Introduction to Dynamical Systems

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LSSU Math 500

1. Low-Dimensional Dynamics
   - Homeomorphisms of the Circle
   - Diffeomorphisms of the Circle
   - Maps of the Interval
   - The Poincaré-Bendixson Theorem

Subsection 1

Homeomorphisms of the Circle

Equivalence Classes of Reals Modulo 1

- Consider the projection $\pi : \mathbb{R} \to S^1$ defined by $\pi(x) = [x]$.
- Consider the equivalence class $[x]$.
- It is represented by its unique representative in the interval $[0, 1)$.
- That is $[x]$ is represented by the number

$$x - \lfloor x \rfloor,$$

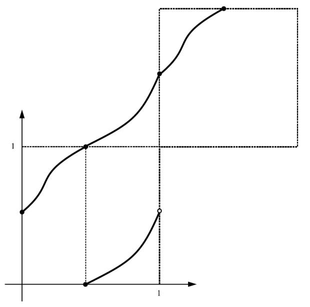
where $\lfloor x \rfloor$ is the integer part of x.

Lifting a Homeomorphism of the Circle

Definition

Let $f : S^1 \to S^1$ be a homeomorphism of the circle.
A continuous function $F : \mathbb{R} \to \mathbb{R}$ is said to be a **lift** of f if

$$f \circ \pi = \pi \circ F$$



Example

- Given $\alpha \in \mathbb{R}$, consider the rotation $R_\alpha : S^1 \to S^1$ given by

$$R_\alpha(x) = x + \alpha \mod 1.$$

Clearly, R_α is a homeomorphism.

Given $k \in \mathbb{Z}$, consider the function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = x + \alpha + k.$$

The function F satisfies

$$\begin{aligned} \pi(F(x)) &= \pi(x + \alpha + k) \\ &= x + \alpha + k \mod 1 \\ &= \pi(x) + \alpha \mod 1 \\ &= R_\alpha(\pi(x)). \end{aligned}$$

Hence, F is a lift of R_α.

Example

- Given $\beta \in \mathbb{R}$, consider the continuous function $f : S^1 \to S^1$ defined by

$$f(x) = x + \beta \sin(2\pi x) \mod 1.$$

Claim: f is a homeomorphism for $|\beta| < \frac{1}{2\pi}$.
Consider he function $F : \mathbb{R} \to \mathbb{R}$ defined by

$$F(x) = x + \beta \sin(2\pi x).$$

We have

$$F'(x) = 1 + 2\pi\beta \cos(2\pi x) \geq 1 - 2\pi|\beta| > 0.$$

So $F(x)$ is increasing.
In particular, for $x \in [0, 1)$, we have $F(x) < F(1) = 1$.
Thus, the function f is one-to-one and onto.

Example (Cont'd)

- Since f is continuous, it maps compact sets to compact sets.

  Thus, it also maps open sets to open sets.

  So its inverse is continuous.

  Hence, it is a homeomorphism.

  Moreover,

$$\begin{array}{rcl} \pi(F(x)) & = & x + \beta \sin(2\pi x) \quad \text{mod } 1 \\ & = & x - \lfloor x \rfloor + \beta \sin(2\pi x) \\ & = & x - \lfloor x \rfloor + \beta \sin(2\pi(x - \lfloor x \rfloor)) \\ & = & f(\pi(x)). \end{array}$$

  So F is a lift of f.

Properties of Lifts

Proposition

Let $f : S^1 \to S^1$ be a homeomorphism. Then:

1. f has lifts;
2. If F and G are lifts of f, then there exists a $k \in \mathbb{Z}$ such that $G - F = k$;
3. Any lift of f is a homeomorphism of $\mathbb{R}$.

- We deal with the case of increasing f. Let $x \in \mathbb{R}$.
  Apply f on the element of S^1 represented by $x - \lfloor x \rfloor$.
  Let $f(x - \lfloor x \rfloor)$ be the representative in the interval $[f(0), f(0) + 1)$.
  Define a function $F : \mathbb{R} \to \mathbb{R}$ by

$$F(x) = f(x - \lfloor x \rfloor) + \lfloor x \rfloor.$$

Now $x - \lfloor x \rfloor$ and $\lfloor x \rfloor$ are continuous on $\mathbb{R} \backslash \mathbb{Z}$. Thus, so too is F.

Properties of Lifts (Cont'd)

- For each $k \in \mathbb{Z}$, we have:

$$\begin{aligned} F(k) &= f(k-\lfloor k \rfloor)+\lfloor k \rfloor = f(k-k)+k = f(0)+k; \\ F(k^-) &= f(k^- - \lfloor k^- \rfloor)+\lfloor k^- \rfloor = f(k^- - k+1)+k-1 \\ &= f(1^-)+k-1 = f(0^+)+1+k-1 = f(0)+k; \\ F(k^+) &= f(k^+ - \lfloor k^+ \rfloor)+\lfloor k^+ \rfloor = f(k^+ - k)+k \\ &= f(0^+)+k = f(0)+k. \end{aligned}$$

Thus, for $k \in \mathbb{Z}$,

$$F(k) = F(k^-) = F(k^+).$$

This shows that the function F is continuous on $\mathbb{R}$.

We also have

$$\pi(F(x)) = \pi(f(x-\lfloor x \rfloor)+\lfloor x \rfloor) = f(x-\lfloor x \rfloor) = f(\pi(x)).$$

Hence, F is a lift of f.

Properties of Lifts (Cont'd)

- Now let F and G be lifts of f. Then

$$\pi \circ F = \pi \circ G = f \circ \pi.$$

By the first identity, for each $x \in \mathbb{R}$, there exists $p(x) \in \mathbb{Z}$, such that

$$G(x) - F(x) = p(x).$$

But F and G are continuous.

So the function $x \mapsto p(x)$ is also continuous.

Moreover, $x \mapsto p(x)$ takes only integer values.

So it must be constant.

Thus, there exists a $k \in \mathbb{Z}$, such that

$$G(x) - F(x) = p(x) = k, \text{ for any } x \in \mathbb{R}.$$

Properties of Lifts (Cont'd)

- By the second property, lifts are unique up to an additive constant. So it is sufficient to show that the lift

$$F(x) = \underbrace{f(x - \lfloor x \rfloor)}_{[f(0), f(0)+1)} + \lfloor x \rfloor$$

is a homeomorphism.

Consider the continuous function $H : \mathbb{R} \to \mathbb{R}$ defined by

$$H(x) = \underbrace{f^{-1}(x - \lfloor x \rfloor)}_{[f^{-1}(0)-1, f^{-1}(0))} + \lfloor x \rfloor,$$

where $f^{-1}(x - \lfloor x \rfloor)$ is the representative in the interval $[0, 1)$.

We can show by examining cases that

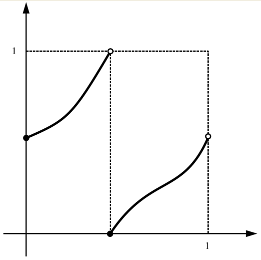
$$F(H(x)) = x \quad \text{and} \quad H(F(x)) = x.$$

Hence, F is a homeomorphism.

Orientation-Preserving Homeomorphisms

Definition

A homeomorphism $f : S^1 \to S^1$ is said to be **orientation-preserving** if it has a lift which is an increasing function.



- It follows from a previous proposition that f is orientation-preserving if and only if all its lifts are increasing functions.

  Examples: The homeomorphisms of the circle considered in the preceding two examples are orientation-preserving since the lifts presented for them are increasing functions.

A Non-Orientation-Preserving Homeomorphism

- Given $\alpha \in \mathbb{R}$, consider the homeomorphism $f : S^1 \to S^1$ defined by
$$f(x) = -x + \alpha \mod 1.$$
One can easily verify that the function $F : \mathbb{R} \to \mathbb{R}$ defined by
$$F(x) = -x + \alpha$$
is a lift of f.
Note that the lift F is decreasing.
So the homeomorphism f is not orientation-preserving.

"Average Speed" of a Lift

Theorem

Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism.
If F is a lift of f, then for each $x \in \mathbb{R}$ the limit

$$\rho(F) = \lim_{n\to\infty} \frac{F^n(x) - x}{n} \in \mathbb{R}_0^+$$

exists and is independent of x.
Moreover, if G is another lift of f, then

$$\rho(G) - \rho(F) \in \mathbb{Z}.$$

"Average Speed" of a Lift (Existence)

- We first assume that $F(x) > x$, for every $x \in \mathbb{R}$.
  Given $x \in \mathbb{R}$, consider the sequence $a_n = F^n(x) - x$.
  For each $m, n \in \mathbb{N}$, we have
  $$a_{m+n} = F^{m+n}(x) - x = F^m(F^n(x)) - F^n(x) + a_n.$$
  Now, since $a_n = F^n(x) - x$,
  $$\lfloor a_n \rfloor \leq F^n(x) - x < \lfloor a_n \rfloor + 1.$$
  That is,
  $$x + \lfloor a_n \rfloor \leq F^n(x) < x + \lfloor a_n \rfloor + 1.$$
  So, by the fact that F is a lifting, we obtain
  $$F^m(F^n(x)) < F^m(x + \lfloor a_n \rfloor) + 1.$$

"Average Speed" of a Lift (Existence Cont'd)

- On the other hand, we have

$$F^m(x+\lfloor a_n \rfloor)-(x+\lfloor a_n \rfloor)=F^m(x)-x=a_m.$$

Using these inequalities, we get

$$\begin{array}{rcl} a_{m+n} & < & F^m(x+\lfloor a_n \rfloor)+1-F^n(x)+a_n \\ & = & a_m+a_n+x+\lfloor a_n \rfloor-F^n(x)+1. \end{array}$$

Since $x+\lfloor a_n \rfloor \leq F^n(x)$, $a_{m+n} \leq a_m+a_n+1$.

So the sequence $c_n=a_n+1$ satisfies the condition $c_{m+n} \leq c_m+c_n$.

By a previous lemma, the following limit exists

$$\lim_{n\to\infty}\frac{F^n(x)-x}{n}=\lim_{n\to\infty}\frac{a_n}{n}=\inf\left\{\frac{a_n}{n}:n\in\mathbb{N}\right\}.$$

Since $a_n=F^n(x)-x>0$ (F is increasing), the limit is finite.

"Average Speed" of a Lift (Independence from x)

- Now we show that the limit is independent of x.

  Given $x, y \in \mathbb{R}$ and $k \in \mathbb{N}$ with $|x-y| \leq k$, we have

  $$F(x) \leq F(y+k) = F(y) + k;$$
  $$F(x) \geq F(y-k) = F(y) - k.$$

  Hence,

  $$|F(x) - F(y)| \leq k.$$

  It follows by induction that, for all $n \in \mathbb{N}$,

  $$|F^n(x) - F^n(y)| \leq k.$$

"Average Speed" of a Lift (Independence from x Cont'd)

- We showed $|x-y| \leq k$ implies $|F^n(x) - F^n(y)| \leq k$, $n \in \mathbb{N}$.

  This implies that

$$\begin{aligned} \left|\frac{F^n(x)-x}{n} - \frac{F^n(y)-y}{n}\right| &= \left|\frac{F^n(x)-F^n(y)}{n} + \frac{y-x}{n}\right| \\ &\leq \frac{2k}{n} \stackrel{n\to\infty}{\to} 0. \end{aligned}$$

  Note that, given $x, y \in \mathbb{R}$, one can always choose $k \in \mathbb{N}$, such that

$$|x-y| \leq k.$$

  Therefore, for $x, y \in \mathbb{R}$,

$$\lim_{n\to\infty} \frac{F^n(x)-x}{n} = \lim_{n\to\infty} \frac{F^n(y)-y}{n}.$$

"Average Speed" of a Lift (Last Property)

- It remains to establish the last property in the theorem.

  By a previous proposition, if F and G are lifts of f, then there exists a $k \in \mathbb{Z}$, such that
  $$G - F = k.$$
  It follows by induction that
  $$G^n(x) = F^n(x) + nk.$$
  Therefore,
  $$\begin{aligned} \rho(G) &= \lim_{n\to\infty} \frac{G^n(x)-x}{n} \\ &= \lim_{n\to\infty} \frac{F^n(x)-x}{n} + k \\ &= \rho(F) + k. \end{aligned}$$

The Rotation Number

Definition

The **rotation number** of an orientation-preserving homeomorphism $f : S^1 \to S^1$ is defined by

$$\rho(f) = \pi(\rho(F)),$$

where F is any lift of f and where $\pi(x) = [x]$.

- It follows from the last property in the theorem that the rotation number is well defined, i.e., $\rho(f)$ does not depend on the lift F.

Example

- Let $\alpha \in \mathbb{R}$ and consider the rotation
$$R_\alpha = x + \alpha \mod 1.$$
Recall the lift
$$F(x) = x + \alpha + k.$$
We obtain
$$\frac{F^n(x) - x}{n} = \frac{x + n(\alpha + k) - x}{n} = \alpha + k.$$
Thus, $\rho(F) = \alpha + k$.
Hence,
$$\rho(R_\alpha) = \pi(\rho(F)) = \alpha \mod 1.$$

Example

- Now we consider the homeomorphism $f : S^1 \to S^1$ defined by

$$f(x) = x + \beta \sin(2\pi x) \mod 1,$$

with $|\beta| < \frac{1}{2\pi}$.
Recall the lift

$$F(x) = x + \beta \sin(2\pi x).$$

By the theorem, $\rho(F) = \lim_{n\to\infty} \frac{F^n(x) - x}{n}$ does not depend on x.
So we have

$$\rho(F) = \lim_{n\to\infty} \frac{F^n(0) - 0}{n} = 0.$$

Homeomorphisms with Rational Rotation Number

- We consider the homeomorphisms with rational rotation number.
- Recall that $x \in S^1$ is said to be a **periodic point** of a map $f : S^1 \to S^1$ if
$$f^q(x) = x, \text{for some } q \in \mathbb{N}.$$

Theorem

Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism. Then $\rho(f) \in \mathbb{Q}$ if and only if f has at least one periodic point.

- We first assume that $\rho(f) = 0$ and we show that f has a fixed point.

  Assume, to the contrary, that f has no fixed points.

  Suppose F is a lift of f.

  Suppose that, for some $x \in \mathbb{R}$, $F(x) - x \in \mathbb{Z}$.

  Then $\pi(x) = \pi(F(x)) = f(\pi(x))$.

  Thus, $\pi(x)$ would be a fixed point of f.

Rational Rotation Number ($\rho(f) = 0$)

- It follows that
$$F(x) - x \in \mathbb{R} \backslash \mathbb{Z}, \quad \text{for } x \in \mathbb{R}.$$
Since F is continuous, there exists a $k \in \mathbb{Z}$, such that
$$k < F(x) - x < k + 1, \quad \text{for } x \in \mathbb{R}.$$
On the other hand, for $x \in \mathbb{R}$,
$$F(x+1) - (x+1) = F(x) - x.$$
Thus, the continuous function $x \mapsto F(x) - x$ is completely determined by its values on the compact interval $[0, 1]$.
It follows from Weierstrass' Theorem that there exists an $\varepsilon > 0$, such that
$$k + \varepsilon \leq F(x) - x \leq k + 1 - \varepsilon, \quad \text{for } x \in \mathbb{R}.$$

Rational Rotation Number ($\rho(f) = 0$ Cont'd)

- We saw that there exists an $\varepsilon > 0$, such that

$$k + \varepsilon \leq F(x) - x \leq k + 1 - \varepsilon, \quad \text{for } x \in \mathbb{R}.$$

But

$$F^n(x) - x = \sum_{i=0}^{n-1} [F(F^i(x)) - F^i(x)].$$

So we get

$$k + \varepsilon \leq \frac{F^n(x) - x}{n} \leq k + 1 - \varepsilon.$$

Thus,

$$\rho(f) = \lim_{n \to \infty} \frac{F^n(x) - x}{n} \mod 1 \in [\varepsilon, 1 - \varepsilon].$$

This contradicts the hypothesis that $\rho(f) = 0$.

Thus, f must have a fixed point.

Homeomorphisms with Rational Rotation Number (Cont'd)

- Now we assume that $\rho(f) = \frac{p}{q} \in \mathbb{Q}$.

  Since F^q is a lift of f^q, we obtain

$$\begin{aligned}
\rho(f^q) &= \lim_{n\to\infty} \frac{(F^q)^n(x) - x}{n} \mod 1 \\
&= q \lim_{n\to\infty} \frac{F^{qn}(x) - x}{qn} \mod 1 \\
&= q\rho(f) \mod 1 \\
&= p \mod 1 \\
&= 0.
\end{aligned}$$

  It follows from the above argument for a zero rotation number that the homeomorphism f^q has a fixed point.

  This fixed point is a periodic point of f.

Rational Rotation Number (Converse)

- For the converse, we assume that f has a periodic point.

  Then there exist $y \in \mathbb{R}$ and $q \in \mathbb{N}$, such that $f^q(\pi(y)) = \pi(y)$.

  By induction, $f^q \circ \pi = \pi \circ F^q$.

  Thus,

$$\pi(F^q(y)) = f^q(\pi(y)) = \pi(y).$$

  Hence, $F^q(y) = y + p$, for some $p \in \mathbb{Z}$.

  On the other hand, $F(x+1) - (x+1) = F(x) - x$.

  So $F(x+p) = F(x) + p$, for $x \in \mathbb{R}$.

  Thus, for $x \in \mathbb{R}$ and $q \in \mathbb{N}$,

$$F^q(x+p) = F^q(x) + p.$$

Rational Rotation Number (Converse)

- We got $F^q(x+p) = F^q(x) + p$, for $x \in \mathbb{R}$ and $q \in \mathbb{N}$.
  In particular, taking $x = y$, we obtain

$$\begin{array}{rcl} F^{2q}(y) & = & F^q(F^q(y)) \\ & = & F^q(y+p) \\ & = & F^q(y)+p \\ & = & y+2p. \end{array}$$

  It follows by induction that

$$F^{nq}(y) = y + np, \quad \text{for } n \in \mathbb{N}.$$

  Thus,

$$\rho(F) = \lim_{n\to\infty} \frac{F^{nq}(y)-y}{nq} = \lim_{n\to\infty} \frac{np}{nq} = \frac{p}{q}.$$

q-Periodic Points

- Consider a homeomorphism $f : S^1 \to S^1$.
- Recall that, given $q \in \mathbb{N}$, a point $x \in S^1$ is said to be a q-**periodic point** of f if
$$f^q(x) = x.$$
- It follows from the proof of the preceding theorem that f^q has a fixed point, that is, f has a q-periodic point, if and only if
$$\rho(f) = \frac{p}{q}, \quad \text{for some } p \in \mathbb{N}.$$
- Thus, f has a periodic point with period q if and only if $\rho(f) = \frac{p}{q}$, with p and q coprime.
- By the previous observation, f has no ℓ-periodic points for any $\ell < q$.

Period of Periodic Points

Theorem

Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism. If $\rho(f) = \frac{p}{q}$ with p and q coprime, then all periodic points of f have period q.

- Let $x \in S^1$ be a periodic point of f.

  By the former discussion, x has period $\ell = dq$, for some $d \in \mathbb{N}$.

  On the other hand, by the proof of the preceding theorem, if F is a lift of f, then

  $$F^\ell(x) = x + dp + m\ell, \quad \text{for some } m \in \mathbb{Z}.$$

  In fact, one can always assume that $m = 0$.

  Let G be another lift of f. Then $F = G + m$, for some $m \in \mathbb{Z}$.

  Thus, $F^\ell = G^\ell + m\ell$. So it is sufficient to replace F by G.

Period of Periodic Points (Cont'd)

Claim: $F^q(x) = x + p$.

Suppose, first, that $F^q(x) > x + p$.

We know that $F^q(x + p) = F^q(x) + p$.

Since F is increasing,

$$F^{2q}(x) > F^q(x + p) = F^q(x) + p > x + 2p.$$

By induction,

$$F^{\ell}(x) = F^{dq}(x) > x + dp.$$

This contradicts $F^{\ell}(x) = x + dp$.

Similarly, $F^q(x) < x + p$ yields a contradiction.

Thus, $F^q(x) = x + p$ and the point x has period q.

Irrational Rotation Number and Ordering

Theorem

Let F be a lift of an orientation-preserving homeomorphism of the circle $f : S^1 \to S^1$ with $\rho(f) \in \mathbb{R} \backslash \mathbb{Q}$. For each $x \in \mathbb{R}$ and $n_1, n_2, m_1, m_2 \in \mathbb{Z}$, we have

$$F^{n_1}(x) + m_1 < F^{n_2}(x) + m_2$$

if and only if

$$n_1 \rho(F) + m_1 < n_2 \rho(F) + m_2.$$

- If $n_1 = n_2$, there is nothing to prove. So suppose $n_1 \neq n_2$.
  Assume, first, that the inequality holds.
  For $n_1 > n_2$, we have $F^{n_1 - n_2}(x) < x + m_2 - m_1$, for $x \in \mathbb{R}$.
  Thus,

$$F^{2(n_1 - n_2)}(x) < F^{n_1 - n_2}(x) + m_2 - m_1 < x + 2(m_2 - m_1).$$

Irrational Rotation Number and Ordering (Cont'd)

- We obtain $F^{2(n_1-n_2)}(x) < x + 2(m_2 - m_1)$.

  By induction,
  $$F^{n(n_1-n_2)}(x) < x + n(m_1 - m_2).$$

  We obtain
  $$\rho(F) = \lim_{n\to\infty} \frac{F^{n(n_1-n_2)}(x) - x}{n(n_1 - n_2)} < \frac{m_2 - m_1}{n_1 - n_2}.$$

  Strict inequality holds, since $\rho(f)$ is irrational.

  This shows that the second inequality holds.

Irrational Rotation Number and Ordering (Cont'd)

- Analogously, for $n_1 < n_2$, we have

$$F^{n_2-n_1}(x) > x + m_1 - m_2, \text{ for } x \in \mathbb{R}.$$

Thus,

$$F^{n(n_2-n_1)}(x) > x + n(m_1 - m_2).$$

Hence,

$$\rho(F) = \lim_{n\to\infty} \frac{F^{n(n_2-n_1)}(x) - x}{n(n_2 - n_1)} > \frac{m_1 - m_2}{n_2 - n_1}.$$

So the second inequality also holds in this case.

Irrational Rotation Number and Ordering (Converse)

- In the other direction, we must show that

$$F^{n_1}(x) + m_1 \geq F^{n_2}(x) + m_2 \quad \text{implies} \quad n_1\rho(F) + m_1 \geq n_2\rho(F) + m_2.$$

By hypothesis, $\rho(f)$ is irrational.

So none of these inequalities can be an equality.

Thus, the implication is equivalent to

$$F^{n_1}(x) + m_1 > F^{n_2}(x) + m_2 \quad \text{implies} \quad n_1\rho(F) + m_1 > n_2\rho(F) + m_2.$$

For this it suffices to reverse all inequalities in the previous argument.

Irrational Rotation Number and Rotation of Circle

Theorem

Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with rotation number $\rho(f) \in \mathbb{R} \backslash \mathbb{Q}$. Then there exists a nondecreasing and onto continuous function $h : S^1 \to S^1$, such that

$$h \circ f = R_{\rho(f)} \circ h.$$

- Let F be a lift of the homeomorphism f and $\rho = \rho(F)$.
  For a point $x \in \mathbb{R}$, consider the sets

  $$A = \{F^n(x) + m : n, m \in \mathbb{Z}\}, \quad B = \{n\rho + m : n, m \in \mathbb{Z}\}.$$

  Define a function $H : \mathbb{R} \to \mathbb{R}$ by

  $$H(y) = \sup \{n\rho + m : F^n(x) + m \leq y\}.$$

  By the preceding theorem, H is nondecreasing.

Irrational Rotation and Rotation of Circle (Lemma)

Claim: H is constant on each interval in the complement of $\overline{A}$.

Suppose $[a, b] \subseteq S^1 \backslash \overline{A}$.

Then, for every $n, m \in \mathbb{Z}$,

$$F^n(x) + m \leq a \quad \text{iff} \quad F^n(x) + m \leq b.$$

Thus, $H(a) = H(b)$.

Lemma: The set B is dense in $\mathbb{R}$.

We have $y \in B$ if and only if $y + m \in B$, for some $m \in \mathbb{Z}$.

So it suffices to show that $B \cap [0, 1]$ is dense in $[0, 1]$.

The set $B \cap [0, 1]$ is infinite.

If not, there would exist pairs $(n_1, m_1) \neq (n_2, m_2)$ in $\mathbb{Z}^2$, such that

$$n_1\rho + m_1 = n_2\rho + m_2.$$

This is impossible, since ρ is irrational (if $n_1 = n_2$, then $m_1 \neq m_2$).

Irrational Rotation and Rotation of Circle (Lemma Cont'd)

- Let then x_n be a sequence in $B \cap [0,1]$ with infinitely many values.

  The interval $[0,1]$ is compact.

  So we can assume that the sequence x_n is convergent.

  Hence, given $\varepsilon > 0$, there exist $m, n \in \mathbb{N}$, such that

  $$0 < |x_n - x_m| < \varepsilon.$$

  Write $x_n = n_1\rho + m_1$ and $x_m = n_2\rho + m_2$.

  We obtain

  $$x_n - x_m = (n_1 - n_2)\rho + (m_1 - m_2) \in B.$$

  This shows that the set $B \supseteq \{k(x_n - x_m) : k \in \mathbb{Z}\}$ is ε-dense in $\mathbb{R}$.

  Since ε is arbitrary, we conclude that B is dense in $\mathbb{R}$.

Irrational Rotation and Rotation of Circle (Cont'd)

- Since ρ is irrational, it follows from the preceding theorem that

$$H(F^n(x) + m) = n\rho + m.$$

  This implies that the function H has no jumps.

  By the preceding equality, $H(\mathbb{R}) \supseteq H(A) = B$.

  By the lemma, the set B is dense in $\mathbb{R}$.

  Since H is monotonic, this implies that it is also continuous.

Irrational Rotation and Rotation of Circle (Cont'd)

- Now we consider the lift $S : \mathbb{R} \to \mathbb{R}$ of R_ρ given by

$$S(x) = x + \rho.$$

By the preceding equality, we have

$$\begin{aligned} (H \circ F)(F^n(x) + m) &= H(F^{n+1}(x) + m) = (n+1)\rho + m; \\ (S \circ H)(F^n(x) + m) &= S(n\rho + m) = (n+1)\rho + m. \end{aligned}$$

Thus, in A,

$$H \circ F = S \circ H.$$

But the maps H, F and S are continuous.

So this identity holds in $\overline{A}$.

But H is constant on each interval in the complement of $\overline{A}$.

So we have $H \circ F = S \circ H$ in $\mathbb{R}$.

Irrational Rotation and Rotation of Circle (Conclusion)

- On the other hand,

$$\begin{aligned} H(y+1) &= \sup\{n\rho+m : F^n(x)+m\leq y+1\} \\ &= \sup\{n\rho+m : F^n(x)+m-1\leq y\} \\ &= \sup\{n\rho+m-1 : F^n(x)+m-1\leq y\}+1 \\ &= H(y)+1. \end{aligned}$$

The function H is also onto: By continuity, we have

$$H(\mathbb{R}) = H([0,1]) \supseteq \overline{B} = \mathbb{R}.$$

Hence, the function $h : S^1 \to S^1$ defined by $h(y) = H(y) \mod 1$ is continuous, nondecreasing and onto.

Moreover, since $H \circ F = S \circ H$, we have $h \circ f = R_\rho \circ h$.

Poincaré's Theorem

- If the homeomorphism has a dense positive semiorbit, which by a previous theorem is equivalent to the existence of a dense orbit, then the preceding theorem can be strengthened as follows:

Theorem (Poincaré)

Let $f : S^1 \to S^1$ be an orientation-preserving homeomorphism with $\rho(f) \in \mathbb{R} \backslash \mathbb{Q}$. If f has a dense positive semiorbit, then it is topologically conjugate to the rotation $R_\rho(f)$, i.e., there exists a homeomorphism $h : S^1 \to S^1$ such that

$$h \circ f = R_\rho(f) \circ h.$$

- Let $x \in S^1$ be a point whose positive semiorbit is dense in S^1. Consider $h : S^1 \to S^1$, as constructed in the preceding theorem. In this case, $A = \{F^n(x) + m : n, m \in \mathbb{Z}\}$ is dense in S^1.

Poincaré's Theorem (Cont'd)

- Thus, the function

$$H(y) = \sup\{n\rho + m : F^n(x) + m \leq y\}$$

is bijective (we recall that H is constant on each interval contained in $\mathbb{R} \backslash \overline{A}$, which now is the empty set).

It follows that the function h is also bijective.

It remains to show that h is open.

That is, that the image $h(U)$ of an open set U is also open.

Since h is continuous, it maps compact sets to compact sets.

Hence, given an open set U, the image

$$h(S^1 \backslash U) = S^1 \backslash h(U)$$

is compact. Thus, $h(U)$ is an open set.

This shows that h is a homeomorphism.

Subsection 2

Diffeomorphisms of the Circle

Diffeomorphisms; Functions of Bounded Variation

- A **diffeomorphism** is a bijective differentiable map with differentiable inverse.
- We show that any sufficiently regular diffeomorphism $f : S^1 \to S^1$ with irrational rotation number is topologically conjugate to a rotation.
- More precisely, there exists a homeomorphism $h : S^1 \to S^1$, such that
$$h \circ f = R_{\rho(f)} \circ h.$$
- Recall that a function $\varphi : S^1 \to \mathbb{R}$ is of **bounded variation** if
$$\text{Var}(\varphi) = \sup \sum_{k=1}^{n} |\varphi(x_k) - \varphi(y_k)| < +\infty,$$
where the supremum is taken over all disjoint open intervals $(x_1, y_1), \ldots, (x_n, y_n)$, with $n \in \mathbb{N}$.

Example

- Let $\varphi : S^1 \to \mathbb{R}$ be a differentiable function with bounded derivative.

  Then there exists a $K > 0$, such that $|\varphi'(x)| \leq K$ for $x \in S^1$.

  If (x_i, y_i), for $i = 1, \ldots, n$, are disjoint open intervals with $y_1 \leq x_2$, $y_2 \leq x_3$, …, $y_{n-1} \leq x_n$, then

$$\begin{aligned} \sum_{i=1}^{n} |\varphi(y_i) - \varphi(x_i)| &= \sum_{i=1}^{n} |\varphi'(z_i)|(y_i - x_i) \\ &\quad \text{(for some } z_i \text{ in } (x_i, y_i)) \\ &\leq \sum_{i=1}^{n} K(y_i - x_i) \leq K. \end{aligned}$$

  Thus, $\text{Var}(\varphi) \leq K$. So φ has bounded variation.

Diffeomorphisms and Rotations

Theorem (Denjoy)

Let $f : S^1 \to S^1$ be an orientation-preserving C^1 diffeomorphism whose derivative has bounded variation. If $\rho(f) \in \mathbb{R} \backslash \mathbb{Q}$, then f is topologically conjugate to the rotation $R_{\rho(f)}$.

- By Poincaré' theorem, it suffices to show that there exists a point $z \in S^1$ whose positive semiorbit is dense.

  Equivalently, we must show that $\omega(z) = S^1$.

  Suppose, to the contrary, that $\omega(z) \neq S^1$.

  Then the set $S^1 \backslash \omega(z)$ is a disjoint union of maximal intervals (an open interval $I \subseteq S^1 \backslash \omega(z)$ is *maximal* if any nonempty open interval J such that $I \subseteq J \subseteq S^1 \backslash \omega(z)$ coincides with I).

  Moreover, since f is a homeomorphism, the set $\omega(z)$ is f-invariant.

  Thus, the image and the preimage of any of these intervals are also maximal intervals.

Diffeomorphisms and Rotations (Cont'd)

- Now let $I \subseteq S^1 \backslash \omega(z)$ be a maximal interval.

  We show that the sets $f^n(I)$, for $n \in \mathbb{Z}$, are pairwise disjoint.

  Suppose there exist integers $m > n$, such that $f^m(I) \cap f^n(I) \neq \emptyset$.

  Then $f^{m-n}(I) \cap I \neq \emptyset$.

  Thus, $f^{m-n}(I) = I$.

  But f is continuous.

  Therefore, $f^{m-n}(\overline{I}) = \overline{I}$.

Diffeomorphisms and Rotations (Lemma 1)

Lemma

Let $g : J \to J$ be a continuous function on some interval $J \subseteq \mathbb{R}$. If $K \subseteq J$ is a compact interval such that $g(K) \supseteq K$, then g has a fixed point in K.

- Write $K = [\alpha, \beta]$. By hypothesis, $g(K) \supseteq K$.
  So there exist $a, b \in K$, with $g(a) = \alpha \leq a$ and $g(b) = \beta \geq b$.
  Now we have $g(a) - a \leq 0$ and $g(b) - b \geq 0$.
  So the continuous function $x \mapsto g(x) - x$ has a zero in K.
- By the lemma that f^{m-n} has a fixed point in I.
  This is impossible since the rotation number is irrational.
  Thus, the intervals $f^n(I)$ are pairwise disjoint.
  Moreover, their lengths λ_n satisfy $\sum_{n \in \mathbb{Z}} \lambda_n \leq 1$.

Diffeomorphisms and Rotations (Lemma 2)

Lemma

There exist infinitely many $n \in \mathbb{N}$, such that, for each $x \in S^1$, the intervals $J = (x, f^{-n}(x))$, $f(J), \ldots, f^n(J)$ are pairwise disjoint.

- Recall that f is orientation-preserving.

  Thus, for each $k = 0, \ldots, n$, $f^k(J) = (f^k(x), f^{k-n}(x))$.

  Hence, the intervals $f^k(J)$ are pairwise disjoint if and only if

  $$f^k(x), f^{k-n}(x) \notin f^\ell(J), \text{ for } k, \ell = 0, \ldots, n, \text{ with } \ell < k.$$

  Equivalently,

  $$f^k(x) \notin J, \quad \text{for } |k| \leq n.$$

  Note that this property only depends on the ordering of the orbit of x.

Diffeomorphisms and Rotations (Lemma 2 Cont'd)

- We noted that
$$f^k(x) \notin J, \quad \text{for } |k| \leq n,$$
only depends on the ordering of the orbit of x.

  By a previous theorem, this is the same as the ordering of the orbits of the rotation R_ρ, where $\rho = \rho(f)$.

  Since ρ is irrational, all negative semiorbits are dense.

  Thus, there exist infinitely many $n \in \mathbb{N}$, such that
$$R_\rho^k(y) \notin (y, R_\rho^{-n}(y)), \quad \text{for } |k| \leq n \text{ and } y \in S^1.$$

Diffeomorphisms and Rotations (Lemma 3)

Lemma

If $J \subseteq S^1$ is an open interval such that the sets $J, f(J), \ldots, f^{n-1}(J)$ are pairwise disjoint, then, for $c = \exp\mathrm{Var}(\log f') < +\infty$,

$$c^{-1} \leq \frac{(f^n)'(y)}{(f^n)'(z)} \leq c, \quad \text{for any } y, z \in \overline{J}.$$

- Note that, since f is orientation preserving, $f' > 0$.

  So we may define a function $\varphi : S^1 \to \mathbb{R}$ by

  $$\varphi = \log f'.$$

  Now the sets $J, \ldots, f^{n-1}(J)$ are pairwise disjoint.

  So given $y, z \in \overline{J}$, the open intervals determined by the pairs of points $f^k(y)$ and $f^k(z)$, for $k = 0, \ldots, n-1$, are also disjoint.

Diffeomorphisms and Rotations (Lemma 3 Cont'd)

- Thus,

$$\begin{array}{rcl} \mathrm{Var}(\varphi) & \geq & \sum_{k=0}^{n-1} |\varphi(f^k(y)) - \varphi(f^k(z))| \\ & \geq & |\sum_{k=0}^{n-1} \varphi(f^k(y)) - \varphi(f^k(z))| \\ & = & \left|\log \prod_{k=0}^{n-1} f'(f^k(y)) - \log \prod_{k=0}^{n-1} f'(f^k(z))\right| \\ & = & \left|\log \frac{(f^n)'(y)}{(f^n)'(z)}\right|. \end{array}$$

  This implies that

$$-\mathrm{Var}(\varphi) \leq \log \frac{(f^n)'(y)}{(f^n)'(z)} \leq \mathrm{Var}(\varphi).$$

  This finishes the proof provided that $\mathrm{Var}(\varphi)$ is finite.

Diffeomorphisms and Rotations (Lemma 3 Cont'd)

- Now S^1 is compact and f' is continuous.

  Therefore, $\inf f' > 0$.

  Hence, for $x, y \in S^1$,

$$|\varphi(y) - \varphi(z)| = |\log f'(y) - \log f'(z)| \leq \frac{|f'(y) - f'(z)|}{\inf f'}.$$

  Also, f' has bounded variation.

  Hence, we obtain

$$\text{Var}(\varphi) \leq \frac{\text{Var}(f')}{\inf f'} < +\infty.$$

  This completes the proof of the lemma.

Diffeomorphisms and Rotations (Cont'd)

- Now apply Lemma 3 to the intervals $J = (x, f^{-n}(x))$ in Lemma 2, with $y = x \in I$ and $z = f^{-n}(x)$ (with n independent of x).
  We conclude that
  $$\frac{1}{c} \leq (f^n)'(x)(f^{-n})'(x) \leq c.$$
  But $a + b \geq \sqrt{ab}$, for $a, b \geq 0$.
  So we obtain, for the integers n given by Lemma 2,
  $$\begin{array}{rcl} \lambda_n + \lambda_{-n} & = & \int_I (f^n)'(x)dx + \int_I (f^{-n})'(x)dx \\ & = & \int_I [(f^n)'(x) + (f^{-n})'(x)]dx \\ & \geq & \int_I \sqrt{(f^n)'(x)(f^{-n})'(x)}dx \\ & \geq & \frac{1}{\sqrt{c}}\lambda_0. \end{array}$$
  This implies $\sum_{m \in \mathbb{Z}} \lambda_m = +\infty$, contradicting $\sum_{n \in \mathbb{Z}} \lambda_n \leq 1$.
  Thus, there exists a point $z \in S^1$ with $\omega(z) = S^1$.

Subsection 3

Maps of the Interval

Covering

- Let $f : I \to I$ be a continuous map of an interval $I \subseteq \mathbb{R}$.

Definition

Given intervals $J, K \subseteq I$, we say that J **covers** K if

$$f(J) \supseteq K.$$

In that case, we write $J \to K$.

Covering and Existence of Periodic Points

Proposition

Let $f: I \to I$ be a continuous map of a compact interval $I \subseteq R$. If there exist closed intervals $I_0, I_1, \ldots, I_{n-1} \subseteq I$, such that

$$I_0 \to I_1 \to I_2 \to \cdots \to I_{n-1} \to I_0,$$

then f has an n-periodic point $x \in I$, such that

$$f^m(x) \in I^m, \quad \text{for } m = 0, 1, \ldots, n-1.$$

Claim: There exists a closed interval $J_0 \subseteq I_0$, such that $f(J_0) = I_1$.
By hypothesis, $f(I_0) \supseteq I_1$.
So there exist $a_0, b_0 \in I_0$ whose images are the endpoints of I_1.
Let J_0 is the closed interval with endpoints a_0 and b_0.
Then $f(J_0) = I_1$.

Proof of the Proposition (Cont'd)

- Assume that we constructed closed intervals $J_0 \supseteq J_1 \supseteq \cdots \supseteq J_{m-1}$ contained in I_0, for some $m < n$, such that

$$f^{k+1}(J_k) = I_{k+1}, \quad \text{for } k = 0, \ldots, m-1.$$

Then $f^{m+1}(J_{m-1}) = f(I_m) \supseteq I_{m+1}$.

By a similar argument there exists a closed interval $J_m \supseteq J_{m-1}$, such that

$$f^{m+1}(J_m) = I_{m+1}.$$

Thus, we obtain closed intervals $J_0 \supseteq J_1 \supseteq \cdots \supseteq J_{n-1}$, such that

$$f^{k+1}(J_k) = I_{k+1}, \quad k = 0, \ldots, n-1, \quad I_n = I_0.$$

Proof of the Proposition (Cont'd)

- In particular, we have
  - $f^n(J_{n-1}) = I_0 \supseteq J_{n-1}$;
  - Each point $x \in J_{n-1}$ satisfies, for $m = 0, \ldots, n-1$,

$$f^m(x) \in f^m(J_{n-1}) \subseteq f^m(J_{m-1}) = I_m.$$

On the other hand, it follows from $f^n(J_{n-1}) = I_0 \supseteq J_{n-1}$ and Lemma 1 in Denjoy's Theorem that f^n has a fixed point in J_{n-1}.

Thus, f has an n-periodic point in J_{n-1}, which also satisfies

$$f^m(x) \in I_m, \quad m = 0, 1, \ldots, n-1.$$

Example

- Given $a > 4$, consider the map $f : [0,1] \to \mathbb{R}$ defined by
$$f(x) = ax(1-x).$$
We have
$$\begin{array}{rcl} f\left([\frac{1}{a}, \frac{1}{2}]\right) & = & [1 - \frac{1}{a}, \frac{a}{4}] \supseteq [1 - \frac{1}{a}, 1]; \\ f\left([1 - \frac{1}{a}, 1]\right) & = & [0, 1 - \frac{1}{a}] \supseteq [\frac{1}{a}, \frac{1}{2}]. \end{array}$$
Notice, also, that
$$\left[\frac{1}{a}, \frac{1}{2}\right] \cap \left[1 - \frac{1}{a}, 1\right] = \emptyset.$$
By the proposition, f has a periodic point in $[\frac{1}{a}, \frac{1}{2}]$ with period 2.

Special Case of Sharkovsky's Theorem

Theorem

Let $f : I \to I$ be a continuous map of a compact interval $I \subseteq \mathbb{R}$. If f has a periodic point with period 3, then it has periodic points with all periods.

- Let $x_1 < x_2 < x_3$ be the elements of the orbit of a periodic point with period 3.
  - Suppose $f(x_2) = x_3$. Then $f^2(x_2) = x_1$.
    Thus,
    $$[x_1, x_2] \leftrightarrow [x_2, x_3]\circlearrowleft$$
  - Suppose $f(x_2) = x_1$.
    Then
    $$[x_2, x_3] \leftrightarrow [x_1, x_2]\circlearrowleft$$

  In the first case, $I \to I$ taking $I = [x_2, x_3]$.

  In the second case, $I \to I$ taking $I = [x_1, x_2]$.

  It follows from the proposition that f has a fixed point.

Special Case of Sharkovsky's Theorem (Cont'd)

- Given an integer $n \geq 2$, with $n \neq 3$, we have

$$\underbrace{I_1 \to I_2 \to I_2 \to \cdots \to I_2 \to I_2 \to I_1}_{n+1 \text{ elements}}$$

  taking, respectively, $I_1 = [x_1, x_2]$ and $I_2 = [x_2, x_3]$ or $I_1 = [x_2, x_3]$ and $I_2 = [x_1, x_2]$.

  By the preceding proposition, f has an n-periodic point $x \in I_1$.

  If it did not have period n, then $x \in I_1 \cap I_2 = \{x_2\}$.

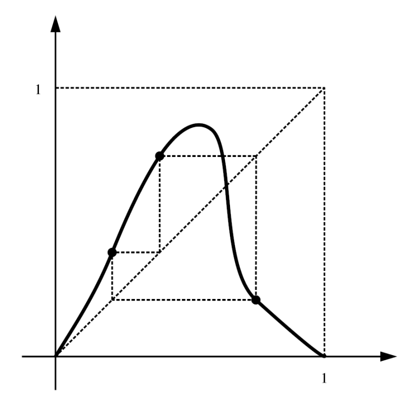
  So $x = x_2$.

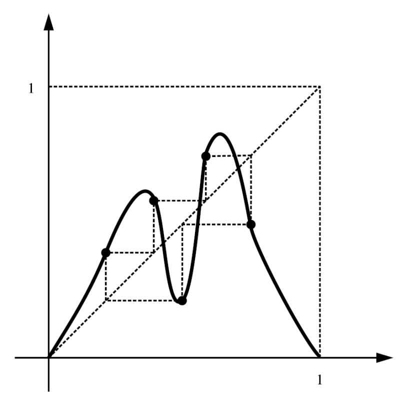
  The orbit of x_2 belongs successively to $I_1 \ I_2 \ I_2 \ I_1 \ I_2 \ I_2 \ I_1 \ldots$.

  Thus, it cannot belong successively to the intervals in the displayed chain unless $n = 3$.

  Since we took $n \neq 3$, the periodic point x has period n.

Examples on $[0, 1]$ with Periods 3 and 5





Ordering Used in Sharkovsky's Theorem

- We consider the ordering $\prec$ on $\mathbb{N}$ defined by

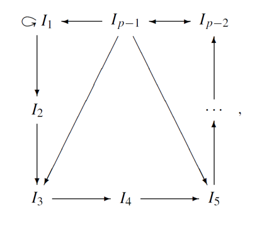
$$\begin{aligned}1 &\prec 2 \prec 2^2 \prec 2^3 \prec \cdots \prec 2^m \prec \cdots \\ &\cdots \\ &\prec \cdots \prec 2^m(2n+1) \prec \cdots \prec 2^m 7 \prec 2^m 5 \prec 2^m 3 \prec \cdots \\ &\cdots \\ &\prec \cdots \prec 2(2n+1) \prec \cdots \prec 2 \cdot 7 \prec 2 \cdot 5 \prec 2 \cdot 3 \prec \cdots \\ &\prec \cdots \prec 2n+1 \prec \cdots \prec 7 \prec 5 \prec 3.\end{aligned}$$

Lemma 1

Lemma

Let $f : I \to I$ be a continuous map of a compact interval $I \subseteq \mathbb{R}$.
Let $x \in I$ be a periodic point with odd period $p > 1$, such that there exist no periodic points with odd period less than p.

Then the intervals determined in I by the orbit of x can be numbered $I_1, \ldots, I_{p-1}$ so that the graph obtained from the covering relations between them contains the subgraph on the right i.e., $I_1 \to I_1 \to I_2 \to \cdots \to I_{p-1}$ and $I_{p-1} \to I_k$ for any odd k.



- Consider $I_1 = [u, v]$, where, for $\gamma(x)$ the orbit of x,

$$u = \max\{y \in \gamma(x) : f(y) > y\};$$
$$v = \min\{y \in \gamma(x) : y > u\}.$$

Lemma 1 (Cont'd)

- By the definition of u, we have $f(v) \leq v$.
  Since x is not a fixed point, $f(v) \neq v$.
  Therefore, we get $f(v) < v$.
  Since $f(u) > u$, by the definition of v, $f(u) \geq v$.
  Since $f(v) < v$, $f(v) < u$.
  Therefore, $I_1 \to I_1$.
  The inclusion $f(I_1) \supseteq I_1$ is proper (otherwise x would have period 2).
  Now $f^p(I_1) \supseteq f^{p-1}(I_1) \supseteq \cdots \supseteq f(I_1) \supseteq I_1$ and x is p-periodic.
  Thus, we have $f^p(I_1) \supseteq \gamma(x)$.
  So $f^p(I_1)$ contains all intervals determined by adjacent points in the orbit of x.

Lemma 1 (Cont'd)

- Let
$$I^- = \gamma(x) \cap (-\infty, u] \quad \text{and} \quad I^+ = \gamma(x) \cap [v, +\infty).$$
Define
$$r = \text{card} I^- \quad \text{and} \quad s = \text{card} I^+.$$
We have $r + s = p$. Since p is odd, $r \neq s$.

  So there exist adjacent points of $\gamma(x)$ in I^- or in I^+, determining an interval J, such that only one of them is mapped by f to the other interval.

  Otherwise, we would have $f(I^-) \subseteq I^+$ and $f(I^+) \subseteq I^-$ (since $f(u) > u$ and $f(v) < v$). This is impossible, since $r \neq s$.

  We also note that $J \to I_1$.

Lemma 1 (Cont'd)

- Now let $I_1 \to I_2 \to \cdots \to I_k \to I_1$ be the shortest cycle of the form $I_1 \to \cdots \to I_1$ that is different from $I_1 \circlearrowleft$ (it follows from the former discussion that such a cycle always exists).

  Clearly, $k \leq p-1$ since the orbit of x determines $p-1$ intervals.

  Let q be the odd element of $\{k, k+1\}$.

  Now we have:
  - $I_1 \to \cdots \to I_k \to I_1$;
  - $I_1 \to \cdots \to I_k \to I_1 \to I_1$.

  So by a previous proposition, f^q has a fixed point y.

  Note that y is not a fixed point of f.

  Otherwise, $y \in I_1 \cap \cdots \cap I_k \subseteq I_1 \cap I_2$ (recall that $k \geq 2$) would be in the orbit of x. This yields a contradiction since x is not a fixed point.

  By the minimality of the odd period p, $q \geq p$. Thus, $k = p-1$.

  This shows that $I_1 \to I_2 \to \cdots \to I_{p-1} \to I_1$ is the shortest cycle of the form $I_1 \to \cdots \to I_1$ that is different from $I_1 \circlearrowleft$.

Lemma 1 (Cont'd)

- Now we show that $I_{p-1} \to I_k$ for k odd.

  This includes $I_{p-1} \to I_{p-2}$ since p is odd.

  We first verify that the intervals I_i are ordered in I in the form $I_{p-1}, I_{p-3}, \ldots, I_2, I_1, I_3, \ldots, I_{p-2}$ (up to orientation).

  We know $I_1 \to \cdots \to I_{p-1} \to I_1$ is the shortest cycle of the form $I_1 \to \cdots \to I_1$ that is different from $I_1 \circlearrowleft$.

  Hence, if $I_k \to I_\ell$, then $\ell \leq k+1$.

  Otherwise, there would exist a shorter cycle of this form.

  This implies that I_1 only covers I_1 and I_2.

  Hence, I_2 is adjacent to I_1 (since $f(I_1)$ is connected).

  Since $I_1 = [u, v]$, we have one of the following:
  - $I_2 = [w, u]$, with $f(u) = v$ (recall that $f(u) > u$) and $f(v) = w$;
  - $I_2 = [v, w]$, with $f(u) = w$ and $f(v) = u$.

Lemma 1 (Conclusion)

- We analyze only the first case.

  The second one is entirely analogous.

  We have $f(u) = v$ and I_2 does not cover I_1.

  Hence, $f(I_2) \subseteq [v, +\infty)$.

  But I_2 covers I_3. We conclude that $I_3 = [v, t]$, with $t = f(w) = f^2(v)$ (I_2 covers no other interval).

  Continuing this procedure yields the claimed ordering.

  This implies that, for $u_i = f^i(u)$,

  $$u_{p-1} < u_{p-3} < \cdots < u_2 < u < u_1 < u_3 < \cdots < u_{p-2}.$$

  Now $f(u_{p-1}) = u$ and $f(u_{p-3}) = u_{p-2}$.

  Thus, we obtain $I_{p-1} = [u_{p-1}, u_{p-3}] \to I_k$, for k odd.

  This completes the proof of the lemma.

Lemma 2

Lemma

Let $f : I \to I$ be a continuous map of a compact interval $I \subseteq \mathbb{R}$. If f has a periodic point with even period, then it has a periodic point with period 2.

- Let x be a periodic point with even period $p > 2$.

  We consider two cases.

Lemma 2 (Case 1)

- We first assume that there are no adjacent points in the orbit of x determining an interval $J \neq I_1$ that covers I_1.

  Let y and z be, respectively, the minimum and maximum of the orbit of x,

  $$y = \min \gamma(x) \quad \text{and} \quad z = \max \gamma(x).$$

  By construction, $f(u) \geq v$.

  Thus, $f([y, u])$ intersects $[v, +\infty)$.

  By hypothesis, the interval $[y, u]$ does not cover I_1.

  Thus, $f([y, u]) \subseteq [v, +\infty)$.

  Similarly, $f([v, z]) \subseteq (-\infty, u]$.

  Since f permutes the points in the orbit of x, we obtain

  $$[y, u] \to [v, z] \to [y, u].$$

  By a previous proposition, f has a periodic point with period 2.

Lemma 2 (Case 2)

- Assume that there are adjacent points in the orbit of x determining an interval $I_k \neq I_1$ that covers I_1.

  Let $I_1 \to \cdots \to I_k \to I_1$ be the shortest cycle of the form $I_1 \to \cdots \to I_1$ that is different from $I_1 \circlearrowleft$.

  Then $k \leq p - 1$.

  Take $q \in \{k, k+1\}$ even.

  Clearly $q \leq p$.

  We have

$$I_1 \to \cdots \to I_k \to I_1,$$
$$I_1 \to \cdots \to I_k \to I_1 \to I_1.$$

  By a previous proposition, f^q has a fixed point y.

Lemma 2 (Case 2 Cont'd)

- We note that y is not a fixed point of f.

  If p was the smallest even period, then $q = p$ and thus $k = p - 1$.

  Proceeding as in the proof of the preceding lemma, one could then show that:

  - The intervals I_i must be ordered in I in the form $I_{p-2}, \ldots, I_2, I_1, I_3, \ldots, I_{p-1}$ (up to orientation);
  - $I_{p-1} \to I_k$ for k even.

  In particular, we would obtain the cycle $I_{p-1} \to I_{p-2} \to I_{p-1}$.

  By a previous proposition, f would have a periodic point with period 2 (since $I_{p-2} \cap I_{p-1} = \emptyset$).

  This contradiction shows that p cannot be the smallest even period.

  So one can consider a periodic point with a smaller even period.

  By repeating the process, we get down to period 2.

Sharkovsky's Theorem

Theorem (Sharkovsky)

Let $f : I \to I$ be a continuous map of a compact interval $I \subseteq \mathbb{R}$.
If f has a periodic point with period p and $q \prec p$, then f has a periodic point with period q.

- We consider four cases.

1. $p = 2^k$ and $q = 2^\ell \prec p$, with $\ell < k$.
   Suppose $\ell > 0$. Let x be a periodic point of f with period p.
   Then x is a periodic point of $f^{q/2}$ with period $2^{k-\ell+1}$.
   But $k - \ell + 1 \geq 2$.
   By Lemma 2, $f^{q/2}$ has a periodic point y with period 2.
   Then y is a periodic point of f with period q.
   Suppose $\ell = 0$. By Lemma 2, f has a periodic point with period 2.
   It determines an interval I_1 in I whose endpoints are permuted by f.
   Since f is continuous, it must have a fixed point in I_1.

Sharkovsky's Theorem (Case 2)

2. $p = 2^k r$ and $q = 2^k s \prec p$ with $r > 1$ odd minimal and s even.
   Note r is the smallest odd period of the periodic points of f^{2k}.
   By Lemma 1, there exists a cycle of length s.
   When $s < r$, we take
   $$I_{r-1} \to I_{r-s} \to \cdots \to I_{r-2} \to I_{r-1}$$
   When $s \geq r$, we take
   $$I_1 \to I_2 \to \cdots \to I_{r-1} \to I_1 \to I_1 \to \cdots \to I_1.$$
   By a previous proposition, f^{2^k} has a periodic point with period s.
   This is a periodic point of f with period $2^k s = q$.

Sharkovsky's Theorem (Case 3)

3. $p = 2^k r$ and $q = 2^\ell \prec p$ with $r > 1$ odd minimal and $\ell = k$.
   Take $s = 2$ in Case 2.
   We obtain a periodic point of f with period $2^k s = 2^{k+1}$.
   Now we revert to Case 1.
   f has a periodic point with period 2^ℓ for each $\ell \leq k$.

Sharkovsky's Theorem (Case 4)

4. $p = 2^k r$ and $q = 2^k s \prec p$ with $r > 1$ odd minimal and $s > r$ odd.
   Again, r is the smallest odd period of the periodic points of f^{2^k}.
   By Lemma 1, we obtain the cycle of length s given by

$$I_1 \to I_2 \to \cdots \to I_{r-1} \to I_1 \to I_1 \to \cdots \to I_1.$$

   By a previous proposition, f^{2^k} has a periodic point x with period s.
   Suppose x is a periodic point of f with period $2^k s$.
   Then the proof is complete.
   Suppose x is not a periodic point of f with period $2^k s$.
   Then x has period $2^\ell s$ for some $\ell < k$.
   Take $\overline{p} = 2^\ell s$ and $\overline{q} = 2^\ell \overline{s} = q$, where $\overline{s} = 2^{k-\ell} s$.
   Now $\overline{s}$ is even.
   Thus, Case 2 yields a periodic point of f with period $\overline{q} = q$.

Subsection 4

The Poincaré-Bendixson Theorem

The Setup

- Given a C^1 function $f : \mathbb{R}^2 \to \mathbb{R}^2$, consider, for each $x_0 \in \mathbb{R}^2$, the initial value problem
$$x' = f(x), \quad x(0) = x_0.$$
- We assume that the unique solution $x(t, x_0)$ of the system is defined for $t \in \mathbb{R}$.
- By a previous proposition the family of maps $\varphi_t : \mathbb{R}^2 \to \mathbb{R}^2$ defined, for each $t \in \mathbb{R}$, by
$$\varphi_t(x_0) = x(t, x_0)$$
is a flow.
- We call a point $x \in \mathbb{R}^2$ with $f(x) = 0$ a **critical point** of f.

Transversals and Crossings

- A line segment $L \subseteq \mathbb{R}^2$ is called a **transversal to** $f : \mathbb{R}^2 \to \mathbb{R}^2$ if, for each $x \in L$, the directions of L and $f(x)$ generate $\mathbb{R}^2$.

Lemma

Let φ_t be a flow determined by a differential equation $x' = f(x)$ for some C^1 function $f : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose $L \subseteq \mathbb{R}^2$ is a transversal to f.

- If $x \in \mathbb{R}^2$ is not periodic and meets L at points x_k at times t_k, with $t_1 < t_2 < \cdots$, then the order of the x_k on L is the same as that of the t_k.
- If x is periodic, then it can meet L in at most one point.

- Assume, first, that x is not periodic.
  Consider the simple closed curve consisting of $\gamma(x)$ between x_0 and x_1 and the segment of L joining x_0 and x_1.
  The orbit cannot cross through the curve, since then it would either be periodic or cause a discontinuity in the vector field.
  Hence, the next crossing occurs beyond x_1.

Transversals and Crossings (Cont'd)

- Next suppose that x is periodic, with least period $T > 0$.

  We express the solution as $f(t, x_0)$ so that the transversal L is constructed at $x_0 = f(0, x_0)$.

  Any other point on the orbit is achieved at a unique $t \in [0, T)$.

  Thus, if the orbit crosses $x_1 \neq x_0$ on L, it does so at $t_1 < T$.

  The orbit cannot return to x_0 across L.

  So it must cross $\gamma(x)$ at some $x_2 = f(t_2, x_0)$, $t_1 < t_2 < T$.

  However x_2 also precedes x_1.

  So we must have $x_2 = f(\tau_2, x_0)$, where $\tau_2 < t_2$.

  But then $\gamma(x)$ is periodic with period $t_2 - \tau_2$.

  This is a positive number less than T.

  This contradicts the assumption that T is the least period.

Transversals and Limit Sets

Lemma

Let φ_t be a flow determined by a differential equation $x' = f(x)$ for some C^1 function $f : \mathbb{R}^2 \to \mathbb{R}^2$. Suppose $L \subseteq \mathbb{R}^2$ is a transversal to f. Then, for each $x \in \mathbb{R}^2$, the set $\omega(x) \cap L$ contains at most one point.

- Suppose $p, q \in \omega(x) \cap L$, with $p \neq q$.

  Then $\gamma(x)$ meets L in more than one point.

  Hence, by the lemma, f is not periodic.

  Thus, $\gamma(x)$ meets L at infinite many points $\{x_k\}$ at times $t_1 < t_2 < \cdots$.

  But there are two different limit points on $\gamma(x) \cap L$.

  Thus, the $\{x_k\}$ cannot be in the order required by the lemma on L.

The Poincaré-Bendixson Theorem

Theorem (Poincaré-Bendixson)

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 function. Consider the flow φ_t determined by the equation $x' = f(x)$. Suppose that:

- The positive semiorbit $\gamma^+(x)$ of a point $x \in \mathbb{R}^2$ is bounded;
- $\omega(x)$ contains no critical points.

Then $\omega(x)$ is a periodic orbit.

- By hypothesis, the positive semiorbit $\gamma^+(x)$ is bounded.
  By a previous proposition, $\omega(x)$ is nonempty.
  Take a point $p \in \omega(x)$.
  Now $\omega(x)$ is contained in the closure of $\gamma^+(x)$.
  By a previous proposition, $\omega(p)$ is nonempty.
  Moreover, by the same proposition, $\omega(p) \subseteq \omega(x)$.
  Now take a point $q \in \omega(p)$.

The Poincaré-Bendixson Theorem (Cont'd)

- By hypothesis, q is not a critical point.

  By the preceding lemma, there exists a line segment L containing q that is a transversal to f.

  But $q \in \omega(p)$.

  Thus, by a previous proposition, there exists a sequence $t_k \nearrow +\infty$ in $\mathbb{R}^+$, such that $\varphi_{t_k}(p) \to q$ when $k \to \infty$.

  One can also assume that $\varphi_{t_k}(p) \in L$, for $k \in \mathbb{N}$.

  On the other hand, since $p \in \omega(x)$, by a previous proposition, $\varphi_{t_k}(p) \in \omega(x)$, for $k \in \mathbb{N}$.

  Now $\varphi_{t_k}(p) \in \omega(x) \cap L$.

  By the preceding lemma, for $k, \ell \in \mathbb{N}$,

  $$\varphi_{t_k}(p) = \varphi_{t_\ell}(p) = q.$$

  This implies that $\gamma(p) \subseteq \omega(x)$ is a periodic orbit.

The Poincaré-Bendixson Theorem (Cont'd)

- Now we show that $\omega(x) = \gamma(p)$.

  Assume that $\omega(x) \backslash \gamma(p) \neq \emptyset$.

  By a previous proposition, $\omega(x)$ is connected.

  So, in each open neighborhood of $\gamma(p)$, there exist points of $\omega(x)$ that are not in $\gamma(p)$.

  Moreover, any sufficiently small open neighborhood of $\gamma(p)$ contains critical points.

  Thus, there exists a transversal L' to f containing one of these points, which is in $\omega(x)$, and a point of $\gamma(p)$.

  Since $\gamma(p) \subseteq \omega(x)$, $\omega(x) \cap L'$ contains at least two points.

  This contradicts the preceding lemma.

  Thus, $\omega(x) = \gamma(p)$ and the ω-limit set of x is a periodic orbit.

Example

- Consider the differential equation

$$\begin{cases} x' &= x(3-2y-x^2-y^2)-y, \\ y' &= y(3-2y-x^2-y^2)+x. \end{cases}$$

Writing in polar coordinates, we get

$$\begin{cases} r' &= r(3-2r\sin\theta-r^2), \\ \theta' &= 1. \end{cases}$$

For any sufficiently small r, we have

$$r' = r(3-2r\sin\theta-r^2) \geq r(3-2r-r^2) > 0.$$

For any sufficiently large r, we have

$$r' = r(3-2r\sin\theta-r^2) \leq r(3+2r-r^2) < 0.$$

Example (Cont'd)

- Now the origin is the only critical point.

  Therefore, for any $r_2 > r_1 > 0$, there are no critical points in the ring

  $$D = \{x \in \mathbb{R}^2 : r_1 < \|x\| < r_2\}.$$

  Moreover, provided that r_1 is sufficiently small and r_2 is sufficiently large, it follows from the preceding inequalities that any positive semiorbit $\gamma^+(x)$ of a point $x \in D$ is contained in D.

  By the theorem, the set $\omega(x) \subseteq D$ is a periodic orbit for each $x \in D$.

  In particular, the flow determined by the differential equation has at least one periodic orbit in the set D.

Poincaré-Bendixson for Bounded Negative Semiorbits

- We have an analogous result to the Poincaré-Bendixson Theorem for bounded negative semiorbits.

Theorem

Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 function. Consider the flow φ_t determined by the equation $x' = f(x)$. Suppose that:

- The negative semiorbit $\gamma^-(x)$ of a point $x \in \mathbb{R}^2$ is bounded;
- $\alpha(x)$ contains no critical points.

Then $\alpha(x)$ is a periodic orbit.