Introduction to Dynamical Systems

George Voutsadakis[1]

[1]Mathematics and Computer Science
Lake Superior State University

LSSU Math 500

1. Hyperbolic Dynamics I
   - Smooth Manifolds
   - Hyperbolic Sets
   - Hyperbolic Sets and Invariant Families of Cones
   - Stability of Hyperbolic Sets

Subsection 1

Smooth Manifolds

Differentiable Structures

Definition

A set M is said to admit a **differentiable structure** of dimension $n \in \mathbb{N}$ if there exist injective maps

$$\varphi_i : U_i \to M \text{ in open sets } U_i \subseteq \mathbb{R}^n,\ i \in I,$$

such that:

1. $\bigcup_{i \in I} \varphi_i(U_i) = M$;
2. For any $i, j \in I$, such that $V = \varphi_i(U_i) \cap \varphi_j(U_j) \neq \emptyset$, the preimages $\varphi_i^{-1}(V)$ and $\varphi_j^{-1}(V)$ are open and the map $\varphi_j^{-1} \circ \varphi_i$ is of class C^1.

- Each map $\varphi_i : U_i \to M$ is called a **chart** or a **coordinate system**.
- Given a differentiable structure on M, we consider the topology on M formed by the sets $A \subseteq M$, such that

$$\varphi_i^{-1} A \subseteq \mathbb{R}^n \text{ is open for every } i \in I.$$

Smooth Manifolds

- A topological space is said to be **Hausdorff** if any distinct points have disjoint open neighborhoods.
- A topological space is said to **have a countable basis** if there exists a countable family of open sets such that each open set can be written as a union of elements of this family.

Definition

A set M is said to be a (**smooth**) **manifold** of dimension n if:

- It admits a differentiable structure of dimension n;
- It is a Hausdorff topological space;
- It has a countable basis.

Example

- Let $\varphi : U \to \mathbb{R}^m$ be a function of class C^1 in an open set $U \subseteq \mathbb{R}^n$. Then the graph

$$M = \{(x, \varphi(x)) : x \in U\} \subseteq \mathbb{R}^n \times \mathbb{R}^m$$

is a manifold of dimension n.

A differentiable structure is given by the single map $\psi : U \to \mathbb{R}^n \times \mathbb{R}^m$ defined by

$$\psi(x) = (x, \varphi(x)).$$

Example

- The set
$$\mathbb{T} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 = 1\}$$
is a manifold of dimension 1.

  A differentiable structure is given by the maps $\varphi_i : (-1, 1) \to \mathbb{T}$, $i = 1, 2, 3, 4$, defined by
$$\varphi_1(x) = (x, \sqrt{1 - x^2}), \quad \varphi_2(x) = (x, -\sqrt{1 - x^2}),$$
$$\varphi_3(x) = (\sqrt{1 - x^2}, x), \quad \varphi_4(x) = (-\sqrt{1 - x^2}, x).$$

- We note that $\mathbb{T}$ can be identified with S^1.

  In particular, the map $\chi : S^1 \to \mathbb{T}$ defined by
$$\chi(x) = (\cos(2\pi x), \sin(2\pi x))$$
is a homeomorphism.

Example

- The torus $\mathbb{T}^n = S^n$ is a manifold of dimension n.

  Recall the maps $\varphi_i : (-1,1) \to \mathbb{T}$, $i = 1,2,3,4$, defined by

$$\varphi_1(x) = (x, \sqrt{1-x^2}), \quad \varphi_2(x) = (x, -\sqrt{1-x^2}),$$
$$\varphi_3(x) = (\sqrt{1-x^2}, x), \quad \varphi_4(x) = (-\sqrt{1-x^2}, x).$$

  A differentiable structure is given by the maps $\psi : (-1,1)^n \to \mathbb{T}^n$, defined by

$$\psi(x_1, \ldots, x_n) = ((\chi^{-1} \circ \psi_1)(x_1), \ldots, (\chi^{-1} \circ \psi_n)(x_n)),$$

  where each ψ_i is any of the functions $\varphi_1, \varphi_2, \varphi_3$ and φ_4.

Differentiable Maps

Definition

A map $f : M \to N$ between manifolds is said to be **differentiable at a point** $x \in M$ if there exist charts

$$\varphi : U \to M \quad \text{and} \quad \psi : V \to N,$$

such that:

1. $x \in \varphi(U)$ and $f(\varphi(U)) \subseteq \psi(V)$;
2. $\psi^{-1} \circ f \circ \varphi$ is differentiable at $\varphi^{-1}(x)$.

Moreover, f is said to be **of class** C^k in an open set $W \subseteq M$ if all maps $\psi^{-1} \circ f \circ \varphi$ are of class C^k in $\varphi^{-1}(W)$.

Tangent Vectors

- Let M be a manifold of dimension n.
- Let D_x be the set of all functions $g : M \to \mathbb{R}$ that are differentiable at $x \in M$.

Definition

The **tangent vector** to a differentiable path $\alpha : (-\varepsilon, \varepsilon) \to M$, with $\alpha(0) = x$ at $t = 0$, is the function $v_\alpha : D_x \to \mathbb{R}$ defined by

$$v_\alpha(g) = \left. \frac{d(g \circ \alpha)}{dt} \right|_{t=0}.$$

We also say that v_α is a **tangent vector** at x.

Tangent Spaces and Tangent Bundles

- One can show that the set $T_x M$ of all tangent vectors at x is a vector space of dimension n.
- The space $T_x M$ is called the **tangent space of** M **at** x.
- Moreover, the set

$$TM = \{(x, v) : x \in M, v \in T_x M\}$$

  is a manifold of dimension $2n$.
- TM is called the **tangent bundle** of M.

Differentiable Structure on the Tangent Bundle

- Let $\varphi : U \to M$ be a chart.
- Let $(x_1, \ldots, x_n)$ be the coordinates in U.
- Let
$$(e_1, \ldots, e_n)$$
is the standard basis of $\mathbb{R}_n$.
- Consider the differentiable paths $\alpha_i : (-\varepsilon, \varepsilon) \to M$ for $i = 1, \ldots, n$, defined by
$$\alpha_i(t) = \varphi(te_i).$$
- The tangent vector to the path α_i at $t = 0$ is denoted by $\frac{\partial}{\partial x_i}$.

Differentiable Structure on the Tangent Bundle (Cont'd)

- One can show that
$$\left(\frac{\partial}{\partial x_1},\ldots,\frac{\partial}{\partial x_n}\right)$$
is a basis of the tangent space $T_{\varphi(0)}M$.
- Moreover, a differentiable structure on
$$TM = \{(x,v) : x \in M, v \in T_xM\}$$
is given by the maps
$$\psi : U \times \mathbb{R}^n \to TM$$
defined by
$$\psi(x_1,\ldots,x_n,y_1,\ldots,y_n) = \left(\varphi(x_1,\ldots,x_n), \sum_{i=1}^n y_i \frac{\partial}{\partial x_i}\right).$$

Subsection 2

Hyperbolic Sets

The Setup

- A **diffeomorphism** of a manifold M is an invertible C^1 map $f : M \to M$, whose inverse is also of class C^1.
- Let $f : M \to M$ be a C^1 diffeomorphism of a manifold M.
- For each $x \in M$, define a linear transformation

$$d_x f : T_x M \to T_{f(x)} M$$

between the tangent spaces $T_x M$ and $T_{f(x)} M$ by

$$d_x f v = v_{f \circ \alpha},$$

for any differentiable path $\alpha : (-\varepsilon, \varepsilon) \to M$, such that $\alpha(0) = x$ and $v_\alpha = v$.
- One can show that the definition does not depend on the path α.

Riemannian Manifolds

- We always assume that M is a Riemannian manifold.
- That is, each tangent space $T_x M$ is equipped with an inner product $\langle \cdot, \cdot \rangle_x$, such that the map

$$TM \ni (x, v) \mapsto \langle v, v \rangle_x$$

  is differentiable.
- This inner product induces the norm

$$\|v\|_x = \langle v, v \rangle_x^{1/2}, \quad v \in T_x M.$$

- For simplicity of notation, we always write $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$, without indicating the dependence on x (deduced from the context).

Hyperbolic Sets

Definition

A compact f-invariant set $\Lambda \subseteq M$ is said to be a **hyperbolic set** for f if there exist $\lambda \in (0,1)$, $c > 0$, and a decomposition

$$T_x M = E^s(x) \oplus E^u(x),$$

for each $x \in \Lambda$, such that:

1. $$d_x f E^s(x) = E^s(f(x)) \quad \text{and} \quad d_x f E^u(x) = E^u(f(x));$$

Hyperbolic Sets (Cont'd)

Definition (Cont'd)

2. If $v \in E^s(x)$ and $n \in \mathbb{N}$, then
$$\|d_x f^n v\| \le c\lambda^n \|v\|;$$
3. If $v \in E^u(x)$ and $n \in \mathbb{N}$, then
$$\|d_x f^{-n} v\| \le c\lambda^n \|v\|.$$

The linear spaces $E^s(x)$ and $E^u(x)$ are called, respectively, the **stable** and **unstable spaces** at the point x.

Example

- Let $a \in (0,1)$ and $b > 1$.

  Define the linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ by

  $$f(x,y) = (ax, by).$$

  We have $f(0) = 0$. Hence, the origin is a fixed point.

  Consider the decomposition $\mathbb{R}^2 = E^s \oplus E^u$, where:
  - E^s is the horizontal axis;
  - E^u is the vertical axes.

  Consider the linear transformation

  $$A = d_0 f = f.$$

Example (Cont'd)

- For the linear transformation

$$A = d_0 f = f$$

we have:

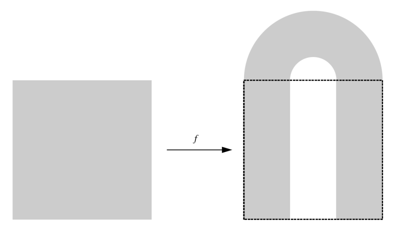
1. $AE^s = E^s$ and $AE^u = E^u$;
2. $\|Av\| \leq a\|v\|$, for $v \in E^s$;
3. $\|A^{-1}v\| \leq b^{-1}\|v\|$, for $v \in E^u$.

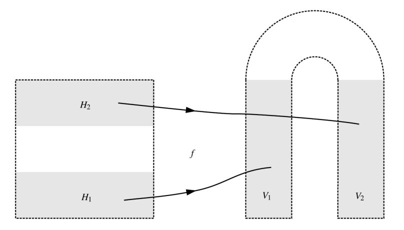
Take $\lambda = \max\{a, b^{-1}\}$ and $c = 1$.

We see that $\{0\} \subseteq \mathbb{R}^2$ is a hyperbolic set for the diffeomorphism f.

The Smale Horseshoe

- Let f be a diffeomorphism on an open neighborhood of the square $Q = [0,1]^2$ with the behavior shown on he left.



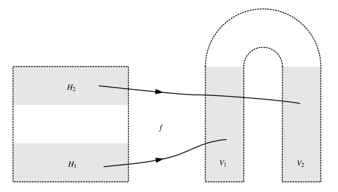


We consider the following horizontal and vertical strips, for some constant $a \in (0, \frac{1}{2})$.

$$H_1 = [0,1] \times [0,a], \quad H_2 = [0,1] \times [1-a,1];$$
$$V_1 = [0,a] \times [0,1], \quad V_2 = [1-a,1] \times [0,1].$$

The Smale Horseshoe (Cont'd)

- We assume that $f(H_1) = V_1$ and $f(H_2) = V_2$.



This yields the identity

$$Q \cap f(Q) = V_1 \cup V_2.$$

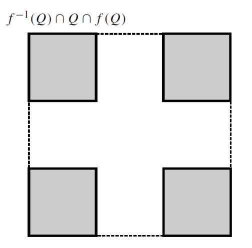
We also assume that the restrictions $f\,|_{H_1}$ and $f\,|_{H_2}$ are affine, with

$$f(x,y) = \begin{cases} (ax, by), & \text{if } (x,y) \in H_1 \\ (-ax+1, -by+b), & \text{if } (x,y) \in H_2 \end{cases}, \quad b = \frac{1}{a}.$$

We shall see that the construction of the Smale horseshoe only depends on the restriction $f\,|_{H_1 \cup H_2}$.

The Smale Horseshoe (Cont'd)

- Now we consider the diffeomorphism f^{-1}.
  We have $f^{-1}(V_1) = H_1$ and $f^{-1}(V_2) = H_2$.
  Taking into account $Q \cap f(Q) = V_1 \cup V_2$, we get
  $$f^{-1}(Q) \cap Q = f^{-1}(V_1) \cup f^{-1}(V_2) = H_1 \cup H_2.$$
  From these two relations, we get
  $$\bigcap_{k=-1}^{1} f^n(Q) = (H_1 \cup H_2) \cap (V_1 \cup V_2).$$
  So $f^{-1}(Q) \cap Q \cap f(Q)$ is the union of four squares of size a.

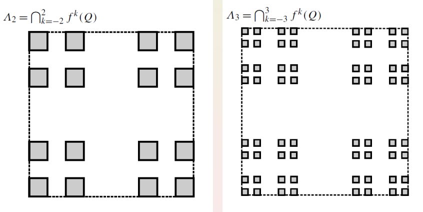


The Smale Horseshoe (Cont'd)

- We iterate this procedure, i.e., consider successively the images $f^n(Q)$ and the preimages $f^{-n}(Q)$. The intersection

$$\Lambda_n = \bigcap_{k=-n}^{n} f^k(Q)$$

is the union of 4^n squares of size a^n.



Now Λ_n is a decreasing sequence of nonempty closed sets.
Thus, the compact set $\Lambda = \bigcap_{n\in\mathbb{N}} \Lambda_n = \bigcap_{k\in\mathbb{Z}} f^k(Q)$ is nonempty.
It is called a **Smale horseshoe** (for f).

The Smale Horseshoe (Cont'd)

- Clearly, the set Λ has no interior points since the diameters of the 4^n squares in Λ_n tend to zero when $n \to \infty$.

  One can also verify that Λ has no isolated points.

  Hence, it is a Cantor set (closed with neither interior nor isolated points).

Hyperbolic Character of the Smale Horseshoe

Proposition

Λ is a hyperbolic set for the diffeomorphism f.

- We have
$$\Lambda = \bigcap_{k \in \mathbb{Z}} f^k(Q).$$
So Λ is f-invariant, i.e., $f^{-1}\Lambda = \Lambda$.
On the other hand, by the definition of f, we have:
  - $d_x f = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$, for $x \in H_1$;
  - $d_x f = \begin{pmatrix} -a & 0 \\ 0 & -b \end{pmatrix}$, for $x \in H_2$.

Hyperbolic Character of the Smale Horseshoe (Cont'd)

- For each $x \in \Lambda$, we consider the decomposition

$$\mathbb{R}^2 = E^s(x) \oplus E^u(x),$$

where $E^s(x)$ is the horizontal and $E^u(x)$ the vertical axis.

The matrices for $d_x f$ are diagonal.

So we get

$$d_x f E^s(x) = E^s(f(x)) \quad \text{and} \quad d_x f E^u(x) = E^u(f(x)).$$

Moreover, by the matrix expressions,

$$\|d_x f v\| = \begin{cases} a\|v\|, & \text{if } v \in E^s(x), \\ b\|v\|, & \text{if } v \in E^u(x). \end{cases}$$

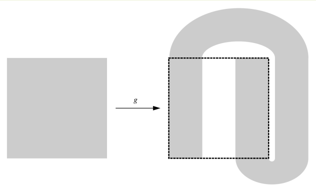
We take $\lambda = a$ and $c = 1$ in the definition of a hyperbolic set.

A Second Construction

- let g be a diffeomorphism on an open neighborhood of the square Q with the behavior shown.

Assume $g(H_1) = V_1$, $g(H_2) = V_2$. Moreover, let

$$g(x,y) = \begin{cases} (\frac{x}{3}, 3y), & \text{if } (x,y) \in H_1, \\ (\frac{x}{3} + \frac{2}{3}, 3y - 2), & \text{if } (x,y) \in H_2. \end{cases}$$



Then the compact g-invariant set $\Lambda_g = \bigcap_{n \in \mathbb{Z}} g^n(Q)$ cincides with the Λ of the Smale's horseshoe.

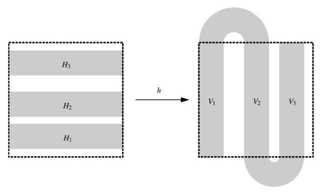
Proposition

Λ_g is a hyperbolic set for the diffeomorphism g.

A More General Construction

- Let h be a diffeomorphism on an open neighborhood of the square Q, such that $Q \cap h(Q)$ has a finite number of connected components.

  More precisely, consider pairwise disjoint closed horizontal strips $H_1, \ldots, H_m \subseteq Q$ (figure shows $m = 3$).

  We assume that the images $V_i = h(H_i)$, for $i = 1, \ldots, m$, are vertical strips in Q (necessarily disjoint since h is invertible).

  Moreover, we assume that $h\mid_{H_i}$ is an affine transformation of the form

  $$h\mid_{H_i}(x, y) = (\lambda_i x + a_i, \mu_i y + b_i),$$

  for $i = 1, \ldots, m$, with $|\lambda_i| < 1$ and $|\mu_i| > 1$.

  Let $\mu = \max\{|\lambda_i|, |\mu_i|^{-1} : i = 1, \ldots, m\}$.

A More General Construction (Cont'd)

- For each $n \in \mathbb{N}$, consider the intersection

$$\Lambda_n^h = \bigcap_{k=-n}^{n} h^k(Q).$$

It is the union of m^{2n} rectangles with sides of length at most μ^n.

Consider, moreover, the compact h-invariant set

$$\Lambda_h = \bigcap_{n \in \mathbb{Z}} h^n(Q).$$

Λ_h has no interior points.

We can also verify that Λ_h has no isolated points.

Proposition

Λ_h is a hyperbolic set for the diffeomorphism h, taking $\lambda = \mu$ and $c = 1$.

Distance Between Subspaces

- Let $E \subseteq \mathbb{R}^p$ and $v \in \mathbb{R}^p$.
- Define
$$d(v, E) = \min \{\|v - w\| : w \in E\}.$$
- Moreover, given subspaces $E, F \subseteq \mathbb{R}^p$, we define
$$d(E, F) = \max \left\{ \max_{v \in E, \|v\|=1} d(v, F), \max_{w \in F, \|w\|=1} d(w, E) \right\}.$$

Example

- Let $E, F \subseteq \mathbb{R}^2$ be subspaces of dimension 1.
  Then
  $$d(E, F) = \sin\alpha,$$
  where $\alpha \in [0, \frac{\pi}{2}]$ is the angle between E and F.
  Indeed, in this case, we have:
  - $$\max_{v \in E, \|v\|=1} d(v, F) = d(v_E, F),$$
    where $v_E \in E$ is any vector with norm 1;
  - $$\max_{v \in F, \|w\|=1} d(w, E) = d(v_F, E),$$
    where $v_F \in F$ is any vector with norm 1.

  These numbers coincide.
  Hence,
  $$d(E, F) = d(v_E, F) = d(v_F, E) = \sin\alpha.$$

Lemma 1: Sublimits of Sequences of Unit Length

Lemma

Let $\Lambda \subseteq \mathbb{R}^p$ be a hyperbolic set and $x \in \Lambda$. Consider the stable and unstable spaces $E^s(x)$ and $E^u(x)$. Let $x_m \in \Lambda$, for all $m \in \mathbb{N}$, such that $x_m \to x$ when $m \to \infty$. Any sublimit of a sequence $v_m \in E^s(x_m) \subseteq \mathbb{R}^p$, with $\|v_m\| = 1$ is in $E^s(x)$.

- Note that the closed unit sphere of $\mathbb{R}^p$ is compact.
  So the sequence v_m has sublimits.
  Since$v_m \in E^s(x_m)$, we have
  $$\|d_{x_m} f^n v_m\| \leq c\lambda^n \|v_m\|, \quad m, n \in \mathbb{N}.$$
  Letting $m \to \infty$, we obtain
  $$\|d_x f^n v\| \leq c\lambda^n \|v\|, \quad n \in \mathbb{N},$$
  where v is any sublimit of the sequence v_m.
  By definition, v has no component in $E^u(x)$. Thus, $v \in E^s(x)$.

Lemma 2: Dimension of Stable and Unstable Spaces

Lemma

Let $\Lambda \subseteq \mathbb{R}^p$ be a hyperbolic set and $x \in \Lambda$. Consider the stable and unstable spaces $E^s(x)$ and $E^u(x)$. Let $x_m \in \Lambda$, for all $m \in \mathbb{N}$, such that $x_m \to x$ when $m \to \infty$. Then, there exists an $m \in \mathbb{N}$, such that, for any $p, q > m$:

- $\dim E^s(x_p) = \dim E^s(x_q)$;
- $\dim E^u(x_p) = \dim E^s(x_q)$.

- The dimensions $\dim E^s(x_m)$ and $\dim E^u(x_m)$ can only take finitely many values.

  So there exists a subsequence y_m of x_m such that the numbers $\dim E^s(y_m)$ and $\dim E^u(y_m)$ are independent of m.

  Let $v_{1m}, \ldots, v_{km} \in E^s(y_m) \subseteq \mathbb{R}^p$ be an orthonormal basis of $E^s(y_m)$, where $k = \dim E^s(y_m)$ (which, by hypothesis, is independent of m).

Lemma 2: Dimension (Cont'd)

- The closed unit sphere of $\mathbb{R}^p$ is compact.
  So the sequence $(v_{1m}, \ldots, v_{km})$ has sublimits.
  Moreover, each sublimit $(v_1, \ldots, v_k)$ is still an orthonormal set.
  By the preceding lemma, $v_1, \ldots, v_k \in E^s(x)$.
  Thus, since $(v_1, \ldots, v_k)$ is an orthonormal set,
  $$\dim E^s(x) \geq k.$$
  Proceeding analogously for the unstable spaces, we obtain
  $$\dim E^u(x) \geq \dim M - k.$$
  But we have $T_x M = E^s(x) \oplus E^u(x)$.
  Therefore,
  $$\dim E^s(x) = k \quad \text{and} \quad \dim E^u(x) = \dim M - k.$$

Lemma 2: Dimension (Cont'd)

- In particular, the vectors $v_1, \ldots, v_k$ generate $E^s(x)$.

  We show that each vector $v \in E^s(x)$ with norm $\|v\| = 1$ is a sublimit of some sequence $v_m \in E^s(y_m)$ with $\|v_m\| = 1$.

  Write $v = \sum_{i=1}^k \alpha_i v_i$ with $\sum_{i=1}^k \alpha_i^2 = 1$.

  Then take
  $$v_m = \frac{\sum_{i=1}^k \alpha_i v_{im}}{\| \sum_{i=1}^k \alpha_i v_{im}\|}.$$

  Suppose z_m is another subsequence of x_m such that the dimensions $\dim E^s(z_m)$ and $\dim E^u(z_m)$ are independent of m.

  Say, $\dim E^s(z_m) = \ell$ and $\dim E^u(z_m) = \dim M - \ell$.

  Then we also have $\dim E^s(x) = \ell$ and $\dim E^u(x) = \dim M - \ell$.

  Thus, $\ell = k$.

  So $\dim E^s(x_m)$ and $\dim E^u(x_m)$ are constant for sufficiently large m.

Lemma 3: Estimating Distance Between Stable Subspaces

Lemma

Let $\Lambda \subseteq \mathbb{R}^p$ be a hyperbolic set and $x \in \Lambda$. Consider the stable and unstable spaces $E^s(x)$ and $E^u(x)$. Let $x_m \in \Lambda$, for all $m \in \mathbb{N}$, such that $x_m \to x$ when $m \to \infty$. Given $\delta > 0$, there exists a $p \in \mathbb{N}$, such that

$$\max_{w \in E^s(x_m), \|w\|=1} d(w, E^s(x)) < \delta, \quad \text{for } m > p.$$

- Note that given $\varepsilon > 0$ and a sequence $w_m \in E^s(x_m)$ with $\|w_m\| = 1$, we have $d(w_m, E^s(x)) < \varepsilon$ for any sufficiently large m.

  Otherwise, there would exist a subsequence w_{k_m}, such that

  $$d(w_{k_m}, E^s(x)) \geq \varepsilon, \quad \text{for } m \in \mathbb{N}.$$

  Then, any sublimit w of w_{k_m} satisfies $d(w, E^s(x)) \geq \varepsilon$.

  But this is impossible since, by Lemma 1, $w \in E^s(x)$.

Lemma 3: Estimating Distance (Cont'd)

- We consider orthonormal bases $(v_{1m},\ldots,v_{km})$ of $E^s(x_m)$ (for each sufficiently large m, such that $\dim E^s(x_m)=k$).
  By the preceding paragraph, there exist integers $p_1,\ldots,p_k\in\mathbb{N}$, such that, for $m>p_i$,
  $$d(v_{im},E^s(x))<\varepsilon.$$
  We also take vectors $w_m\in E^s(x_m)$, with norm $\|w_m\|=1$, and we write $w_m=\sum_{i=1}^k\alpha_{im}v_{im}$ with $\sum_{i=1}^k\alpha_{im}^2=1$.
  Now, for each $i=1,\ldots,k$ and $m>p:=\max\{p_1,\ldots,p_k\}$, there exists a $w_{im}\in E^s(x)$, such that $\|v_{im}-w_{im}\|<\varepsilon$. Then
  $$\begin{array}{rcl} d(w_m,E^s(x)) & \leq & \|w_m-\sum_{i=1}^k\alpha_{im}w_{im}\| \\ & \leq & \sum_{i=1}^k|\alpha_{im}|\cdot\|v_{im}-w_{im}\| \\ & < & k\epsilon. \end{array}$$
  Hence, $\max\limits_{w\in E^s(x_m),\|w\|=1}d(w,E^s(x))<k\varepsilon$, for $m>p$.

Lemma 4: Estimating Distance Between Stable Subspaces

Lemma

Let $\Lambda \subseteq \mathbb{R}^p$ be a hyperbolic set and $x \in \Lambda$. Consider the stable and unstable spaces $E^s(x)$ and $E^u(x)$. Let $x_m \in \Lambda$, for all $m \in \mathbb{N}$, such that $x_m \to x$ when $m \to \infty$. Given $\delta > 0$, there exists a $q \in \mathbb{N}$, such that

$$\max_{v \in E^s(x), \|v\|=1} d(v, E^s(x_m)) < \delta, \quad \text{for } m > q.$$

- Let $\varepsilon > 0$ and $v \in E^s(x)$.
  We show that $d(v, E^s(x_m)) < \varepsilon$, for any sufficiently large m.
  Otherwise, there would exist a sequence x_{k_m}, such that
  $$d(v, E^s(x_{k_m})) \geq \varepsilon, \quad \text{for } m \in \mathbb{N}.$$
  Consider $w_m \in E^s(x_{k_m})$ with $\|w_m\| = 1$, having v as a sublimit (each $v \in E^s(x)$ is obtained as a sublimit of such a sequence).
  Then $\|v - w_m\| \geq \varepsilon$, $m \in \mathbb{N}$. So $0 = \|v - v\| \geq \varepsilon$, a contradiction.

Lemma 4: Estimating Distance (Cont'd)

- Now consider an orthonormal basis $v_1, \ldots, v_k$ of $E^s(x)$. Take integers $q_1, \ldots, q_k \in \mathbb{N}$, such that
$$d(v_i, E^s(x_m)) < \varepsilon, \quad \text{for } m > q_i.$$
For each i, there exists a $v_{im} \in E^s(x_m)$ with $\|v_i - v_{im}\| < \varepsilon$. Given $v \in E^s(x)$, with $\|v\| = 1$, write
$$v = \sum_{i=1}^k \alpha_i v_i, \quad \sum_{i=1}^k \alpha_i^2 = 1.$$
Then, for $m > q := \max\{q_1, \ldots, q_k\}$,
$$\begin{array}{rcl} d(v, E^s(x_m)) & \leq & \|v - \sum_{i=1}^k \alpha_i v_{im}\| \\ & \leq & \sum_{i=1}^k |\alpha_i| \cdot \|v_i - v_{im}\| \\ & < & k\epsilon. \end{array}$$
Hence, $\max\limits_{v \in E^s(x), \|v\|=1} d(v, E^s(x_m)) < k\varepsilon$, for $m > q$.

Convergence and Distance

Theorem

If $\Lambda \subseteq \mathbb{R}^p$ is a hyperbolic set, then the spaces $E^s(x)$ and $E^u(x)$ vary continuously with $x \in \Lambda$. That is, if $x_m \to x$ when $m \to \infty$, with $x_m, x \in \Lambda$, for each $m \in \mathbb{N}$, then

$$d(E^s(x_m), E^s(x)) \to 0 \quad \text{when } m \to \infty;$$
$$d(E^u(x_m), E^u(x)) \to 0 \quad \text{when } m \to \infty.$$

- Let $(x_m)_{m \in \mathbb{N}}$ be a sequence as in the statement of the theorem. By Lemmas 3 and 4, given $\delta > 0$, there exist $p, q \in \mathbb{N}$, such that

  $$d(E^s(x_m), E^s(x)) < 2\delta, \text{ for } m > \max\{p, q\}.$$

  The result for unstable spaces is obtained similarly.

Subsection 3

Hyperbolic Sets and Invariant Families of Cones

The Setup

- Let $f : M \to M$ be a C^1 diffeomorphism.
- Let $\Lambda \subseteq M$ be a compact f-invariant set.
- For each $x \in \Lambda$, we consider a decomposition

$$T_x M = F^s(x) \oplus F^u(x)$$

  and an inner product $\langle \cdot, \cdot \rangle' = \langle \cdot, \cdot \rangle'_x$ in $T_x M$.
- We emphasize that this may not be the original inner product.
- We always assume that the dimensions $\dim F^s(x)$ and $\dim F^u(x)$ are independent of x.
- On the other hand, we do not require that

$$d_x f F^s(x) = F^s(f(x)), \quad d_x f F^u(x) = F^u(f(x)), \quad \text{for } x \in \Lambda.$$

The Cones

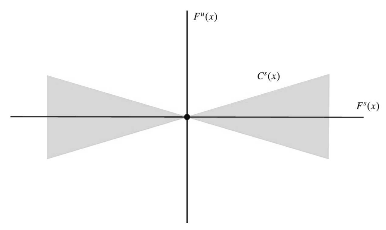
Definition

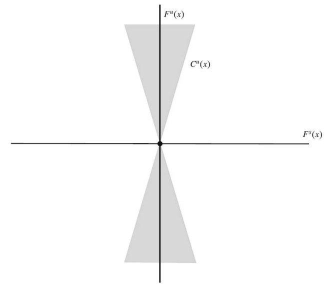
Given $\gamma \in (0,1)$ and $x \in \Lambda$, we define the **cones**

$$C^s(x) = \{(v,w) \in F^s(x) \oplus F^u(x) : \|w\|' < \gamma \|v\|'\} \cup \{0\}$$

and

$$C^u(x) = \{(v,w) \in F^s(x) \oplus F^u(x) : \|v\|' < \gamma \|w\|'\} \cup \{0\}.$$





Characterization of Hyperbolic Sets

Theorem

Let $f : M \to M$ be a C^1 diffeomorphism. Let $\Lambda \subseteq M$ be a compact f-invariant set. Then Λ is a hyperbolic set for f if and only if there exist a decomposition $T_xM = F^s(x) \oplus F^u(x)$ and an inner product $\langle \cdot, \cdot \rangle'_x$ in T_xM, for each $x \in \Lambda$, and constants $\mu, \gamma \in (0, 1)$ such that:

1. For any $x \in \Lambda$,
$$d_x f \overline{C^u(x)} \subseteq C^u(f(x)) \quad \text{and} \quad d_x f^{-1} \overline{C^s(x)} \subseteq C^s(f^{-1}(x));$$
2. For any $x \in \Lambda$,
$$\begin{aligned} \|d_x f v\|' &\geq \frac{1}{\mu}\|v\|', && \text{for } v \in C^u(x); \\ \|d_x f^{-1} v\|' &\geq \frac{1}{\mu}\|v\|', && \text{for } v \in C^s(x). \end{aligned}$$

- The theorem follows from the next two theorems.

Existence of Invariant Families of Cones

Theorem

Let $f : M \to M$ be a C^1 diffeomorphism. Let $\Lambda \subseteq M$ be a hyperbolic set for f. Then there exist an inner product $\langle \cdot, \cdot \rangle'_x$ in $T_x M$ varying continuously with $x \in \Lambda$ and constants $\mu, \gamma \in (0,1)$, such that

$$\begin{array}{rcl} C^s(x) & = & \{(v,w) \in E^s(x) \oplus E^u(x) : \|w\|' < \gamma \|v\|'\} \cup \{0\}, \\ C^u(x) & = & \{(v,w) \in E^s(x) \oplus E^u(x) : \|v\|' < \gamma \|w\|'\} \cup \{0\}, \end{array}$$

satisfy, for any $x \in \Lambda$,

1. $d_x f \overline{C^u(x)} \subseteq C^u(f(x))$ and $d_x f^{-1} \overline{C^s(x)} \subseteq C^s(f^{-1}(x))$;
2. $\|d_x f v\|' \geq \frac{1}{\mu} \|v\|'$, for $v \in C^u(x)$;
3. $\|d_x f^{-1} v\|' \geq \frac{1}{\mu} \|v\|'$, for $v \in C^s(x)$.

- We divide the proof into steps.

Existence of Invariant Families of Cones (Inner Product)

- Take $m \in \mathbb{N}$ such that $c\lambda^m < 1$.

  Given $v, w \in E^s(x)$, we define

  $$\langle v, w \rangle' = \sum_{n=0}^{m-1} \langle d_x f^n v, d_x f^n w \rangle.$$

  For each $v \in E^s(x)$, we have

  $$\begin{array}{rcl} (\|d_x f v\|')^2 & = & \sum_{n=0}^{m-1} \|d_x f^{n+1} v\|^2 \\ & = & \sum_{n=0}^{m-1} \|d_x f^n v\|^2 - \|v\|^2 + \|d_x f^m v\|^2 \\ & \leq & (\|v\|')^2 - (1 - c^2\lambda^{2m})\|v\|^2. \end{array}$$

Existence of Invariant Families of Cones (Cont'd)

- We got, for each $v \in E^s(x)$,

$$(\|d_x f v\|')^2 \leq (\|v\|')^2 - (1 - c^2\lambda^{2m})\|v\|^2.$$

On the other hand,

$$(\|v\|')^2 \leq \sum_{n=0}^{m-1} c^2\lambda^{2n}\|v\|^2 \leq c^2 m\|v\|^2.$$

Thus, $\|d_x f v\|' \leq \tau\|v\|'$, where $\tau = \sqrt{1 - \frac{1-c^2\lambda^{2m}}{c^2 m}} < 1$.
Analogously, given $v, w \in E^u(x)$, we define

$$\langle v, w\rangle' = \sum_{n=0}^{m-1} \langle d_x f^{-n} v, d_x f^{-n} w\rangle.$$

We verify similarly that $\|d_x f^{-1} v\|' \leq \tau\|v\|'$, for $v \in E^u(x)$.

Existence of Invariant Families (Inner Product Cont'd)

- Now we consider an inner product $\langle\cdot,\cdot\rangle = \langle\cdot,\cdot\rangle_x$ in T_xM.

  Let $v, w \in T_xM$, where

  $$v = v^s + v^u \quad \text{and} \quad w = w^s + w^u,$$

  with $v^s, w^s \in E^s(x)$ and $v^u, w^u \in E^u(x)$.

  Then we set

  $$\langle v, w\rangle' = \langle v^s, w^s\rangle' + \langle v^u, w^u\rangle'.$$

  Consider, next, the cones $C^s(x)$ and $C^u(x)$, with the norm $\|\cdot\|'$ induced from the inner product $\langle\cdot,\cdot\rangle'$.

Existence of Invariant Families (Inner Product Cont'd)

- Given $(v, w) \in \overline{C^u(x)}$, we have, by definition, $\|v\|' \leq \gamma \|w\|'$. We also have $d_x f E^s(x) = E^s(f(x))$ and $d_x f E^u(x) = E^u(f(x))$. Hence,

$$d_x f(v, w) = (d_x f v, d_x f w) \in E^s(f(x)) \oplus E^u(f(x)).$$

We know $\|d_x f v\|' \leq \tau \|v\|'$ and $\|d_x f^{-1} v\|' \leq \tau \|v\|'$. These give

$$\|d_x f v\|' \leq \tau \|v\|' \leq \tau\gamma \|w\|' \leq \tau^2 \gamma \|d_x f w\|'.$$

Thus, $d_x f(v, w) \in C^u(f(x))$.
Analogously, given $(v, w) \in \overline{C^s(x)}$, we have $\|w\|' \leq \gamma \|v\|'$. Thus,

$$\|d_x f^{-1} w\|' \leq \tau \|w\|' \leq \tau\gamma \|v\|' \leq \tau^2 \gamma \|d_x f^{-1} v\|'.$$

This shows that $d_x f^{-1}(v, w) \in C^s(f^{-1}(x))$ proving Part 1.

Existence of Invariant Families (Estimates Inside Cones)

- Let $(v, w) \in C^u(x)$.
  We have $\|d_x f v\|' \leq \tau \|v\|'$ and $\|d_x f^{-1} w\|' \leq \tau \|w\|'$.
  Therefore,
  $$\begin{array}{rcl}\|d_x f(v, w)\|' & \geq & \|d_x f w\|' - \|d_x f v\|' \\ & \geq & \tau^{-1}\|w\|' - \tau\|v\|' \\ & \geq & \tau^{-1}\|w\|' - \tau\gamma\|w\|'.\end{array}$$
  But $\|(v, w)\|' < (1 + \gamma)\|w\|'$.
  So we have
  $$\|d_x f(v, w)\|' \geq \frac{\tau^{-1} - \tau\gamma}{1 + \gamma}\|(v, w)\|'.$$
  Choose γ sufficiently small so that $\mu := \left(\frac{\tau^{-1} - \tau\gamma}{1+\gamma}\right)^{-1} > 1$.
  Then we obtain $\|d_x f v\|' \geq \mu^{-1}\|v\|'$.

Existence of Invariant Families (Cont'd)

- Analogously, let $(v, w) \in C^s(x)$.
  Again, $\|d_x f v\|' \leq \tau \|v\|'$ and $\|d_x f^{-1} w\|' \leq \tau \|w\|'$.
  We get that

$$\begin{array}{rcl} \|d_x f^{-1}(v, w)\|' & \geq & \|d_x f^{-1} v\|' - \|d_x f^{-1} w\|' \\ & \geq & \tau^{-1} \|v\|' - \tau \|w\|' \\ & \geq & \frac{\tau^{-1} - \tau\gamma}{1+\gamma} \|(v, w)\|' \\ & = & \mu^{-1} \|(v, w)\|'. \end{array}$$

  This completes the proof.

Criterion for Hyperbolicity

Theorem

Let $f : M \to M$ be a C^1 diffeomorphism. Let $\Lambda \subseteq M$ be a compact f-invariant set. Suppose there exist a decomposition

$$T_x M = F^s(x) \oplus F^u(x)$$

and an inner product $\langle \cdot, \cdot \rangle_x'$ in $T_x M$, for each $x \in \Lambda$, and constants $\mu, \gamma \in (0,1)$, such that the cones $C^s(x)$ and $C^u(x)$ satisfy, for any $x \in \Lambda$:

1. $d_x f \overline{C^u(x)} \subseteq C^u(f(x))$ and $d_x f^{-1} \overline{C^s(x)} \subseteq C^s(f^{-1}(x))$;
2. $\|d_x f v\|' \geq \frac{1}{\mu} \|v\|'$, for $v \in C^u(x)$;
3. $\|d_x f^{-1} v\|' \geq \frac{1}{\mu} \|v\|'$, for $v \in C^s(x)$.

Criterion for Hyperbolicity (Cont'd)

Theorem (Cont'd)

Then Λ is a hyperbolic set for f, taking $\lambda = \mu$ and $c = 1$. Moreover, the stable and unstable spaces are given by

$$E^s(x) = \bigcap_{n=0}^{\infty} d_{f^n(x)} \overline{C^s(f^n(x))}, \quad E^u(x) = \bigcap_{n=0}^{\infty} d_{f^{-n}(x)} \overline{C^u(f^{-n}(x))}.$$

- We divide the proof into steps.

Criterion (Construction of Invariant Sets)

- For each $x \in \Lambda$, we consider the sets

$$\begin{array}{rcl} G^s(x) & = & \bigcap_{n=0}^{\infty} d_{f^n(x)} f^{-n} \overline{C^s(f^n(x))}; \\ G^u(x) & = & \bigcap_{n=0}^{\infty} d_{f^{-n}(x)} f^n \overline{C^u(f^{-n}(x))}. \end{array}$$

By hypothesis,

$$d_x f \overline{C^u(x)} \subseteq C^u(f(x)) \quad \text{and} \quad d_x f^{-1} \overline{C^s(x)} \subseteq C^s(f^{-1}(x)).$$

So we have

$$\begin{array}{rcl} G^s(x) & = & \bigcap_{n=0}^{\infty} d_{f^n(x)} f^{-n} \overline{C^s(f^n(x))} \\ & \subseteq & \bigcap_{n=0}^{\infty} C^s(f^{-n}(f^n(x))) \\ & = & C^s(x). \end{array}$$

Similarly,

$$G^u(x) \subseteq C^u(x).$$

Criterion (Construction Cont'd)

- It now follows that
$$d_x f^{-1} G^s(x) \subseteq C^s(f^{-1}(x)) \quad \text{and} \quad d_x f G^u(x) \subseteq C^u(f(x)).$$
Writing $y = f^{-1}(x)$, we obtain
$$\begin{array}{rcl} d_x f^{-1} G^s(x) & = & \overline{C^s(y)} \cap d_x f^{-1} G^s(x) \\ & = & \overline{C^s(y)} \cap \bigcap_{n=0}^\infty d_{f^n(x)} f^{-(n+1)} \overline{C^s(f^n(x))} \\ & = & \overline{C^s(y)} \cap \bigcap_{n=0}^\infty d_{f^{n+1}(y)} f^{-(n+1)} \overline{C^s(f^{n+1}(y))} \\ & = & G^s(y). \end{array}$$
Analogously, $d_x f G^u(x) = G^u(f(x))$.

Criterion (Construction of Stable and Unstable Spaces)

- By hypothesis, the dimensions $k = \dim F^s(x)$ and $\ell = \dim F^u(x)$ are independent of x.

  So, for each $m \in \mathbb{N}$, the sets

$$\begin{aligned} \bigcap_{n=0}^{m} d_{f^n(x)} f^{-n} \overline{C^s(f^n(x))} &= d_{f^m(x)} f^{-m} \overline{C^s(f^m(x))}, \\ \bigcap_{n=0}^{m} d_{f^{-n}(x)} f^{n} \overline{C^u(f^{-n}(x))} &= d_{f^{-m}(x)} f^{m} \overline{C^u(f^{-m}(x))} \end{aligned}$$

  contain subspaces $E_m^s(x)$ and $E_m^u(x)$, respectively, of dimensions $\dim E_m^s(x) = k$ and $\dim E_m^u(x) = \ell$.

  For each $m \in \mathbb{N}$, let $v_{1m}, \ldots, v_{km}$ be an orthonormal basis of $E_m^s(x)$.

  Then there exists a convergent subsequence, say with limits $v_1, \ldots, v_k$ that also form an orthonormal set.

  This shows that $G^s(x)$ contains a subspace $E^s(x)$ of dimension k (generated by $v_1, \ldots, v_k$).

  Similarly, $G^u(x)$ contains a subspace $E^u(x)$ of dimension ℓ.

Criterion (Construction of Stable/Unstable Spaces Cont'd)

- Recall we have
$$G^s(x) \subseteq C^s(x) \quad \text{and} \quad G^u(x) \subseteq C^u(x).$$
Thus, we get
$$\begin{array}{rcl} E^s(x) \cap E^u(x) & \subseteq & G^s(x) \cap G^u(x) \\ & \subseteq & C^s(x) \cap C^u(x) \\ & = & \{0\}. \quad (\text{since } \gamma < 1.) \end{array}$$
Moreover, by hypothesis, $T_xM = F^s(x) \oplus F^u(x)$. Hence,
$$\begin{array}{rcl} \dim M & = & \dim F^s(x) + \dim F^u(x) \\ & = & k + \ell \\ & = & \dim E^s(x) + \dim E^u(x). \end{array}$$
Thus, the spaces $E^s(x)$ and $E^u(x)$ generate T_xM.
Hence, we obtain the direct sum $T_xM = E^s(x) \oplus E^u(x)$.

Criterion (Estimates on Spaces $E^s(x)$ and $E^u(x)$)

- Recall we have, for all $x \in \Lambda$,

$$\begin{array}{ll} d_x f^{-1}\overline{C^s(x)} = C^s(f^{-1}(x)), & d_x f\overline{C^u(x)} = C^u(f(x)); \\ G^s(x) \subseteq C^s(x), & G^u(x) \subseteq C^u(x); \\ d_x f^{-1} G^s(x) = G^s(f^{-1}(x)), & d_x f G^u(x) = G^u(f(x)). \end{array}$$

Let $v \in E^s(x)$ and $n \in \mathbb{N}$.

We get, for $k = 0, \ldots, n$,

$$d_x f^k v \in d_x f^k E^s(x) \subseteq d_x f^k G^s(x) = G^s(f^k(x)) \subseteq C^s(f^k(x)).$$

But we know $\|d_x f^{-1} v\|' \geq \mu^{-1}\|v\|'$, $v \in C^s(x)$.

Hence, $\|d_x f^n v\|' \leq \mu^n \|v\|'$.

Let, similarly, $v \in E^u(x)$ and $n \in \mathbb{N}$.

We know $\|d_x f v\|' \geq \mu^{-1}\|v\|'$, $v \in C^u(x)$.

It follows that $\|d_x f^{-n} v\|' \leq \mu^n \|v\|'$.

Criterion (Estimates on Spaces $E^s(x)$ and $E^u(x)$ Cont'd)

- Now we show that $E^s(x) = G^s(x)$ and $E^u(x) = G^u(x)$ for any $x \in \Lambda$.
  Suppose there existed a $v \in G^s(x) \backslash E^s(x) \subseteq C^s(x)$.
  Then $v = v^s + v^u$, where $v^s \in E^s(x)$ and $v^u \in E^u(x) \backslash \{0\}$.
  For each $n \in \mathbb{N}$, we would have

$$\begin{array}{rcl} \mu^{-n}\|v^u\|' & \leq & \|d_x f^n v^n\|' \\ & \leq & \|d_x f^n v\|' + \|d_x f^n v^s\|' \\ & \leq & \mu^n(\|v\|' + \|v^s\|'). \end{array}$$

  This implies that $\|v^u\|' \leq \mu^{2n}(\|v\|' + \|v^s\|') \to 0$ when $n \to \infty$.
  Thus $v^u = 0$. This contradiction shows that $E^s(x) = G^s(x)$.
  One can show in an analogous manner that $E^u(x) = G^u(x)$.
  But $d_x f^{-1} G^s(x) = G^s(f^{-1}(x))$ and $d_x f G^u(x) = G^u(f(x))$.
  So $d_x f^{-1} E^s(x) = E^s(f^{-1}(x))$ and $d_x f E^u(x) = E^u(f(x))$.
  Therefore, Λ is a hyperbolic set, taking $\lambda = \mu$ and $c = 1$.

Subsection 4

Stability of Hyperbolic Sets

Stability of Hyperbolic Sets

- Given differentiable maps $f, g : M \to M$, we define

$$d(f,g) = \sup_{x \in M} d(f(x), g(x)) + \sup_{x \in M} \|d_x f - d_x g\|.$$

- Recall Tietze's Extension Theorem from Analysis:

  Suppose $f : A \to \mathbb{R}$ is a continuous function in a closed subset $A \subseteq X$ of a normal space (a space such that any two disjoint closed sets have disjoint open neighborhoods). Then there exists a continuous function $g : X \to \mathbb{R}$, such that $g|_A = f$.

Theorem

Let Λ be a hyperbolic set for a C^1 diffeomorphism $f : M \to M$. Then there exist $\varepsilon > 0$ and an open set $U \supseteq \Lambda$, such that, if $g : M \to M$ is a C^1 diffeomorphism with $d(f,g) < \varepsilon$ and $\Lambda' \subseteq U$ is a compact g-invariant set, then Λ' is a hyperbolic set for g.

Proof of Stability

- By a previous theorem, the stable and unstable spaces $E^s(x)$ and $E^u(x)$ vary continuously with $x \in \Lambda$.

  We apply Tietze's Extension Theorem.

  We obtain continuous extensions $F^s(x)$ and $F^u(x)$, respectively, of $E^s(x)$ and $E^u(x)$, for x in some open neighborhood U of Λ, such that

  $$T_x M = F^s(x) \oplus F^u(x) \quad \text{for } x \in U.$$

  Let $\gamma > 0$ be given.

  Let $C^s(x)$ and $C^u(x)$ be the cones associated to this decomposition.

  By the Existence Theorem for Invariant Families of Cones, there exist constants $\mu, \gamma \in (0, 1)$ and an inner product $\langle \cdot, \cdot \rangle' = \langle \cdot, \cdot \rangle'_x$ in $T_x M$, varying continuously with x, such that, for each $x \in \Lambda$:

  1. $d_x f \overline{C^u(x)} \subsetneq C^u(f(x))$ and $d_x f^{-1} \overline{C^s(x)} \subsetneq C^s(f^{-1}(x))$;
  2. $\|d_x f v\|' > \mu^{-1} \|v\|'$, for $v \in \overline{C^u(x)} \setminus \{0\}$;
  3. $\|d_x f^{-1} v\|' > \mu^{-1} \|v\|'$, for $v \in \overline{C^s(x)} \setminus \{0\}$.

Proof of Stability (Cont'd)

- Let S_x be the closed unit sphere in T_xM (with respect to $\|\cdot\| = \|\cdot\|'_x$).

  These properties are equivalent, for each $x \in \Lambda$, to:
  1. $d_xf(S_x \cap \overline{C^u(x)}) \subsetneq C^u(f(x))$ and $d_xf^{-1}(S_x \cap \overline{C^s(x)}) \subsetneq C^s(f^{-1}(x))$;
  2. $\|d_xfv\|' > \mu^{-1}$, for $v \in S_x \cap \overline{C^u(x)}$;
  3. $\|d_xf^{-1}v\|' > \mu^{-1}$, for $v \in S_x \cap \overline{C^s(x)}$.

  The product $\langle \cdot, \cdot \rangle'_x$ and, thus, also $\|\cdot\|'_x$, vary continuously with x.

  So the set $\{(x, v) \in \Lambda \times T_xM : \|v\|'_x = 1\}$ is compact.

  For any sufficiently small open neighborhood $U \subseteq \Lambda$, the properties above hold for any $x \in U$ (and some continuous extension of the inner product).

  Moreover, for any sufficiently small ε the same properties also hold for any $x \in U$ with f replaced by g.

  By the preceding theorem, any compact g-invariant set $\Lambda' \subseteq U$ is a hyperbolic set for g.

The Case of Anosov Diffeomorphisms

Definition

A diffeomorphism $f : M \to M$ of a compact manifold M is called an **Anosov diffeomorphism** if M is a hyperbolic set for f.

Example: Any automorphism of the torus induced by a matrix without eigenvalues with modulus 1 (called a hyperbolic automorphism of the torus) is an Anosov diffeomorphism.

- The following result is an immediate consequence of the preceding theorem.

Theorem

The set of Anosov diffeomorphisms of class C^1 of a compact manifold M is open with respect to the topology induced by the distance d.