Introduction to Dynamical Systems

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Subsection 1

Basic Notions

The Shift Map

- Let $k > 1$ be an integer.
- Consider the set
$$\Sigma_k^+ = \{1, \ldots, k\}^{\mathbb{N}}$$
of sequences
$$\omega = (i_1(\omega) i_2(\omega) \cdots),$$
where $i_n(\omega) \in \{1, \ldots, k\}$, for all $n \in \mathbb{N}$.

Definition

The **shift map** $\sigma : \Sigma_k^+ \to \Sigma_k^+$ is defined by
$$\sigma(\omega) = (i_2(\omega) i_3(\omega) \cdots).$$

- Clearly, the map σ is not invertible.

Number of m-Periodic Points

- Given $m \in \mathbb{N}$, we compute the number of m-periodic points of σ.
- These are the sequences $\omega \in \Sigma_k^+$, such that $\sigma^m(\omega) = \omega$.
- By definition of σ, ω is m-periodic if and only if

$$i_{n+m}(\omega) = i_n(\omega), \quad \text{for } n \in \mathbb{N}.$$

- Equivalently, the first m elements of σ are repeated indefinitely.
- Thus, in order to specify an m-periodic point it is sufficient to specify its first m elements.
- Conversely, consider integers $j_1, \ldots, j_m \in \{1, \ldots, k\}$.
- Let $\omega \in \Sigma_k^+$ be such that:
  - $i_n(\omega) = j_n$, for $n = 1, \ldots, m$;
  - $i_{n+m}(\omega) = i_n(\omega)$, for $n \in \mathbb{N}$.
- ω is an m-periodic point.
- So the number of m-periodic points is $\text{card}(\{1, \ldots, k\}^m) = k^m$.

Distance and Topology on Σ_k^+

- Fix $\beta > 1$.
- Consider $\omega, \omega' \in \Sigma_k^+$.
- Denote by
$$n = n(\omega, \omega') \in \mathbb{N}$$
the smallest positive integer such that $i_n(\omega) \neq i_n(\omega')$.
- Define, for all $\omega, \omega' \in \Sigma_k^+$,
$$d(\omega, \omega') = \begin{cases} \beta^{-n}, & \text{if } \omega \neq \omega', \\ 0, & \text{if } \omega = \omega'. \end{cases}$$

Distance, Topology and Shift

Proposition

For each $\beta > 1$, the following properties hold:

1. d is a distance on Σ_k^+;
2. (Σ_k^+, d) is a compact metric space;
3. The shift map $\sigma : \Sigma_k^+ \to \Sigma_k^+$ is continuous.

- By the definition of d,

$$d(\omega', \omega) = d(\omega, \omega').$$

Moreover, $d(\omega, \omega') = 0$ if and only if $\omega = \omega'$.

Proof (The Triangle Inequality)

- Let $\omega, \omega', \omega'' \in \Sigma_k^+$.
  We have
  $$d(\omega,\omega'') = \beta^{-n_1}, \quad d(\omega,\omega') = \beta^{-n_2}, \quad d(\omega',\omega'') = \beta^{-n_3},$$
  where n_1, n_2 and n_3 are, respectively, the smallest positive integers such that
  $$i_{n_1}(\omega) \neq i_{n_1}(\omega''), \quad i_{n_2}(\omega) \neq i_{n_2}(\omega'), \quad i_{n_3}(\omega') \neq i_{n_3}(\omega'').$$
  If $n_2 > n_1$ and $n_3 > n_1$, then $i_{n_1}(\omega) = i_{n_1}(\omega') = i_{n_1}(\omega'')$.
  This contradicts the preceding inequations.
  Hence, $n_2 \leq n_1$ or $n_3 \leq n_1$.
  Thus, $\beta^{-n_1} \leq \beta^{-n_2}$ or $\beta^{-n_1} \leq \beta^{-n_3}$.
  This establishes the triangle inequality.

Proof (Compactness)

- We now show that Σ_k^+ is compact.

  Consider, for $j_1, \ldots, j_m \in \{1, \ldots, k\}$, the sets

  $$C_{j_1 \cdots j_m} = \{\omega \in \Sigma_k^+ : i_n(\omega) = j_n, \text{ for } n = 1, \ldots, m\}.$$

  Those are exactly the d-open balls.

  Equip $\{1, \ldots, k\}$ with the discrete topology (in which all subsets of $\{1, \ldots, k\}$ are open).

  The product topology on $\Sigma_k^+ = \{1, \ldots, k\}^{\mathbb{N}}$ coincides with the topology generated by the open balls $C_{j_1 \cdots j_m}$.

  In other words, it coincides with the topology induced by d.

  So (Σ_k^+, d) is the product of compact topological spaces, with the product topology.

  By Tychonoff's Theorem, (Σ_k^+, d) is a compact topological space.

Proof (Continuity)

- Finally, we show that $\sigma : \Sigma_k^+ \to \Sigma_k^+$ is continuous.
  Suppose $d(\omega, \omega') = \beta^{-n}$.
  Then
  $$d(\sigma(\omega), \sigma(\omega')) \le \beta^{-(n-1)} = \beta d(\omega, \omega').$$
  So the shift map is continuous.
- Note that, from the proof of the proposition, we have
  $$d(\omega, \omega'') \le \max\{d(\omega, \omega'), d(\omega', \omega'')\}.$$

Topological Entropy Revisited

- Let $f : X \to X$ be a continuous map of a compact metric space (X, d).
- For each $n \in N$, we introduced a new distance on X by

$$d_n(x, y) = \max\{d(f^k(x), f^k(y)) : 0 \leq k \leq n-1\}.$$

- Denote by $N(n, \varepsilon)$ the largest number of points $p_1, \ldots, p_m \in X$ such that

$$d_n(p_i, p_j) \geq \varepsilon, \quad \text{for } i \neq j.$$

- The topological entropy of f was defined by

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log N(n, \varepsilon).$$

- By the preceding proposition, $\sigma : \Sigma_k^+ \to \Sigma_k^+$ is a continuous map of a compact metric space.
- Hence, its topological entropy is well defined.

Topological Entropy of the Shift Map

Proposition

We have $h(\sigma |_{\Sigma_k^+}) = \log k$.

- Let $m, p \in \mathbb{N}$ and $\omega, \omega' \in \Sigma_k^+$.
  We have
  $$d_m(\omega, \omega') = \max\{d(\sigma^j(\omega), \sigma^j(\omega')) : j = 0, \ldots, m-1\}.$$
  Clearly,
  $$d(\sigma^j(\omega), \sigma^j(\omega')) \geq \beta^{-p} \quad \text{iff} \quad n = n(\omega, \omega') \in \{1+j, \ldots, p+j\}.$$
  Thus, $d_m(\omega, \omega') \geq \beta^{-p}$ if and only if $n \leq p + m - 1$.
  The largest number of distinct sequences in Σ_k^+ that differ in some of their first $p + m - 1$ elements is k^{p+m-1}.
  Therefore, $N(m, \beta^{-p}) \leq k^{p+m-1}$.

Topological Entropy of the Shift Map (Cont'd)

- The number of $(p+m-1)$-periodic points of σ is k^{p+m-1}.

  Let ω and ω' be two of these points.

  Then $n(\omega, \omega') \in \{1, \ldots, p+m-1\}$.

  Therefore,

$$d_m(\omega, \omega') = \max\{d(\sigma^j(\omega), \sigma^j(\omega')) : j = 0, \ldots, m-1\} \geq \beta^{-p}.$$

  Hence, $N(m, \beta^{-p}) \geq k^{p+m-1}$.

  We conclude $N(m, \beta^{-p}) = k^{p+m-1}$.

  Finally,

$$\begin{aligned} h(\sigma \mid_{\Sigma_k^+}) &= \lim_{p\to\infty} \lim_{m\to\infty} \frac{1}{m} \log N(m, \beta^{-p}) \\ &= \lim_{p\to\infty} \lim_{m\to\infty} \frac{p+m-1}{m} \log k \\ &= \log k. \end{aligned}$$

Two-Sided Sequences

- We consider in an analogous manner the case of two-sided sequences.
- Given an integer $k > 1$, consider the set $\Sigma_k = \{1, \ldots, k\}^{\mathbb{Z}}$ of sequences

$$\omega = (\cdots i_{-1}(\omega) i_0(\omega) i_1(\omega) \cdots).$$

Definition

The shift map $\sigma : \Sigma_k \to \Sigma_k$ is defined by $\sigma(\omega) = \omega'$, where

$$i_n(\omega') = i_{n+1}(\omega), \quad \text{for } n \in \mathbb{Z}.$$

- Note that the shift map on Σ_k is invertible.

Number of Periodic Points

- Given $m \in \mathbb{N}$, a point $\omega \in \Sigma_k$ is *m*-periodic if and only if
$$i_{n+m}(\omega) = i_n(\omega), \quad \text{for } n \in \mathbb{Z}.$$
- Hence, in order to specify an *m*-periodic point $\omega \in \Sigma_k$ it is sufficient to specify the elements $i_1(\omega), \ldots, i_m(\omega)$.
- On the other hand, let $j_1, \ldots, j_m \in \{1, \ldots, k\}$.
- Consider the sequence $\omega \in \Sigma_k$, with:
    - $i_n(\omega) = j_n$, for $n = 1, \ldots, m$;
    - $i_{n+m}(\omega) = i_n(\omega)$, for $n \in \mathbb{Z}$.
- It is an *m*-periodic point.
- So the number of *m*-periodic points of $\sigma|_{\Sigma_k}$ is k^m.

Distance and Topology on Σ_k

- We introduce a distance and, thus, also a topology on Σ_k.
- Let $\beta > 1$ and $\omega, \omega' \in \Sigma_k$.
- Denote by $n = n(\omega, \omega') \in \mathbb{N}$ the smallest integer such that

$$i_n(\omega) \neq i_n(\omega') \quad \text{or} \quad i_{-n}(\omega) \neq i_{-n}(\omega').$$

- Define

$$d(\omega, \omega') = \begin{cases} \beta^{-n}, & \text{if } \omega \neq \omega', \\ 0, & \text{if } \omega = \omega', \end{cases}$$

- One can verify that d is a distance on Σ_k.

Subsection 2

Examples of Codings

Codings

- A **coding** is a symbolic dynamics, i.e., a shift map on some space Σ_k^+ or Σ_k.
- We illustrate how one can naturally associate a coding to several dynamical systems introduced in the former chapters.

Example

- Consider the expanding map $E_2 : S^1 \to S^1$.

  Write
  $$x = 0.x_1 x_2 \ldots \in S^1$$
  in base 2 (with $x_n \in \{0,1\}$ for each n).

  Then we have
  $$E_2(0.x_1 x_2 \ldots) = 0.x_2 x_3 \ldots .$$

  This is the behavior observed in $\sigma \mid_{\Sigma_k^+}$.

  So one may expect some relation between E_2 and $\sigma \mid_{\Sigma_2^+}$.

  We define a function $H : \Sigma_2^+ \to S^1$ by
  $$H(i_1 i_2 \ldots) = \sum_{n=1}^{\infty} (i_n - 1) 2^{-n} = 0.(i_1 - 1)(i_2 - 1) \cdots .$$

Example (Cont'd)

- Then

$$\begin{array}{rcl}(H \circ \sigma)(i_1 i_2 \cdots) & = & H(i_2 i_3 \cdots) \\ & = & \sum_{n=1}^{\infty}(i_{n+1} - 1)2^{-n} \\ & = & 0.(i_2 - 1)(i_3 - 1) \cdots \\ & = & E_2(0.(i_1 - 1)(i_2 - 1) \cdots) \\ & = & (E_2 \circ H)(i_1 i_2 \cdots).\end{array}$$

We discovered that

$$H \circ \sigma = E_2 \circ H \text{ in } \Sigma_2^+.$$

Example (Cont'd)

- The map H is not one-to-one, since, for any $i_1, \ldots, i_n \in \{1, 2\}$,

$$H(i_1 \cdots i_n 211 \cdots) = H(i_1 \cdots i_n 122 \cdots).$$

On the other hand, let $B \subseteq \Sigma_2^+$ be the subset of all sequences with infinitely many consecutive 2's.

Then the map

$$H\mid_{\Sigma_2^+ \setminus B} : \Sigma_2^+ \setminus B \to S^1$$

is bijective.

Periodic Points

- We use the preceding example to find the number of m-periodic points of the expanding map E_2.

  By a previous example, the number of m-periodic points of the shift map $\sigma_{\Sigma_2^+}$ is 2^m.

  Only one of them belongs to B, namely the constant sequence

  $$(22\ldots).$$

  Thus, the number of m-periodic points of $\sigma\mid_{\Sigma_2^+\setminus B}$ is $2^m - 1$.

  Note that the set $\Sigma_2^+\setminus B$ is forward σ-invariant.

  Hence, the orbits of these points are in fact in $\Sigma_2^+\setminus B$.

Periodic Points (Cont'd)

- We know $H \circ \sigma = E_2 \circ H$.
  It follows that, in Σ_2^+, for each $m \in \mathbb{N}$,
  $$H \circ \sigma^m = E_2^m \circ H.$$
  Now take $\omega \in \Sigma_2^+ \backslash B$ and $m \in \mathbb{N}$.
  The set $\Sigma_2^+ \backslash B$ is forward σ-invariant.
  So we have $\sigma^m(\omega) \in \Sigma_2^+ \backslash B$.
  Moreover, the function $H\,|_{\Sigma_2^+ \backslash B}$ is bijective.
  It follows that
  $$\sigma^m(\omega) = \omega \quad \text{iff} \quad H(\omega) = H(\sigma^m(\omega)) = E_2^m(H(\omega)).$$
  Thus, $\omega \in \Sigma_2^+ \backslash B$ is an m-periodic point of σ if and only if $H(\omega)$ is an m-periodic point of E_2.
  So the number of m-periodic points of E_2 is $2^m - 1$.

Periodic Points (Conclusion)

- The $2^m - 1$ m-periodic points of the expanding map E_2 are

$$x_{i_1 \ldots i_m} = H(i_1 \ldots i_m i_1 \ldots i_m \ldots) \in S^1,$$

for $(i_1, \ldots, i_m) \in \{1, \ldots, k\}^m \backslash \{(2, \ldots, 2)\}$.
It follows from the definition of H that

$$\begin{aligned} x_{i_1 \ldots i_m} &= \textstyle\sum_{n=1}^m (i_n - 1) 2^{-n} (1 + 2^{-m} + 2^{-2m} + \cdots) \\ &= \tfrac{1}{1-2^{-m}} \textstyle\sum_{n=1}^m (i_n - 1) 2^{-n} \\ &= \tfrac{1}{2^m - 1} \textstyle\sum_{n=1}^m (i_n - 1) 2^{m-n}. \end{aligned}$$

The sum

$$\sum_{n=1}^m (i_n - 1) 2^{m-n}$$

takes the values $0, 1, \ldots, 2^m - 1$ since $(i_1, \ldots, i_m) \neq (2, \ldots, 2)$.
Hence, we recover the periodic points already obtained previously.

Example

- Let A be the compact forward E_4-invariant set

$$A = \bigcap_{n \geq 0} E_4^{-n} \left(\left[0, \frac{1}{4}\right] \cup \left[\frac{2}{4}, \frac{3}{4}\right] \right).$$

Consider the restriction of the map E_4,

$$E_4 \mid_A : A \to A.$$

Write

$$x = 0.x_1 x_2 \ldots \in S^1$$

in base 4, with $x_n \in \{0, 1, 2, 3\}$, for each $n \in \mathbb{N}$.

We have

$$E_4(0.x_1 x_2 \ldots) = 0.x_2 x_3 \ldots.$$

Example (Cont'd)

- Define a function $H : \Sigma_2^+ \to S^1$ by

$$H(i_1 i_2 \ldots) = \sum_{n=1}^{\infty} 2(i_1 - 1)4^{-n} = 0.j_1 j_2 \ldots,$$

also in base 4, where $j_n = 2(i_n - 1) \in \{0, 2\}$, for $n \in \mathbb{N}$.
We have

$$(H \circ \sigma)(i_1 i_2 \ldots) = H(i_2 i_3 \ldots) = \sum_{n=1}^{\infty} 2(i_{n+1} - 1)4^{-n}.$$

Also

$$(E_4 \circ H)(i_1 i_2 \ldots) = E_4(0.j_1 j_2 \ldots) = 0.j_2 j_3 \ldots.$$

It follows that, in Σ_2^+,

$$H \circ \sigma = E_4 \circ H.$$

Example (Cont'd)

- We note that the map H is one-to-one.
  It is also a homeomorphism onto its image $H(\Sigma_2^+) = A$.
  Indeed, let $\omega, \omega' \in \Sigma_2^+$ with $\omega \neq \omega'$.
  Let $n = n(\omega, \omega') \in \mathbb{N}$ be the smallest integer such that $i_n(\omega) \neq i_n(\omega')$.
  Let, also, d_{S^1} be the distance on S^1.
  Then we have

$$\begin{array}{rcl} d_{S^1}(H(\omega), H(\omega')) & \leq & \sum_{m=n}^{\infty} 2 \cdot 4^{-m} \\ & = & 2\frac{1}{4^n}\frac{1}{1-\frac{1}{4}} \\ & = & \frac{8 \cdot 4^{-n}}{3} \\ & = & \frac{8}{3}(\beta^{-n})^{\log 4/\log \beta} \\ & = & \frac{8}{3}d(\omega, \omega')^{\log 4/\log \beta}. \end{array}$$

Example (Cont'd)

- On the other hand, let

$$x = 0.j_1 j_2 \dots \quad \text{and} \quad x' = 0.j_1' j_2' \dots \in A.$$

Equivalently, $(j_1 j_2 \dots), (j_1' j_2' \dots) \in \{0, 2\}^{\mathbb{N}}$.
Then we have

$$d(H^{-1}(x), H^{-1}(x')) = d\left(\sum_{n=1}^{\infty} 2(i_n - 1)4^{-n}, \sum_{n=1}^{\infty} 2(i_n' - 1)4^{-n}\right),$$

with $j_n = 2(i_n - 1)$ and $j_n' = 2(i_n' - 1)$ for $n \in \mathbb{N}$.
Now consider $x \neq x'$, such that

$$d_{S^1}(x, x') = |x - x'|.$$

Suppose $n \in \mathbb{N}$ is the smallest integer such that $j_n \neq j_n'$ or, equivalently, $i_n \neq i_n'$.

Example (Conclusion)

- Then
$$d_{S^1}(x,x') \geq 2\cdot 4^{-n} - \sum_{m=n+1}^{\infty} 2\cdot 4^{-m} = \frac{1}{3}4^{-n+1}.$$
Also,
$$\begin{array}{rcl} d(H^{-1}(x),H^{-1}(x')) & = & \beta^{-n} \\ & = & 4^{-n\log\beta/\log 4} \\ & = & (\frac{3}{4}\cdot\frac{1}{3}4^{-n+1})^{\log\beta/\log 4} \\ & \leq & (\frac{3}{4}d_{S^1}(x,x'))^{\log\beta/\log 4}. \end{array}$$
This shows that $H:\Sigma_2^+\to A$ is a homeomorphism.
Finally, it follows from a previous theorem together with the preceding proposition that $h(E_4\mid_A) = h(\sigma\mid_{\Sigma_2^+}) = \log 2$.

Example: A Quadratic Map

- Let $a > 4$.
  Consider the quadratic map $f : [0,1] \to \mathbb{R}$, defined by
  $$f(x) = ax(1-x).$$
  Let $X \subseteq [0,1]$ be the forward f-invariant set
  $$X = \bigcap_{n=0}^{\infty} f^{-n}[0,1].$$
  We also consider the restriction $f|_X : X \to X$.
  We define a function $H : \Sigma_2^+ \to X$ by
  $$H(i_1 i_2 \ldots) = \bigcap_{n=1}^{\infty} f^{-n+1} I_{i_n},$$
  where $I_1 = \left[0, \frac{1-\sqrt{1-4/a}}{2}\right]$ and $I_2 = \left[\frac{1+\sqrt{1-4/a}}{2}, 1\right]$.

Example: A Quadratic Map (Cont'd)

Claim: For any sufficiently large a, the map H is well defined.

I.e., the intersection in its definition contains exactly one point for each sequence $(i_1 i_2 \ldots) \in \Sigma_2^+$.

Let $a > 2 + \sqrt{5}$.

Set $\lambda = a\sqrt{1 - 4/a}$.

We have, for all $x \in I_1 \cup I_2$,

$$|f'(x)| = a|1 - 2x| \geq \lambda > 1.$$

Hence, each interval

$$I_{i_1 \cdots i_m} = \bigcap_{n=1}^{m} f^{-n+1} I_{i_n}$$

has length at most $\lambda^{-(m-1)}$.

So each intersection in the definition of H has exactly one point.

Example: A Quadratic Map (Cont'd)

- We know that $f^{-1}[0,1] = I_1 \cup I_2$.
  By definition $X = \bigcap_{n=0}^{\infty} f^{-n}[0,1]$.
  It follows that

$$X = \bigcap_{n=0}^{\infty} f^{-n}(I_1 \cup I_2) = \bigcup_{(i_1 i_2 \ldots) \in \Sigma_2^+} H(i_1 i_2 \ldots).$$

  So the map H is onto.
  It is also invertible.
  Given $x \in X$, let $i_n = j$ when $f^{n-1}(x) \in I_j$, for each $n \in \mathbb{N}$.
  Then its inverse given by

$$H^{-1}(x) = (i_1 i_2 \ldots).$$

Example: A Quadratic Map (Cont'd)

- We show that H is a homeomorphism.
  Let $\omega, \omega' \in \Sigma_2^+$ be distinct points, with $n = n(\omega, \omega') > 1$.
  We have
  $$|H(\omega) - H(\omega')| = a_{i_1 \ldots i_{n-1}},$$
  where $a_{i_1 \ldots i_{n-1}}$ is the length of the interval $I_{i_1 \ldots i_{n-1}}$.
  But $|f'(x)| > 1$.
  So
  $$|H(\omega) - H(\omega')| \leq \lambda^{-(n-2)} \to 0, \quad \text{when } n \to \infty.$$
  This shows that the map H is continuous.

Example: A Quadratic Map (Cont'd)

- Let $x, x' \in X$ be distinct points.
  There exists an $n \in \mathbb{N}$, such that
  - $I_{i_1 \ldots i_{n-1}} = I_{i'_1 \ldots i'_{n-1}}$;
  - $I_{i_1 \ldots i_n} \cap I_{i'_1 \ldots i'_n} = \emptyset$,

  where
  $$H^{-1}(x) = (i_1 i_2 \ldots) \quad \text{and} \quad H^{-1}(x') = (i'_1 i'_2 \ldots).$$
  Then
  $$d(H^{-1}(x), H^{-1}(x')) = \beta^{-n} \to 0 \quad \text{when } n \to \infty.$$
  It follows that $|x - x'| \geq \lambda^{-(n-1)}$.
  Thus, if $x' \to x$, then $n \to \infty$.
  This shows that the map H^{-1} is continuous.
  Since $H : \Sigma_2^+ \to X$ is a homeomorphism, it follows by a previous theorem together with the preceding proposition that
  $$h(f \mid_X) = h(\sigma \mid_{\Sigma_2^+}) = \log 2.$$

Example: The Smale Horseshoe

- Let $\Lambda \subseteq [0,1]^2$ be the Smale horseshoe constructed from a diffeomorphism f defined in an open neighborhood of $[0,1]^2$.

  We consider again the vertical strips

  $$V_1 = [0,a] \times [0,1] \quad \text{and} \quad V_2 = [1-a,1] \times [0,1].$$

  We define a function $H : \Sigma_2 \to \Lambda$ by

  $$H(\ldots i_{-1} i_0 i_1 \ldots) = \bigcap_{n \in \mathbb{Z}} f^{-n} V_{i_n}.$$

  We verify that H is well defined.

Example: The Smale Horseshoe (Cont'd)

- For every $\omega = (\ldots i_{-1} i_0 i_1 \ldots)$, consider the sets

$$R_n(\omega) = \bigcap_{k=-n}^{n} f^{-k} V_{i_k}.$$

Each $R_n(\omega)$ is contained in a square of size a^n.

Thus, $\operatorname{diam} R_n(\omega) \to 0$ when $n \to \infty$.

This implies that each intersection

$$\bigcap_{n \in \mathbb{Z}} f^{-n} V_{i_n} = \bigcap_{n \in \mathbb{Z}} R_n(\omega)$$

has at most one point.

But $R_n(\omega)$ is a decreasing sequence of nonempty closed sets.

So the intersection $\bigcap_{n \in \mathbb{N}} R_n(\omega)$ has at least one point.

This shows that $\operatorname{card} H(\omega) = 1$, for each $\omega \in \Sigma_2$.

Hence, the function H is well defined.

Example: The Smale Horseshoe (Cont'd)

- By the construction of the Smale horseshoe, we have

$$\Lambda = \bigcap_{n\in\mathbb{Z}} f^{-n}(V_1 \cup V_2) = \bigcup_{\omega\in\Sigma_2} \bigcap_{n\in\mathbb{Z}} f^{-n} V_{i_n} = \bigcup_{\omega\in\Sigma_2} H(\omega).$$

  Thus, the map H is onto.

  We show that it is also one-to-one.

  Consider sequences $\omega, \omega' \in \Sigma_2$, with $\omega \neq \omega'$.

  Then, there exists an $m \in \mathbb{Z}$, such that $i_m(\omega) \neq i_m(\omega')$.

  Thus, we also have $V_{i_m(\omega)} \cap V_{i_m(\omega')} = \emptyset$.

  Hence,

$$H(\omega) \cap H(\omega') = \left(\bigcap_{n\in\mathbb{Z}} f^{-n} V_{i_n(\omega)}\right) \cap \left(\bigcap_{n\in\mathbb{Z}} f^{-n} V_{i_n(\omega')}\right) = \emptyset.$$

  This shows that $H(\omega) \neq H(\omega')$ and the map H is one-to-one.

Example: The Smale Horseshoe (Conclusion)

- We also have

$$H(\sigma(\omega)) = \bigcap_{n\in\mathbb{Z}} f^{-n}V_{i_{n+1}(\omega)} = \bigcap_{n\in\mathbb{Z}} f^{1-n}V_{i_n(\omega)} = f(H(\omega)).$$

I.e., $H \circ \sigma = f \circ H$ in Σ_2.

Given $m \in \mathbb{N}$ and $\omega \in \Sigma_2$, we obtain

$$H(\sigma^m(\omega)) = f^m(H(\omega)).$$

This implies that ω is an m-periodic point of σ if and only if $H(\omega)$ is an m-periodic point of $f\mid_\Lambda$.

Moreover, ω is a periodic point of σ with period m if and only if $H(\omega)$ is a periodic point of $f\mid_\Lambda$ with period m.

In particular, it follows from a previous example that the number of m-periodic points of $f\mid_\Lambda$ is 2^m.

Subsection 3

Topological Markov Chains

Topological Markov Chains

- Let $k > 1$ be an integer.
- Let $A = (a_{ij})$ be a $k \times k$ matrix with entries $a_{ij} \in \{0, 1\}$.
- We consider the subset of Σ_k^+ defined by

$$\Sigma_A^+ = \{\omega \in \Sigma_k^+ : a_{i_n(\omega) i_{n+1}(\omega)} = 1, \text{ for } n \in \mathbb{N}\}.$$

- Clearly, $\sigma(\Sigma_A^+) \subseteq \Sigma_A^+$.

Definition

The restriction $\sigma |_{\Sigma_A^+}: \Sigma_A^+ \to \Sigma_A^+$ is called the **topological Markov chain** with **transition matrix** A.

- A topological Markov chain is also called **(sub)shift of finite type**.

Example

- Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

We have

$$\begin{aligned} \Sigma_A^+ &= \{\omega \in \Sigma_2^+ : a_{i_n(\omega) i_{n+1}(\omega)} = 1, \text{ for } n \in \mathbb{N}\} \\ &= \{\omega \in \Sigma_2^+ : (i_n(\omega), i_{n+1}(\omega)) \neq (1,1) \text{ for } n \in \mathbb{N}\}. \end{aligned}$$

In other words, Σ_A^+ is the subset of all sequences in Σ_2^+ in which the symbol 1, whenever it occurs, is always isolated.

Two-Sided Topological Markov Chains

- One can consider also the case of two-sided sequences.
- Let $k > 1$ be an integer.
- Let $A = (a_{ij})$ be a $k \times k$ matrix with entries $a_{ij} \in \{0, 1\}$.
- We consider the subset of Σ_k defined by

$$\Sigma_A = \{\omega \in \Sigma_k : a_{i_n(\omega) i_{n+1}(\omega)} = 1, \text{ for } n \in \mathbb{Z}\}.$$

- We have $\sigma(\Sigma_A) = \Sigma_A$.

Definition

The restriction $\sigma |_{\Sigma_A} : \Sigma_A \to \Sigma_A$ is called the (**two-sided**) **topological Markov chain** with **transition matrix** A.

Example

- Consider the matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

We have

$$\begin{aligned} \Sigma_A &= \{\omega \in \Sigma_2 : a_{i_n(\omega) i_{n+1}(\omega)} = 1, \text{ for } n \in \mathbb{Z}\} \\ &= \{\omega \in \Sigma_2 : i_n(\omega) \neq i_{n+1}(\omega), \text{ for } n \in \mathbb{Z}\}. \end{aligned}$$

Hence, the set Σ_A has exactly two sequences:

  - The first is $\omega_1 = (\ldots i_0 \ldots)$, where

$$i_n = \begin{cases} 1, & \text{if } n \text{ is even}, \\ 2, & \text{if } n \text{ is odd}; \end{cases}$$

Example (Cont'd)

- The second is $\omega_2 = (\ldots j_0 \ldots)$, where

$$j_n = \begin{cases} 2, & \text{if } n \text{ is even,} \\ 1, & \text{if } n \text{ is odd.} \end{cases}$$

We note that $\sigma(\omega_1) = \omega_2$ and $\sigma(\omega_2) = \omega_1$.

Thus, $\Sigma_A = \{\omega_1, \omega_2\}$ is a periodic orbit with period 2.

Example

- Let $\Sigma \subseteq \Sigma_2$ be the subset of all sequences in Σ_2 in which the symbol 1 occurs finitely many times and always in pairs (when it occurs).

  Clearly, $\sigma(\Sigma) = \Sigma$.

  So one can consider the restriction $\sigma\mid_\Sigma : \Sigma \to \Sigma$.

  Claim: $\sigma\mid_\Sigma$ is not a topological Markov chain.

  Consider the sequence

  $$\omega = (\ldots i_0 \ldots),$$

  with $i_0 = i_1 = 1$ and $i_j = 2$, for $j \notin \{0, 1\}$.

  We note that $\omega \in \Sigma$.

  If $\sigma\mid_\Sigma$ was a topological Markov chain, then we would have $\Sigma = \Sigma_2$.

  Indeed, the sequence ω contains the transitions

  $$1 \mapsto 1, \quad 1 \mapsto 2, \quad 2 \mapsto 1, \quad 2 \mapsto 2.$$

  However, $\Sigma \neq \Sigma_2$. So $\sigma\mid_\Sigma$ is not a topological Markov chain.

Example: Periodic Points

- Let $\sigma \mid_{\Sigma_A^+}$ be the topological Markov chain with the transition matrix

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$

We compute the number of m-periodic points for $m = 1, 2$.

Consider, first, the case $m = 1$.

We have $a_{11} = 0$.

So the sequence $(22\ldots)$ is the only fixed point of $\sigma \mid_{\Sigma_A^+}$.

Example: Periodic Points (Cont'd)

- Now consider the case $m = 2$.

  We note that a point $\omega \in \Sigma_A^+$ is m-periodic if and only if

  $$i_{n+m}(\omega) = i_n(\omega), \quad \text{for } n \in \mathbb{N}.$$

  We have to find the number of sequences in Σ_A^+, with this property. This coincides with the number of vectors $(i, j) \in \{1, 2\}^2$, such that

  the transitions $i \to j \to i$ are allowed.

  This condition is equivalent to $a_{ij} = a_{ji} = 1$.

  Thus, the number of 2-periodic points of $\sigma \mid_{\Sigma_A^+}$ is equal to

  $$\sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}a_{ji} = \sum_{i=1}^{2}(A^2)_{ii} = \operatorname{tr}(A^2),$$

  where $(A^2)_{ii}$ is the entry (i, i) of the matrix A^2.

m-Periodic Points of Markov Chains

Proposition

For each $m \in \mathbb{N}$, the number of m-periodic points of the topological Markov chain $\sigma |_{\Sigma_A^+}$ is equal to $\text{tr}(A^m)$.

- $\omega \in \Sigma_A^+$ is m-periodic iff $i_{n+m}(\omega) = i_n(\omega)$, for $n \in \mathbb{N}$.
  We have to find the number of sequences in Σ_A^+ with this property.
  This is the number of vectors $(i_1, \ldots, i_m) \in \{1, \ldots, k\}^m$, such that

  the transitions $i_1 \to i_2 \to \cdots \to i_m \to i_1$ are allowed.

  This condition is equivalent to $a_{i_1 i_2} = a_{i_2 i_3} = \cdots = a_{i_{m-1} i_m} = a_{i_m i_1} = 1$.
  Thus, the number of m-periodic points of $\sigma |_{\Sigma_A^+}$ is equal to

$$\sum_{(i_1, \ldots, i_m) \in \{1, \ldots, k\}^m} a_{i_1 i_2} a_{i_2 i_3} \cdots a_{i_m i_1} = \sum_{i_1 \in \{1, \ldots, k\}} (A^m)_{i_1 i_1} = \text{tr}(A^m).$$

Example

- Let
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}.$$
By the proposition, for each $m \in \mathbb{N}$, the number of m-periodic points of $\sigma\mid_{\Sigma_A^+}$ is equal to $\mathrm{tr}(A^m)$.
Using diagonalization, we have
$$A = S \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} S^{-1}, \ S = \begin{pmatrix} \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}.$$

Example (Cont'd)

- We found

$$A = S \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix} S^{-1}, \ S = \begin{pmatrix} \frac{-1+\sqrt{5}}{2} & \frac{-1-\sqrt{5}}{2} \\ 1 & 1 \end{pmatrix}.$$

It follows that

$$A^m = S \begin{pmatrix} \frac{1+\sqrt{5}}{2} & 0 \\ 0 & \frac{1-\sqrt{5}}{2} \end{pmatrix}^m S^{-1}.$$

Hence,

$$\operatorname{tr}(A^m) = \left(\frac{1+\sqrt{5}}{2}\right)^m + \left(\frac{1-\sqrt{5}}{2}\right)^m.$$

This is the number of m-periodic points of $\sigma \mid_{\Sigma_A^+}$.

- Incidentally, this shows that this number is an integer.

Topological Entropy of a Topological Markov Chain

Theorem

We have $h(\sigma\mid_{\Sigma_A^+}) = \log\rho(A)$, where $\rho(A)$ is the spectral radius of A.

- The map $\sigma\mid_{\Sigma_k^+}$ is expansive.
  So the same happens to the topological Markov chain $\sigma\mid_{\Sigma_A^+}$.
  Thus, we can apply, for any sufficiently small $\alpha > 0$,
  $$h(f) = \lim_{n\to\infty}\frac{1}{n}\log N(n,\alpha).$$
  Let $m, p \in \mathbb{N}$ and $\omega, \omega' \in \Sigma_k^+$.
  We know that
  $$d_m(\omega,\omega') \geq \beta^{-p} \quad \text{iff} \quad n \leq p+m-1.$$
  So we have
  $$d_m(\omega,\omega') \geq \beta^{-p} \quad \text{iff} \quad n = n(\omega,\omega') \leq p+m-1.$$

Topological Entropy (Cont'd)

- Hence, for $q = p + m - 1$,

$$N(m, \beta^{-p}) \leq \sum_{(i_1,\ldots,i_q)\in\{1,\ldots,k\}^q} a_{i_1 i_2} \cdots a_{i_{q-1} i_q} = \sum_{i_1=1}^{k} \sum_{i_q=1}^{k} (A^{q-1})_{i_1 i_q}.$$

Using the Jordan form of A, we conclude that there exists a polynomial $c(q)$, such that

$$\sum_{i_1=1}^{k} \sum_{i_q=1}^{k} (A^{q-1})_{i_1 i_q} \leq c(q)\rho(A)^{q-1}.$$

Now we have

$$\begin{array}{rcl} h(\sigma \mid_{\Sigma_A^+}) & = & \lim_{m\to\infty} \frac{1}{m} \log N(m, \beta^{-p}) \\ & \leq & \lim_{m\to\infty} \frac{1}{m} \log \left[c(q)\rho(A)^{p+m-2}\right] \\ & = & \log \rho(A). \end{array}$$

Topological Entropy (Cont'd)

- On the other hand, by the preceding proposition, the number of q-periodic points of $\sigma\mid_{\Sigma_A^+}$ is equal to $\mathrm{tr}(A^q)$.

  But we know that, if ω and ω' are two of these points, then
  $$d_m(\omega,\omega') = \max\{d(\sigma^j(\omega),\sigma^j(\omega')) : j = 0,\ldots,m-1\} \geq \beta^{-p}.$$
  Hence,
  $$N(m,\beta^{-p}) \geq \mathrm{tr}(A^q).$$
  It follows by a previous theorem that
  $$\begin{aligned} h(\sigma\mid_{\Sigma_A^+}) &= \lim_{m\to\infty}\tfrac{1}{m}\log N(m,\beta^{-p}) \\ &\geq \lim_{m\to\infty}\tfrac{1}{m}\log\mathrm{tr}(A^{p+m-1}) \\ &= \lim_{m\to\infty}\tfrac{1}{m}\log\mathrm{tr}(A^m). \end{aligned}$$

Topological Entropy (Conclusion)

- Now let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of A, counted with their multiplicities.

  We have

  $$\operatorname{tr}(A^m) = \sum_{i=1}^{k} \lambda_i^m.$$

  So we obtain

  $$\begin{array}{rcl} h(\sigma \mid_{\Sigma_A^+}) & \geq & \lim_{m\to\infty} \frac{1}{m} \log \sum_{i=1}^{k} \lambda_i^m \\ & = & \log \lim_{m\to\infty} (\mid \sum_{i=1}^{k} \lambda_i^m \mid^{1/m}) \\ & = & \log \max \{ |\lambda_i| : i = 1, \ldots, k \} \\ & = & \log \rho(A). \end{array}$$

Irreducible and Transitive Matrices

Definition

A $k \times k$ matrix A is called:

1. **Irreducible** if, for each $i, j \in \{1, \ldots, k\}$, there exists an $m = m(i, j) \in \mathbb{N}$, such that the (i, j)-th entry of A^m is positive;
2. **Transitive** if, there exists an $m \in \mathbb{N}$, such that all entries of the matrix A^m are positive.

- Clearly, any transitive matrix is irreducible.
- However, an irreducible matrix may not be transitive.

Example

- Let
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
No power of A has all entries positive.

  So A is not transitive.

  However, $A^2 = \text{Id}$.

  Thus, for each pair (i, j), either A or A^2 has positive (i, j)-th entry.

  Hence, the matrix A is irreducible.

Irreducibility and Topological Transitivity

Proposition

If the matrix A is irreducible, then the topological Markov chain $\sigma\mid_{\Sigma_A^+}$ is topologically transitive.

- Consider the sets

$$\begin{aligned} D_{j_1\dots j_n} &= C_{j_1\dots j_n} \cap \Sigma_A^+ \\ &= \{\omega \in \Sigma_A^+ : i_m(\omega) = j_m, \text{ for } m = 1, \dots, n\}. \end{aligned}$$

  They generate the (induced) topology of Σ_A^+.

  Hence, it is sufficient to consider only these sets in the definition of topological transitivity.

  Take two nonempty sets $D_{j_1\dots j_n}, D_{k_1\dots k_n} \subseteq \Sigma_A^+$.

  We must find $m \in \mathbb{N}$, such that $\sigma^{-m} D_{j_1\dots j_n} \cap D_{k_1\dots k_n} \neq \emptyset$.

Irreducibility and Topological Transitivity (Cont'd)

- We first verify that there exists an $m \geq n$, such that the (k_n, j_1)-th entry of the matrix A^{m-n+1} is positive.

  By hypothesis, the matrix A is irreducible. So, there exist positive integers m_1 and m_2, such that $(A^{m_1})_{k_n j_1} > 0$ and $(A^{m_2})_{j_1 k_n} > 0$.

  Then, for $\ell \in \mathbb{N}$,

$$\begin{array}{rcl} (A^{(m_1+m_2)\ell+m_1})_{k_n j_1} & = & \sum_{p=1}^{k} (A^{(m_1+m_2)\ell})_{k_n p} (A^{m_1})_{p j_1} \\ & \geq & (A^{(m_1+m_2)\ell})_{k_n k_n} (A^{m_1})_{k_n j_1} \\ & \geq & (A^{m_1+m_2})_{k_n k_n}^{\ell} (A^{m_1})_{k_n j_1} \\ & \geq & (A^{m_1})_{k_n j_1}^{\ell} (A^{m_2})_{j_1 k_n}^{\ell} (A^{m_1})_{k_n j_1} > 0. \end{array}$$

  This shows that there exists a transition from k_n to j_1 in $q = (m_1 + m_2)\ell + m_1$ steps.

  Taking $m = q + n - 1$, we obtain the desired result.

Irreducibility and Topological Transitivity (Conclusion)

- Hence, given a sequence $(i_1 i_2 \ldots) \in D_{j_1 \ldots j_n}$, there exist $\ell_1, \ldots, \ell_{m-n} \in \{1, \ldots, k\}$, such that

$$\omega = (k_1 \ldots k_n \ell_1 \ldots \ell_{m-n} i_1 i_2 \ldots) \in \Sigma_A^+.$$

We note that $\omega \in D_{k_1 \ldots k_n}$ and that

$$\sigma^m(\omega) = (i_1 i_2 \ldots) \in D_{j_1 \ldots j_n}.$$

Therefore,

$$\omega \in \sigma^{-m} D_{j_1 \ldots j_n} \cap D_{k_1 \ldots k_n} \neq \emptyset.$$

This shows that the topological Markov chain $\sigma \mid_{\Sigma_A^+}$ is topologically transitive.

Transitivity and Powers

Lemma

Suppose that the matrix A is transitive. If all entries of the matrix A^m are positive, then for each $p \geq m$, all entries of the matrix A^p are positive.

- For each $j \in \{1, \ldots, k\}$, there exists an $r = r(j) \in \{1, \ldots, k\}$, such that $a_{rj} = 1$.

  Otherwise, $(A^p)_{ij} = 0$, for any $p \in \mathbb{N}$ and $i \in \{1, \ldots, k\}$.

  Thus, the matrix A would not be transitive.

  Now we use induction on p.

  Suppose, for some $p \geq m$, A^p has only positive entries.

  Then

  $$(A^{p+1})_{ij} = \sum_{\ell=1}^{k} (A^p)_{i\ell} a_{\ell j} \geq (A^p)_{ir} a_{rj} > 0.$$

  This completes the proof.

Transitivity and Topological Mixing

Proposition

Assume the matrix A is transitive. Then the topological Markov chain $\sigma\mid_{\Sigma_A^+}$ is topologically mixing.

- Suppose $D_{j_1\ldots j_n}, D_{k_1\ldots k_n} \subseteq \Sigma_A^+$ are nonempty sets.
  We show there exists $q \in \mathbb{N}$, such that, for all $p \geq q$,

$$\sigma^{-p} D_{j_1\ldots j_n} \cap D_{k_1\ldots k_n} \neq \emptyset.$$

  By the lemma, for each $p \in \mathbb{N}$, with $p \geq m+n-1$, given nonempty $D_{j_1\ldots j_n}, D_{k_1\ldots k_n} \subseteq \Sigma_A^+$, there exist $\ell_1, \ldots, \ell_{p-n} \in \{1, \ldots, k\}$, such that, for any sequence $(i_1 i_2 \ldots) \in D_{j_1\ldots j_n}$,

$$\omega = (k_1 \ldots k_n \ell_1 \ldots \ell_{p-n} i_1 i_2 \ldots) \in \Sigma_A^+.$$

  Therefore, $\omega \in \sigma^{-p} D_{j_1\ldots j_n} \cap D_{k_1\ldots k_n} \neq \emptyset$, for $p \geq m+n-1$.
  So the topological Markov chain $\sigma\mid_{\Sigma_A^+}$ is topologically mixing.

A Version of the Perron-Frobenius Theorem

Theorem

Any square matrix, with all entries in $\mathbb{N}$, has a real eigenvalue > 1.

- Consider the set

$$S = \{v \in (\mathbb{R}_0^+)^k : \|v\| = 1\},$$

where $v = (v_1, \ldots, v_k)$ and $\|v\| = \sum_{i=1}^k |v_i|$.

Let B be a $k \times k$ matrix with all entries b_{ij} in $\mathbb{N}$.

We define a function $F : S \to S$ by

$$F(v) = \frac{Bv}{\|Bv\|}.$$

The set S is homeomorphic to the closed unit ball of $\mathbb{R}^{k-1}$.

Moreover, the function F is continuous.

A Version of the Perron-Frobenius Theorem (Cont'd)

- So, by Brouwer's Fixed Point Theorem, F has a fixed point $v \in S$.
  Hence, $Bv = \|Bv\| v$.
  So v is an eigenvector of B associated to the real eigenvalue

$$\begin{array}{rcl} \lambda & = & \|Bv\| \\ & = & \sum_{i=1}^{k} (Bv)_i \\ & = & \sum_{i=1}^{k} \sum_{j=1}^{k} b_{ij} v_j \\ & \geq & \sum_{i=1}^{k} \sum_{j=1}^{k} v_j \\ & = & k \sum_{j=1}^{k} v_j \\ & = & k \\ & > & 1. \end{array}$$

Transitivity and Topological Entropy

Proposition

If the matrix A is transitive, then $h(\sigma \mid_{\Sigma_A^+}) > 0$.

- Take $m \in \mathbb{N}$, such that A^m has only positive entries.

  By the preceding theorem, A^m has a real eigenvalue $\lambda > 1$.

  Hence, by a previous theorem,

$$\begin{array}{rcl} h(\sigma \mid_{\Sigma_A^+}) & = & \log \rho(A) \\ & = & \frac{1}{m} \log \rho(A^m) \\ & \geq & \frac{1}{m} \log \lambda \\ & > & 0. \end{array}$$

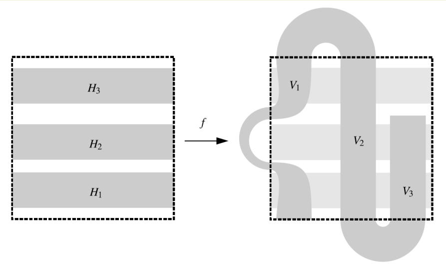
  This completes the proof of the proposition.

Subsection 4

Horseshoes and Topological Markov Chains

Example

- Let f be a diffeomorphism in an open neighborhood of the square $[0,1]^2$ with the behavior shown in the figure.



We can choose the sizes of H_i and of $V_i = f(H_i)$, for $i = 1, 2, 3$, as well as the diffeomorphism, so that

$$\Lambda = \bigcap_{n \in \mathbb{Z}} f^n(H_1 \cup H_2 \cup H_3)$$

is a hyperbolic set for f.

Example (Cont'd)

- Now we consider the 3×3 matrix $A = (a_{ij})$ with entries

$$a_{ij} = \begin{cases} 1, & \text{if } f(H_i) \cap H_j \neq \emptyset, \\ 0, & \text{if } f(H_i) \cap H_j = \emptyset. \end{cases}$$

  This is the matrix

$$A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

  We also consider the set $\Sigma_A \subseteq \Sigma_3$ induced by this matrix.

  We define

$$H(\omega) = \bigcap_{n \in \mathbb{Z}} f^{-n} H_{i_n}(\omega).$$

Example (Cont'd)

Proposition

The function $H : \Sigma_A \to \Lambda$ is well defined and

$$f \circ H = H \circ \sigma \quad \text{in } \Sigma_A.$$

- As in a previous example, $\text{card} H(\omega) \leq 1$ for $\omega \in \Sigma_A$.

  Now we show that $\text{card} H(\omega) \geq 1$ for $\omega \in \Sigma_A$.

  We first note that the following *Markov property* holds:

  1. If $f(H_i) \cap H_j \neq \emptyset$, then the image $f(H_i)$ intersects H_j along the whole unstable direction;
  2. If $f^{-1}(H_i) \cap H_j \neq \emptyset$, then the preimage $f^{-1}(H_i)$ intersects H_j along the whole stable direction.

Example (Cont'd)

- Let H_i, H_j and H_k be rectangles such that

$$f(H_i) \cap H_j \neq \emptyset \quad \text{and} \quad f(H_j) \cap H_k \neq \emptyset.$$

By the Markov property, we conclude that $f(H_i)$ intersects H_j along the whole unstable direction.
Thus, $f^2(H_i)$ also intersects $f(H_j)$ along the whole unstable direction.
But $f(H_j)$ intersects H_k along the whole unstable direction.
This implies that $f^2(H_i) \cap f(H_j) \cap H_k \neq \emptyset$.
Now take $\omega \in \Sigma_A$. By the definition of A, for each $n \in \mathbb{Z}$,

$$f(H_{i_n(\omega)}) \cap H_{i_{n+1}(\omega)} \neq \emptyset.$$

By induction, it follows that

$$\bigcap_{k=-n}^{n} f^{n-k}(H_{i_k(\omega)}) \neq \emptyset \quad \text{and} \quad K_n := \bigcap_{k=-n}^{n} f^{-k}(H_{i_k(\omega)}) \neq \emptyset.$$

Example (Cont'd)

- The sets K_n are closed and nonempty.

  So the intersection $H(\omega) = \bigcap_{n\in\mathbb{N}} K_n$ is also nonempty and

  $$\text{card} H(\omega) = \text{card} \bigcap_{n\in\mathbb{N}} K_n \geq 1.$$

  We conclude that the function H is well defined.

  To get $f \circ H = H \circ \sigma$ in Σ_A, we note that

  $$\begin{aligned} H(\sigma(\omega)) &= \bigcap_{n\in\mathbb{Z}} f^{-n}(H_{i_{n+1}(\omega)}) \\ &= \bigcap_{n\in\mathbb{Z}} f^{1-n}(H_{i_n(\omega)}) \\ &= f(H(\omega)). \end{aligned}$$

Subsection 5

Zeta Functions

The Zeta Function of a Map

Definition

Given a map $f : X \to X$, with

$$a_n := \text{card}\{x \in X : f^n(x) = x\} < \infty,$$

for each $n \in \mathbb{N}$, its **zeta function** is defined by

$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{a_n z^n}{n},$$

for each $z \in \mathbb{C}$ such that the series converges.

Convergence of the Zeta Function

- We recall that the radius of convergence of the power series is given by
$$R = \left(\limsup_{n \to \infty} \sqrt[n]{\frac{a_n}{n}} \right)^{-1} = \left(\limsup_{n \to \infty} \sqrt[n]{a_n} \right)^{-1}.$$
- In particular, the series converges for $|z| < R$.
- The function ζ is holomorphic on the ball $B(0, R) \subseteq \mathbb{C}$.
- ζ is uniquely determined by $(a_n)_{n \in \mathbb{N}}$ and vice versa.

Example

- Let $\sigma |_{\Sigma_A^+} : \Sigma_A^+ \to \Sigma_A^+$ be a topological Markov chain defined by a $k \times k$ matrix A with spectral radius $\rho(A) > 0$.
  By a previous proposition that the sequence $(a_n)_{n \in \mathbb{N}}$ is
  $$a_n = \mathrm{tr}(A^n).$$
  Let $\lambda_1, \ldots, \lambda_k$ be the eigenvalues of A, with multiplicities.
  We have
  $$a_n = \mathrm{tr}(A^n) = \sum_{i=1}^{k} \lambda_i^n.$$
  Let log be the principal branch of the logarithm.
  Recall that
  $$\log(1 + w) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} w^n, \quad \text{for } |w| < 1.$$

Example (Cont'd)

- Now we have

$$\begin{aligned}\zeta(z) &= \exp \sum_{i=1}^{k} \sum_{n=1}^{\infty} \frac{\lambda_i^n z^n}{n} \\ &= \exp \sum_{i=1}^{k} -\log(1-\lambda_i z) \\ &= \exp \sum_{i=1}^{k} \log \frac{1}{1-\lambda_i z} \\ &= \prod_{i=1}^{k} \frac{1}{1-\lambda_i z}.\end{aligned}$$

On the other hand, the complex numbers $1-\lambda_i z$ are the eigenvalues of the matrix $\mathrm{Id} - zA$, counted with their multiplicities.

Thus, for $|z| < \min\left\{\frac{1}{|\lambda_i|} : i = 1, \ldots, k\right\} = \frac{1}{\rho(A)}$,

$$\zeta(z) = \frac{1}{\det(\mathrm{Id} - zA)}.$$

Example(Cont'd)

- The shift map
$$\sigma : \Sigma_k^+ \to \Sigma_k^+$$
coincides with the topological Markov chain defined by the $k \times k$ matrix $A = A_k$ with all entries equal to 1.
It follows from $\zeta(z) = \frac{1}{\det(\mathrm{Id} - zA)}$ that, for $|z| < \frac{1}{\rho(A_k)} = \frac{1}{k}$,
$$\zeta(z) = \frac{1}{\det(\mathrm{Id} - zA_k)}.$$

Example(Cont'd)

- Subtracting the first row of $\mathrm{Id} - zA_k$ from the other rows and then expanding the determinant along the second column, we obtain

$$\begin{aligned}
&\det(\mathrm{Id} - zA_k) \\
&= \det \begin{pmatrix} 1-z & -z & \cdots & -z \\ -1 & & & \\ \vdots & & \mathrm{Id} & \\ -1 & & & \end{pmatrix} \\
&= z\det \begin{pmatrix} -1 & 0 & \cdots & 0 \\ -1 & & & \\ \vdots & & \mathrm{Id} & \\ -1 & & & \end{pmatrix} + \det \begin{pmatrix} 1-z & -z & \cdots & -z \\ -1 & & & \\ \vdots & & \mathrm{Id} & \\ -1 & & & \end{pmatrix} \\
&= -z + \det(\mathrm{Id} - zA_{k-1}).
\end{aligned}$$

But $\det(\mathrm{Id} - zA_1) = 1 - z$. By induction, $\det(\mathrm{Id} - zA_k) = 1 - kz$.

Thus, $\zeta(z) = \frac{1}{1-kz}$, for $|z| < \frac{1}{k}$.

Example (Cont'd)

- Alternatively, the number of n-periodic points of $\sigma \mid_{\Sigma_k^+}$ is k^n. Thus,

$$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{k^n z^n}{n}.$$

Now, for $|z| < \frac{1}{k}$,

$$\left(\sum_{n=1}^{\infty} \frac{k^n z^n}{n}\right)' = \sum_{n=1}^{\infty} k^n z^{n-1} = \frac{k}{1-kz}.$$

We conclude that, for $|z| < \frac{1}{k}$,

$$\zeta(z) = \exp\left[-\log(1-kz)\right] = \frac{1}{1-kz}.$$

Example

- Now we consider the expanding map $E_2 : S^1 \to S^1$.
  We know that the number of n-periodic points of E_2 is $2^n - 1$.
  Hence,
  $$\zeta(z) = \exp \sum_{n=1}^{\infty} \frac{(2^n - 1)z^n}{n}.$$
  We have, for $|z| < \frac{1}{2}$,
  $$\left(\sum_{n=1}^{\infty} \frac{(2^n - 1)z^n}{n} \right)' = \sum_{n=1}^{\infty} (2^n - 1)z^{n-1} = \frac{2}{1 - 2z} - \frac{1}{1 - z}.$$
  So we obtain, for $|z| < \frac{1}{2}$,
  $$\zeta(z) = \exp\left[- \log(1 - 2z) + \log(1 - z)\right] = \frac{1 - z}{1 - 2z}.$$