)) = d(x, y), \quad x, y \in S^1.$$

  Thus, $d_n = d_1 = d$, for $n \in \mathbb{N}$.

  Now we get

  $$h(R_\alpha) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(1, \varepsilon) = 0.$$

# Example

- Consider the expanding map $E_2 : S^1 \to S^1$.

  The function $\varepsilon \mapsto \limsup_{n\to\infty} \frac{1}{n} \log N(n, \varepsilon)$ is nonincreasing.

  So, for any sequence $(a_k)_{k\in\mathbb{N}} \subseteq \mathbb{R}^+$, such that $a_k \to 0$,

  $$h(E_2) = \lim_{k\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log N(n, a_k).$$

  Let us take $a_k = \frac{1}{2^{k+1}}$.

  Claim: $N(n, \frac{1}{2^{k+1}}) = 2^{n+k}$, for $n, k \in \mathbb{N}$.

  Suppose, first, $d(x, y) < \frac{1}{2^n}$.

  Then

  $$d_n(x, y) = d(E_2^{n-1}(x), E_2^{n-1}(y)) = 2^{n-1} d(x, y).$$

# Example (Cont'd)

- Now consider the points $p_i = \frac{i}{2^{n+k}}$, for $i = 0, \ldots, 2^{n+k} - 1$.

  We get

  $$d_n(p_i, p_{i+1}) = \frac{1}{2^{k+1}}, \quad i = 0, \ldots, 2^{n+k} - 1.$$

  But there is no point $p_j$ between $p_i$ and $p_{i+1}$.

  So $d_n(p_i, p_j) \geq \frac{1}{2^{k+1}}$, for $i \neq j$. Thus, $N(n, \frac{1}{2^{k+1}}) \geq \frac{1}{2^{n+k}}$.

  Now consider a set $A \subseteq S^1$ with cardinality at least $2^{n+k} + 1$.

  Clearly, there exist points $x, y \in A$, with $x \neq y$, such that

  $$d(x, y) < \frac{1}{2^{n+k}}.$$

  This implies that $d_n(x, y) < \frac{1}{2^{k+1}}$. Hence, $N(n, \frac{1}{2^{k+1}}) \leq 2^{n+k}$.

# Example (Cont'd)

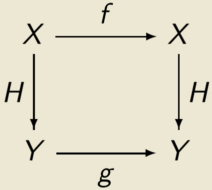
- Finally, using the claim, we get

$$\begin{aligned} h(E_2) &= \lim_{k\to\infty} \limsup_{n\to\infty} \frac{1}{n} \log N(n, \tfrac{1}{2^{k+1}}) \\ &= \lim_{k\to\infty} \limsup_{n\to\infty} \frac{n+k}{n} \log 2 \\ &= \log 2. \end{aligned}$$

# Topologically Conjugate Maps

## Definition

Two maps $f : X \to X$ and $g : Y \to Y$, where $X$ and $Y$ are topological spaces, are said to be **topologically conjugate** if there exists a homeomorphism $H : X \to Y$ such that $H \circ f = g \circ H$.




Then $H$ is called a **topological conjugacy**.

# Example

- Consider the map $f : R \to R$ defined by

$$f(z) = z^2$$

on the set $R = \{z \in \mathbb{C} : |z| = 1\}$.
Consider, also, the continuous map $H : S^1 \to R$ defined by

$$H(x) = e^{2\pi ix}.$$

$H$ is a homeomorphism, with inverse given by

$$H^{-1}(z) = \frac{\arg z}{2\pi} \mod 1.$$

We have

$$\begin{array}{rcl}(f \circ H)(x) & = & f(e^{2\pi ix}) = e^{4\pi ix}; \\ (H \circ E_2)(x) & = & H(2x) = e^{4\pi ix}.\end{array}$$

This shows that $H \circ E_2 = f \circ H$.
Thus, the maps $E_2$ and $f$ are topologically conjugate.

# Topological Invariance

- We say that a certain quantity, such as, for example, topological entropy, is a **topological invariant** if it takes the same value for topologically conjugate dynamical systems.

**Theorem**

Let $f : X \to X$ and $g : Y \to Y$ be continuous maps of compact metric spaces. If $f$ and $g$ are topologically conjugate, then

$$h(f) = h(g).$$

# Topological Invariance of Entropy

- Let $H : X \to Y$ be a homeomorphism such that $H \circ f = g \circ H$.

  The map $H$ is uniformly continuous.

  So, given $\varepsilon > 0$, there exists a $\delta > 0$, such that

  $$d_X(x,y) < \delta \quad \text{implies} \quad d_Y(H(x), H(y)) < \varepsilon,$$

  where $d_X$ and $d_Y$ are, respectively, the distances on $X$ and $Y$.

  We note that when $\varepsilon \to 0$, $\delta \to 0$.

  On the other hand, for $m \in \mathbb{N}$ and $x \in X$, $H(f^m(x)) = g^m(H(x))$.

  Hence, if $p_1, \ldots, p_m \in Y$, with $q_i = H(p_i)$, are such that

  $$\max\{d_Y(g^m(q_i), g^m(q_j)) : m = 0, \ldots, n-1\} \geq \varepsilon, \quad i \neq j,$$

  then $\max\{d_X(f^m(p_i), f^m(p_j)) : m = 0, \ldots, n-1\} \geq \delta$, for $i \neq j$.

  This shows that $N_f(n, \delta) \geq N_g(n, \varepsilon)$.

# Topological Invariance of Entropy

- We showed that $N_f(n,\delta) \geq N_g(n,\varepsilon)$.
  It follows that
  $$\limsup_{n\to\infty} \frac{1}{n} \log N_f(n,\delta) \geq \limsup_{n\to\infty} \frac{1}{n} \log N_g(n,\varepsilon),$$
  for each $\varepsilon > 0$. Letting $\varepsilon \to 0$, we have $\delta \to 0$. Thus, $h(f) \geq h(g)$.
  Now we rewrite $H \circ f = g \circ H$ in the form
  $$H^{-1} \circ g = f \circ H^{-1}.$$
  The previous argument, with $H$ replaced by $H^{-1}$, yields $h(g) \geq h(f)$.
  Therefore, $h(f) = h(g)$.

# Example

- Recall the preceding example.

  We considered the maps:
  - $f : R \to R$ defined by
    $$f(z) = z^2$$
    on the set $R = \{z \in \mathbb{C} : |z| = 1\}$;
  - The expanding map $E_2 : S^1 \to S^1$.

  We showed that $f$ is topologically conjugate to $E_2$.

  By the theorem and a previous example, we get
  $$h(f) = h(E_2) = \log 2.$$

# The Sets $M(n, \varepsilon)$ and $C(n, \varepsilon)$

**Definition**

Given $n \in \mathbb{N}$ and $\varepsilon > 0$, we denote by $M(n, \varepsilon)$ the least number of points $p_1, \ldots, p_m \in X$, such that each $x \in X$ satisfies $d_n(x, p_i) < \varepsilon$, for some $i$.

**Definition**

Given $n \in \mathbb{N}$ and $\varepsilon > 0$, we denote by $C(n, \varepsilon)$ the least number of elements of a cover of $X$ by sets $U_1, \ldots, U_m$ with

$$\sup \{d_n(x, y) : x, y \in U_i\} < \varepsilon, \text{ for } i = 1, \ldots, m.$$

- The supremum appearing in the definition of $C(n, \varepsilon)$ is called the $d_n$-**diameter** of $U_i$.

# Relations Between $M(n,\varepsilon)$, $C(n,\varepsilon)$ and $N(n,\varepsilon)$

## Proposition

For each $n \in \mathbb{N}$ and $\varepsilon > 0$, we have

$$C(n,2\varepsilon) \le M(n,\varepsilon) \le N(n,\varepsilon) \le M\left(n,\frac{\varepsilon}{2}\right) \le C\left(n,\frac{\varepsilon}{2}\right).$$

- We establish successively each of the inequalities:

1. For $m = M(n,\varepsilon)$, take points $p_1,\ldots,p_m \in X$, such that each $x \in X$ satisfies $d_n(x,p_i) < \varepsilon$, for some $i$.

   Then, the following $d_n$-open balls cover $X$,

   $$B_n(p_i,\varepsilon) = \{x \in X : d_n(x,p_i) < \varepsilon\}.$$

   But $B_n(p_i,\varepsilon)$ has $d_n$-diameter $2\varepsilon$.

   Therefore, $m \ge C(n,2\varepsilon)$.

# Relations Between $M(n,\varepsilon)$, $C(n,\varepsilon)$ and $N(n,\varepsilon)$ (Cont'd)

2. ($M(n,\varepsilon) \le N(n,\varepsilon)$) For $m = N(n,\varepsilon)$, let $p_1,\ldots,p_m \in X$ be such that
$$d_n(p_i,p_j) \ge \varepsilon, \quad i \ne j.$$
But each $x \in X \backslash \{p_1,\ldots,p_m\}$ satisfies $d_n(x,p_i) < \varepsilon$, for some $i$.
Hence, $M(n,\varepsilon) \le m$.
3. ($N(n,\varepsilon) \le M(n,\frac{\varepsilon}{2})$) Note that no $d_n$-open ball of radius $\frac{\varepsilon}{2}$ contains two points at a $d_n$-distance $\varepsilon$. Thus, $N(n,\varepsilon) \le M(n,\frac{\varepsilon}{2})$.
4. ($M(n,\frac{\varepsilon}{2}) \le C(n,\frac{\varepsilon}{2})$) For $m = C(n,\frac{\varepsilon}{2})$, let $U_1,\ldots,U_m$ be a cover of $X$ by sets of $d_n$-diameter less than $\frac{\varepsilon}{2}$.
Take a point $p_i \in U_i$ for each $i$. Clearly, $B_n(p_i,\frac{\varepsilon}{2}) \supseteq U_i$.
Now these $d_n$-balls form a cover of $X$.
Hence, $M(n,\frac{\varepsilon}{2}) \le C(n,\frac{\varepsilon}{2})$.

# A Property of $C(n,\varepsilon)$

## Lemma

Let $f : X \to X$ be a continuous map of a compact metric space. Given $m, n \in \mathbb{N}$ and $\varepsilon > 0$, we have

$$C(m+n,\varepsilon) \leq C(m,\varepsilon)C(n,\varepsilon).$$

- Let $U_1,\ldots,U_k$ be a cover of $X$ by sets of $d_n$-diameter less than $\varepsilon$, where $k = C(n,\varepsilon)$. Let $V_1,\ldots,V_\ell$ be a cover of $X$ by sets of $d_m$-diameter less than $\varepsilon$, where $\ell = C(m,\varepsilon)$.
  Note that, for all $x, y \in X$,

  $$d_{m+n}(x,y) = \max\{d_n(x,y), d_m(f^n(x), f^n(y))\}.$$

  Thus, the sets $U_i \cap f^{-n}V_j$, $i = 1,\ldots,k$, $j = 1,\ldots,\ell$, form a cover of $X$ and have $d_{m+n}$-diameter less than $\varepsilon$.
  It follows that $C(m+n,\varepsilon) \leq \ell k = C(m,\varepsilon)C(n,\varepsilon)$.

# An Auxiliary Lemma

## Lemma

If $(c_n)_{n\in\mathbb{N}}$ is a sequence of real numbers such that

$$c_{m+n} \leq c_m + c_{,} \quad m, n \in \mathbb{N},$$

then the limit

$$\lim_{n\to\infty} \frac{c_n}{n} = \inf\left\{\frac{c_n}{n} : n \in \mathbb{N}\right\}$$

exists.

# An Auxiliary Lemma (Cont'd)

- Given integers $n, k \in \mathbb{N}$, write

$$n = qk + r, \quad q \in \mathbb{N} \cup \{0\}, \ r \in \{0, \ldots, k-1\}.$$

Now we have

$$\frac{c_n}{n} \leq \frac{c_{qk} + c_r}{qk + r} \leq \frac{qc_k + c_r}{qk + r}.$$

Since $q \to \infty$ when $n \to \infty$ (for a fixed $k$),

$$\limsup_{n\to\infty} \frac{c_n}{n} \leq \frac{c_k}{k}.$$

Since $k$ is arbitrary, this implies that

$$\limsup_{n\to\infty} \frac{c_n}{n} \leq \inf\left\{\frac{c_k}{k} : k \in \mathbb{N}\right\} \leq \liminf_{n\to\infty} \frac{c_n}{n}.$$

# Alternative Formulas for the Topological Entropy

## Theorem

If $f: X \to X$ is a continuous map of a compact metric space, then

$$\begin{aligned} h(f) &= \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon) \\ &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log M(n, \varepsilon) \\ &= \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log M(n, \varepsilon) \\ &= \lim_{\varepsilon \to 0} \lim_{n \to \infty} \frac{1}{n} \log C(n, \varepsilon). \end{aligned}$$

# Alternative Formulas for the Topological Entropy (Cont'd)

- By the two preceding lemmas, the following limit exists,

$$\lim_{n\to\infty}\frac{1}{n}\log C(n,\varepsilon)=\inf\left\{\frac{1}{n}\log C(n,\varepsilon):n\in\mathbb{N}\right\}.$$

Using the inequalities of the preceding proposition, we get

$$\begin{aligned}\lim_{n\to\infty}\tfrac{1}{n}\log C(n,2\varepsilon) &\leq \liminf_{n\to\infty}\tfrac{1}{n}\log M(n,\varepsilon)\\ &\leq \liminf_{n\to\infty}\tfrac{1}{n}\log N(n,\varepsilon)\\ &\leq \limsup_{n\to\infty}\tfrac{1}{n}\log N(n,\varepsilon)\\ &\leq \limsup_{n\to\infty}\tfrac{1}{n}\log M(n,\tfrac{\varepsilon}{2})\\ &\leq \lim_{n\to\infty}\tfrac{1}{n}\log C(n,\tfrac{\varepsilon}{2}).\end{aligned}$$

# Alternative Formulas for the Topological Entropy (Cont'd)

- Letting $\varepsilon \to 0$ yields the inequalities

$$\begin{array}{rcl} \lim\limits_{\varepsilon \to 0} \lim\limits_{n \to \infty} \frac{1}{n} \log C(n, 2\varepsilon) & \leq & \lim\limits_{\varepsilon \to 0} \liminf\limits_{n \to \infty} \frac{1}{n} \log M(n, \varepsilon) \\ & \leq & \lim\limits_{\varepsilon \to 0} \liminf\limits_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon) \\ & \leq & h(f) \\ & \leq & \lim\limits_{\varepsilon \to 0} \limsup\limits_{n \to \infty} \frac{1}{n} \log M(n, \frac{\varepsilon}{2}) \\ & \leq & \lim\limits_{\varepsilon \to 0} \lim\limits_{n \to \infty} \frac{1}{n} \log C(n, \frac{\varepsilon}{2}). \end{array}$$

The equality of the first and the last terms establishes the desired result.

# Example: Automorphisms of the Torus

- Let $T_A : \mathbb{T}^2 \to \mathbb{T}^2$ be an automorphism of the torus.

  We recall that along the eigendirections of $A$ the distances are multiplied by $\lambda$ or $\lambda^{-1}$, for some $\lambda > 1$.

  Now we consider a cover of $\mathbb{T}^2$ by $d_n$-open balls $B_n(p_i, \varepsilon)$.

  We have

$$B_n(p_i, \varepsilon) = \bigcap_{k=0}^{n-1} T_A^{-k} B(T_A^k(p_i), \varepsilon).$$

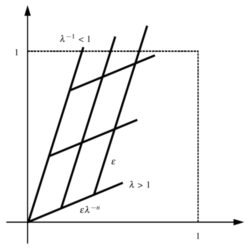
  Thus, there exists a $C > 0$ (independent of $n$, $\varepsilon$ and $i$), such that the area of $B_n(p_i, \varepsilon)$ is at most $C\lambda^{-n}\varepsilon^2$. Hence, $M(n, \varepsilon) \geq \frac{1}{C\lambda^{-n}\varepsilon^2}$.

  It follows from the theorem that

$$h(f) = \lim_{\varepsilon \to 0} \liminf_{n \to \infty} \frac{1}{n} \log M(n, \varepsilon) \geq \log \lambda.$$

# Example: Automorphisms of the Torus (Cont'd)

- We also consider partitions of $\mathbb{T}^2$ by parallelograms with sides parallel to the eigendirections of $A$.

  More precisely, we consider a partition of $\mathbb{T}^2$ by parallelograms $P_i$ with sides of length $\varepsilon\lambda^{-n}$ and $\varepsilon$, up to a multiplicative constant, along the eigendirections of $\lambda$ and $\lambda^{-1}$, respectively.

  Then there exists a $D > 1$ (independent of $n$, $\varepsilon$ and $i$), such that each $P_i$ has area at least $D^{-1}\lambda^{-n}\varepsilon^2$ and $d_n$-diameter less than $D\varepsilon$.

  Thus, $C(n, D\varepsilon) \leq \frac{1}{D^{-1}\lambda^{-n}\varepsilon^2}$.

  By the theorem, we have

  $$h(f) = \lim_{\varepsilon\to 0} \lim_{n\to\infty} \frac{1}{n} \log C(n,\varepsilon) \leq \log\lambda.$$

  This shows that $h(f) = \log\lambda$.

# Expansive Maps

## Definition

A map $f : X \to X$ is called (**positively**) **expansive** if there exists a $\delta > 0$, such that

$$d(f^n(x), f^n(y)) < \delta, \text{ for all } n \geq 0, \text{ implies } x = y.$$

Example: The expanding map $E_m : S^1 \to S^1$ is expansive.

Suppose $d(x, y) < \frac{1}{m^2}$ and $x \neq y$.

Then there exists an $n \in \mathbb{N}$, such that

$$d(E_m^n(x), E_m^n(y)) = m^n d(x, y) \geq \frac{1}{m^2}.$$

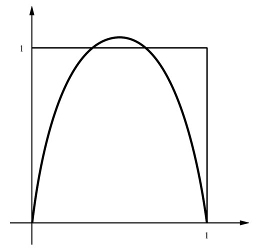
Thus, if $d(E_m^n(x), E_m^n(y)) < \frac{1}{m^2}$, for all $n \geq 0$, then $x = y$.

So the expanding map $E_m$ is expansive.

# Example

- Given $a > 4$, let $f : [0,1] \to \mathbb{R}$ be the quadratic map

$$f(x) = ax(1-x).$$




The set

$$X = \bigcap_{n=0}^{\infty} f^{-n}[0,1]$$

is compact and forward $f$-invariant.
In particular, one can consider the restriction $f \mid X : X \to X$.
We have $f(x) = 1$, for $x = \frac{1 \pm c}{2}$, where $c = \sqrt{1 - \frac{4}{a}}$.
Therefore, for $x \in X$,

$$|f'(x)| = a|1-2x| \geq ac.$$

# Example (Cont'd)

- Assume that $a > 4$ is so large that $ac > 1$.
  Equivalently, assume that
  $$a > 2 + \sqrt{5}.$$
  Let $x, y \in X$ be such that
  $$|f^k(x) - f^k(y)| < c, \quad \text{for } k \in \mathbb{N} \cup \{0\}.$$
  Then, for
  $$I_1 = \left[0, \frac{1-c}{2}\right] \quad \text{and} \quad I_2 = \left[\frac{1+c}{2}, 1\right],$$
  we have
  $$f^k(x), f^k(y) \in I_1 \quad \text{or} \quad f^k(x), f^k(y) \in I_2.$$
  Using the derivative inequality, for $k \in \mathbb{N}$,
  $$c > |f^k(x) - f^k(y)| \geq (ac)^k |x - y|.$$
  Since $ac > 1$, we get $x = y$. So $f \mid X$ is expansive.

# Entropy Formula for Expansive Maps

**Theorem**

Let $f: X \to X$ be a continuous expansive map of a compact metric space. Then

$$\begin{aligned} h(f) &= \lim_{n\to\infty} \frac{1}{n} \log N(n,\alpha) \\ &= \lim_{n\to\infty} \frac{1}{n} \log M(n,\alpha) \\ &= \lim_{n\to\infty} \frac{1}{n} \log C(n,\alpha), \end{aligned}$$

for any sufficiently small $\alpha > 0$.

- Let $\delta$ be the constant in the expansive property.

  Take constants $\varepsilon, \alpha > 0$, such that $0 < \varepsilon < \alpha < \delta$.

  Let $A \subseteq X$ be a set with $\text{card} A = N(n,\varepsilon)$, such that

  $$d_n(x,y) \geq \varepsilon, \quad \text{for all } x, y \in A, \text{ with } x \neq y.$$

# Entropy Formula for Expansive Maps (Cont'd)

- Claim: There exists an $m = m(\varepsilon, \alpha) \in \mathbb{N}$, such that, if $d(x, y) \geq \varepsilon$, then

$$d(f^i(x), f^i(y)) > \alpha, \quad \text{for some } i \in \{0, \ldots, m\}.$$

Let

$$q \in K := \{(x, y) \in X \times X : d(x, y) \geq \varepsilon\}.$$

Now $f$ is continuous and expansive.

So there exist an open ball $B(q) \subseteq X \times X$ centered at $q$ and an integer $i = i(q) \in \mathbb{N} \cup \{0\}$, such that

$$(x, y) \in B(q) \quad \text{implies} \quad d(f^i(x), f^i(y)) > \delta > \alpha.$$

The balls $B(q)$ cover the compact set $K$.

Hence, there exists a finite subcover $B(q_j)$, with $j = 1, \ldots, p$.

Take $m = \max\{i(q_j) : j = 1, \ldots, p\}$.

We obtain the claimed property for $(x, y) \in K$.

# Entropy Formula for Expansive Maps (Cont'd)

- So, when $d_n(x,y) \geq \varepsilon$ and hence, for $x,y \in A$, with $x \neq y$,

$$d_n(f^j(x), f^j(y)) > \alpha, \quad \text{for some } j \in \{0,\ldots,m\}.$$

Thus, for $z,w \in f^{-m}A$, with $f^m(z) \neq f^m(w)$, we have

$$\begin{array}{rcl} d_{n+2m}(z,w) & \geq & \max\{d_n(f^i(z), f^i(w)) : i = m,\ldots,2m\} \\ & = & \max\{d_n(f^{j+m}(z), f^{j+m}(w)) : j = 0,\ldots,m\} \\ & > & \alpha, \end{array}$$

since $f^m(z), f^m(w) \in A$.

This yields the inequality

$$N(n+2m,\alpha) \geq N(n,\epsilon).$$

# Entropy Formula for Expansive Maps (Cont'd)

- It follows from a previous proposition that

$$\begin{aligned} N(n,\varepsilon) &\leq N(n+2m,\alpha) \\ &\leq M(n+2m,\tfrac{\alpha}{2}) \\ &\leq C(n+2m,\tfrac{\alpha}{2}) \\ &\leq C(n+2m,\tfrac{\varepsilon}{2}). \end{aligned}$$

Thus, applying the preceding theorem, we conclude that

$$\begin{aligned} \limsup_{n\to\infty}\frac{1}{n}\log N(n,\varepsilon) &\leq \limsup_{n\to\infty}\frac{1}{n}\log N(n+2m,\alpha) \\ &\leq \limsup_{n\to\infty}\frac{1}{n}\log M(n+2m,\tfrac{\alpha}{2}) \\ &\leq \limsup_{n\to\infty}\frac{1}{n}\log C(n+2m,\tfrac{\alpha}{2}) \\ &\leq \limsup_{n\to\infty}\frac{1}{n}\log C(n+2m,\tfrac{\varepsilon}{2}). \end{aligned}$$

# Entropy Formula for Expansive Maps (Conclusion)

- Letting $\varepsilon \to 0$ yields the inequalities

$$\begin{aligned} h(f) &\leq \limsup_{n\to\infty} \frac{1}{n} \log N(n, \alpha) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log M(n, \tfrac{\alpha}{2}) \\ &\leq \limsup_{n\to\infty} \frac{1}{n} \log C(n, \tfrac{\alpha}{2}) \leq h(f). \end{aligned}$$

  One can also replace each $\limsup$ by $\liminf$.

  Then, letting $\varepsilon \to 0$, we obtain

$$\begin{aligned} h(f) &\leq \liminf_{n\to\infty} \frac{1}{n} \log N(n, \alpha) \\ &\leq \liminf_{n\to\infty} \frac{1}{n} \log M(n, \tfrac{\alpha}{2}) \\ &\leq \liminf_{n\to\infty} \frac{1}{n} \log C(n, \tfrac{\alpha}{2}) \leq h(f). \end{aligned}$$

  The identities now follow from these two chains of inequalities.

# Example

- Consider the restriction $E_4 \mid A : A \to A$, where $A$ is the compact forward $E_4$-invariant set

$$A = \bigcap_{n \geq 0} E_4^{-n} \left( \left[0, \frac{1}{4}\right] \cup \left[\frac{2}{4}, \frac{3}{4}\right] \right).$$

Note that if $d(x, y) < \frac{1}{4^n}$, then

$$d_n(x, y) = d(E_4^{n-1}(x), E_4^{n-1}(y)) = 4^{n-1} d(x, y).$$

Given $k \in \mathbb{N}$, consider the $2^{n+k+1}$ points $x_i$ on the boundary of

$$\bigcap_{m=0}^{n+k-1} E_4^{-m} \left( \left[0, \frac{1}{4}\right] \cup \left[\frac{2}{4}, \frac{3}{4}\right] \right).$$

From the relation between the distances, for $i \neq j$,

$$d_n(x_i, x_j) \geq 4^{n-1} \cdot \frac{1}{4^{n+k}} = \frac{1}{4^{k+1}}.$$

# Example (Cont'd)

- We conclude that $N(n, \frac{1}{4^{k+1}}) \geq 2^{n+k+1}$.

  On the other hand, given a set $B \subseteq A$ with at least $2^{n+k+1}+1$ points, there exist $x, y \in B$, with $x \neq y$, such that $d(x,y) < \frac{1}{4^{n+k}}$. Thus,

  $$d_n(x,y) < \frac{1}{4^{k+1}}.$$

  This implies that

  $$N\left(n, \frac{1}{4^{k+1}}\right) = 2^{n+k+1}, \text{ for } n, k \in \mathbb{N}.$$

  Since $E_4$ is expansive, the same happens to the restriction $E_4 \mid A$. It then follows from the preceding theorem that

  $$\begin{array}{rcl} h(E_4 \mid A) & = & \lim_{n\to\infty} \frac{1}{n} \log N(n, \frac{1}{4^{k+1}}) \\ & = & \lim_{n\to\infty} \frac{n+k+1}{n} \log 2 = \log 2. \end{array}$$